R-matrix Analysis (I)

TALENT Course 6
Theory for exploring nuclear reaction experiments

GANIL 1st-19th July

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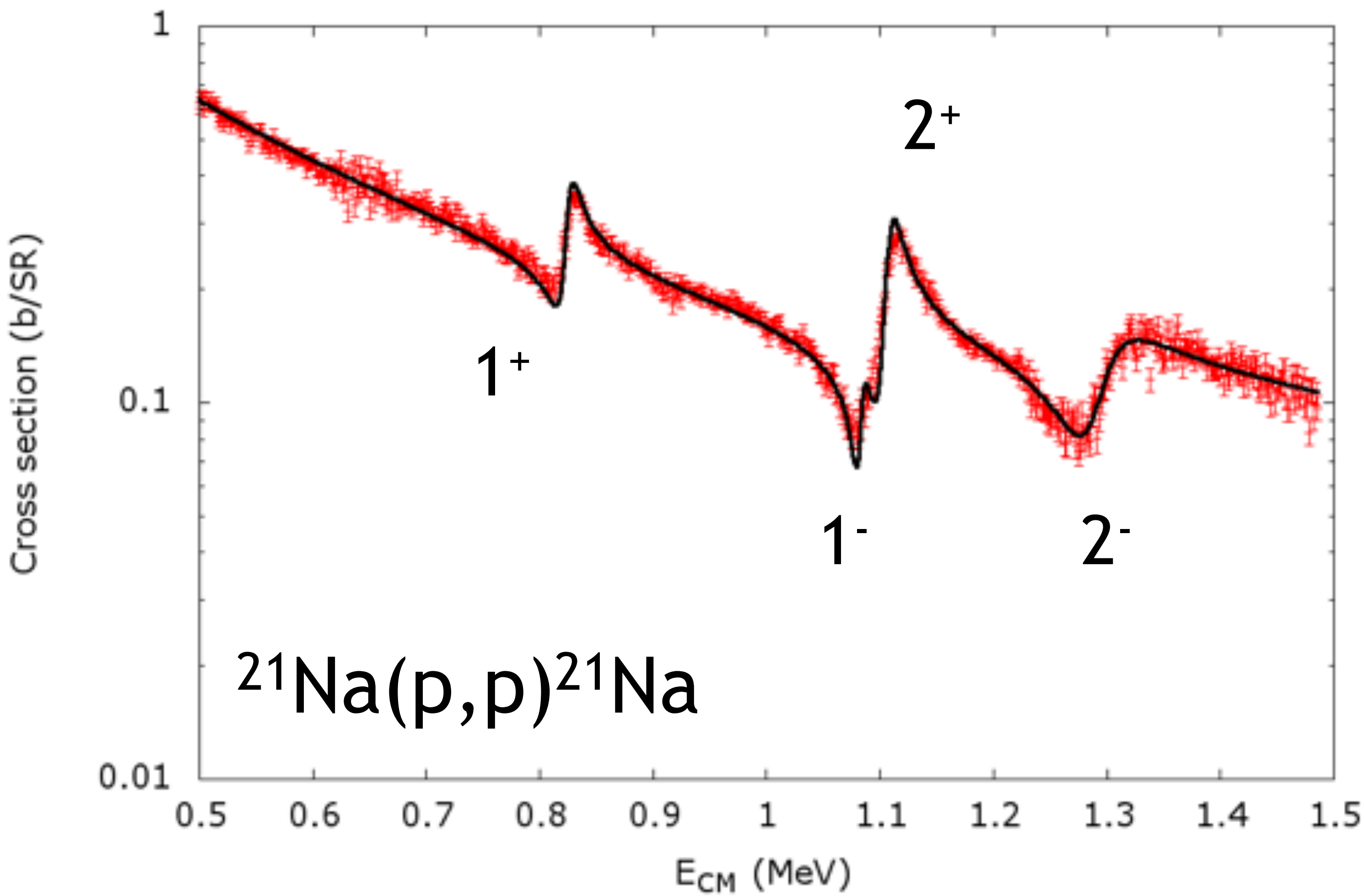
Introduction

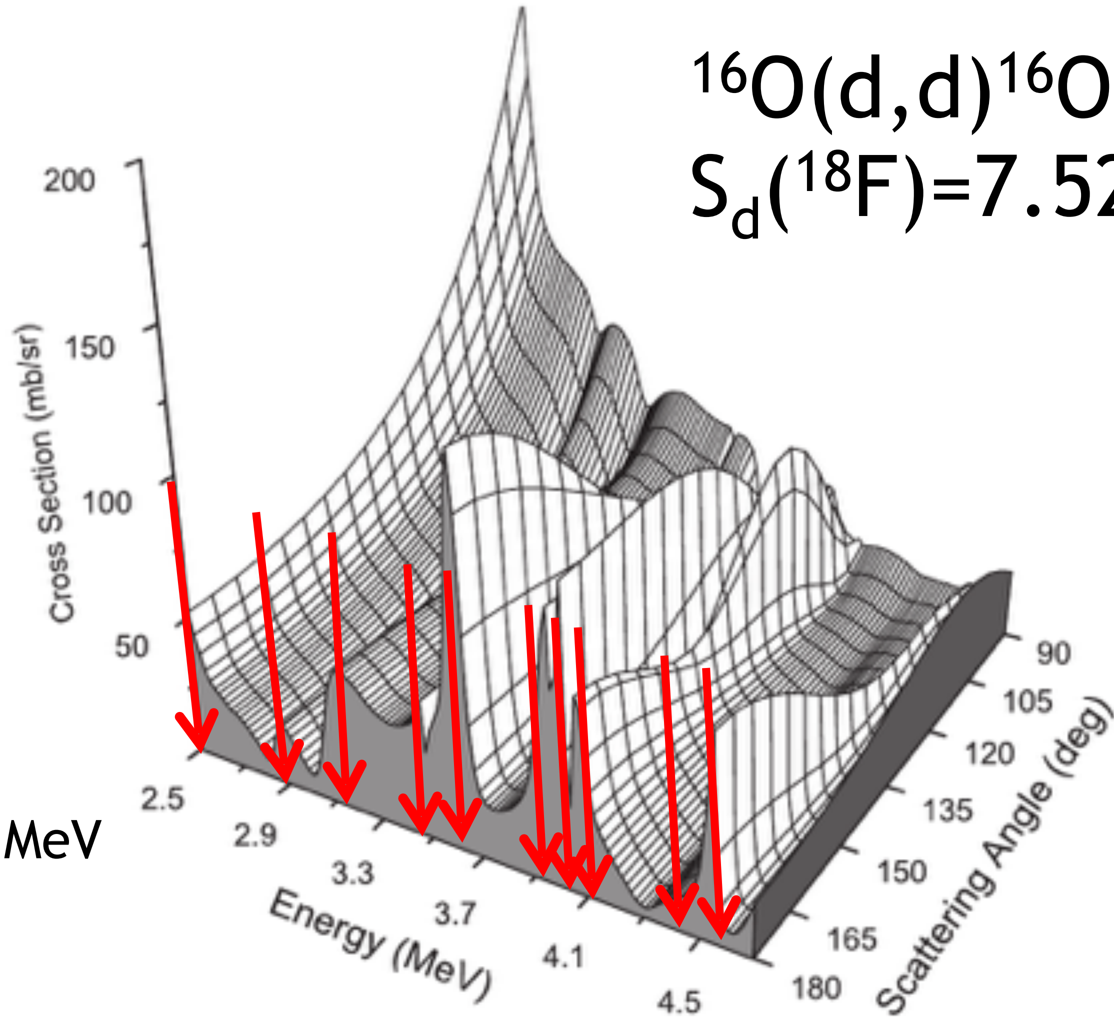
Why do we need R-matrix phenomenology?

Determining nuclear structure



Ruiz *et al.*, Phys. Rev. C 71, 025802 (2005)

