Algebra

Notes

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Contents

1 Introduction

This document contains concise notes and worked examples.

2 Algebra Basics

2.1 Constants

A **constant** is a fixed value that does not change. Unlike a **variable**, which can represent different numbers, a **constant** always has the same value.

Examples of **constant** include:

- Specific numbers like 2, -7, or $\frac{1}{2}$
- Mathematical constants like $\pi \approx 3.1416$ or $e \approx 2.718$

In the equation:

$$x + 3 = 7$$

the number 3 and 7 are constants — they stay the same, while x is the variable we solve for.

2.2 Variables

A **variable** is a symbol, usually a letter like x, y, or z, that represents an unknown or changeable value.

For example, in the equation:

$$x + 3 = 7$$

the variable x represents a number. Solving the equation means finding the value of x that makes the equation true in this case, $x = 4$.

Variables are fundamental in algebra because they allow us to generalize problems and create formulas.

2.3 Coefficients

A **coefficient** is the numerical factor multiplied by a variable in an algebraic expression.

For example, in the term:

$$5x$$

the number 5 is the coefficient of the variable x. It tells us how many times x is being counted or scaled.

More examples:

- In $-3y$, the coefficient is -3
- In $\frac{1}{2}a$, the coefficient is $\frac{1}{2}$
- In z, the coefficient is implicitly 1, since $z = 1 \cdot z$

Coefficients help determine the slope of a line in linear equations and play a major role in simplifying and solving expressions.

3 Equations vs. Expressions

Understanding the difference between expressions and equations is essential in algebra.

3.1 Expressions

An **expression** is a combination of numbers, variables, and operations (like addition or multiplication), but it does **not** contain an equals sign.

Examples:

$$2x + 5, \quad 3a^2 - 4, \quad \frac{1}{2}y$$

Expressions represent a value, but not a complete statement to solve. You can simplify or evaluate expressions, but you cannot "solve" them unless they're part of an equation.

3.2 Equations

An **equation** is a mathematical statement that two expressions are equal. It always contains an equals sign (=) and usually involves finding the value of a variable that makes the equation true.

Examples:

$$2x + 5 = 11, \quad a^2 = 16, \quad \frac{1}{2}y = 3$$

Solving an **equation** means determining the value(s) of the variable(s) that make both sides equal.

Summary Table

Expression	**Equation**
No equals sign	Has an equals sign
Represents a value	Represents a relationship
Can be simplified or evaluated	Can be solved
Example: $3x + 2$	Example: $3x + 2 = 11$

4 The Associative Property

The **associative property** refers to the grouping of terms using parentheses in addition or multiplication. It tells us that the way numbers are grouped does not change the result — only the order of operations inside parentheses changes, not the outcome.

4.1 Associative Property of Addition

The associative property of addition states:

$$(a + b) + c = a + (b + c)$$

You can add numbers in any grouping, and the sum will stay the same.

Example:

$$(2 + 3) + 4 = 5 + 4 = 9$$

$$2 + (3 + 4) = 2 + 7 = 9$$

So, $(2 + 3) + 4 = 2 + (3 + 4)$.

4.2 Associative Property of Multiplication

The associative property of multiplication states:

$$(a \times b) \times c = a \times (b \times c)$$

You can multiply in any grouping, and the product remains unchanged.

Example:

$$(2 \times 3) \times 4 = 6 \times 4 = 24$$
$$2 \times (3 \times 4) = 2 \times 12 = 24$$

So, $(2 \times 3) \times 4 = 2 \times (3 \times 4)$.

5 The Commutative Property

The **commutative property** describes how the order of numbers does not affect the result when adding or multiplying. It applies only to **addition and multiplication** — not subtraction or division.

5.1 Commutative Property of Addition

The commutative property of addition states:

$$a + b = b + a$$

You can change the order of the numbers being added without changing the sum.

Example:

$$4 + 7 = 11 \quad \text{and} \quad 7 + 4 = 11$$

So, $4 + 7 = 7 + 4$

5.2 Commutative Property of Multiplication

The commutative property of multiplication states:

$$a \times b = b \times a$$

You can change the order of the numbers being multiplied without changing the product.

Example:

$$6 \times 5 = 30 \quad \text{and} \quad 5 \times 6 = 30$$

So, $6 \times 5 = 5 \times 6$

6 Like Terms

Like terms are terms that have the same variable(s) raised to the same power(s). Only the numerical coefficients can be different. Like terms can be combined using addition or subtraction.

6.1 Addition and Subtraction of Like Terms

To combine like terms, simply add or subtract their coefficients.

Example 1:

$$3x + 5x = (3 + 5)x = 8x$$

Example 2:

$$7a^2 - 2a^2 = 5a^2$$

Note: You *cannot* combine terms that are not like terms.

$$4x + 2x^2 \neq 6x^2 \quad \text{(not like terms)}$$

6.2 Multiplication and Division of Like Terms

When multiplying or dividing like terms, you combine coefficients and apply exponent rules to the variables.

Multiplication Example:

$$(3x)(2x) = 6x^2$$

$$(4a^2)(-2a^3) = -8a^5$$

Division Example:

$$\frac{10x^3}{2x} = 5x^2$$

$$\frac{-6y^4}{3y^2} = -2y^2$$

Key Idea

- **Addition/Subtraction:** Combine only like terms (same variables and exponents)
- **Multiplication/Division:** Use exponent rules, even for unlike terms

7 The Distributive Property

The **distributive property** connects multiplication and addition or subtraction. It allows you to multiply a number or variable by each term inside parentheses.

$$a(b + c) = ab + ac \quad \text{and} \quad a(b - c) = ab - ac$$

This property is used frequently in algebra to expand expressions and solve equations.

Examples

Example 1 (with numbers):

$$3(4+5) = 3 \cdot 9 = 27$$

$$3 \cdot 4 + 3 \cdot 5 = 12 + 15 = 27$$

Example 2 (with variables):

$$x(2+y) = 2x + xy$$

Example 3 (with subtraction):

$$5(a-3) = 5a - 15$$

Why It Matters

The distributive property helps you:

- Expand expressions like $2(x+3)$
- Simplify algebraic expressions
- Solve equations more efficiently
- Factor expressions in reverse

More on the Distributive Property

The distributive property is also essential when working with variables, negative numbers, and factoring expressions.

Example with Variables and Negatives

$$-2(x-4) = -2 \cdot x + (-2) \cdot (-4) = -2x + 8$$

Notice:

- The negative sign distributes to both terms
- Be careful with signs: $-2 \cdot -4 = +8$

Common Mistake to Avoid

Incorrect:

$$3(x+2) = 3x + 2 \quad \text{(Only distributed to } x\text{)}$$

Correct:

$$3(x+2) = 3x + 6$$

Always distribute to *every* term inside the parentheses.

Using the Distributive Property to Factor

The distributive property also works *in reverse*, which is how we factor expressions.

$$6x + 12 = 6(x + 2)$$

Here, we pulled out the common factor of 6 — essentially undoing the distribution.

Factoring is the process of writing an expression as a product using the distributive property in reverse.

Tip: Look for a greatest common factor (GCF) before factoring!

8 Polynomials

A **polynomial** is an expression made up of variables, constants, and exponents, combined using addition, subtraction, and multiplication — but no variables in the denominator or under radicals.

Examples of Polynomials

$$3x^2 + 2x - 5, \quad x^3 - 4x + 7, \quad 2a^2b + 3ab^2$$

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8.1 Addition and Subtraction of Polynomials

To add or subtract polynomials:

- Combine like terms (same variables raised to the same powers)
- Add/subtract the coefficients of like terms

Example 1 — Addition:

$$(2x^2 + 3x + 1) + (x^2 + 4x - 5)$$

$$= (2x^2 + x^2) + (3x + 4x) + (1 - 5) = 3x^2 + 7x - 4$$

Example 2 — Subtraction:

$$(5x^2 - 2x + 6) - (3x^2 + x - 4)$$

$$= (5x^2 - 3x^2) + (-2x - x) + (6 + 4) = 2x^2 - 3x + 10$$

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8.2 Multiplication of Polynomials

To multiply polynomials:

- Use the distributive property (FOIL for binomials)
- Multiply each term in the first polynomial by each term in the second
- Combine like terms

Example:

$$(x+2)(x+5) = x(x+5) + 2(x+5) = x^2 + 5x + 2x + 10 = x^2 + 7x + 10$$

Another Example:

$$(2x-3)(x^2+x-4) = 2x(x^2+x-4) - 3(x^2+x-4)$$

$$= 2x^3 + 2x^2 - 8x - 3x^2 - 3x + 12 = 2x^3 - x^2 - 11x + 12$$

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8.3 Division of Polynomials (Intro)

Dividing polynomials can be done using:

- Long division
- Synthetic division (when dividing by linear terms like $x - a$)

Basic Example:

$$\frac{6x^2 + 9x}{3x} = \frac{6x^2}{3x} + \frac{9x}{3x} = 2x + 3$$

Note: More complex division techniques will be covered in a later section.

Polynomial Operations Summary

Polynomials are expressions with variables and constants using only addition, subtraction, and multiplication (no variables in denominators or exponents).

Addition/Subtraction:
- Combine like terms (same variable and exponent)
- Only coefficients are added or subtracted

Multiplication:
- Use the distributive property or FOIL
- Multiply each term in one polynomial by each term in the other
- Combine like terms

Division:
- Simplify each term if possible
- For complex division, use long division or synthetic division (covered later)

8.4 Long Division

Long division is a step-by-step method for dividing numbers or algebraic expressions.

- **Dividend:** the number or expression being divided
- **Divisor:** the number or expression you are dividing by
- **Quotient:** the result of the division (goes on top)

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Example 1: Long Division with Whole Numbers

Divide:

$$125 \div 5$$

Here:

- Dividend = 125
- Divisor = 5
- Quotient = 25

$$\begin{array}{c|c} 5 & 125 \\ \hline & 25 \end{array}$$

Steps:

1. Divide 12 by 5 → 2 (since $5 \times 2 = 10$)
2. Subtract: $12 - 10 = 2$, bring down the 5 → 25
3. Divide 25 by 5 → 5 (since $5 \times 5 = 25$)
4. Final result: 25

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Example 2: Polynomial Long Division

Divide:

$$\frac{x^2 + 3x + 2}{x + 1}$$

Here:

- Dividend = $x^2 + 3x + 2$
- Divisor = $x + 1$
- Quotient = the expression that results from division

Step 1: Divide leading terms:

$$x^2 \div x = x$$

Step 2: Multiply and subtract:

$$(x^2 + 3x + 2) - (x)(x + 1) = x^2 + 3x + 2 - (x^2 + x) = 2x + 2$$

Step 3: Divide leading terms again:

$$2x \div x = 2$$

Step 4: Multiply and subtract:

$$(2x + 2) - 2(x + 1) = 2x + 2 - (2x + 2) = 0$$

So the division is exact.

Final Answer:

$$\frac{x^2 + 3x + 2}{x + 1} = x + 2$$

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Terminology Review

- **Dividend:** what you're dividing (e.g., $x^2 + 3x + 2$)
- **Divisor:** what you're dividing by (e.g., $x + 1$)
- **Quotient:** result of the division (e.g., $x + 2$)

8.5 Multivariable Polynomial Long Division

Polynomial long division can also be performed with expressions that include more than one variable. The process is similar, but care must be taken to match like terms correctly and order terms consistently by degree.

