On the Theory of Aplanatic Aspheric Systems

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On the Theory of Aplanatic Aspheric Systems

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ABSTRACT. Methods are described for the design of two aspheric surfaces for any given centred system, so as to achieve exact aplanatism. The practical application of the methods is illustrated by the design of a reflecting microscope.

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§ 1. INTRODUCTION

The design of aspherical systems raises a number of questions. Foremost among these is the problem of how to determine for any given system the aspheric profiles so as to obtain good definition over the whole of a finite field. To the approximation of third order (Seidel) optics this can be accomplished by the elegant method of plate diagram analysis (Burch 1942, Linfoot 1944). As a rule, however, higher aberrations must be taken into account. A reasonable way of doing this is to supplement the preliminary Seidel design by further analysis. General methods using differential corrections were described by Volosov (1947), but their form makes them difficult to handle. Frequently, however, it is sufficient to confine the further analysis to the attainment of either axial stigmatism or aplanatism.‡ The former has been discussed by Herzberger and Hoadley (1946) and by Wolf (1948). No corresponding analysis for the case of aplanatism appears to have been made. However, a simple method adequate for some systems was given by Linfoot (1943). Particular systems were considered by various authors; for instance, Schwarzschild (1905) and Chrétien (1922) derived exact equations for two-mirror aplanatic telescopes. Their methods, although valuable for the practical design of such aplanats, are not very suitable for application to more complex systems.§

The aim of the present paper is to provide methods for the design of two aspheric surfaces for any given centred system, so as to achieve exact aplanatism. We derive two simultaneous first-order differential equations for their profiles. These equations contain certain functions which characterize two-ray congruences in the spaces immediately before and after the two surfaces. These are easily obtained from a ray trace. Our differential equations permit of a rigorous solution of the problem and can be solved to any desired accuracy by standard numerical methods.

§ 2. EQUATIONS FOR THE DESIGN OF TWO SURFACES TO ENSURE APLANATISM

2.1. In this section we derive formulae for the design of two correcting surfaces so as to obtain simultaneously: (*a*) axial stigmatism, (*b*) exact satisfaction of the sine condition. Our results apply to any centred system in which the two

‡ We use the term aplanatism to imply axial stigmatism together with the satisfaction of the exact sine condition.

§ This is illustrated, for example, by the analysis of Bureau and Swings (1934) who applied Schwarzschild's methods to the design of two-mirror aplanats with both foci finite.

aspheric surfaces are optical neighbours. The two surfaces in question may, however, be separated from the object or image points by any number of spherical or aspherical surfaces. We shall only be concerned with the final corrections of the system, and we assume all design data, save the profiles of the two correctors, to be known.

We introduce two sets of Cartesian coordinate systems (Figure 1) with origins at the poles O and O′ (assumed to be known) of the two correcting surfaces Σ and Σ', with their x axes along the axis of the system; and take $OO'=d$. Let us denote by σ and σ' the spaces immediately preceding Σ and following Σ', and by σ^* the space between them. Finally, let n, n^* and n' be the corresponding refractive indices.

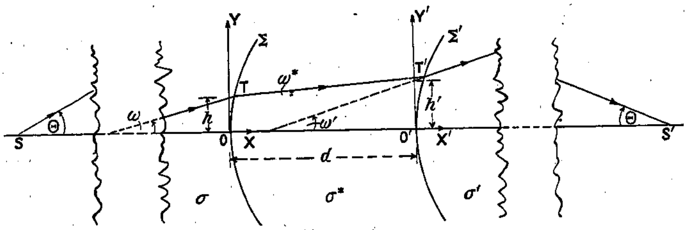

Figure 1.