Materials analysis



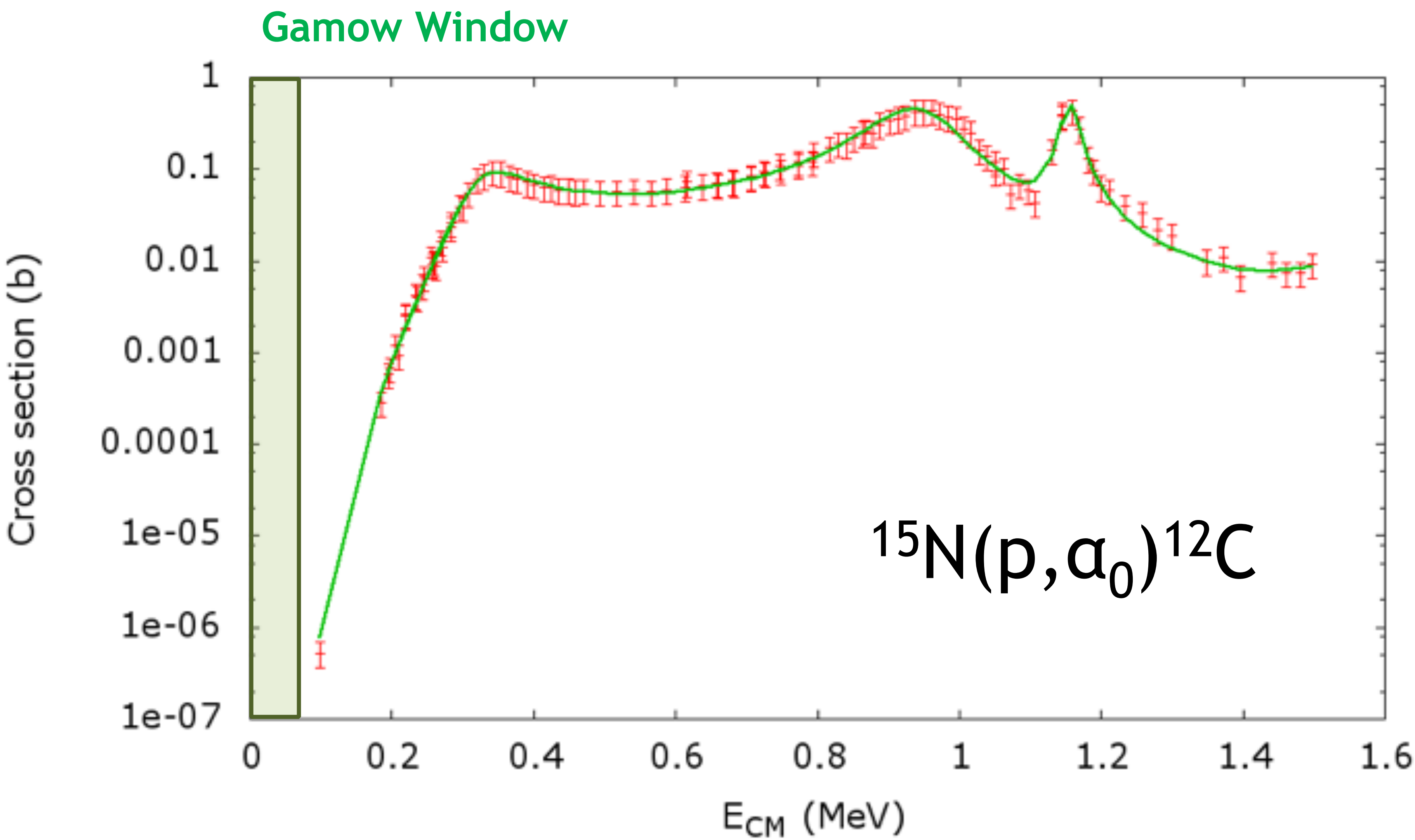
$^{16}O(d,d)^{16}O$

$S_d(^{18}F)=7.525$ MeV

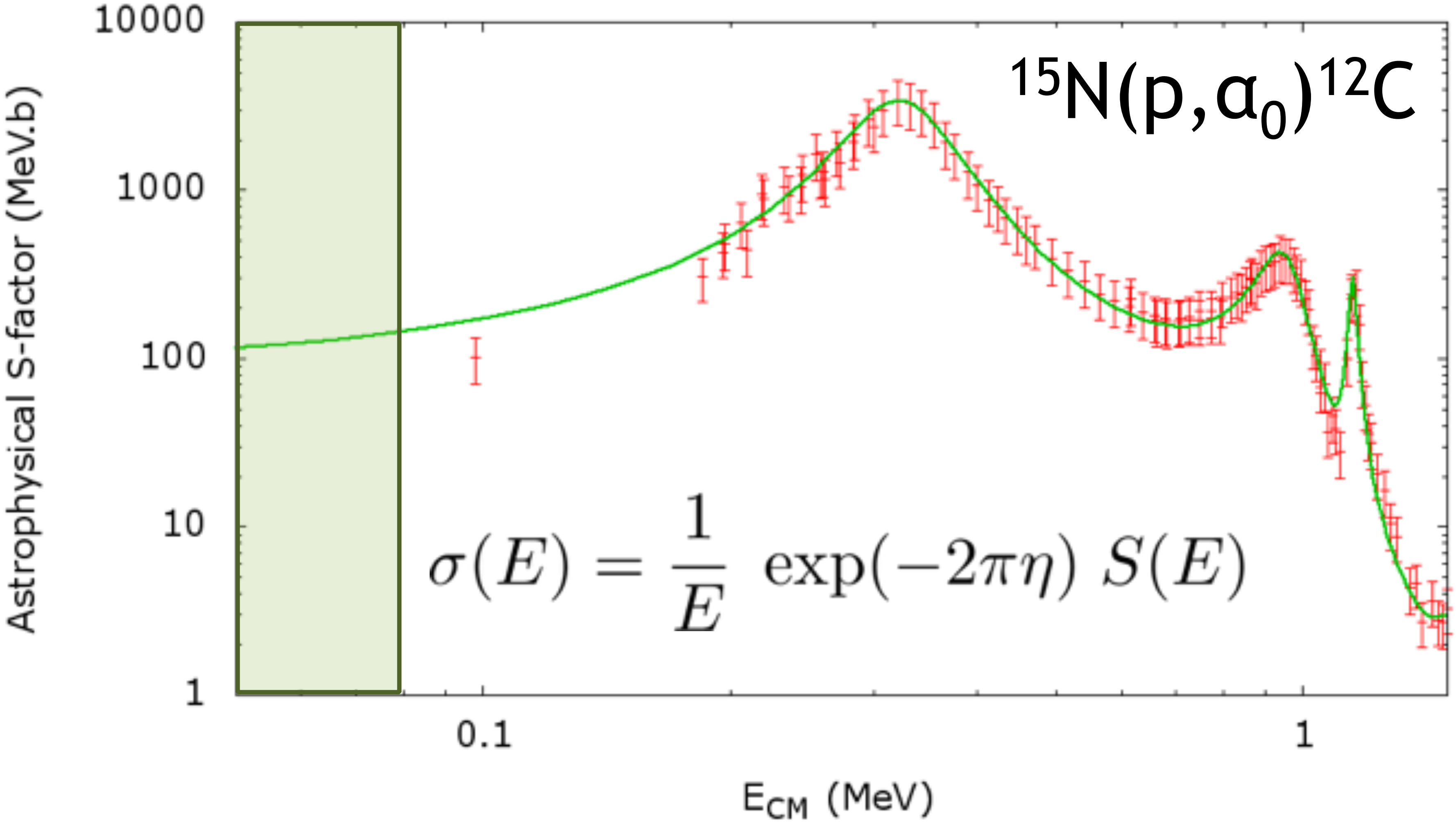
$E_x=9.747$ MeV

$E_x=11.525$ MeV

Nuclear astrophysics



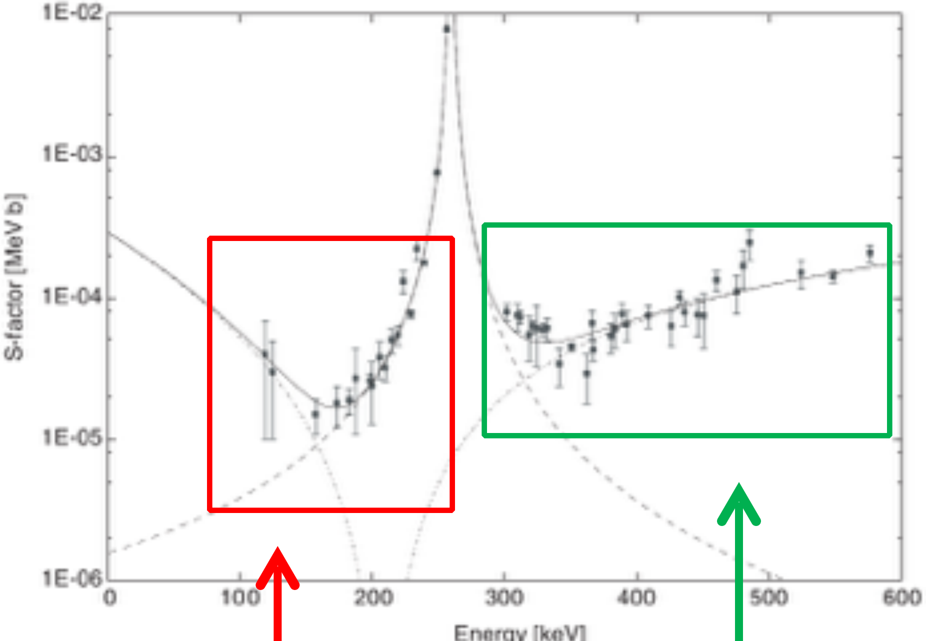
The astrophysical S-factor

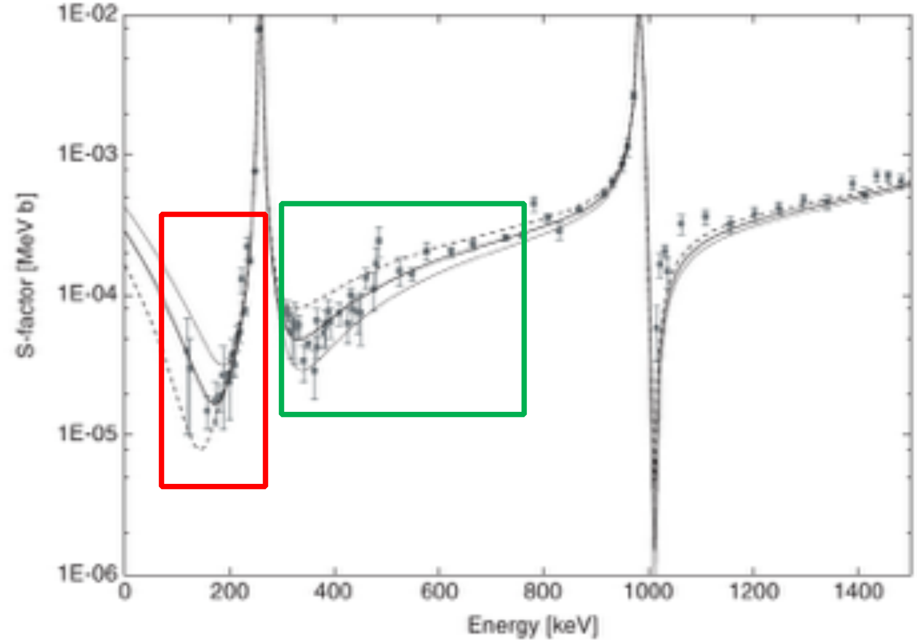


$$\sigma(E) = \frac{1}{E}\ \exp(-2\pi\eta)\ S(E)$$

Guiding experiment

$^{14}N(p,\gamma)^{15}O$





LUNA
(under a mountain)
Circa 2005

Münster/Toronto
Circa 1987

Runkle et al., Phys. Rev Letts. 94, 082503 (2005)
Trautvetter *et al.*, J. Phys. G 35, 014019 (2008)
Marta *et al.*, Phys. Rev. C 78, 022802(R) (2008)
Adelberger *et al.*, Rev. Mod. Phys. 83, 195 (2012)

Low energy cross sections

- **We (usually) cannot measure the cross section...**
- **What determines the low-energy cross section?**
  - Tails of higher lying states?
  - Sub-threshold resonances?
  - Non-compound nuclear processes?
  - (Unobserved) resonances in the Gamow window?
- **What would help constrain this most?**
  - Direct measurements at higher energies?
  - Resonance widths? (From a different reaction?)
  - ANC measurements from transfer (for capture)?
  - Lifetime (and ANC) measurements for sub-threshold states?
  - Structure information from mirror nuclei?
  - New direct measurements of related channels?
  - New low energy direct measurements.........?

