limits.pdf

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1 Introduction: Two common limits

Let $y_1 = 12\left(\frac{x-\sin(x)}{x^3}\right)$. When taking the limit as x approaches zero, we have:

$$\lim_{x\to 0} 12\left(\frac{x-\sin(x)}{x^3}\right) = 12\left(\frac{0-\sin(0)}{0^3}\right) = 12\left(\frac{0}{0}\right)$$

This is an indeterminate form (0/0), therefore we can use L'Hôpital's Rule to potentially find the limit.

Taking the derivative of the numerator and the denominator, we get the first derivative y_1':

$$y_1' = 12\left(\frac{\frac{d}{dx}(x-\sin(x))}{\frac{d}{dx}(x^3)}\right) = 12\left(\frac{1-\cos(x)}{3x^2}\right)$$

Evaluating the limit of y_1' as $x \to 0$:

$$\lim_{x\to 0} 12\left(\frac{1-\cos(x)}{3x^2}\right) = 12\left(\frac{1-\cos(0)}{3(0)^2}\right) = 12\left(\frac{1-1}{0}\right) = 12\left(\frac{0}{0}\right)$$

This is still an indeterminate form. We must iterate L'Hôpital's Rule again.

Taking the derivative once more, we find the second derivative y_1'':

$$y_1'' = 12\left(\frac{\frac{d}{dx}(1-\cos(x))}{\frac{d}{dx}(3x^2)}\right) = 12\left(\frac{\sin(x)}{6x}\right)$$

Evaluating the limit of y_1'' as $x \to 0$:

$$\lim_{x\to 0} 12\left(\frac{\sin(x)}{6x}\right) = 12\left(\frac{\sin(0)}{6(0)}\right) = 12\left(\frac{0}{0}\right)$$

This is still an indeterminate form. We apply L'Hôpital's Rule for a third time.

Taking the derivative again, we obtain the third derivative y_1''':

$$y_1''' = 12\left(\frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(6x)}\right) = 12\left(\frac{\cos(x)}{6}\right)$$

Now, evaluating the limit of y_1''' as $x \to 0$:

$$\lim_{x\to 0} 12\left(\frac{\cos(x)}{6}\right) = 12\left(\frac{\cos(0)}{6}\right) = 12\left(\frac{1}{6}\right) = 2$$

Since this form is no longer indeterminate and the function is continuous at this point, we may evaluate y_1''' at $x = 0$ to obtain the limit. Therefore, the limit of the original function as x approaches zero is 2.

For the second question, $y_2 = \frac{e^x - e^{-x} - 2}{x - \sin(x)}$ is an indeterminate form of $-2/0$ but does not fall into the category of indeterminates for L'Hôpital's rule. We can easily use algebra on power series of each function to simplify the numerator and denominator of y_2. Since the power series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and e^{-x} is $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$, where both series begin at $n = 0$ until infinity, the first term is $2x$ when these two series are summed, plus more terms with x. We can rewrite the series in the numerator and write the numerator of the problem as $2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} - 2$. Similarly, in the denominator, writing the $\sin(x)$ function as its power series representation then subtracting from x, the series in the denominator becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x)^{2n+1}}{(2n+1)!}$. Now, as x approaches zero from the left, the numerator becomes -2 since all the x terms will become zero. Meanwhile, the denominator is very close to zero and a negative number. Thus, as x approaches zero from the left, the limit tends towards positive infinity, whereas as x approaches zero from the right, y approaches negative infinity using the same logic. We conclude that the limit of y_2 does not exist as x approaches zero. Q.E.D.