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Example: Divide $6x^2y + 9xy^2$ by $3xy$

Here:

- Dividend: $6x^2y + 9xy^2$
- Divisor: $3xy$

Step 1: Divide the first term of the dividend by the first term of the divisor

$$\frac{6x^2y}{3xy} = 2x$$

Step 2: Multiply the entire divisor by $2x$

$$2x \cdot (3xy) = 6x^2y$$

Step 3: Subtract

$$(6x^2y + 9xy^2) - 6x^2y = 9xy^2$$

Step 4: Divide next term

$$\frac{9xy^2}{3xy} = 3y$$

Step 5: Multiply and subtract

$$3y \cdot (3xy) = 9xy^2$$

$$9xy^2 - 9xy^2 = 0$$

Final Answer:

$$\frac{6x^2y + 9xy^2}{3xy} = 2x + 3y$$

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Tips for Multivariable Long Division

- Organize terms in descending order of one variable (typically x)
- Divide one term at a time, matching both variable parts and coefficients
- Use standard subtraction to cancel each step before proceeding

9 Quadratic Polynomials

A **quadratic polynomial** is a polynomial of degree 2, meaning the highest power of the variable is 2.

General Form

$$ax^2 + bx + c$$

where:

- a, b, c are real numbers and $a \neq 0$
- a is the **leading coefficient**
- b is the **linear coefficient**
- c is the **constant term**

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Examples

- $x^2 + 5x + 6$
- $3x^2 - 2x + 1$
- $-x^2 + 4$

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Graphing Quadratics

Quadratic functions graph as a **parabola**. The shape of the parabola depends on the sign of a:

- If $a > 0$: the parabola opens **upward** (like a smile)
- If $a < 0$: the parabola opens **downward** (like a frown)

The highest or lowest point on the parabola is called the **vertex**.

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Solving Quadratic Equations

Quadratics can be solved using several methods:

- **Factoring**
- **Completing the square**
- **Quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Quick Facts About Quadratics

- Degree = 2 (because of x^2)
- Graph = parabola
- May have 0, 1, or 2 real roots depending on the discriminant $b^2 - 4ac$
- Coefficients tell you the shape and position of the graph

9.1 Factoring a Quadratic Polynomial

Factoring is one of the most common ways to solve a quadratic equation when it can be written as a product of two binomials.

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Example: Factor $x^2 + 5x + 6$

Step 1: Look for two numbers that multiply to 6 (the constant term) and add to 5 (the linear coefficient).

$$\text{Factors of 6: } (1, 6),\ (2, 3)$$
$$2 + 3 = 5 \quad \Rightarrow \quad \text{Use 2 and 3}$$

Step 2: Write the factored form:

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Step 3: Check by expanding:

$$(x + 2)(x + 3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6$$

✓Confirmed!

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Factoring Tips

- Always look for a greatest common factor (GCF) first
- Use a factoring method like:
  - Simple guess-and-check
  - Box or area method
  - AC method (for trinomials where $a \neq 1$)
- Always double-check by expanding!

9.2 Difference of Squares

The **difference of squares** is a special factoring pattern that applies when a binomial consists of two perfect squares being subtracted.

$$a^2 - b^2 = (a - b)(a + b)$$

This works because when expanded:

$$(a - b)(a + b) = a^2 + ab - ab - b^2 = a^2 - b^2$$

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Examples

- $x^2 - 9 = (x - 3)(x + 3)$
- $4x^2 - 25 = (2x - 5)(2x + 5)$
- $49y^2 - 1 = (7y - 1)(7y + 1)$
- $x^4 - 16 = (x^2)^2 - 4^2 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

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Important Notes:

- Works only with **subtraction**, not addition
- Both terms must be perfect squares
- Often used to simplify expressions or solve equations

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Difference of Squares Summary

$$a^2 - b^2 = (a - b)(a + b)$$

Examples:

- $x^2 - 16 = (x - 4)(x + 4)$
- $9a^2 - 1 = (3a - 1)(3a + 1)$

9.3 Completing the Square

To solve a quadratic equation of the form:

$$ax^2 + bx + c = 0$$

we can complete the square — a method that rewrites the expression as a perfect square trinomial.

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What is a Trinomial?

A **trinomial** is a polynomial with three terms.
In quadratic form:

$$ax^2 + bx + c$$

is a trinomial.
When completing the square, we turn:

$$x^2 + bx \quad \text{(2 terms)}$$

into:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 \quad \text{(3 terms — a perfect square trinomial)}$$

Steps to Complete the Square (when $a = 1$)

1. Move the constant to the other side:

$$x^2 + bx = -c$$

2. Take half of the coefficient of x, square it, and add to both sides:

$$\left(\frac{b}{2}\right)^2$$

3. Now the left-hand side is a perfect square:

$$\left(x + \frac{b}{2}\right)^2$$

4. Solve by taking the square root of both sides

5. Solve for x

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Example: Solve $x^2 + 6x + 4 = 0$

$$\begin{aligned} x^2 + 6x + 4 &= 0 \\ x^2 + 6x &= -4 \\ \left(\frac{6}{2}\right)^2 &= 9 \\ x^2 + 6x + 9 &= -4 + 9 \\ (x+3)^2 &= 5 \\ x + 3 &= \pm\sqrt{5} \\ x &= -3 \pm \sqrt{5} \end{aligned}$$

So the solution is:

$$x = -3 \pm \sqrt{5}$$

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Tip for Completing the Square

When $a \neq 1$, divide the whole equation by a first. Completing the square is also how we derive the quadratic formula!

Nature of Solutions: Real vs Complex

After completing the square (when $a = 1$), we get:

$$x^2 + bx + c = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$$

The number and type of solutions depend on the value of:

$$\left(\frac{b}{2}\right)^2 - c$$

- If $\left(\frac{b}{2}\right)^2 - c > 0$, then x has **two real** solutions
- If $\left(\frac{b}{2}\right)^2 - c = 0$, then x has **one real** solution (a repeated root)
- If $\left(\frac{b}{2}\right)^2 - c < 0$, then x has **two complex** (nonreal) solutions

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Real vs Complex Solutions from Completing the Square

Given:

$$x^2 + bx + c = 0 \Rightarrow \left(x + \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$$

Then:

- $\left(\frac{b}{2}\right)^2 - c > 0 \Rightarrow 2$ distinct real solutions
- $\left(\frac{b}{2}\right)^2 - c = 0 \Rightarrow 1$ real solution
- $\left(\frac{b}{2}\right)^2 - c < 0 \Rightarrow 2$ complex solutions

Example: Completing the Square with Imaginary Solutions

Solve the equation:

$$x^2 + 4x + 8 = 0$$

Step 1: Move the constant to the other side.

$$x^2 + 4x = -8$$

Step 2: Complete the square. Take half of 4 and square it:

$$\left(\frac{4}{2}\right)^2 = 4$$

Add 4 to both sides:

$$x^2 + 4x + 4 = -8 + 4$$

$$(x + 2)^2 = -4$$

Step 3: Take the square root of both sides.

$$x + 2 = \pm\sqrt{-4}$$

$$x + 2 = \pm 2i$$

Step 4: Solve for x.

$$x = -2 \pm 2i$$

Final Answer:

$$\boxed{x = -2 \pm 2i}$$

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Imaginary Solutions

When completing the square leads to a negative number under the square root, the equation has **two complex conjugate solutions**, involving the imaginary unit $i = \sqrt{-1}$.

Complex Conjugate Solutions and the Discriminant (When $a = 1$)

Consider the quadratic equation:

$$x^2 + 2x + 5 = 0$$

Completing the square:

$$x^2 + 2x = -5$$

Take half of 2 and square it:

$$\left(\frac{2}{2}\right)^2 = 1$$

Add 1 to both sides:

$$x^2 + 2x + 1 = -5 + 1$$

$$(x + 1)^2 = -4$$

Taking the square root:

$$x + 1 = \pm\sqrt{-4} = \pm 2i$$

Therefore,

$$x = -1 \pm 2i$$

Complex Conjugates: These solutions come in pairs called complex conjugates:

$$a + bi \quad \text{and} \quad a - bi$$

Here, $a = -1$ and $b = 2$.

Discriminant: The discriminant is:

$$\Delta = b^2 - 4ac$$

For the equation:

$$a = 1, \quad b = 2, \quad c = 5$$

$$\Delta = 2^2 - 4 \cdot 1 \cdot 5 = 4 - 20 = -16 < 0$$

Since $\Delta < 0$, the solutions are complex conjugates.

Summary

- $\Delta > 0$: Two distinct real solutions
- $\Delta = 0$: One repeated real solution
- $\Delta < 0$: Two complex conjugate solutions

The Quadratic Formula and the Discriminant

Any quadratic equation of the form:

$$ax^2 + bx + c = 0$$

can be solved using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant: The expression under the square root, denoted by:

$$\Delta = b^2 - 4ac$$

is called the **discriminant**. It determines the type of solutions:

Discriminant Summary

- If $\Delta > 0$: two distinct real solutions
- If $\Delta = 0$: one real solution (a repeated root)
- If $\Delta < 0$: two complex conjugate solutions

Example: Solve $x^2 - 2x - 3 = 0$ using the quadratic formula.

Here, $a = 1$, $b = -2$, and $c = -3$. Compute the discriminant:

$$\Delta = (-2)^2 - 4(1)(-3) = 4 + 12 = 16$$

Now apply the formula:

$$x = \frac{-(-2) \pm \sqrt{16}}{2(1)} = \frac{2 \pm 4}{2}$$

So the two solutions are:

$$x = \frac{2+4}{2} = 3 \quad \text{and} \quad x = \frac{2-4}{2} = -1$$

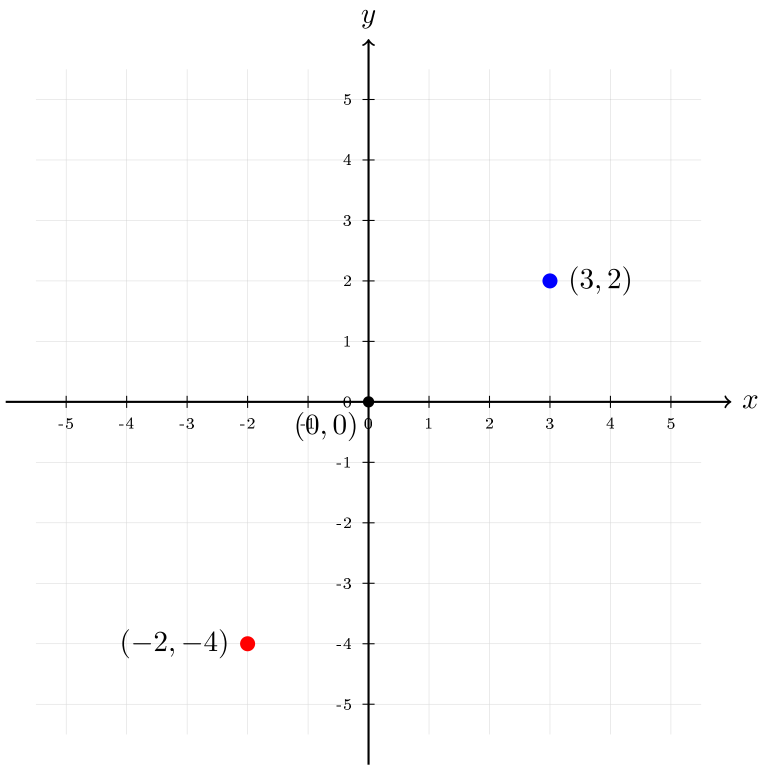
Conclusion: The quadratic formula is a universal method for solving any quadratic equation, and the discriminant tells you what kind of solutions to expect.