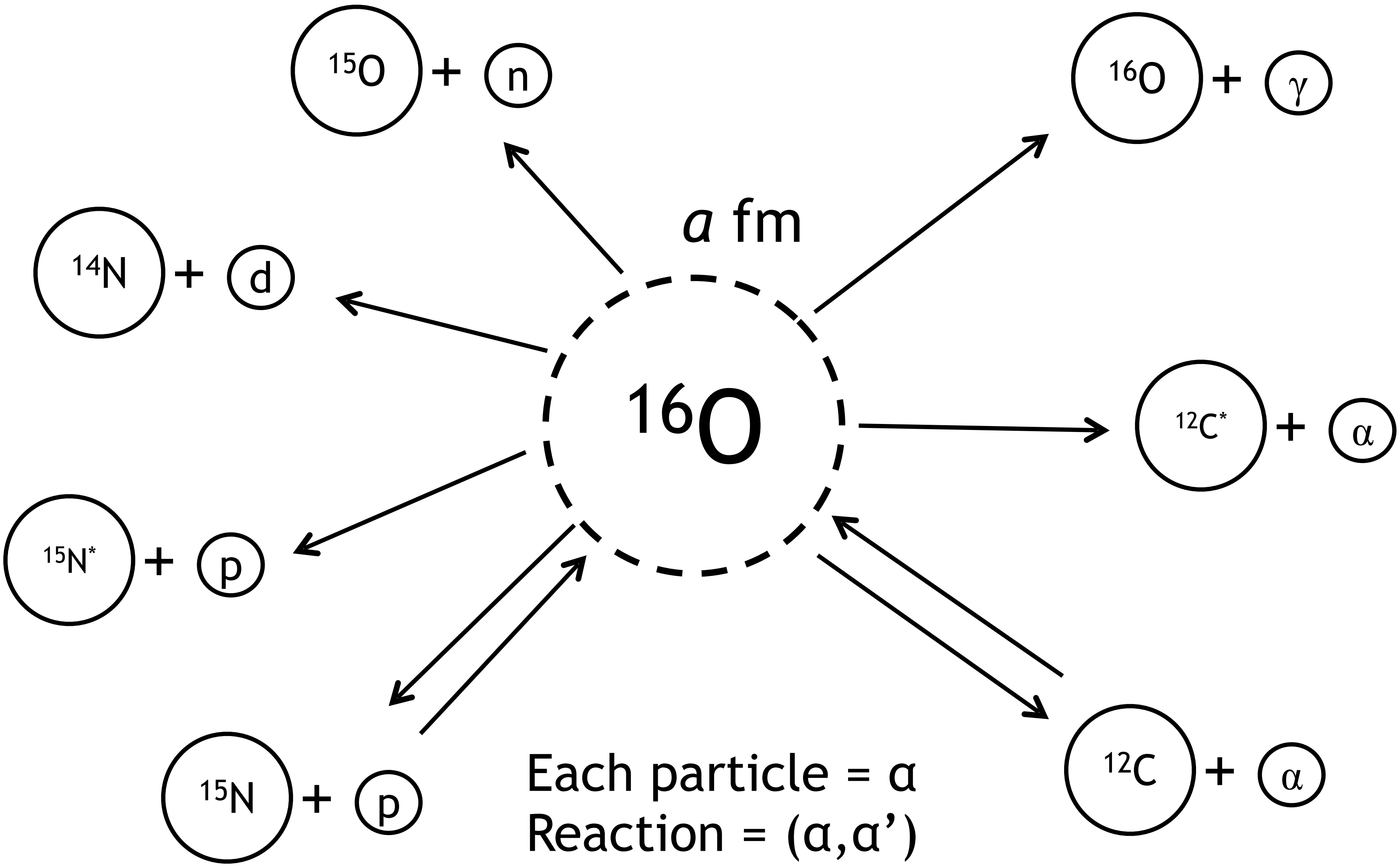
Materials analysis with ion beams

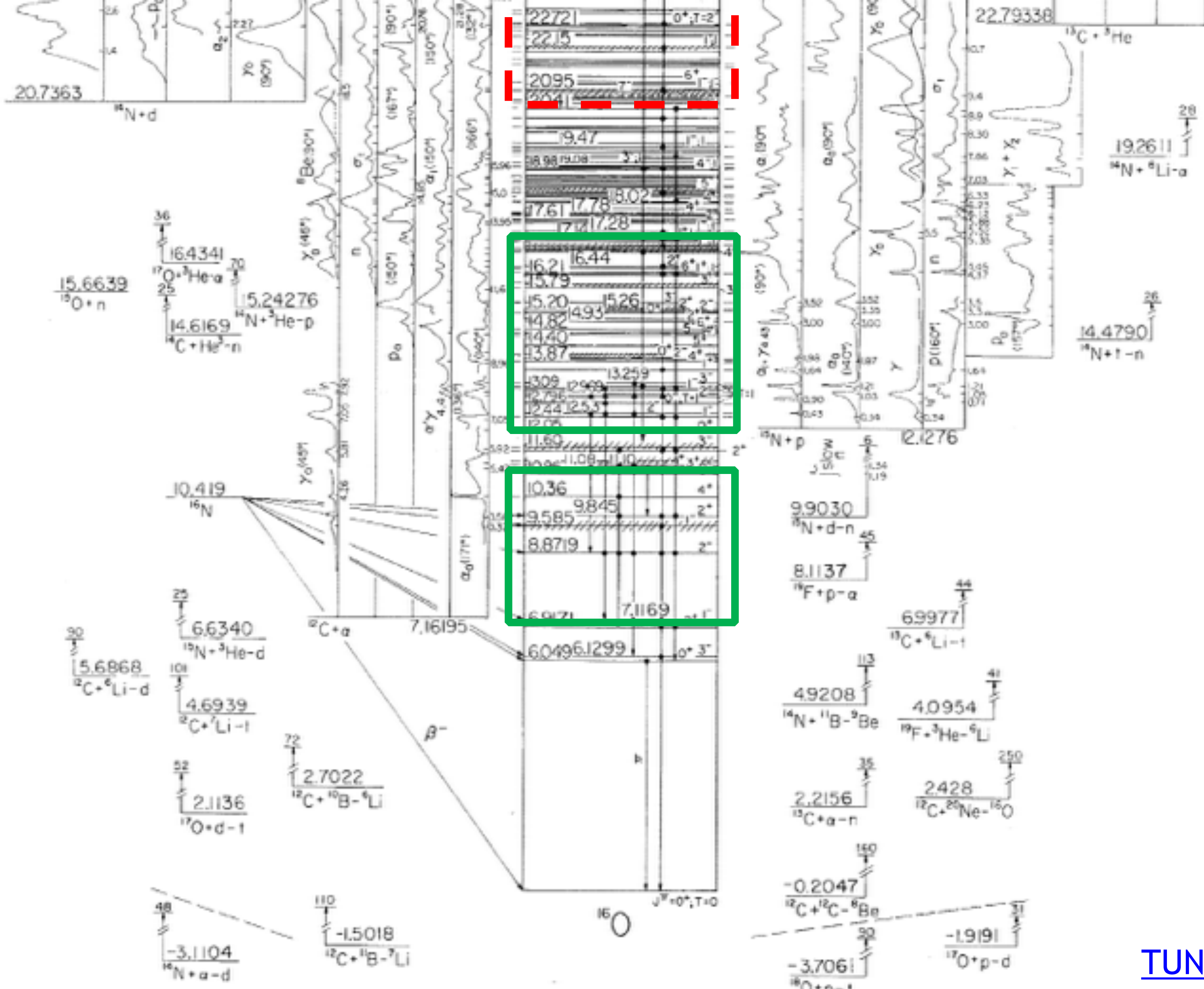
- **Scattering and reactions...**
  - Backward angle scattering cross sections
  - Reaction and capture cross sections
  - Wide variety of light-ion reactions on light-nuclei
  - Need to know at any energy and angle
- **What do we need?**
  - Scattering and reaction $\sigma(E,\theta)$...
  - Which angles and energies are most useful?
  - Which reactions are most important?
  - Understanding historic data sets? Verification experiments?
  - What nuclear data (energies and spin-parities) is available? Is this sufficient?
  - New scattering and reactions measurements?

Resonance reactions and R-matrix phenomenology

General approach, definitions and resonance parameters

Example: ^{16}O compound reactions





20.7363
$^{14}N+d$
22.79338
$^{13}C+{}^{3}He$
19.2611
$^{14}N+{}^{6}Li-\alpha$
16.4341
$^{17}O+{}^{3}He-\alpha$
15.6639
$^{15}O+n$
15.24276
$^{14}N+{}^{3}He-p$
14.6169
$^{14}C+{}^{3}He-n$
14.4790
$^{14}N+t-n$
12.1276
$^{15}N+p$
10.419
^{16}N
9.9030
$^{15}N+d-n$
8.1137
$^{19}F+p-\alpha$
6.9977
$^{13}C+{}^{6}Li-t$
7.16195
$^{12}C+\alpha$
6.6340
$^{15}N+{}^{3}He-d$
5.6868
$^{12}C+{}^{6}Li-d$
4.6939
$^{12}C+{}^{7}Li-t$
4.9208
$^{14}N+{}^{11}B-{}^{9}Be$
4.0954
$^{19}F+{}^{3}He-{}^{6}Li$
2.7022
$^{12}C+{}^{10}B-{}^{6}Li$
2.1136
$^{17}O+d-t$
2.2156
$^{13}C+\alpha-n$
2.428
$^{12}C+{}^{20}Ne-{}^{16}O$
-0.2047
$^{12}C+{}^{12}C-{}^{8}Be$
-1.5018
$^{12}C+{}^{11}B-{}^{7}Li$
-3.1104
$^{14}N+\alpha-d$
-1.9191
$^{17}O+p-d$
-3.7061
$^{18}O+p-t$
β^-
22.721
22.15
20.95
19.47
17.61
17.78
18.02
17.28
16.44
16.21
15.79
15.20
15.26
14.93
14.82
14.40
13.87
13.259
13.09
12.44
12.53
12.05
11.60
11.08
11.10
10.36
9.845
9.585
8.8719
7.1169
6.9171
6.049
6.1299
$J^{\pi}=0^{+};T=0$
^{16}O

TUNL

Definitions

- α = Particle pair
- s = channel spin
- ℓ = relative angular momentum
- $c = \alpha s\ell$ = channel, a particular particle pair with channel spin s and angular momentum ℓ
- λ = compound nucleus state
- $\gamma_{\lambda c}$ = reduced width amplitude
- *a* = R-matrix radius, marks the division between internal and external regions

Lane and Thomas, page 265-266, Rev. Mod. Phys. 30, 257 (1958)

Channel spin and angular momentum

$$I_1 = \text{Angular momentum of particle 1}$$

$$I_2 = \text{Angular momentum of particle 2}$$

$$\vec{s} = \vec{I}_1 + \vec{I}_2$$

$$s = |I_1 - I_2| \ldots I_1 + I_2$$

$$\vec{J} = \vec{s} + \vec{\ell}$$

$$J = |s - \ell| \ldots s + \ell$$

where $\pi = \pi_1 \pi_2 (-1)^\ell$ is parity of resonance

Lane and Thomas, page 265-266, [Rev. Mod. Phys. 30, 257 (1958)]

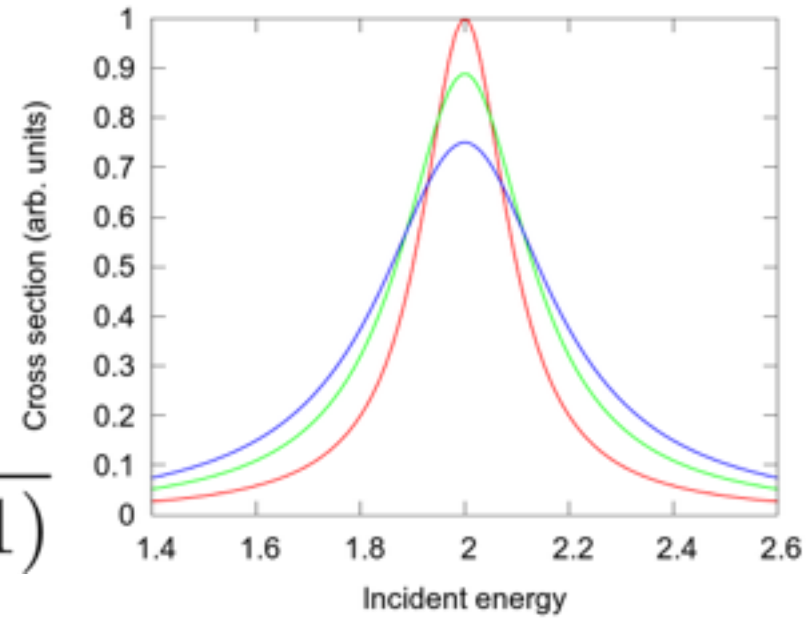
Breit-Wigner resonance

$$\sigma_{cc'}^{BW}(E) = \frac{\pi g_J}{k_c^2} \frac{\Gamma_c^o \Gamma_{c'}^o}{(E_R - E)^2 + (\Gamma^o/2)^2}$$

$$\Gamma^o = \sum_k \Gamma_k^o$$

$$\tau = \hbar/\Gamma^o$$

$$g_J = \frac{2J+1}{(2I_1+1)(2I_2+1)}$$



- Lifetime = $\hbar/\Gamma^o$ = 10^{-16} – 10^{-21} seconds (Γ^o= *eV – MeV*)
- Direct reactions timescales ~10^{-22} seconds

Breit-Wigner limits

- Half maximum at $(E_R \pm \Gamma^o/2)$
- If one width narrow e.g.