The rays which proceed from the axial object point S are refracted (or reflected) at the successive surfaces and form normal rectilinear congruence Γ in the space σ. Let h denote the height at which a typical ray of Γ meets OY (if produced far enough) and let ω denote the angle which the rays make with the x axis (measured anticlockwise from the positive x direction to the direction along which the light advances). Γ can then be completely specified by two relations:

$$\omega=\omega(t); \qquad h=h(t), \qquad \ldots\ldots(2.1)$$

where t is a free parameter. If the axial object point S is at a finite distance we choose

$$t=\sin\Theta, \qquad \ldots\ldots(2.2)$$

where Θ is the angle which the corresponding ray in the object space makes with the x axis (Figure 1). If, on the other hand, the object is at infinity, we choose $t=H$ where H is the distance of the corresponding ray in the object space from the x axis (Figure 2 (a)).

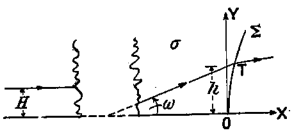

Figure 2 (a).

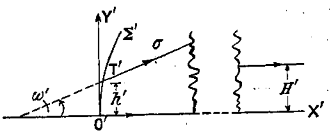

Figure 2 (b).

The ray specified by the parameter t will be referred to as the ray t and its intersection with Σ will be denoted by T. As the relations (2.1) can rarely, in practice, be obtained in analytical form, we form a table of ω and h based on a ray trace for different values of t. Some designers would probably favour polynomial approximations to $\omega(t)$ and $h(t)$ at this stage; but the merits of this expedient are doubtful and will be discussed later.

Similarly the rays which are traced backwards from the axial image point S′ are refracted (or reflected) in succession by all surfaces which follow Σ' and form a normal rectilinear congruence in the space σ'. Let Γ' be the congruence in σ' which is obtained by reversing the sense of every ray of the "back-traced" congruence, i.e. we regard the rays of Γ' as though they proceed from Σ' towards the axial image point (Figure 1). Γ' can then be specified by two relations:

$$\omega' = \omega'(t'); \qquad h' = h'(t'), \qquad \ldots\ldots (2.3)$$

where ω' and h' are now referred to axes at O′ and the parameter t' has a significance analogous to that of t (Figures 1 and 2(b)). The quantities ω' and h' are obtained for selected values of t' by tracing rays backwards from the axial image point S′ into the space σ'.

The conditions (a) and (b) for aplanatism require that the ray t of Γ be transformed by two successive refractions at Σ and Σ' into the ray t' of Γ' where t and t' are related by the sine condition, which takes the form

$$t/t' = \text{constant}. \qquad \ldots\ldots (2.4)$$

Equation (2.4) defines uniquely the ray t' corresponding to a given ray t, so that our problem can be regarded as involving only the free parameter t. To avoid cumbersome suffixes we shall use undashed and dashed symbols to refer to quantities associated with a pair of corresponding rays.

2.2. Let (x, y) and (x', y') be the coordinates, referred to axes at O and O′ respectively, of the points T and T′ (Figure 1). We have, by Snell's law, for the refraction at Σ

$$n(\cos\omega\, dx/dt + \sin\omega\, dy/dt) = n^*(\cos\omega^* dx/dt + \sin\omega^* dy/dt), \qquad \ldots\ldots (2.5)$$

where ω^* is the angle which TT′ makes with the axis.

We have from Figure 1

$$\sin\omega^* = R_y/R; \qquad \cos\omega^* = R_x/R, \qquad \ldots\ldots (2.6)$$

where

$$R_x = x' - x + d, \quad R_y = y' - y, \quad \text{and} \quad R^2 = R_x{}^2 + R_y{}^2. \qquad \ldots\ldots (2.7)$$

Also, from the figure,

$$y = h + x\tan\omega \qquad \ldots\ldots (2.8)$$

and

$$y' = h' + x'\tan\omega'. \qquad \ldots\ldots (2.9)$$

Substituting for $\cos\omega^*$ and $\sin\omega^*$ from (2.6) and for dy/dt from (2.8) into (2.5) it follows that

$$\frac{dx}{dt} = -\left[\frac{(n^*R_x - nR\cos\omega)}{(n^*R_y - nR\sin\omega)} + \tan\omega\right]^{-1}\left[\frac{dh}{dt} + x\frac{d}{dt}\tan\omega\right]. \qquad \ldots\ldots (2.10)$$