10 The Cartesian Plane

The Cartesian Plane is a two-dimensional coordinate system defined by a horizontal number line called the **x-axis**, and a vertical number line called the **y-axis**. These axes intersect at the **origin**, denoted as $(0, 0)$.

Each point on the plane is represented by an ordered pair (x, y), where:

- x is the horizontal value (left/right),
- y is the vertical value (up/down).



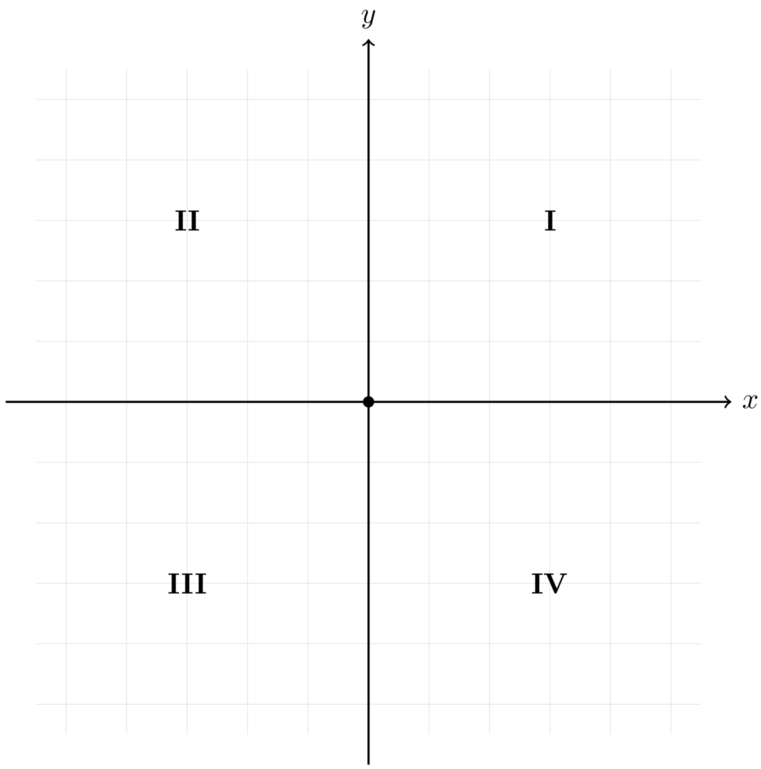
Summary

- The Cartesian Plane has two axes: x-axis (horizontal), y-axis (vertical).
- The point where they meet is the origin $(0, 0)$.
- Each point is identified by an ordered pair (Called a coordinate) (x, y).

10.1 The Four Quadrants

The Cartesian Plane is divided into four regions called **quadrants**. These are numbered in a counterclockwise direction starting from the upper-right:

- **Quadrant I**: $(+x, +y)$
- **Quadrant II**: $(-x, +y)$
- **Quadrant III**: $(-x, -y)$
- **Quadrant IV**: $(+x, -y)$



Summary of Quadrants

- Points in each quadrant have characteristic signs for x and y.
- The quadrants are labeled I through IV, moving counterclockwise.

10.2 Slope of a Line

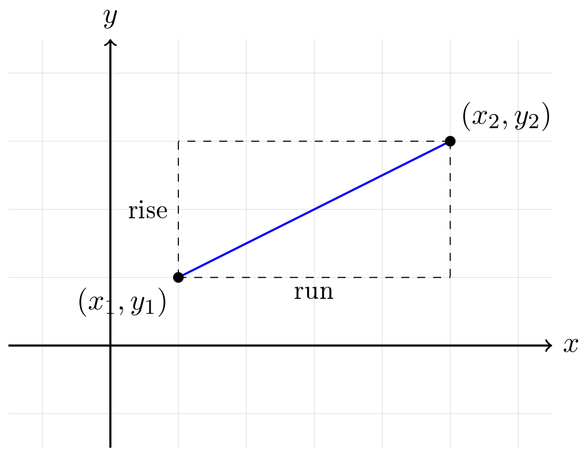
The **slope** of a line is a measure of its steepness. It tells us how much the y-value changes for a given change in the x-value.

Given two points on a line, (x_1, y_1) and (x_2, y_2), the slope m is calculated using the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This is also known as "rise over run":

- **Rise**: the vertical change $(y_2 - y_1)$
- **Run**: the horizontal change $(x_2 - x_1)$



Summary: Slope of a Line

- Slope measures how steep a line is.
- Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to calculate it.
- A positive slope rises left to right; negative slope falls.

10.3 Point-Slope Form of a Line

If you know the slope m of a line and a point (x_1, y_1) it passes through, you can write the line's equation using the **point-slope form**:

$$y - y_1 = m(x - x_1)$$

Requirements:

- A single point (x_1, y_1)
- The slope m

Example: Given the point $(2, 3)$ and slope $m = 4$, the equation becomes:

$$y - 3 = 4(x - 2)$$

This can be left in point-slope form or simplified to slope-intercept form.

Point-Slope Summary

- Use when you know one point and the slope.
- Formula: $y - y_1 = m(x - x_1)$

10.4 Slope-Intercept Form of a Line

The **slope-intercept form** expresses a linear equation as:

$$y = mx + b$$

Where:

- m is the slope of the line.
- b is the **y-intercept** — the value of y when $x = 0$.

Requirements:

- Slope m
- Y-intercept b

Example: If a line has slope $m = -2$ and y-intercept $b = 5$, the equation is:

$$y = -2x + 5$$

This form is especially useful for graphing.

Slope-Intercept Summary

- Use when slope and y-intercept are known.
- Formula: $y = mx + b$
- b tells you where the line crosses the y-axis.

10.5 Converting Between Forms of a Linear Equation

Linear equations can be written in multiple forms. It's often helpful to convert between them depending on the context (e.g., graphing, solving, or analyzing).

1. Point-Slope to Slope-Intercept

Start with the point-slope form:

$$y - y_1 = m(x - x_1)$$

Distribute the slope and solve for y to convert to slope-intercept form:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y &= m(x - x_1) + y_1 \quad \text{(Slope-Intercept Form: } y = mx + b\text{)} \end{aligned}$$

Example: Convert $y - 2 = 3(x + 1)$ to slope-intercept form:

$$\begin{aligned} y - 2 &= 3(x + 1) \\ y - 2 &= 3x + 3 \\ y &= 3x + 5 \end{aligned}$$

2. Two Points to Any Form

Given two points (x_1, y_1) and (x_2, y_2):

1. Find the slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. Use point-slope form with one point
3. Convert to slope-intercept form if needed

Summary: Converting Forms

- Point-slope to slope-intercept: distribute and solve for y
- Two points → slope → point-slope → slope-intercept
- Use the form that best suits the problem (graphing, solving, etc.)

10.6 Undefined Slope and Vertical Lines

A **vertical line** goes straight up and down and has an **undefined slope**. This is because its run (change in x) is zero:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

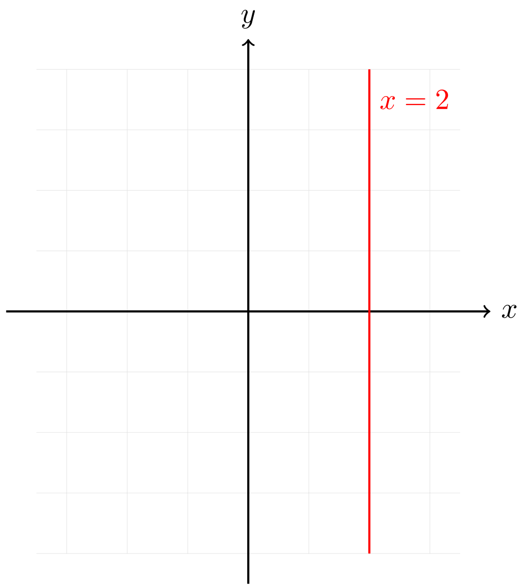
If $x_2 = x_1$, then the denominator becomes 0, and division by zero is undefined.

Equation of a Vertical Line:

A vertical line through $x = a$ is written as:

$$x = a$$

- It has an **undefined slope**.
- It has an **x-intercept only** — it does not cross the y-axis.



Important Notes on Vertical Lines

- Vertical lines have the form $x = a$
- Their slope is undefined.
- They do not cross the y-axis — no y-intercept exists.

11 Graphing Linear Equations

To graph a linear equation, such as:

$$y = mx + b$$

you can create a **table of values** by plugging in values for x and solving for y. This gives you a set of points you can plot on the Cartesian plane.

Steps to Graph a Line

1. Choose three x-values (commonly: $-1, 0, 1$)
2. Plug them into the equation to find corresponding y-values
3. Plot the points (x, y) on the graph
4. Draw a straight line through the points

Example: Graph $y = 2x + 1$

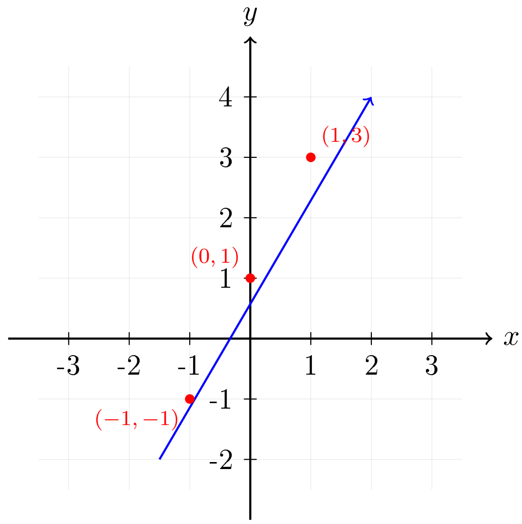
Step 1: Create a table

x	$y = 2x + 1$
-1	$2(-1) + 1 = -2 + 1 = -1$
0	$2(0) + 1 = 0 + 1 = 1$
1	$2(1) + 1 = 2 + 1 = 3$

Step 2: Plot points

$$(-1, -1), \quad (0, 1), \quad (1, 3)$$

Step 3: Draw a straight line through the points



Graphing Summary

- Pick easy x-values: $-1, 0, 1$
- Solve for y, make a table
- Plot at least 3 points and draw the line

12 Functions

A **function** is a relation where each input has exactly one output. The input is called the **independent variable** (usually x), and the output is the **dependent variable** (usually y).

Formally, a function f from a set A (domain) to a set B (codomain) assigns each element $x \in A$ to exactly one element $y \in B$, written as:

$$y = f(x)$$

Key points about functions:

- For every x, there is *only one* corresponding y.
- The value of y depends on the choice of x.
- Functions can be represented by equations, tables, graphs, or mappings.

Example: Consider the function

$$f(x) = 2x + 3.$$

To find the output when $x = 4$, substitute 4 into the function:

$$f(4) = 2(4) + 3 = 8 + 3 = 11.$$

So, the function value at $x = 4$ is 11.

Function notation:

$$f(x) = y$$

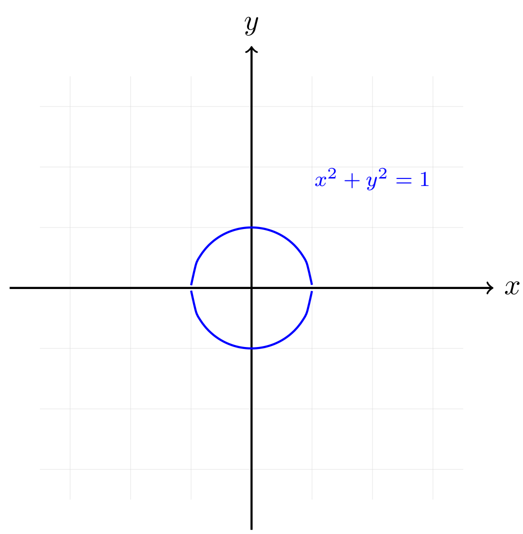
means "the value of the function f at input x is y."