  $\Gamma_c = \Gamma_p$ = particle width

  $\Gamma_{c'} = \Gamma_g$ = gamma-ray width

$$\text{If } \Gamma_c^o >> \Gamma_{c'}^o \quad \sigma_{cc'}^{BW}(E_R) \propto \frac{\Gamma_c^o \Gamma_{c'}^o}{(\Gamma_c^o + \Gamma_{c'}^o)^2} \approx \frac{\Gamma_{c'}^o}{\Gamma_c^o}$$

- In fitting, widths play different roles

  Γ_c determines shape and width

  $\Gamma_{c'}$ determines peak cross section

- Other channels (i.e. $\Gamma_{total} = \Gamma_c + \Gamma_{c'} + \Gamma_X$) broaden the resonance and reduce the peak cross section

R-matrix parameters

- Level energies and reduced width amplitudes

$$R^J_{cc'} = \sum_\lambda^\infty \frac{\gamma_{\lambda c' J}\gamma_{\lambda c J}}{E_\lambda - E}$$

Treat **energies** and **widths** as parameters and fit to reproduce data

$$\gamma_{\lambda c J} = \left(\frac{\hbar^2}{2\mu_c a_c}\right)^{1/2} \int dS\, \varphi_c^* X_{\lambda J M}$$

$$H X_{\lambda J M} = E_{\lambda J} X_{\lambda J M}$$

- Radius a_c [and (arbitrary) boundary condition B_c]
- How can we relate to Breit-Wigner-type parameters?

Lane and Thomas, page 276, Rev. Mod. Phys. 30, 257 (1958)

R-matrix cross section

- Angle-integrated cross section:

$$\sigma_{\alpha\alpha'} = \frac{\pi g_J}{k_\alpha^2} \sum_{J\ell\ell' ss'} \left| e^{2i\omega_c} \delta_{cc'} - U_{cc'} \right|^2$$

- Matrix expression for collision (scattering)matrix:

$$\boldsymbol{U} = \boldsymbol{\Omega} \left[\boldsymbol{1} + \boldsymbol{P}^{\frac{1}{2}} \frac{\boldsymbol{R}}{\boldsymbol{1} - \boldsymbol{R}\boldsymbol{L}^0} \boldsymbol{P}^{\frac{1}{2}} \boldsymbol{w} \right] \boldsymbol{\Omega}$$

$$\boldsymbol{L}^0 = \boldsymbol{S} + i\boldsymbol{P} - \boldsymbol{B}$$

Lane and Thomas, page 271, 89, 293, Rev. Mod. Phys. 30, 257 (1958)

Single resonance XS (via A-matrix)

$$\sigma_{cc'}(E) = \frac{\pi g_J}{k_c^2} \frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{(E_\lambda + \Delta_\lambda - E)^2 + (\Gamma_\lambda/2)^2}$$

$$\Gamma_{\lambda c} = 2P_c\gamma_{\lambda c}^2$$

$$\Delta_\lambda = -\sum_k \gamma_{\lambda k}^2 [\boxed{S_k(E)} - \boxed{B_k}]$$

Arbitrary boundary condition

Shift-function

$$\Gamma_\lambda = \sum_k \Gamma_{\lambda k}$$

$$g_J = \frac{2J+1}{(2I_1+1)(2I_2+1)}$$

Lane and Thomas, page 327, [Rev. Mod. Phys. 30, 257 (1958)]

Carl Brune [http://arxiv.org/abs/nucl-th/0502087]

Decouvemont and Baye, [Rep. Prog. Phys 73, 036301 (2010)]

Observed and formal parameters

- Thomas approximation: choose $B_k=S(E_\lambda)$, then Taylor expand $S(E)$ to define one relationship between physical (BW) and formal (R-matrix) parameters

Formal

$$\Gamma^o_{\lambda c} = \frac{2P_c\tilde{\gamma}^2_{\lambda c}}{1+\sum_k \tilde{\gamma}^2_{\lambda k}\left(\frac{dS_k}{dE}\right)_{\tilde{E}_\lambda}}$$

Observed

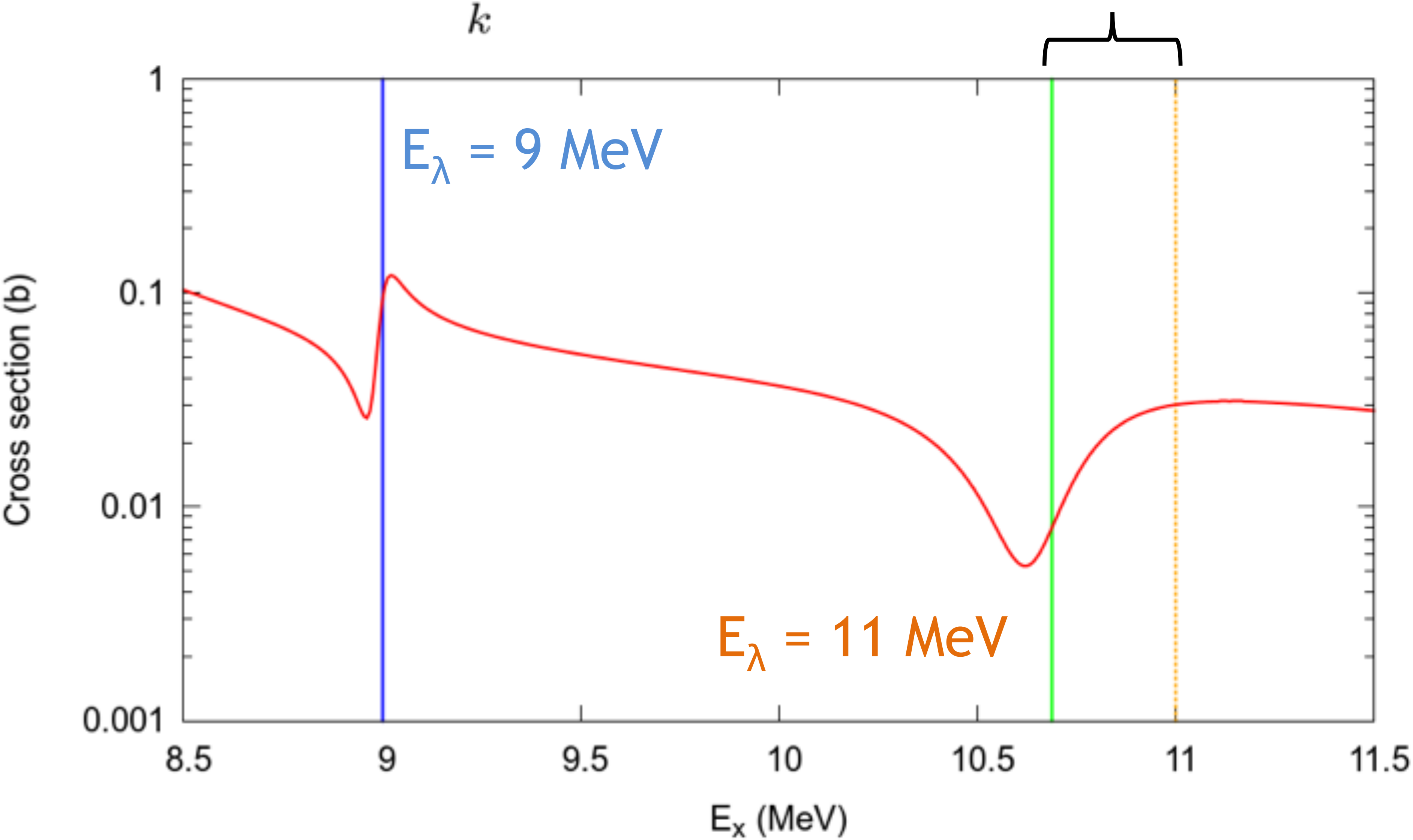
$$\tilde{\gamma}^2_{\lambda c} = \frac{\Gamma^o_{\lambda c}}{P_c}\left[2-\sum_k \frac{\Gamma^o_{\lambda k}}{P_k}\left(\frac{dS_k}{dE}\right)_{\tilde{E}_\lambda}\right]^{-1}$$

“On-resonance” parameters

Lane and Thomas, page 327, [Rev. Mod. Phys. 30, 257 (1958)]

Carl Brune http://arxiv.org/abs/nucl-th/0502087

$$\Delta_\lambda = -\sum_k \tilde{\gamma}^2_{\lambda k}[S_k(E) - B_k] \approx -0.31 \text{ MeV}$$



Alternative parameterization

- Alternative parameterization

$$\{\tilde{E}_\lambda, \tilde{\gamma}_{\lambda c}\} \rightarrow \{E_\lambda, \gamma_{\lambda c}, B_c\}$$

$$\uparrow$$

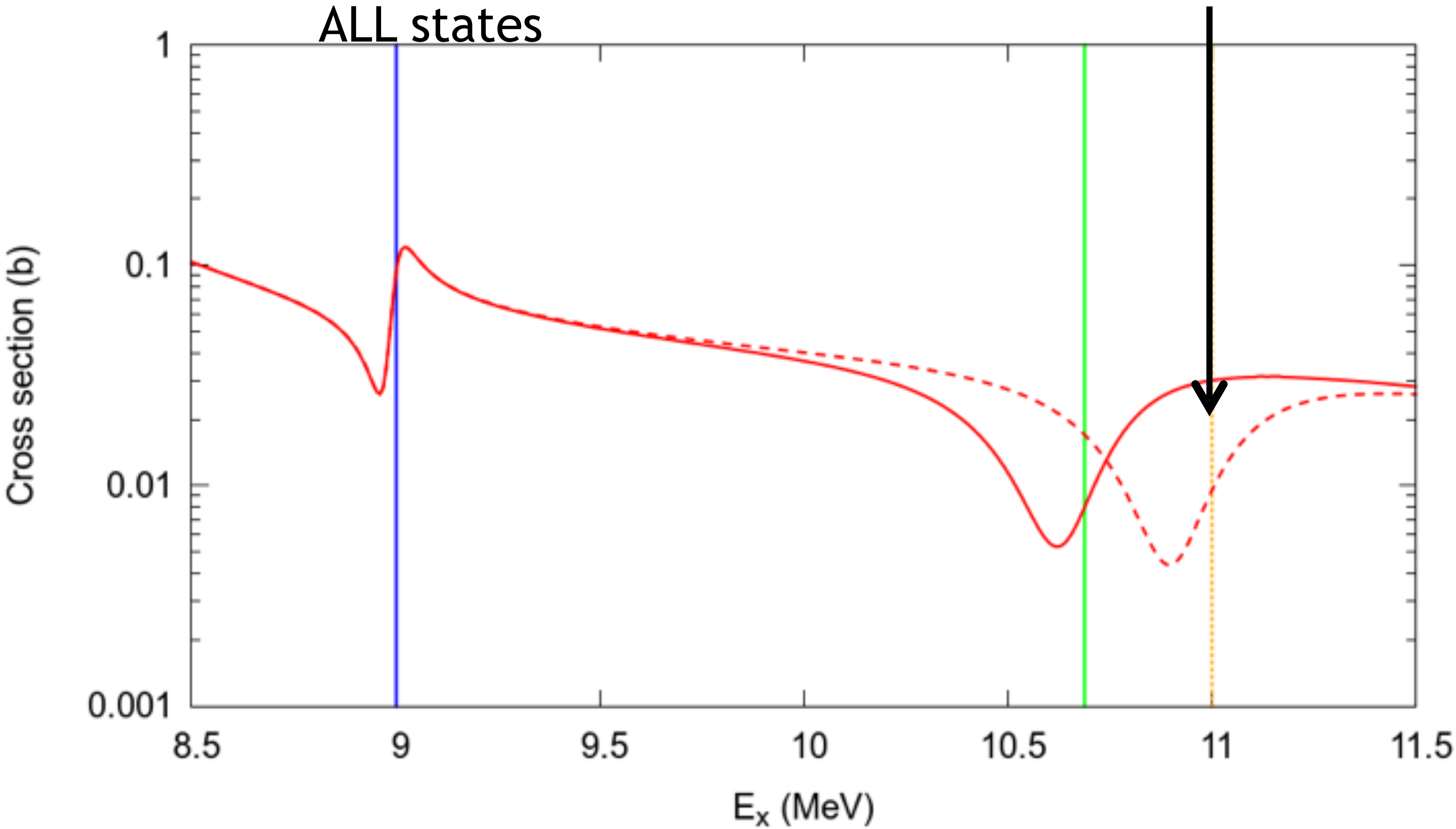
- Inputting   far easier
- Also write cross section in terms of alternative parameters – no boundary condition
- In fitting, can now fix some energies and widths