Similarly we obtain

$$\frac{dx'}{dt} = -\left[\frac{(n^*R_x - n'R\cos\omega')}{(n^*R_y - n'R\sin\omega')} + \tan\omega'\right]^{-1}\left[\frac{dh'}{dt} + x'\frac{d}{dt}\tan\omega'\right]. \qquad \ldots\ldots (2.11)$$

Equations (2.10) and (2.11) permit together with (2.7), (2.8) and (2.9) a complete computation of Σ and Σ'. For by means of (2.7), (2.8) and (2.9) we could eliminate

y and y' from (2.10) and (2.11) and thus obtain two simultaneous first-order differential equations of the type

$$dx/dt=f(x,x',t); \quad \text{and} \quad dx'/dt=g(x,x',t). \qquad \ldots\ldots(2.12)$$

These may be integrated numerically by standard methods. But as we require y and y' as well as x and x' for a range of values of the parameter t, it is preferable to avoid elimination and solve for the unknown quantities step by step. This will be discussed in more detail in the next section. Equation (2.12), subject to the boundary conditions $x=x'=0$ for $t=0$ show that if there exists a physical solution of our problem it will be unique.

2.3. If the correcting surface Σ is a mirror, the following rule should be applied: Put $n=-n^*=1$ and change R_x and R_y in (2.10) into $-R_x$ and $-R_y$. Similarly if Σ' is a mirror, put $n'=-n^*=1$ and change the sign of R_x and R_y in (2.11).

§ 3. METHODS OF COMPUTATION AND SOME APPLICATIONS

3.1. We shall first summarize the procedure for computing the two correcting surfaces.

(*a*) Rays are traced from the axial object point into the space σ for a set of selected values of the parameter t. For each value of t the quantities ω and h are determined. Following this we calculate the values of $d\tan\omega/dt$ and dh/dt by numerical differentiation or otherwise. In simple cases (e.g. when Σ is the first surface of the system) the differentiation may be performed analytically.

(*b*) For each value of t the corresponding value of t' is calculated from (2.4). The quantities ω' and h' are now determined by tracing rays backwards from the axial image point into the space σ'. The differential coefficients $d\tan\omega'/dt$ and dh'/dt are subsequently determined as above.

(*c*) Next, equations (2.10) and (2.11) are integrated numerically step by step, for every value of x and x' the corresponding values of y and y' are calculated at every step from (2.8) and (2.9). The integration is performed in two stages. The first few values can be obtained by the application of the methods of Runge and Kutta (Runge and König 1924, p. 286). The integration can then be continued by Adam's method (Whittaker and Robinson 1946, p. 363). In this way we obtain the two profiles in tabulated form.

Some opticians would probably employ throughout the design polynomial approximations, based on the tracings of a small number of rays. Although polynomial approximations form an important tool in preliminary design they are, in our opinion, not necessarily of advantage in the final computation of an aspheric system. Our doubts of the expediency of polynomial approximations arise first of all in connection with the high accuracy which is frequently required in the design of an aspheric surface. (In astrographic cameras for example, it is generally desirable to determine the profiles down to a fraction of a fringe.) On account of slow convergence (cf. Martin 1944, p. 108) this may necessitate the evaluation of a large number of coefficients. As a rule, however, it is difficult to obtain more than the first three or four coefficients in terms of the design parameters, so that the expansions have to be supplemented in particular cases. Moreover, if we base the expansion on the tracing of a small number of rays we encounter further difficulties as it is generally not possible to determine in advance the number of terms required for a satisfactory expansion (Herzberger and Hoadley 1946, p. 339).

In view of these considerations it is our opinion that polynomial approximations are not very suitable in the solution of this type of problem and should be replaced by tabulated functions based on tracings of dense fans of rays. Linfoot (1943, p. 494) also advocates this procedure.