12.1 Not a Function

An equation is **not a function** if one input has *multiple outputs*. You can test this with the **vertical line test** — if a vertical line touches the graph in more than one place, it is *not* a function.

Examples of equations that are not functions:

- $x = 3$ — a vertical line (undefined slope)
- $x^2 + y^2 = 1$ — a circle (one x leads to two y values)



Function Summary

- A function gives exactly one output for each input.
- It passes the **vertical line test**.
- Not all equations are functions!

12.2 Domain and Range

Domain: The set of all possible input values (x) for which the function is defined.

Range: The set of all possible output values (y) that the function can produce.

How to find Domain and Range:

- **Given a function $f(x)$:** Analyze the equation to determine which x-values are allowed. For example, avoid division by zero or square roots of negative numbers.

- **Given a set of points:** The domain is the set of all x-coordinates, and the range is the set of all y-coordinates.

- **Given a graph:** - The domain corresponds to all x-values covered by the graph (horizontally). - The range corresponds to all y-values covered by the graph (vertically).

Example:

Consider the function

$$f(x) = \sqrt{x-2}.$$

- **Domain:** Since the expression inside the square root must be non-negative,

$$x - 2 \geq 0 \implies x \geq 2,$$

  so the domain is $[2, \infty)$.

- **Range:** The square root outputs non-negative values, so

$$y \geq 0,$$

  and the range is $[0, \infty)$.

12.3 Is It a Function? (Testing Sets of Values)

To determine if a relation (a set of ordered pairs or an equation) is a function, use the following rule:

Function Test

A relation is a **function** if *every input* (x-value) corresponds to *exactly one output* (y-value).

Example 1 (Function — Set of Points):

$$\{(1,2),(2,3),(3,4)\}$$

Each input is unique — this **is** a function.

Example 2 (Not a Function — Set of Points):

$$\{(1,2),(1,3),(2,4)\}$$

The input 1 maps to two different outputs (2 and 3) — this is **not** a function.

Example 3 (Algebraic Test):

Consider the equation:

$$x^2 + y^2 = 1$$

Solve for y:

$$y^2 = 1 - x^2 \Rightarrow y = \pm\sqrt{1 - x^2}$$

This gives two y-values for a single x-value. **Conclusion:** This is **not** a function.

Steps to check:

1. List all x-values or solve for y.
2. If any x-value gives more than one y, it's **not** a function.

13 Operations with Functions

13.1 Sum of Functions

The **sum of two functions** means adding their outputs for the same input value.

Sum of Functions

If $f(x)$ and $g(x)$ are two functions, then:

$$(f + g)(x) = f(x) + g(x)$$

This means you add the two expressions for each value of x.

Example

Let:

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = x^2 - 1$$

Then:

$$(f + g)(x) = f(x) + g(x) = (2x + 3) + (x^2 - 1)$$

Simplify:

$$(f + g)(x) = x^2 + 2x + 2$$

Evaluation Example

Find $(f + g)(2)$:

$$f(2) = 2(2) + 3 = 7$$
$$g(2) = (2)^2 - 1 = 3$$
$$(f + g)(2) = f(2) + g(2) = 7 + 3 = 10$$

13.2 Adding Functions from Sets of Points

You can also add two functions given as sets of ordered pairs. Just add the y-values for matching x-values.

Adding Point-Based Functions

If:

$$f = \{(1, 2), (2, 4), (3, 6)\}, \quad g = \{(1, 5), (2, 1), (3, -2)\}$$

Then:

$$(f + g)(x) = \{(1, 2 + 5), (2, 4 + 1), (3, 6 + (-2))\} = \{(1, 7), (2, 5), (3, 4)\}$$

Note: This only works when both functions share the same x-values.

13.3 Product of Functions

The **product of two functions** means multiplying their outputs for each input value.

Product of Functions

If $f(x)$ and $g(x)$ are functions, then:

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

This means you multiply their expressions together.

Example (Algebraic Expressions)

Let:

$$f(x) = 2x + 1, \quad g(x) = x - 3$$

Then:

$$(f \cdot g)(x) = f(x) \cdot g(x) = (2x + 1)(x - 3)$$

Use distribution (FOIL):

$$(f \cdot g)(x) = 2x^2 - 6x + x - 3 = 2x^2 - 5x - 3$$

Evaluation Example

Find $(f \cdot g)(2)$:

$$f(2) = 2(2) + 1 = 5$$
$$g(2) = 2 - 3 = -1$$
$$(f \cdot g)(2) = 5 \cdot (-1) = -5$$

13.3.1 Product from Sets of Points

If:

$$f = \{(1,2),(2,4),(3,6)\}, \quad g = \{(1,5),(2,1),(3,-2)\}$$

Then:

$$(f \cdot g)(x) = \{(1, 2\cdot 5),(2, 4\cdot 1),(3, 6\cdot(-2))\} = \{(1,10),(2,4),(3,-12)\}$$

Note: Only valid if the x-values match in both functions.

13.4 Even, Odd, or Neither

To classify a function as **even**, **odd**, or **neither**, evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$:

Function Symmetry Rules

- **Even:** $f(-x) = f(x)$ — symmetric about the **y-axis**
- **Odd:** $f(-x) = -f(x)$ — symmetric about the **origin**
- **Neither:** $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

—

Even Function — $f(x) = x^2$

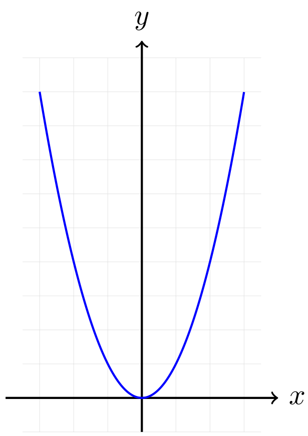


Table of values:

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

—

Odd Function — $f(x) = x^3$

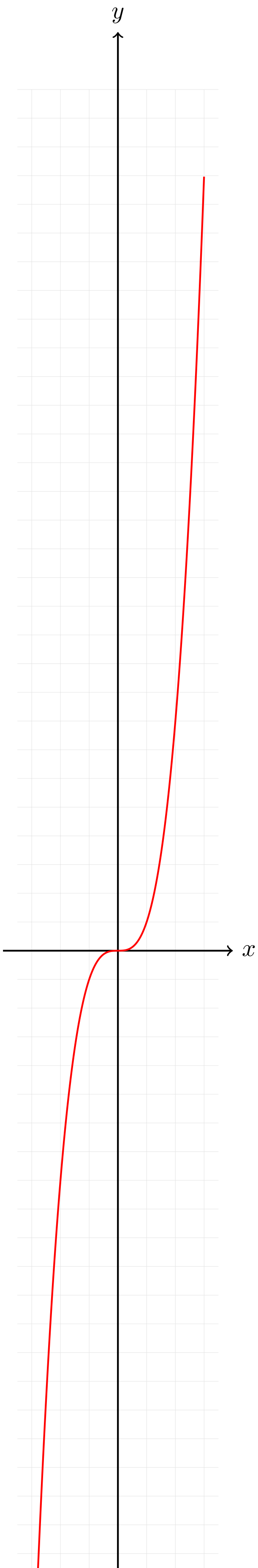
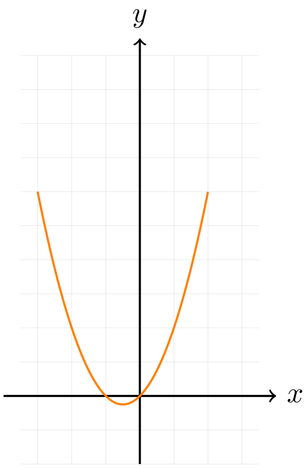


Table of values:

x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8

—

Neither — $f(x) = x^2 + x$



Why is it neither?

$$f(-x) = x^2 - x \neq f(x) \text{ and } \neq -f(x)$$

No symmetry about the y-axis or origin.

—

Quick Summary

- **Even:** Symmetric about the y-axis (e.g., x^2)
- **Odd:** Symmetric about the origin (e.g., x^3)
- **Neither:** Fails both tests (e.g., $x^2 + x$)

14 Trichotomy

The Trichotomy Property

The **Trichotomy Property** states that for any real number a, exactly one of the following is true:

- $a > 0$ (positive)
- $a = 0$ (zero)
- $a < 0$ (negative)

This property applies to all real numbers and ensures that any number must fall into one — and only one — of these categories.

Trichotomy Summary

For any real number a, exactly one of the following holds:

$$a > 0 \quad \text{or} \quad a = 0 \quad \text{or} \quad a < 0$$

Only one condition can be true at a time.

Examples

- $7 > 0$: Positive
- $0 = 0$: Zero
- $-5 < 0$: Negative

Why It Matters

Trichotomy is fundamental in:

- Solving inequalities
- Proving properties in algebra and calculus
- Understanding number line positioning

Inequalities and Negative Numbers

14.1 Working with Inequalities

Inequalities show relationships between two values using symbols:

- $<$: less than
- $>$: greater than
- $\leq$: less than or equal to
- $\geq$: greater than or equal to

14.2 Multiplying or Dividing by Negatives

When you **multiply or divide** both sides of an inequality by a **negative number**, you must **reverse the direction** of the inequality.

Important Rule

If you multiply or divide both sides of an inequality by a negative number, the inequality sign flips:

$$\text{If } a < b, \text{ then } -a > -b$$

Examples

- Original: $3 < 5$ Multiply both sides by -1: $-3 > -5$
- Solve: $-2x > 6$ Divide both sides by -2: $x < -3$ (Notice the sign flip!)

Why This Happens

Negative numbers reverse order on the number line. For example:

$$-3 > -5 \quad \text{because } -3 \text{ is to the right of } -5$$

Summary

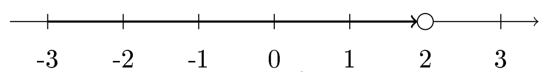
- Inequalities describe relative size.
- Operations like addition and subtraction don't flip the sign.
- Multiplying or dividing by a **negative** *does*.

14.3 Graphing Inequalities on a Number Line

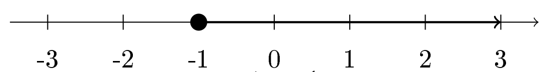
To graph inequalities, we use a number line with:

- An **open circle** for $<$ or $>$ (value is *not* included)
- A **closed (filled) circle** for $\leq$ or $\geq$ (value *is* included)
- An arrow to indicate all values greater than or less than the point

Examples



$x < 2$



$x \geq -1$

Summary

- Use **open circles** for strict inequalities ($<$, $>$)
- Use **closed circles** for inclusive inequalities ($\leq$, $\geq$)
- Shade to the left for "less than", and to the right for "greater than"

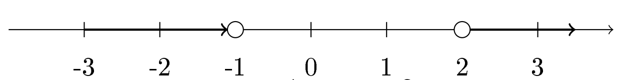
14.4 Graphing Disjunctions (OR)

A **disjunction** connects two inequalities with "or" (symbol: $\vee$). The solution includes values that satisfy *either* inequality.