Angulo and Descouvemont, Phys. Rev. C 61, 064611 (2000)

Carl Brune, Phys. Rev. C 66, 044611 (2002)

Decouvemont and Baye, page 28/29, Rep. Prog. Phys 73, 036301 (2010)

Alternative parameterization allows you to input on-resonance energies and widths for ALL states



Parameter Input in AZURE2

$$\{\tilde{E}_\lambda, \Gamma^o_{\lambda c}\}$$

Sign not determined

$$\{\tilde{E}_\lambda, \tilde{\gamma}_{\lambda c}\}$$

Calculation

$$\{E_\lambda, \gamma_{\lambda c}\}$$

Where a boundary condition required it is always set to the shift function at energy of the lowest level for each spin-parity

Fixing parameters

Only if all parameters of a given resonance are fixed will they all not change in fitting

$$\Gamma^o_{\lambda c} = \frac{2P_c\tilde{\gamma}^2_{\lambda c}}{1 + \boxed{\sum_k \tilde{\gamma}^2_{\lambda k}\left(\frac{dS_k}{dE}\right)_{\tilde{E}_\lambda}}}$$

$$\tilde{\gamma}^2_{\lambda c} = \frac{\Gamma^o_{\lambda c}}{P_c}\left[2 - \sum_k \frac{\Gamma^o_{\lambda k}}{P_k}\left(\frac{dS_k}{dE}\right)_{\tilde{E}_\lambda}\right]^{-1}$$

But you should be able to more easily establish a good set of starting parameters

The Wigner limit

- Defines maximum value of formal reduced width amplitude

$$\gamma_{Wc}^2 = \frac{3\hbar^2}{2\mu_c a_c^2} \quad \sim \text{MeV}$$

$$\theta_{\lambda c}^2 = \gamma_{\lambda c}^2 / \gamma_{Wc}^2$$

- Values of θ near 1 (i.e. γ large fraction of γ_W) indicates cluster-type states, normally much smaller
- Dependent on the R-matrix radius *a*
- Γ_W varies with energy due to penetrability

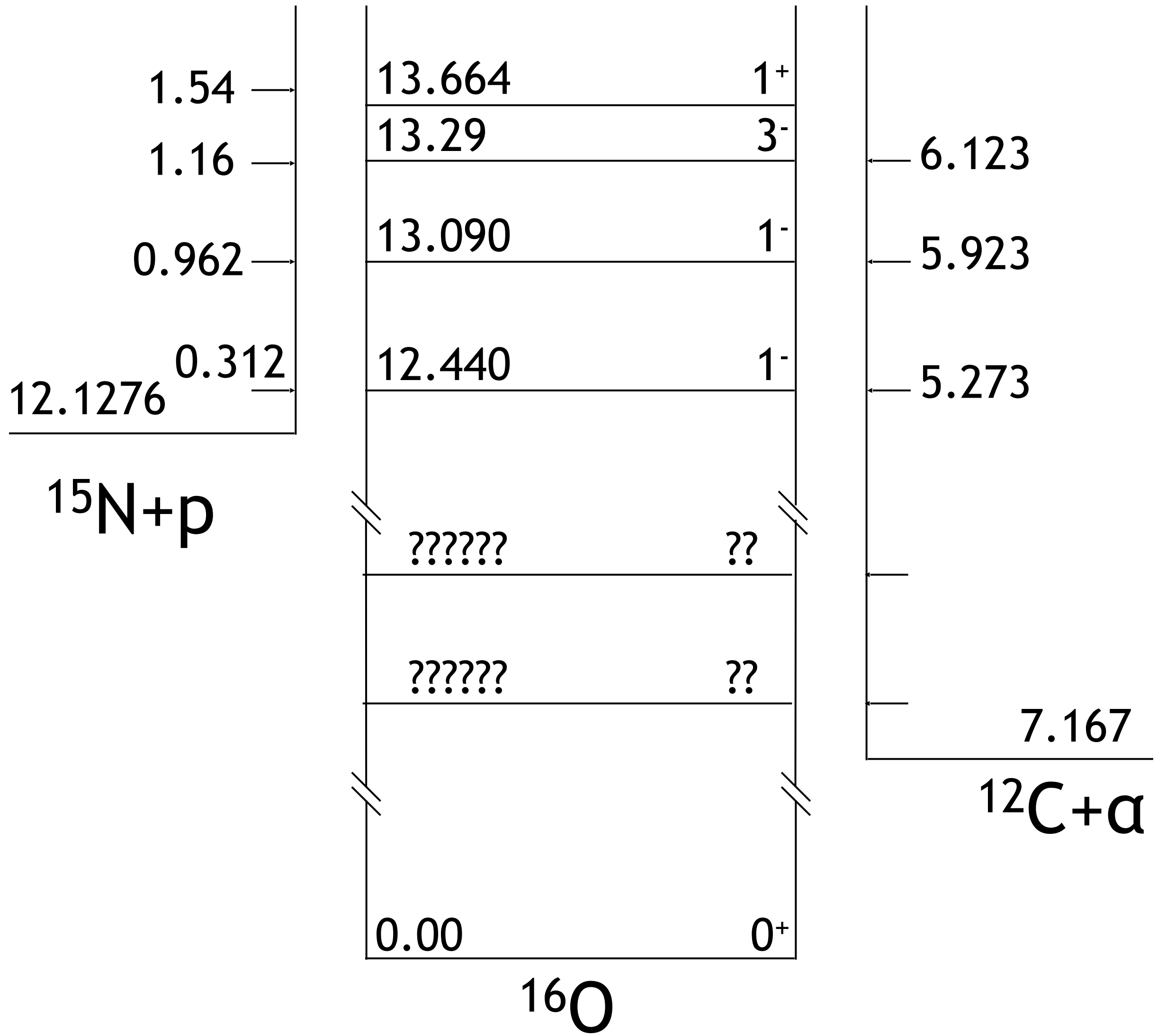
Teichmann and Wigner, Phys. Rev. 87, 123 (1952)
Decouvemont and Baye, Rep. Prog. Phys 73, 036301 (2010)
Freer *et al.*, Phys. Rev. C 85, 014304 (2012)

Analysis of elastic scattering and reactions

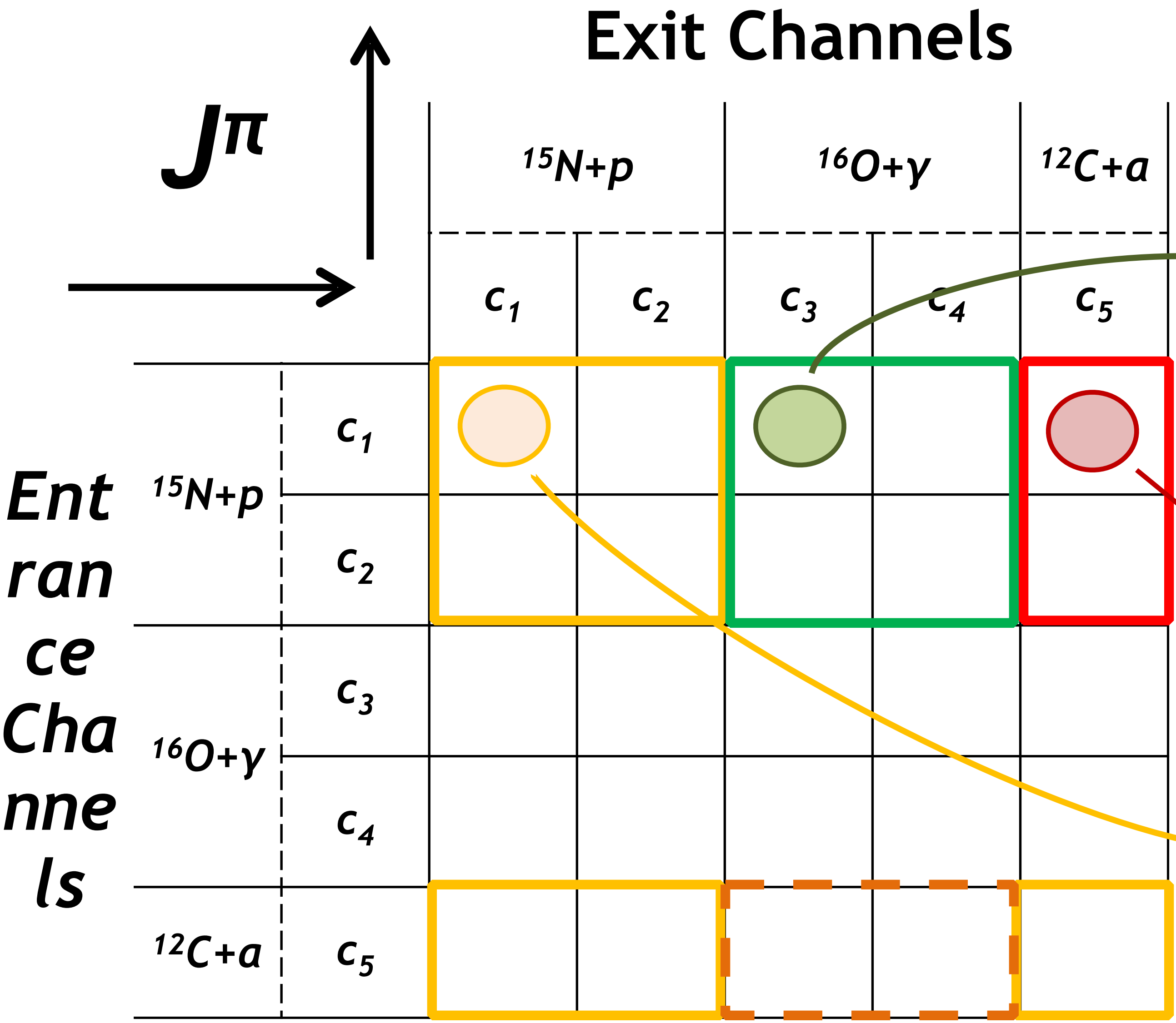
Alpha-particle scattering on ^{12}C

Closing the CNO cycle: $^{15}N(p,\alpha)^{12}C$

^{16}O thresholds: $^{12}C(\alpha,\alpha)^{12}C$, $^{15}N(p,\alpha)^{12}C$



^{16}O R-matrix: ^{15}N(p,γ)^{16}O



$$R_{c_3 c_1} = \sum_{\lambda} \frac{\gamma_{\lambda c_3} \gamma_{\lambda c_1}}{E_\lambda - E}$$

$$R_{c_5 c_1} = \sum_{\lambda} \frac{\gamma_{\lambda c_5} \gamma_{\lambda c_1}}{E_\lambda - E}$$

$$R_{c_1 c_1} = \sum_{\lambda} \frac{\gamma_{\lambda c_1} \gamma_{\lambda c_1}}{E_\lambda - E}$$