3.2. To test the analysis we applied our methods to the design of an aplanatic reflecting microscope and compared the results with those obtained from the analytical solutions of Schwarzschild (1905) and Chrétien (1922) for two-mirror aplanats. Microscopes of this type were constructed and discussed by Burch (1947).

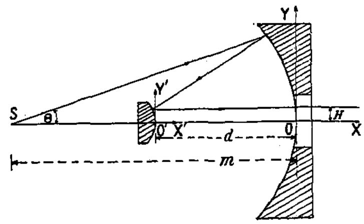

Figure 3.

The objective (Figure 3) consists of a concave primary and a convex secondary mirror. These are separated by a distance d and image parallel light at a distance m from the pole of the primary (m and d are negative in Figure 3). In this case the ray traces are very simple. In the previous notation we have

$$\omega = \Theta, \quad h = -m \tan \Theta; \qquad \omega' = 0, \quad h' = H'. \qquad \text{......(3.1)}$$

Corresponding parameters satisfy the sine condition which now takes the form

$$t' = ft, \qquad \text{......(3.2)}$$

where

$$t = \sin \Theta; \qquad t' = H', \qquad \text{......(3.3)}$$

and f is the focal length of the system. On applying the rule for mirrors of § 2.3 and making use of the above relations, equations (2.10) and (2.11) take the form

$$\frac{dx}{dt} = -\left[\frac{(R\cos\Theta - R_x)}{(R\sin\Theta - R_y)} + \tan\Theta\right]^{-1}(x-m)\sec^3\Theta, \qquad \text{......(3.4)}$$

$$dx'/dt = fR_y/(R - R_x), \qquad \text{......(3.5)}$$

where as before

$$R_x = x' - x + d, \quad R_y = y' - y, \quad \text{and} \quad R^2 = R_x^2 + R_y^2. \qquad \text{......(3.6)}$$

Also (2.8) and (2.9) become

$$y = (x - m)\tan\Theta \qquad \text{......(3.7)}$$

and

$$y' = f\sin\Theta. \qquad \text{......(3.8)}$$

We computed the system with design data $m = -6$ cm., $d = -3$ cm., $f = 0{\cdot}75$ cm. and numerical aperture (N.A.) $= 0{\cdot}58$.

Twenty equally spaced values of the parameter t were selected, viz.

$$t_r = 0{\cdot}029r \quad (r = 1, 2, \ldots 20).$$

First the values of x_r and x_r' with the corresponding values of y_r and y_r' were computed for $r = 1$ and $r = 2$ by Runge's method. The integration was then continued by Adam's method. In calculating the first few differences occurring in Adam's formulae use was made of the fact that $x_{-r} = x_r$ and $x_{-r} = x_r'$. Results for the first six values of x are shown in table 2, where Δ denotes the difference between these solutions and those obtained from Schwarzschild's formulae.

Owing to the inaccuracies of Runge's method an almost constant error of 8×10^{-6} cm. occurred in all values of x_r for $r>2$. The maximum error in x_r' was 2×10^{-6} cm.

Table 1. Reflecting Microscope, $m=-6$ cm., $d=-3$ cm., $f=0{\cdot}75$ cm., N.A. $=0{\cdot}58$.