- Graph both inequalities separately.
- Shade all values that satisfy *either* one.
- The graph is the **union** of both solution sets.

Example:

$x < -1 \quad \vee \quad x > 2$



$x < -1$ or $x > 2$

14.5 Graphing Conjunctions (AND)

A **conjunction** connects two inequalities with "and" (symbol: $\wedge$). The solution includes values that satisfy *both* inequalities simultaneously.

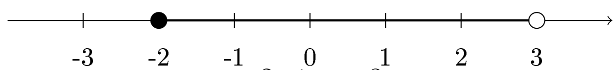
- Graph both inequalities separately.
- The solution is the **intersection** of the two graphs.
- Shade only where both overlap.

Example:

$$-2 \le x < 3$$

This can be written as the conjunction:

$$x \ge -2 \quad \wedge \quad x < 3$$



$$-2 \le x < 3$$

Summary

- **Disjunction (OR)** solutions combine all values satisfying either inequality.
- **Conjunction (AND)** solutions are values satisfying both inequalities simultaneously.
- On number lines, OR means shading the union, AND means shading the intersection.

14.6 Graphing Inequalities in the Plane

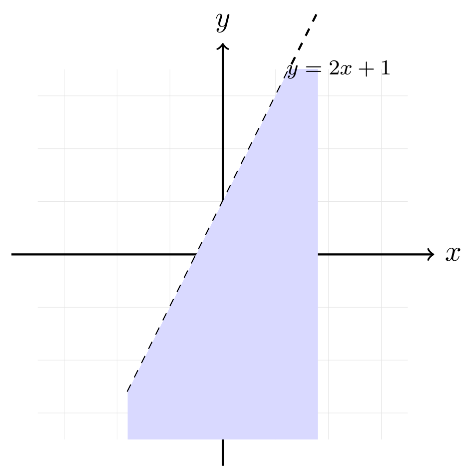
To graph a linear inequality in two variables (like $y < 2x + 1$), follow these steps:

1. **Graph the boundary line:** Replace the inequality symbol with an equals sign and graph the resulting line:
$$y = 2x + 1$$
Use a **dashed line** for $<$ or $>$, and a **solid line** for $\le$ or $\ge$.

2. **Choose a test point:** Often the point $(0, 0)$ is easiest. Plug it into the inequality to check if it makes the inequality true.

3. **Shade the correct region:** If the test point satisfies the inequality, shade the side of the line that includes it. Otherwise, shade the opposite side.

Example: $y < 2x + 1$



Tips

- Dashed line: inequality does **not include** the boundary (e.g., $<$, $>$)
- Solid line: inequality **includes** the boundary (e.g., $\leq$, $\geq$)
- Shade the region where the inequality holds true.

15 Absolute Value Equations

Absolute value equations involve expressions within absolute value bars, such as:

$$|f(x)| = a$$

General Cases

- If $a > 0$, then there are **two solutions**: $f(x) = a$ or $f(x) = -a$
- If $a = 0$, then there is **one solution**: $f(x) = 0$
- If $a < 0$, then there are **no solutions**, because absolute value is never negative.

Example 1: $|x - 4| = 6$

Solve:

$$x - 4 = 6 \quad \text{or} \quad x - 4 = -6$$
$$x = 10 \quad \text{or} \quad x = -2$$

Example 2: $|2x + 5| = 0$

$$2x + 5 = 0$$
$$x = -\frac{5}{2}$$

Example 3: $|x+3| = -7$

No solution, because the absolute value can't equal a negative number.

Summary

- Isolate the absolute value expression.
- Set up two equations: one positive, one negative.
- Check for invalid cases (e.g., equals a negative number).

15.1 Absolute Value Inequalities

Absolute value inequalities involve comparisons such as:

$$|f(x)| < a \quad \text{or} \quad |f(x)| > a$$

Key Cases

1. $|f(x)| <$ **(negative)**
   No solution — absolute value is always non-negative.
2. $|f(x)| >$ **(negative)**
   All real numbers — all absolute values are greater than any negative number.
3. $|f(x)| < a$ (where $a > 0$)
   Conjunction: Rewrite as a compound inequality:
   $$-a < f(x) < a$$
4. $|f(x)| > a$ (where $a > 0$)
   Disjunction: Split into two separate inequalities:
   $$f(x) < -a \quad \text{or} \quad f(x) > a$$

Example 1: $|x-2| < 5$

$$\begin{aligned} -5 &< x - 2 < 5 \\ -3 &< x < 7 \end{aligned}$$

Example 2: $|x+4| > 3$

$$\begin{aligned} x + 4 < -3 &\quad \text{or} \quad x + 4 > 3 \\ x < -7 &\quad \text{or} \quad x > -1 \end{aligned}$$

16 Solving Systems of Equations

A **system of equations** consists of two or more equations with the same set of variables. A solution to the system is the point(s) where the equations intersect — that is, the values that satisfy all equations simultaneously.

There are three common methods to solve a system of linear equations:

16.1 1. Substitution Method

Steps:

1. Solve one equation for one variable.
2. Substitute this expression into the other equation.
3. Solve for the remaining variable.
4. Plug back in to find the other variable.

Example:

$$y = 2x + 1$$
$$3x + y = 13$$

Substitute $y = 2x + 1$ into the second equation:

$$3x + (2x + 1) = 13 \Rightarrow 5x = 12 \Rightarrow x = \frac{12}{5}$$

Then find y:

$$y = 2\left(\frac{12}{5}\right) + 1 = \frac{24}{5} + \frac{5}{5} = \frac{29}{5}$$

16.2 2. Elimination Method

Steps:

1. Align equations and eliminate one variable by addition or subtraction.
2. Solve for the remaining variable.
3. Substitute back to find the other variable.

Example:

$$2x + 3y = 7$$
$$4x - 3y = 5$$

Add the two equations:

$$(2x + 3y) + (4x - 3y) = 7 + 5 \Rightarrow 6x = 12 \Rightarrow x = 2$$

Substitute $x = 2$ into the first equation:

$$2(2) + 3y = 7 \Rightarrow 4 + 3y = 7 \Rightarrow y = 1$$

16.3 3. Graphing Method

Steps:

1. Graph both equations on the same coordinate plane.
2. Identify the point of intersection.

Example:

$$y = x + 1$$
$$y = -x + 5$$

Graph both lines. The intersection point is:

$$x + 1 = -x + 5 \Rightarrow 2x = 4 \Rightarrow x = 2, \quad y = 3$$

Solution: $(2, 3)$

System of Equations Summary

- **Substitution:** Use when one equation is already solved for a variable.
- **Elimination:** Use when variables align easily for cancellation.
- **Graphing:** Use to visualize solutions — intersection point is the solution.

16.4 Solving Systems of Linear Inequalities (Graphing)

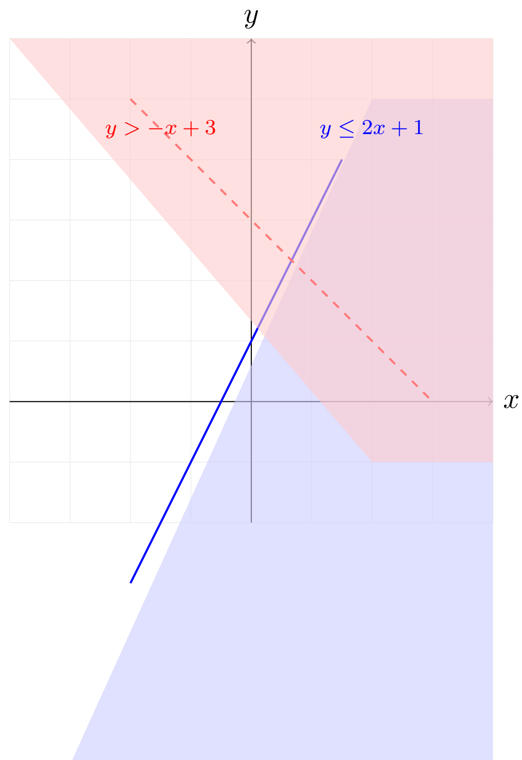
A **system of linear inequalities** consists of two or more inequalities. The solution is the region where all the shaded areas (solutions to each inequality) **overlap**.

Steps to Graph:

1. Graph each inequality as if it were an equation ($y = mx + b$).
2. Use a **dashed line** for $<$ or $>$, and a **solid line** for $\leq$ or $\geq$.
3. Shade the correct side of the line:
   - Use a test point (like $(0, 0)$) to see which side satisfies the inequality.
4. Repeat for each inequality.
5. The **solution region** is where all shaded areas overlap.

Example:

$$\begin{cases} y \leq 2x + 1 \\ y > -x + 3 \end{cases}$$



Graphing Systems of Inequalities

- The **solution set** is the region where all shaded areas **intersect**.
- Use **solid lines** for $\leq, \geq$; **dashed lines** for $<, >$.
- Always test with a point if you're unsure which side to shade.

17 Exponents and Special Powers

Powers of Negative Bases

When a negative number is raised to a power, the result depends on whether the exponent is even or odd:

- **Even Exponent:** Negative base raised to an even power gives a **positive** result.

$$(-2)^4 = 16$$

- **Odd Exponent:** Negative base raised to an odd power gives a **negative** result.

$$(-2)^3 = -8$$

- **Important:** Be careful with parentheses.

$$-2^4 = -(2^4) = -16 \quad \text{(Not the same as } (-2)^4\text{)}$$

Negative Exponents

A negative exponent represents a reciprocal:

$$a^{-n} = \frac{1}{a^n}, \quad a \neq 0$$

- Example:
$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

- Applies to variables as well:
$$x^{-2} = \frac{1}{x^2}, \quad x \neq 0$$

Zero Exponent

Any nonzero number raised to the power of zero equals 1:

$$a^0 = 1, \quad a \neq 0$$

- Example:
$$5^0 = 1 \quad \text{and} \quad (-3)^0 = 1$$

- Note: 0^0 is considered **undefined**.

Exponent Rules Summary

- $(-a)^n$: Positive if n is even, negative if n is odd.
- $a^{-n} = \frac{1}{a^n}$ — Negative exponent means reciprocal.
- $a^0 = 1$, as long as $a \neq 0$.

18 Fractional Exponents

What is a Fractional Exponent?

A fractional exponent represents a **root**. The general rule is:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Special case:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

- The **denominator** (n) of the fraction tells you the **root**.
- The **numerator** (m) tells you the **power**.

Examples

$$9^{\frac{1}{2}} = \sqrt{9} = 3$$
$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$
$$16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$$
$$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$$

Fractional Exponents Summary

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
- Fractional exponents are just another way to express roots and powers.

18.1 Product Property of Radicals

The **Product Property of Radicals** states that:

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \quad \text{where } a \geq 0,\ b \geq 0$$

This property allows you to multiply square roots by combining the radicands into a single radical.

Product Property of Radicals

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \quad (\text{for } a, b \geq 0 \text{ and } n \in \mathbb{N},\ n \geq 2)$$

Examples:

- $\sqrt{3} \cdot \sqrt{12} = \sqrt{36} = 6$
- $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$

Note: This rule does **not** apply to negative radicands when working within the real numbers.

19 Rationalizing the Denominator

Sometimes, we encounter expressions where the denominator contains a square root or an irrational number. Rationalizing the denominator means rewriting the expression so that there are **no radicals in the denominator**.

Why Rationalize?

Rationalizing helps simplify expressions for further operations and provides a standard form that's easier to compare or graph.