Channels for $^{12}C+\alpha$

- Calculated automatically by AZURE2
- For $^{12}C+\alpha$ things are simple:

$$
\begin{aligned}
I_1 =& 0 \qquad & \pi_1 =& 1 \\
I_2 =& 0 \qquad & \pi_2 =& 1 \\
\therefore s =& 0 \\
\text{and } J =& \ell & \pi =& \pi_1 \pi_2 (-1)^\ell = (-1)^\ell
\end{aligned}
$$

- One channel per J, only natural parity states allowed $\pi=(-1)^J$
- In general, many channels allowed per resonance

Hard-sphere phase shift

- For elastic scattering, must include all hard sphere contributions

$$U = \mathbf{\Omega}\left[1 + \mathbf{P}^{\frac{1}{2}} \frac{\mathbf{R}}{1 - \mathbf{R}\mathbf{L}^0} \mathbf{P}^{\frac{1}{2}} \mathbf{w}\right] \mathbf{\Omega}$$

$$\Omega_c^+ = \exp[i(\omega_c - \phi_c^+)]$$

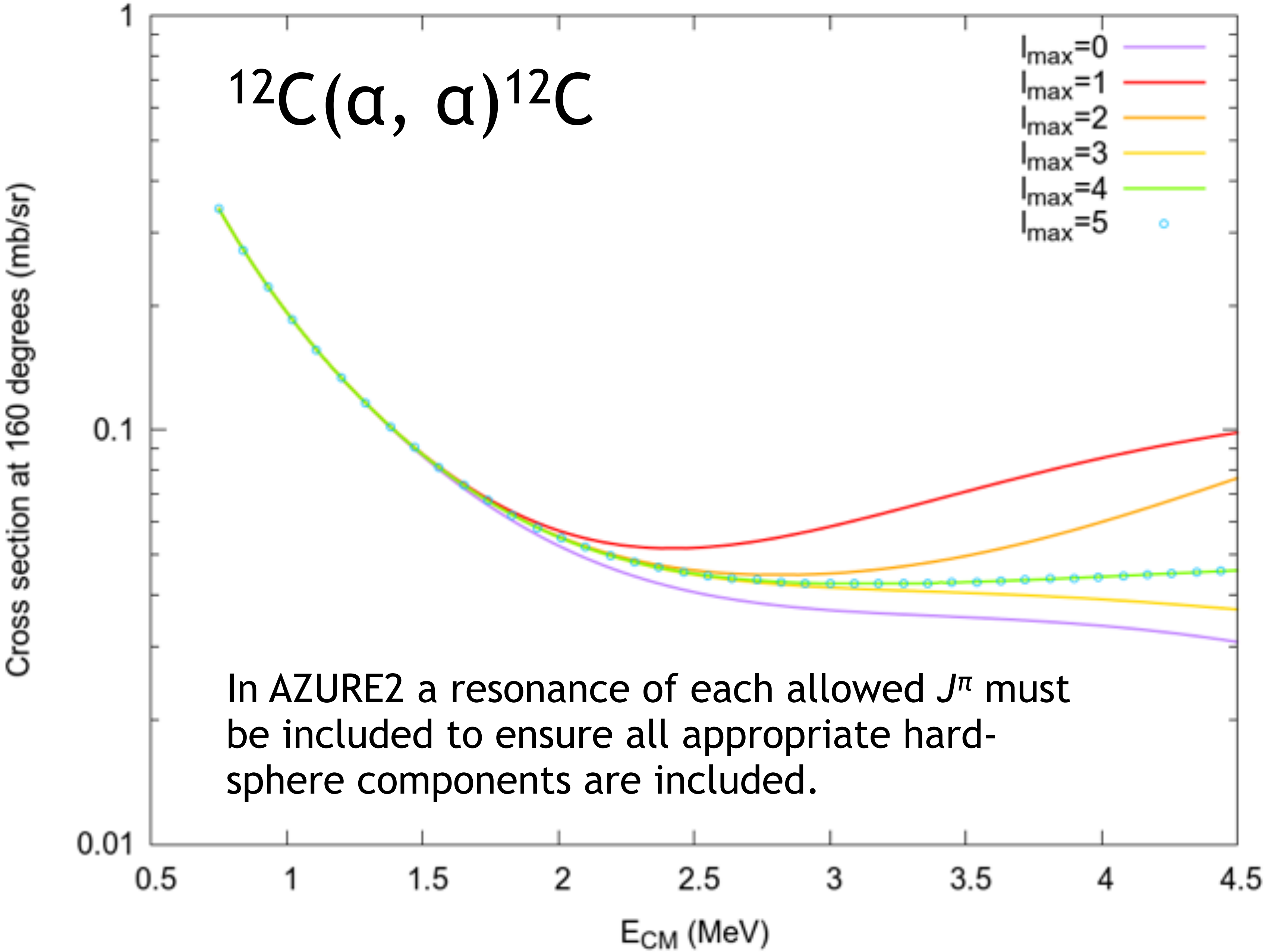
$$\delta_{HS} = -\phi_c^+ = -\tan^{-1}(F_c/G_c)$$

L. V. Namjoshi *et al.*, PRC 13, 915 (1976)

Ruiz *et al.*, PRC 71, 025802 (2005)

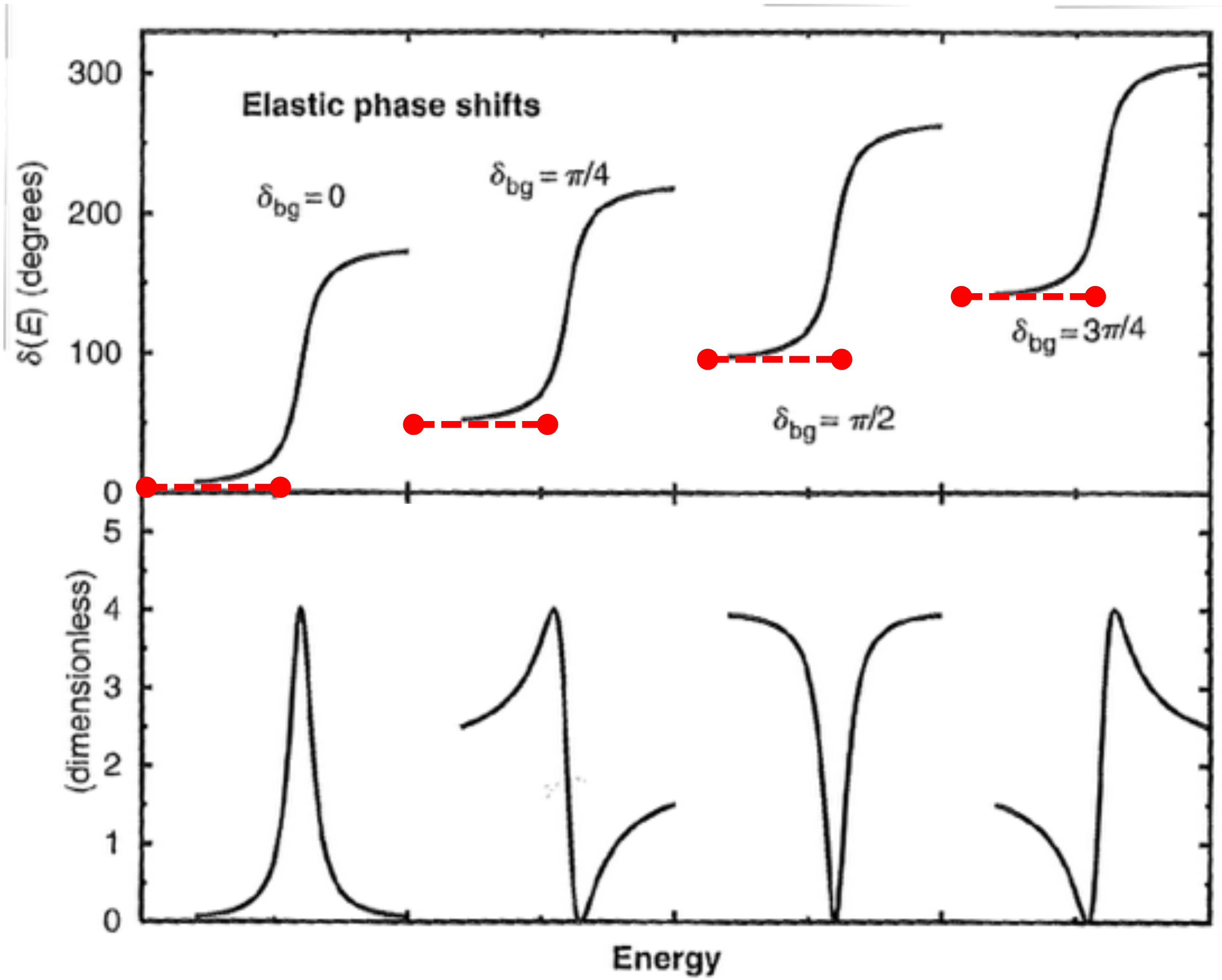
Lane and Thomas, page 271 and 289, Rev. Mod. Phys. 30, 257 (1958)

$^{12}C(\alpha, \alpha)^{12}C$



In AZURE2 a resonance of each allowed J^{π} must be included to ensure all appropriate hard-sphere components are included.