		Concave primary (in cm.)				Convex secondary (in cm.)		
r	$t=\sin\Theta_r$	$-x_r$	$10^6\Delta x_r$	y_r	$10^6\Delta y_r$	$-x_r'$	$10^6\Delta x_r'$	$y_r'=H'$
1	0·029	0·003469	0	0·173973	0	0·000276	0	0·02175
2	0·058	0·013880	0	0·347781	0	0·001104	1	0·04350
3	0·087	0·031240	0	0·521258	1	0·002486	0	0·06525
4	0·116	0·055562	1	0·694241	1	0·004424	0	0·08700
5	0·145	0·086866	1	0·866562	1	0·006920	0	0·10875
6	0·174	0·125175	0	1·038054	0	0·009978	1	0·13050
7	0·203	0·170523	0	1·208547	0	0·013603	1	0·15225
8	0·232	0·222949	1	1·377870	0	0·017799	−1	0·17400
9	0·261	0·282499	1	1·545849	0	0·022575	−1	0·19575
10	0·290	0·349231	0	1·712307	0	0·027937	−1	0·21750
11	0·319	0·423213	0	1·877063	0	0·033894	0	0·23925
12	0·348	0·504524	0	2·039933	0	0·040456	0	0·26100
13	0·377	0·593259	0	2·200726	0	0·047634	0	0·28275
14	0·406	0·689530	0	2·359245	0	0·055441	0	0·30450
15	0·435	0·793468	−1	2·515288	−1	0·063892	1	0·32625
16	0·464	0·905230	0	2·668638	−1	0·073002	2	0·34800
17	0·493	1·024999	−1	2·819072	0	0·082790	1	0·36975
18	0·522	1·152996	0	2·966351	0	0·093277	−1	0·39150
19	0·551	1·289480	−2	3·110222	−1	0·104487	0	0·41325
20	0·580	1·434771	−1	3·250403	1	0·116445	−1	0·43500

Δ denotes difference between results obtained from Schwarzschild's formulae and those calculated by the present methods.

Table 2

r	1	2	3	4	5	6
$-x_r$	0·003471	0·013884	0·031248	0·055570	0·086874	0·125183
$10^6\Delta x_r$	2	4	8	9	9	8

To obtain higher accuracy we repeated the integration for the first two points. This was carried out by substituting the values of dx_r/dt and dx_r'/dt $(r=0, 1, 2)$ previously obtained into the formulae of Newton-Cotes (Whittaker and Robinson 1946, p. 152). Very close agreement with Schwarzschild's results was obtained. The integration was then continued as before with results given in Table 1. It can be seen that Δ never exceeds 2×10^{-6} cm., i.e. about 1/20 of a reflection fringe. Such high accuracy in the design data is not needed by the practical optician, but it is a necessary preliminary to a numerical analysis of the off-axis errors, on which any theoretical assessment of the value of the design must be based.

The computation was carried out with the aid of Gifford's 8-figure tables of trigonometric functions (interpolating up to 1/100″) and Milne-Thomson's 8-figure tables of square roots.

Since, in the derivation of our equations, no use was made of any approximations, any inaccuracies in the final solution are due to small computational errors.

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The Refractive Index in Electron Optics and the Principles of Dynamics

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ABSTRACT. In view of mis-statements made in the literature, the origin of the refractive index in electron optics is discussed in some detail, and the uniqueness of an expression previously given is demonstrated. On this basis, some general properties of electron optics are investigated.

A relation between ray direction and wave normal is obtained. Whereas the refractive index is unique in terms of the magnetic vector potential **A**, this itself is arbitrary to some extent. It is shown that **A** must, for purposes of electron optics, be chosen so as to satisfy Stokes' theorem and that, if it does, no observable effects result from the arbitrariness of **A**. An expression for the optical path difference is given in terms of the magnetic flux enclosed. The results are applied to a number of questions, viz. the differential equations for trajectories, the focusing properties of an axially symmetric field and the interference pattern produced by two converging bundles of rays which enclose a magnetic flux.

§ 1. INTRODUCTION

Whereas in light optics the refractive index of a medium is in the first instance an experimental datum, and its accurate value is the basis of any detailed discussion of the performance of optical instruments, all geometrical electron optics is entirely contained in Lorentz's equation for the forces acting on a moving charge. As a result, all equations for trajectories can be derived directly from that equation by specifying the electric and magnetic fields. Thus the rôle of the refractive index in electron optics is far less obvious than that of its counterpart for light, and in fact different authors have proposed essentially different values of the refractive index for the same field without arousing much perturbation (Glaser 1933, 1937, Opatowski 1943). Only through the persistent