Case 1: Single Radical in the Denominator

Example: Single Radical

Simplify:

$$\frac{3}{\sqrt{2}}$$

Multiply numerator and denominator by $\sqrt{2}$:

$$\frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Case 2: Binomial with a Radical (Use the Conjugate)

When the denominator is a binomial with a radical, multiply by the conjugate:

$$\text{Conjugate of } a + \sqrt{b} \text{ is } a - \sqrt{b}$$

Example: Binomial Denominator

Simplify:

$$\frac{5}{3 + \sqrt{2}}$$

Multiply by the conjugate:

$$\frac{5}{3 + \sqrt{2}} \cdot \frac{3 - \sqrt{2}}{3 - \sqrt{2}} = \frac{5(3 - \sqrt{2})}{(3 + \sqrt{2})(3 - \sqrt{2})}$$

$$= \frac{15 - 5\sqrt{2}}{9 - 2} = \frac{15 - 5\sqrt{2}}{7}$$

Summary

Rationalizing Rules

- Multiply by the radical if it's a single term.
- Multiply by the conjugate if it's a binomial.
- Use: $(a + b)(a - b) = a^2 - b^2$ to eliminate radicals.

19.1 Quotient Theorem

The **Quotient Theorem** for exponents helps simplify expressions involving division with the same base.

Quotient Rule of Exponents

If $a \neq 0$, then:

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 1: Basic Exponents

$$\frac{x^5}{x^2} = x^{5-2} = x^3$$

Example 2: Negative Result

$$\frac{y^3}{y^7} = y^{3-7} = y^{-4} = \frac{1}{y^4}$$

Quotient Rule for Radicals

Quotient Rule of Radicals

For $a, b > 0$, and $n \in \mathbb{N}$:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Example 3: Radical Quotient

$$\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$$

Summary

- Subtract exponents when dividing powers with the same base.
- Simplify radicals by dividing under the root, or split into two roots.
- Remember: $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$

19.2 Simplifying Radical Expressions (Rationalizing the Denominator)

When simplifying radical expressions with square roots in the denominator, we must **rationalize the denominator** — that is, eliminate the square root from the bottom of the fraction.

Example 1:

$$2\sqrt{\frac{3}{5}} - 6\sqrt{\frac{5}{3}}$$

Step 1: Use the identity $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$$= 2 \cdot \frac{\sqrt{3}}{\sqrt{5}} - 6 \cdot \frac{\sqrt{5}}{\sqrt{3}}$$

Step 2: Rationalize each term

$$= 2 \cdot \frac{\sqrt{3} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} - 6 \cdot \frac{\sqrt{5} \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = 2 \cdot \frac{\sqrt{15}}{5} - 6 \cdot \frac{\sqrt{15}}{3}$$

Step 3: Simplify the coefficients

$$= \frac{2\sqrt{15}}{5} - 2\sqrt{15} = \frac{2\sqrt{15}}{5} - \frac{10\sqrt{15}}{5} = \frac{-8\sqrt{15}}{5}$$

$$\boxed{\frac{-8\sqrt{15}}{5}}$$

Example 2:

$$2\sqrt{\frac{3}{5}} - 5\sqrt{\frac{5}{3}} + \sqrt{135}$$

Break and simplify:

$$2\sqrt{\frac{3}{5}} = 2 \cdot \frac{\sqrt{15}}{5} = \frac{2\sqrt{15}}{5}$$
$$5\sqrt{\frac{5}{3}} = 5 \cdot \frac{\sqrt{15}}{3} = \frac{5\sqrt{15}}{3}$$
$$\sqrt{135} = \sqrt{9 \cdot 15} = 3\sqrt{15}$$

$$\frac{2\sqrt{15}}{5} - \frac{5\sqrt{15}}{3} + 3\sqrt{15} = \left(\frac{2}{5} - \frac{5}{3} + 3\right)\sqrt{15}$$

Convert to common denominators:

$$= \left(\frac{2}{5} - \frac{5}{3} + \frac{9}{3}\right)\sqrt{15} = \left(\frac{2}{5} + \frac{4}{3}\right)\sqrt{15} = \frac{26}{15}\sqrt{15}$$

$$\boxed{\frac{26\sqrt{15}}{15}}$$

19.3 Rationalizing with the Conjugate

Sometimes, a radical appears in the denominator as part of a **binomial**, like $\frac{1}{\sqrt{a}+\sqrt{b}}$. In these cases, we **rationalize the denominator** by multiplying the numerator and denominator by the **conjugate** of the denominator.

The Conjugate: The conjugate of $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$ (and vice versa). Multiplying conjugates uses the identity:

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$$

Example:

$$\frac{1}{\sqrt{3}+\sqrt{2}}$$

Multiply numerator and denominator by the conjugate $\sqrt{3}-\sqrt{2}$:

$$\frac{1}{\sqrt{3}+\sqrt{2}} \cdot \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

Simplify the denominator using the difference of squares:

$$= \frac{\sqrt{3}-\sqrt{2}}{3-2} = \sqrt{3}-\sqrt{2}$$

$$\boxed{\frac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{3}-\sqrt{2}}$$

Key Idea: Use conjugates to eliminate radicals in binomial denominators. This technique helps make expressions easier to work with — especially in higher-level algebra.

20 Imaginary Numbers

Imaginary numbers are numbers that involve the square root of a negative number. In mathematics, the unit imaginary number is denoted as i, where:

$$i = \sqrt{-1}$$

This means:

$$i^2 = -1$$

20.1 Examples of Imaginary Numbers

- $\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i$
- $\sqrt{-9} = 3i$
- $7i$ is an imaginary number

20.2 Complex Numbers

A number that includes both a real and an imaginary part is called a **complex number**:

$$a + bi \quad \text{where } a, b \in \mathbb{R}$$

20.3 Operations with Imaginary Numbers

- **Addition/Subtraction:** Combine like terms.

$$(3+2i)+(1-5i)=4-3i$$

- **Multiplication:** Use distributive property and simplify using $i^2=-1$.

$$(2+3i)(1-4i)=2-8i+3i-12i^2=2-5i+12=14-5i$$

Summary

- Imaginary unit: $i=\sqrt{-1}$
- $i^2=-1$
- Complex numbers: $a+bi$
- $\sqrt{-x}=i\sqrt{x}$

20.4 Using the Conjugate to Simplify Expressions

When an expression contains a binomial with a radical or imaginary number in the denominator, we **multiply by the conjugate** to rationalize or simplify.

Definition: The **conjugate** of a binomial $a+b$ is $a-b$, and vice versa.

Why it works: Multiplying a binomial by its conjugate results in a **difference of squares**, eliminating the radical or imaginary part:

$$(a+b)(a-b)=a^2-b^2$$

Example 1 (with a radical):

$$\frac{1}{\sqrt{5}+\sqrt{3}}\cdot\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{5}-\sqrt{3}}{(\sqrt{5})^2-(\sqrt{3})^2}=\frac{\sqrt{5}-\sqrt{3}}{2}$$

Example 2 (with imaginary numbers):

$$\frac{3}{2+i}\cdot\frac{2-i}{2-i}=\frac{3(2-i)}{(2+i)(2-i)}=\frac{6-3i}{4+1}=\frac{6-3i}{5}$$

Final Answer:

$$\frac{6-3i}{5}=\frac{6}{5}-\frac{3}{5}i$$

Key Tip

Always multiply by the conjugate of the denominator when simplifying expressions that contain square roots or imaginary numbers in the denominator.

21 Coefficients in Quadratic Equations

A **quadratic equation** is an equation of the form:

$$ax^2 + bx + c = 0$$

where:

- a, b, and c are **coefficients**
- x is the variable
- $a \neq 0$

Each coefficient has a role:

- a is the **leading coefficient** — it determines the *direction* and *width* of the parabola.
  - If $a > 0$, the parabola opens **upward**.
  - If $a < 0$, it opens **downward**.
  - A larger $|a|$ makes the parabola narrower; a smaller $|a|$ makes it wider.
- b affects the **location of the vertex** along the x-axis.
- c is the **constant term**, and it represents the **y-intercept** — where the graph crosses the y-axis.

Example:

$$y = 2x^2 - 4x + 1$$

- $a = 2$: opens upward and is narrower than x^2
- $b = -4$
- $c = 1$: y-intercept at $(0, 1)$

Summary

The coefficients in a quadratic equation control the shape, direction, and position of the parabola on the graph.

21.1 Coefficients in Quadratics (Factoring)

When factoring a quadratic expression of the form:

$$ax^2 + bx + c$$

the **coefficients** a, b, and c play a key role in determining how to factor it.

Cases:

1. **When $a = 1$:** Factor the expression by finding two numbers that:
   - Multiply to c

- Add to b

Example:

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

2. **When $a \neq 1$:** Use the **AC method** (also called *factoring by grouping*):
   - Multiply $a \cdot c$
   - Find two numbers that multiply to ac and add to b
   - Rewrite the middle term using those numbers
   - Factor by grouping

Example:

$$6x^2 + 11x + 3$$

Multiply $a \cdot c = 6 \cdot 3 = 18$

Find numbers that multiply to 18 and add to 11: 9 and 2

$$6x^2 + 9x + 2x + 3 = 3x(2x + 3) + 1(2x + 3) = (3x + 1)(2x + 3)$$

Key Tip

Always check for a greatest common factor (GCF) before factoring!

22 Difference of Cubes

A **difference of cubes** is an expression of the form:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

This identity allows us to factor expressions where both terms are perfect cubes.

Key Formula

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example: Factor $x^3 - 8$

Step 1: Recognize both terms as perfect cubes.

$$x^3 - 8 = x^3 - 2^3$$

Step 2: Apply the difference of cubes formula.

$$x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$$

Another Example: Factor $27y^3 - 64$

Step 1: Write each term as a cube.

$$27y^3 - 64 = (3y)^3 - 4^3$$

Step 2: Use the formula:

$$(3y - 4)\left((3y)^2 + 3y \cdot 4 + 4^2\right) = (3y - 4)(9y^2 + 12y + 16)$$

Another Example: Factor $x^3 - 64m^6r^9$

Step 1: Recognize each term as a cube:

$$x^3 - (4m^2r^3)^3$$

Step 2: Apply the formula:

$$x^3 - (4m^2r^3)^3 = \left(x - 4m^2r^3\right)\left(x^2 + x \cdot 4m^2r^3 + (4m^2r^3)^2\right)$$

Step 3: Simplify:

$$= \left(x - 4m^2r^3\right)\left(x^2 + 4m^2r^3x + 16m^4r^6\right)$$

Final Answer

$$x^3 - 64m^6r^9 = \left(x - 4m^2r^3\right)\left(x^2 + 4m^2r^3x + 16m^4r^6\right)$$

Tip: Always double-check

Make sure:

- Both terms are perfect cubes.
- The pattern is subtraction. For addition, use the sum of cubes formula.

23 Sum of Cubes

A binomial in the form:

$$a^3 + b^3$$

can be factored using the identity:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Sum of Cubes Formula

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Steps:

1. Identify the cube root of each term.
2. Apply the formula $(a + b)(a^2 - ab + b^2)$.
3. Simplify each part of the second factor.

Example: Factor $x^3 + 27$

$$\begin{aligned} x^3 + 27 &= x^3 + 3^3 \\ &= (x + 3)(x^2 - 3x + 9) \end{aligned}$$

Remember:

- Sum of cubes uses $(a + b)(a^2 - ab + b^2)$
- The middle term in the trinomial is always $-ab$, even though the binomial is a sum.

24 Simplifying Rational Functions

A **rational function** is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $Q(x) \neq 0$.