Resonance shapes

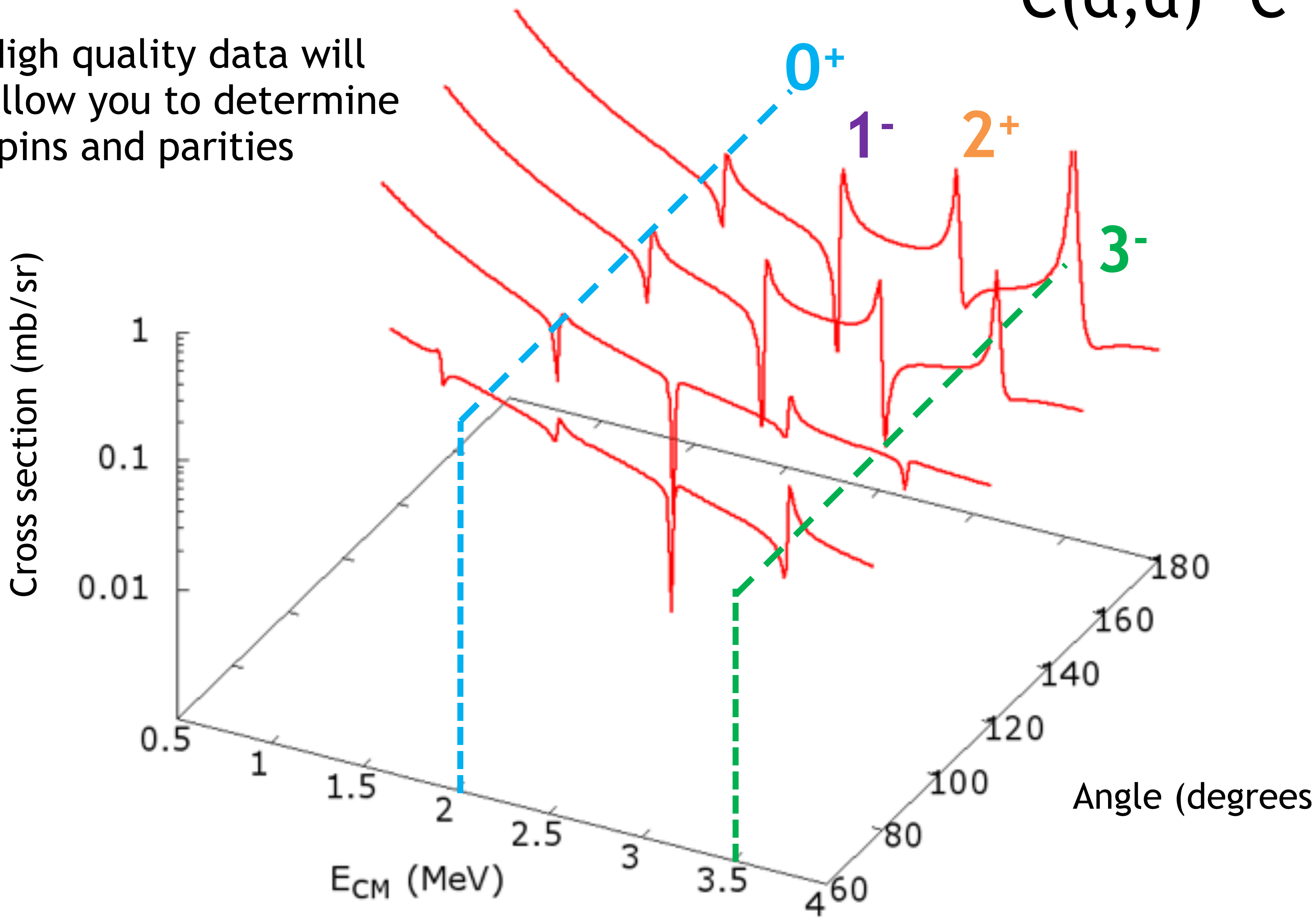


Thompson and Nunes, *Nuclear Reactions for Astrophysics*, Cambridge University Press (2009)

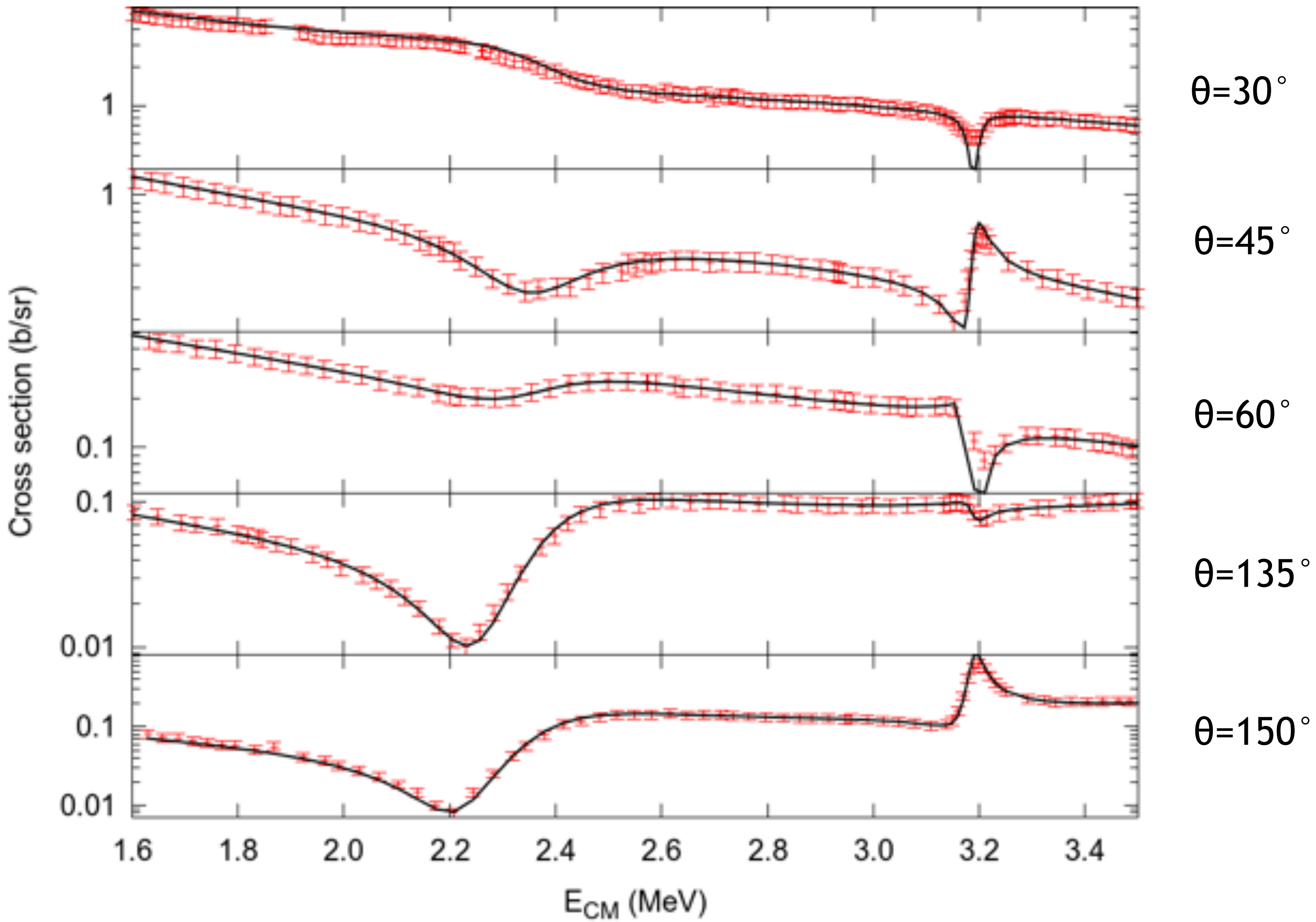
Angular distributions

$^{12}C(\alpha,\alpha)^{12}C$

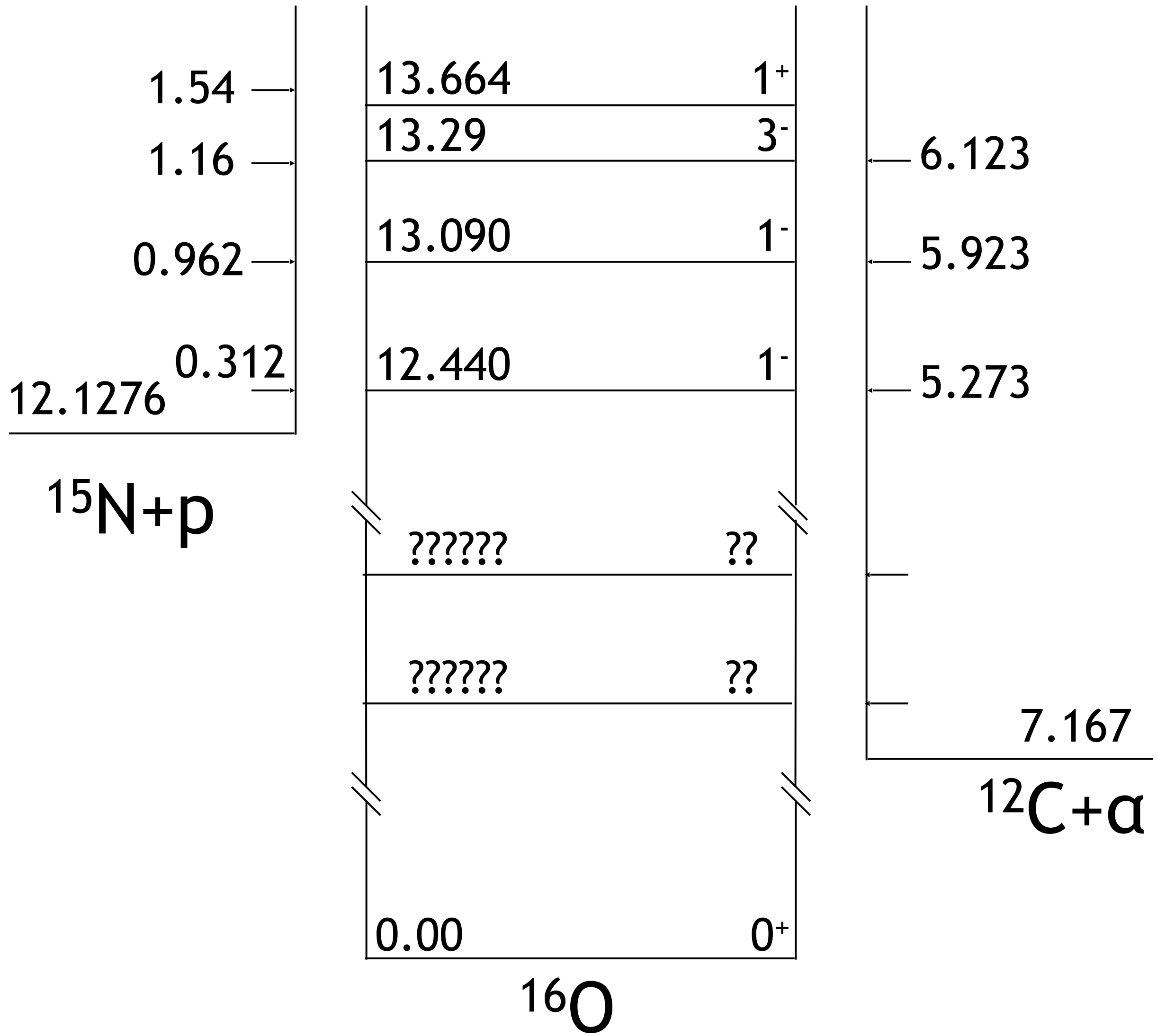
High quality data will allow you to determine spins and parities