Steps to Simplify a Rational Function:

1. Factor both the numerator and the denominator completely.
2. Cancel any common factors.
3. State any restrictions (values that make the denominator zero).

Simplify:

$$\frac{3x^4 - 9x^3}{6x^2}$$

Step 1: Factor both numerator and denominator.
Numerator: $3x^4 - 9x^3 = 3x^3(x-3)$
Denominator: $6x^2$
Step 2: Cancel common factors.

$$\frac{3x^3(x-3)}{6x^2} = \frac{x(x-3)}{2}$$

Step 3: State the restriction.
$x \neq 0$ (the original denominator can't be zero)
Final Answer:

$$\frac{x(x-3)}{2}, \quad x \neq 0$$

Tips

- Always factor before simplifying.
- Be careful not to cancel terms — only factors.
- Remember to state domain restrictions.

24.1 Adding and Subtracting Rational Functions

To **add or subtract rational functions**, you combine them over a common denominator — just like with numerical fractions.

Steps:

1. Factor all denominators.
2. Find the **least common denominator** (LCD).
3. Rewrite each rational function with the LCD.
4. Combine the numerators (add or subtract).
5. Simplify the result if possible.
6. State any restrictions on the variable.

Add:

$$\frac{2}{x} + \frac{3}{x+1}$$

Step 1: The denominators are already factored.
Step 2: LCD $= x(x+1)$
Step 3: Rewrite each with LCD:

$$\frac{2(x+1)}{x(x+1)} + \frac{3x}{x(x+1)}$$

Step 4: Combine:

$$\frac{2(x+1)+3x}{x(x+1)} = \frac{2x+2+3x}{x(x+1)} = \frac{5x+2}{x(x+1)}$$

Restrictions: $x \neq 0, x \neq -1$

Subtract:

$$\frac{x}{x-2} - \frac{3}{x}$$

Step 1: Denominators are already factored.
Step 2: LCD $= x(x-2)$
Step 3: Rewrite each:

$$\frac{x \cdot x}{x(x-2)} - \frac{3(x-2)}{x(x-2)}$$

Step 4: Combine:

$$\frac{x^2-3(x-2)}{x(x-2)} = \frac{x^2-3x+6}{x(x-2)}$$

Restrictions: $x \neq 0, x \neq 2$

Note

- Always factor denominators to help find the LCD.
- Don't forget to distribute negative signs when subtracting!
- Include domain restrictions from all original denominators.

24.2 Factoring to Find a Common Denominator

When adding or subtracting rational expressions, you must first express each fraction with the **least common denominator** (LCD). This often requires factoring the denominators.

Steps:

1. Factor each denominator completely.
2. Identify all unique factors — include each factor the greatest number of times it appears in any denominator.
3. Multiply these factors to get the LCD.

4. Rewrite each expression with the LCD as the new denominator.

Add:

$$\frac{3}{x^2-x}+\frac{2}{x^2-1}$$

Step 1: Factor each denominator:

$$x^2-x=x(x-1), \quad x^2-1=(x-1)(x+1)$$

Step 2: LCD $=\ x(x-1)(x+1)$
Step 3: Rewrite each fraction:

$$\frac{3(x+1)}{x(x-1)(x+1)}+\frac{2x}{x(x-1)(x+1)}$$

Step 4: Combine:

$$\frac{3(x+1)+2x}{x(x-1)(x+1)}=\frac{3x+3+2x}{x(x-1)(x+1)}=\frac{5x+3}{x(x-1)(x+1)}$$

Restrictions: $x \neq 0, \pm 1$

Tips for Factoring

- Always factor completely before identifying the LCD.
- Look for common patterns: differences of squares, trinomials, and factoring by grouping.
- Once the LCD is found, rewrite each fraction using equivalent expressions.

24.3 Multiplying Rational Functions

To multiply rational functions, follow these steps:

1. **Factor all numerators and denominators**, if possible.
2. **Multiply the numerators** together and multiply the denominators together.
3. **Cancel any common factors** that appear in both the numerator and the denominator.

Example:

$$\frac{3x^2}{x^2-9}\cdot\frac{x^2-3x}{6x}$$

Step 1: Factor where possible

$$=\frac{3x^2}{(x-3)(x+3)}\cdot\frac{x(x-3)}{6x}$$

Step 2: Multiply across

$$=\frac{3x^2\cdot x(x-3)}{(x-3)(x+3)\cdot 6x}$$

Step 3: Cancel common factors

- Cancel x from numerator and denominator
- Cancel $x-3$

$$= \frac{3x \cdot x}{6(x+3)} = \frac{3x^2}{6(x+3)} = \frac{x^2}{2(x+3)}$$

Key Tip

Always factor before multiplying to identify common factors that can be cancelled.

24.4 Dividing Rational Functions

A **rational function** is a quotient of two polynomials, i.e.,

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

To **divide** one rational function by another, say

$$\frac{P_1(x)}{Q_1(x)} \div \frac{P_2(x)}{Q_2(x)},$$

we multiply the first function by the reciprocal of the second:

$$\frac{P_1(x)}{Q_1(x)} \times \frac{Q_2(x)}{P_2(x)} = \frac{P_1(x) \cdot Q_2(x)}{Q_1(x) \cdot P_2(x)}.$$

Note: Make sure that $P_2(x) \neq 0$ to avoid division by zero.

Example: Divide

$$\frac{2x^2-3}{x+1} \div \frac{x-4}{x^2+x}.$$

Solution:

$$\frac{2x^2-3}{x+1} \times \frac{x^2+x}{x-4} = \frac{(2x^2-3)(x^2+x)}{(x+1)(x-4)}.$$

You can simplify further by factoring polynomials if possible.

24.5 Dividing Rational Functions with Domain Restrictions

Given two rational functions

$$f(x) = \frac{P_1(x)}{Q_1(x)} \quad \text{and} \quad g(x) = \frac{P_2(x)}{Q_2(x)},$$

where $Q_1(x) \neq 0$ and $Q_2(x) \neq 0$, their division is defined as

$$\frac{f(x)}{g(x)} = \frac{\frac{P_1(x)}{Q_1(x)}}{\frac{P_2(x)}{Q_2(x)}} = \frac{P_1(x)}{Q_1(x)} \times \frac{Q_2(x)}{P_2(x)} = \frac{P_1(x) \cdot Q_2(x)}{Q_1(x) \cdot P_2(x)},$$

with the restriction that $P_2(x) \neq 0$ to avoid division by zero.

Important: The domain of the resulting function excludes all values of x that make any denominator zero, including:

$$Q_1(x) = 0, \quad Q_2(x) = 0, \quad \text{and} \quad P_2(x) = 0.$$

Example:

Divide

$$\frac{3x+2}{x^2-4} \div \frac{x-1}{x+3}.$$

Step 1: Rewrite as multiplication by the reciprocal:

$$\frac{3x+2}{x^2-4} \times \frac{x+3}{x-1} = \frac{(3x+2)(x+3)}{(x^2-4)(x-1)}.$$

Step 2: Factor denominators where possible:

$$x^2 - 4 = (x-2)(x+2).$$

So,

$$\frac{(3x+2)(x+3)}{(x-2)(x+2)(x-1)}.$$

Step 3: Determine excluded values (domain restrictions):

- From original denominators:

$$x^2 - 4 = 0 \implies x = 2, -2,$$

and

$$x + 3 = 0 \implies x = -3.$$

- From the denominator of the reciprocal's numerator (which is $x - 1$ in the denominator after multiplication):

$$x - 1 = 0 \implies x = 1.$$

Therefore, the domain excludes:

$$x \neq -3, -2, 1, 2.$$

25 Advanced Equations

25.1 Direct Variation

In a direct variation, two variables change in the same direction: as one increases, so does the other. The relationship can be written as:

$$y = kx$$

Where:

- y is the dependent variable
- x is the independent variable
- k is the constant of variation (slope)

Key Characteristics

- The graph is a straight line passing through the origin (0, 0).
- If $k > 0$, y increases with x.
- If $k < 0$, y decreases with x.

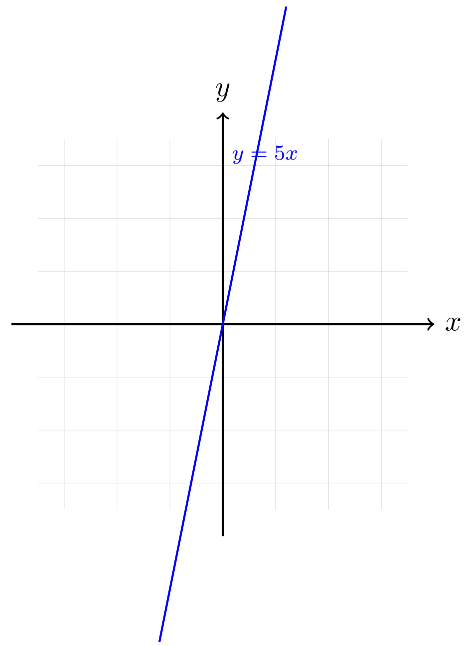
Example: Suppose y varies directly with x, and $y = 10$ when $x = 2$. Find the equation.

$$\begin{aligned} y &= kx \\ 10 &= k(2) \\ k &= 5 \end{aligned}$$

So the equation is:

$$y = 5x$$

Check: When $x = 4$, $y = 5 \cdot 4 = 20$



25.2 Inverse Variation

In an **inverse variation**, two variables are related such that as one increases, the other decreases proportionally. This relationship is modeled by the equation:

$$y = \frac{k}{x}$$

where k is a constant and $x \neq 0$. Here, y varies inversely with x.

Key features:

- The product $xy = k$ is constant.
- The graph is a **hyperbola**, not a straight line.
- The curve never touches the x- or y-axis (asymptotes).

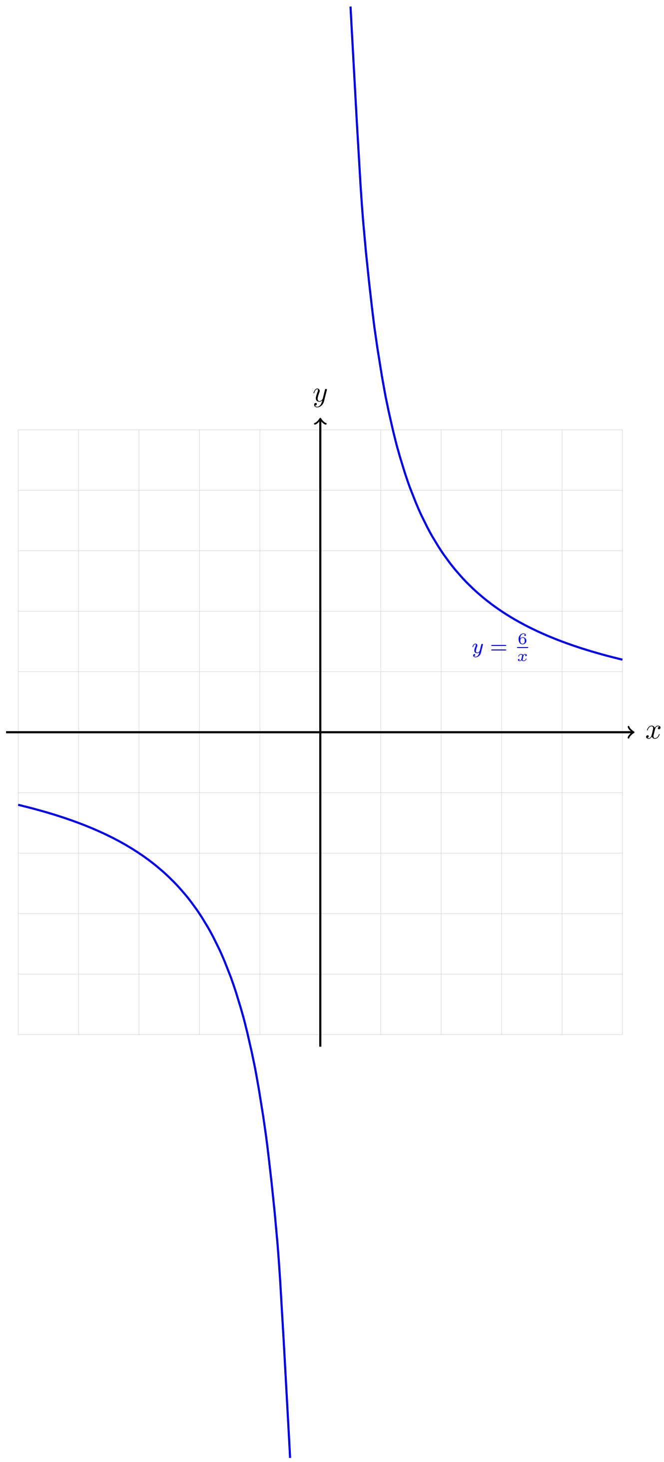
Example:

If y varies inversely with x, and $y = 2$ when $x = 3$, find the equation relating x and y.

$$xy = k \quad \Rightarrow \quad 2 \cdot 3 = 6 \Rightarrow k = 6$$

So, the equation is:

$$y = \frac{6}{x}$$



25.3 Decimal Equations

Decimal equations contain decimal coefficients and constants. A common method to solve them is to eliminate decimals by multiplying the entire equation by a power of 10, converting all decimals to whole numbers. This often simplifies the solving process.

Example: Solve the equation

$$0.5x + 1.2 = 3.7$$

Solution: Step 1: Identify the smallest power of 10 that will eliminate the decimals. Here, the decimals go up to the tenths place, so multiply by 10:

$$10 \times (0.5x + 1.2) = 10 \times 3.7$$

which gives

$$5x + 12 = 37$$

Step 2: Solve the resulting whole number equation:

$$5x = 37 - 12 = 25$$

$$x = \frac{25}{5} = 5$$

Summary: Multiplying to clear decimals turns the original decimal equation into a simpler integer equation, making algebraic manipulation easier and reducing calculation errors.

25.4 Fractional Equations

Fractional equations are equations that contain one or more fractions with variables in the numerator, denominator, or both. To solve these equations, it is often helpful to eliminate the fractions by multiplying both sides of the equation by the least common denominator (LCD) of all fractions involved.

Example: Solve the equation

$$\frac{2x}{3} - \frac{5}{6} = \frac{x}{2}$$

Solution: Step 1: Identify the least common denominator (LCD) of the fractions: the denominators are 3, 6, and 2. The LCD is 6.

Step 2: Multiply every term on both sides of the equation by 6 to eliminate denominators:

$$6 \times \frac{2x}{3} - 6 \times \frac{5}{6} = 6 \times \frac{x}{2}$$

which simplifies to:

$$4x - 5 = 3x$$

Step 3: Solve the resulting equation:

$$4x - 3x = 5 \quad \Rightarrow \quad x = 5$$

Summary: Multiplying both sides of a fractional equation by the LCD is a reliable method to clear denominators, simplifying the equation to one with whole numbers that can be solved using standard algebraic techniques.

25.5 Rational Equations

Rational equations are equations that contain one or more rational expressions—fractions in which the numerator and/or denominator contains a variable. To solve rational equations, it is usually best to eliminate denominators by multiplying both sides of the equation by the least common denominator (LCD).

Example: Solve the equation

$$\frac{4}{x} + \frac{6}{x+a} = 1$$

Solution: Step 1: Identify the least common denominator (LCD). The denominators are x and $x+a$, so the LCD is:

$$\text{LCD} = x(x+a)$$

Step 2: Multiply each term in the equation by the LCD:

$$x(x+a) \cdot \left(\frac{4}{x} + \frac{6}{x+a} \right) = x(x+a) \cdot 1$$

Step 3: Simplify each term:

$$4(x+a) + 6x = x(x+a)$$

Step 4: Expand both sides:

$$4x + 4a + 6x = x^2 + ax$$

Step 5: Combine like terms:

$$10x + 4a = x^2 + ax$$

Step 6: Bring all terms to one side:

$$10x + 4a - x^2 - ax = 0$$

Or, rewritten in standard form:

$$-x^2 + (10-a)x + 4a = 0$$

This is now a quadratic equation in terms of x. You can solve it using factoring, completing the square, or the quadratic formula depending on the values of a.

Summary: To solve rational equations:

- Identify the least common denominator (LCD).
- Multiply every term by the LCD to eliminate fractions.
- Simplify and solve the resulting equation.
- Always check for extraneous solutions—values that make any original denominator zero.

25.6 Radical Equations

Radical equations are equations that contain variables within a radical, most commonly a square root. To solve these equations, the general strategy is to isolate the radical and then eliminate it by raising both sides of the equation to a power.

Example: Solve the equation

$$\sqrt{x+6} + x = 0$$

Solution: Step 1: Isolate the radical expression:

$$\sqrt{x+6} = -x$$

Step 2: Square both sides to eliminate the square root:

$$(\sqrt{x+6})^2 = (-x)^2 \quad \Rightarrow \quad x + 6 = x^2$$

Step 3: Rearrange into standard quadratic form:

$$x^2 - x - 6 = 0$$

Step 4: Factor the quadratic:

$$(x-3)(x+2) = 0$$

Step 5: Solve for x:

$$x = 3 \quad \text{or} \quad x = -2$$

Step 6: Check for extraneous solutions: We must verify each solution in the **original** equation because squaring both sides can introduce invalid (extraneous) solutions.

Check $x = 3$:

$$\sqrt{3+6} + 3 = \sqrt{9} + 3 = 3 + 3 = 6 \neq 0 \quad \text{Not valid}$$

Check $x = -2$:

$$\sqrt{-2+6} + (-2) = \sqrt{4} - 2 = 2 - 2 = 0 \quad \text{Valid}$$

Final Answer:

$$\boxed{x = -2}$$

Summary: To solve radical equations:

- Isolate the radical expression.
- Eliminate the radical by squaring both sides.
- Solve the resulting equation.
- Always check for extraneous solutions in the original equation.

25.7 Multivariable Equations

Multivariable equations contain two or more variables. These types of equations are often manipulated to solve for one variable in terms of the others. This is useful in algebra, physics, and systems of equations.

Example: Solve for z in the equation:

$$3x - y + 2z = 12$$

Solution: Step 1: Isolate the term containing z:

$$2z = 12 - 3x + y$$

Step 2: Divide both sides by 2 to solve for z:

$$z = \frac{12 - 3x + y}{2}$$

Step 3 (Optional): Rearranged for clarity:

$$z = \frac{-3x + y + 12}{2} \quad \text{or} \quad z = \frac{-3x}{2} + \frac{y}{2} + 6$$

Summary: To solve a multivariable equation for a specific variable:

- Move all other terms to the opposite side.
- Isolate the desired variable using inverse operations (e.g., divide or subtract).
- Optionally simplify or rearrange the result for readability.

25.8 Multivariable Rational Equations

Multivariable rational equations involve rational expressions with two or more variables. A common method for solving such equations is to eliminate all denominators by multiplying through by the least common denominator (LCD). This avoids dealing with complex fractions.

Example: Solve for x in the equation:

$$\frac{1}{x} - \frac{m}{y} = p$$

Solution: Step 1: Identify the least common denominator (LCD). The denominators are x and y, so:

$$\text{LCD} = xy$$

Step 2: Multiply every term by the LCD to eliminate denominators:

$$xy \cdot \left(\frac{1}{x} - \frac{m}{y} \right) = xy \cdot p$$

Step 3: Simplify each term:

$$y - mx = pxy$$

Step 4: Solve for x: First, move the y to the other side:

$$-mx = pxy - y$$

Then divide both sides by $-m$:

$$x = \frac{y - pxy}{m}$$

Step 5 (Optional): Rearranged for clarity:

$$x = \frac{y(1 - px)}{m}$$

This form expresses x implicitly (it appears on both sides). You may stop here unless solving explicitly for x is required (which would involve further algebra, possibly leading to a quadratic).

Summary: To solve multivariable rational equations:

- Find the least common denominator (LCD) and multiply through to eliminate all fractions.
- Simplify the resulting equation and isolate the desired variable.
- Watch for variables that appear on both sides—further algebraic manipulation may be necessary.
- Always check for restrictions (denominators cannot be zero).

26 Advanced Systems of Equations

26.1 Systems with Subscripts

Systems of equations may include variables with subscripts, which often arise in applied problems such as physics, chemistry, or engineering. Here's an example of such a system:

Use any method to find the unique solution to the system of equations:

$$\begin{aligned} R_M T_M &= 200 \\ R_P &= 5R_M \\ R_P T_P &= 500 \\ T_P &= 3 - T_M \end{aligned}$$

Solution: We will solve this system using substitution.

Step 1: From the last equation,

$$T_P = 3 - T_M$$

Step 2: Substitute into the third equation:

$$R_P(3 - T_M) = 500$$

Step 3: From the second equation, substitute $R_P = 5R_M$:

$$5R_M(3 - T_M) = 500$$

Step 4: Expand:

$$15R_M - 5R_M T_M = 500$$

Step 5: From the first equation, solve for R_M:

$$R_M = \frac{200}{T_M}$$

Step 6: Substitute into the previous expression:

$$15 \cdot \frac{200}{T_M} - 5 \cdot \frac{200}{T_M} \cdot T_M = 500$$

Simplify both terms:

$$\frac{3000}{T_M} - 1000 = 500$$

Step 7: Solve for T_M:

$$\frac{3000}{T_M} = 1500 \quad \Rightarrow \quad T_M = \frac{3000}{1500} = 2$$

Step 8: Back-substitute to find the remaining variables:

$$R_M = \frac{200}{2} = 100, \quad R_P = 5 \cdot 100 = 500, \quad T_P = 3 - 2 = 1$$

Final Answer:

$$\boxed{T_M = 2, \quad R_M = 100, \quad R_P = 500, \quad T_P = 1}$$

26.2 Uniform Motion

In uniform motion problems, distance is calculated using the formula:

$$\text{Distance} = \text{Rate} \times \text{Time}$$

Example:

A ball flies horizontally toward a gym wall at $30\,\text{m/s}$. After bouncing off the wall, it returns horizontally to its starting point at $20\,\text{m/s}$. The total time of flight is $2\,\text{seconds}$. How far is the wall from the starting place?

Let:

- d: distance from the starting point to the wall
- t_1: time to reach the wall
- t_2: time to return to the starting point

We know:

$$d = 30t_1$$
$$d = 20t_2$$
$$t_1 + t_2 = 2$$

Substitute $t_2 = 2 - t_1$ into the second equation:

$$30t_1 = 20(2 - t_1)$$

$$30t_1 = 40 - 20t_1$$

$$50t_1 = 40 \quad \Rightarrow \quad t_1 = \frac{40}{50} = 0.8$$

Now find the distance:

$$d = 30 \cdot 0.8 = \boxed{24\,\mathrm{m}}$$

Answer: The wall is $\boxed{\text{24 meters}}$ away from the starting place.