$^{12}C(\alpha,\alpha)^{12}C$



^{16}O thresholds: $^{12}C(\alpha,\alpha)^{12}C$, $^{15}N(p,\alpha)^{12}C$



Proton channels in $^{15}N(p,\alpha)^{12}C$

$$I_1 = 1/2, \quad \pi_1 = +$$

$$I_2 = 1/2, \quad \pi_2 = -$$

$$\therefore s = 0, 1$$

^{15}N+p

$$\text{e.g.} \quad J^\pi = 0^+, \quad \boxed{s = 1, \ell = 1}$$

$$J^\pi = 1^-, \quad \boxed{s = 1, \ell = 0}$$

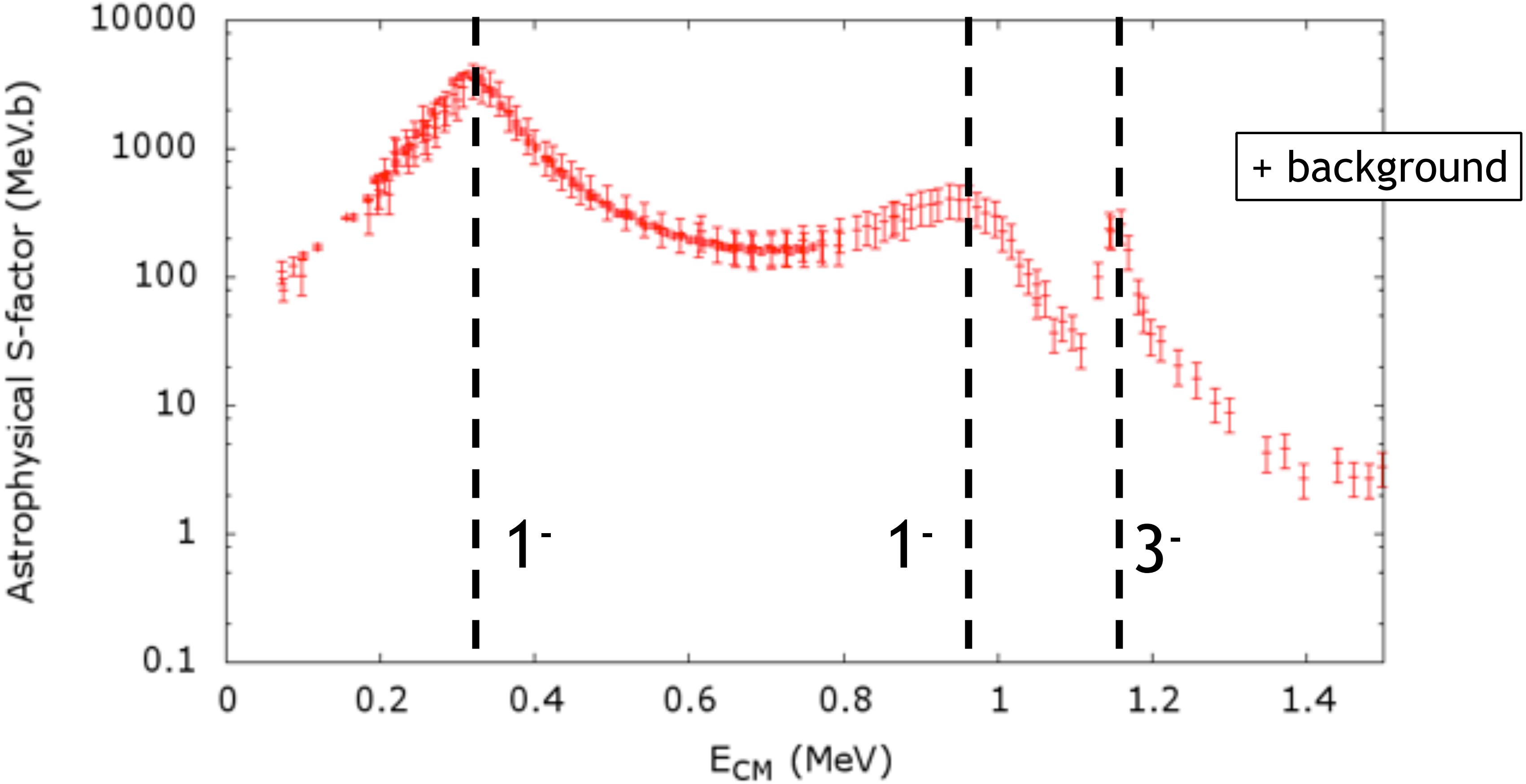
$$\boxed{s = 1, \ell = 2}$$

$$J^\pi = 2^+, \quad \boxed{s = 1, \ell = 1}$$

$$\boxed{s = 1, \ell = 3}$$

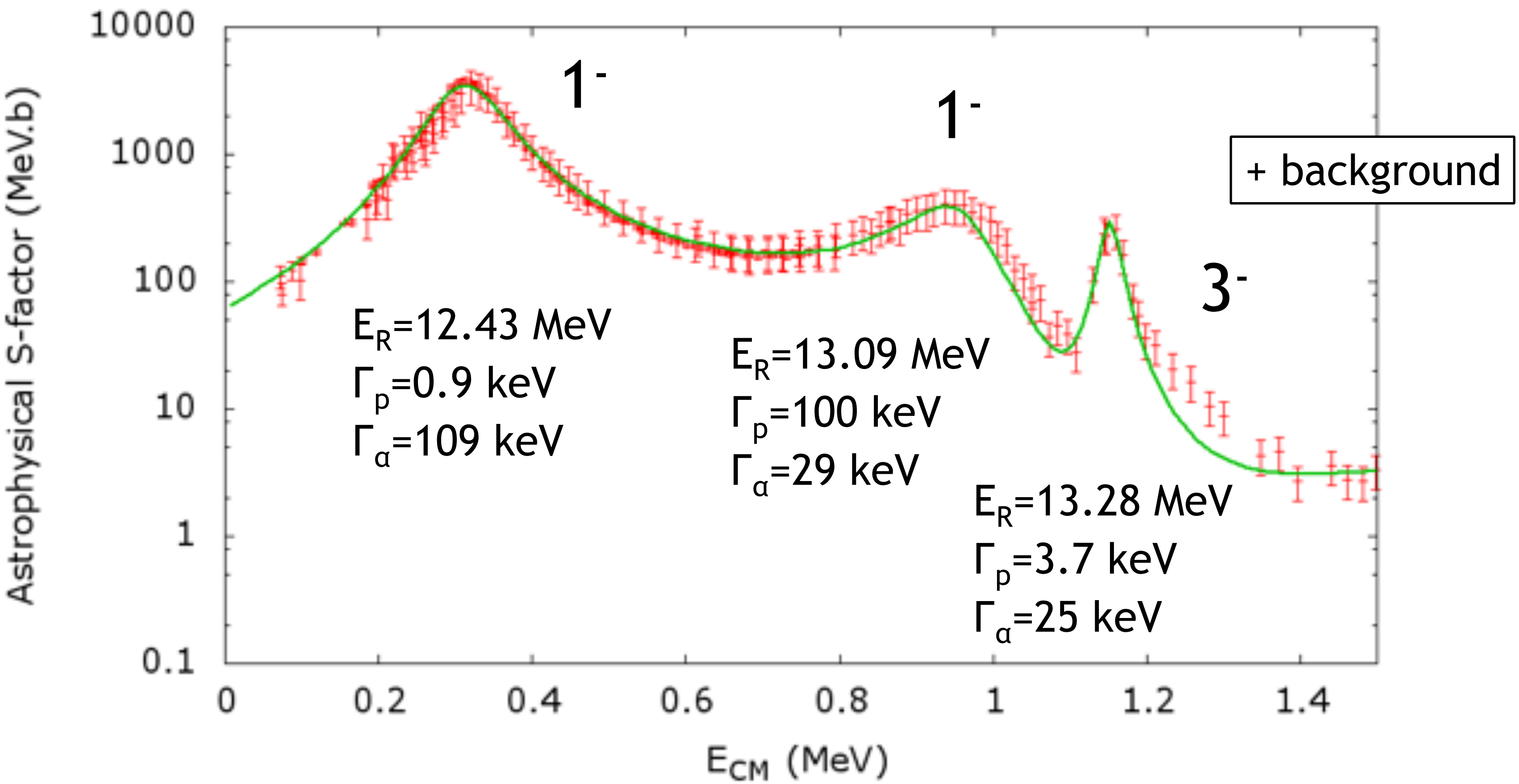
- ***s=0*** doesn't occur for natural parity resonances

$^{15}N(p,\alpha)^{12}C$

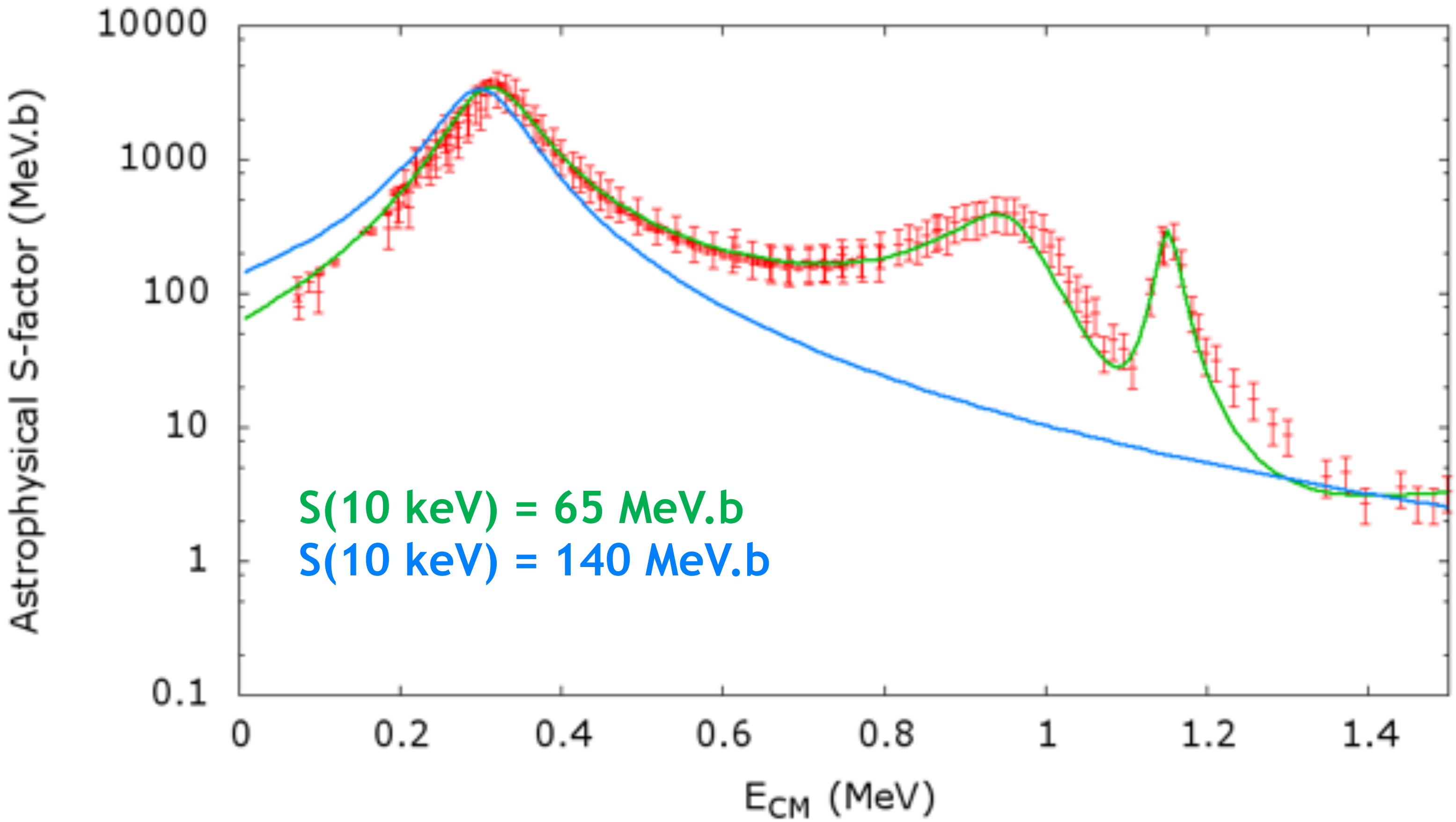


$^{15}N(p,\alpha)^{12}C$

3 states, 9 parameters (assuming lowest ℓ)
+ background resonance parameters

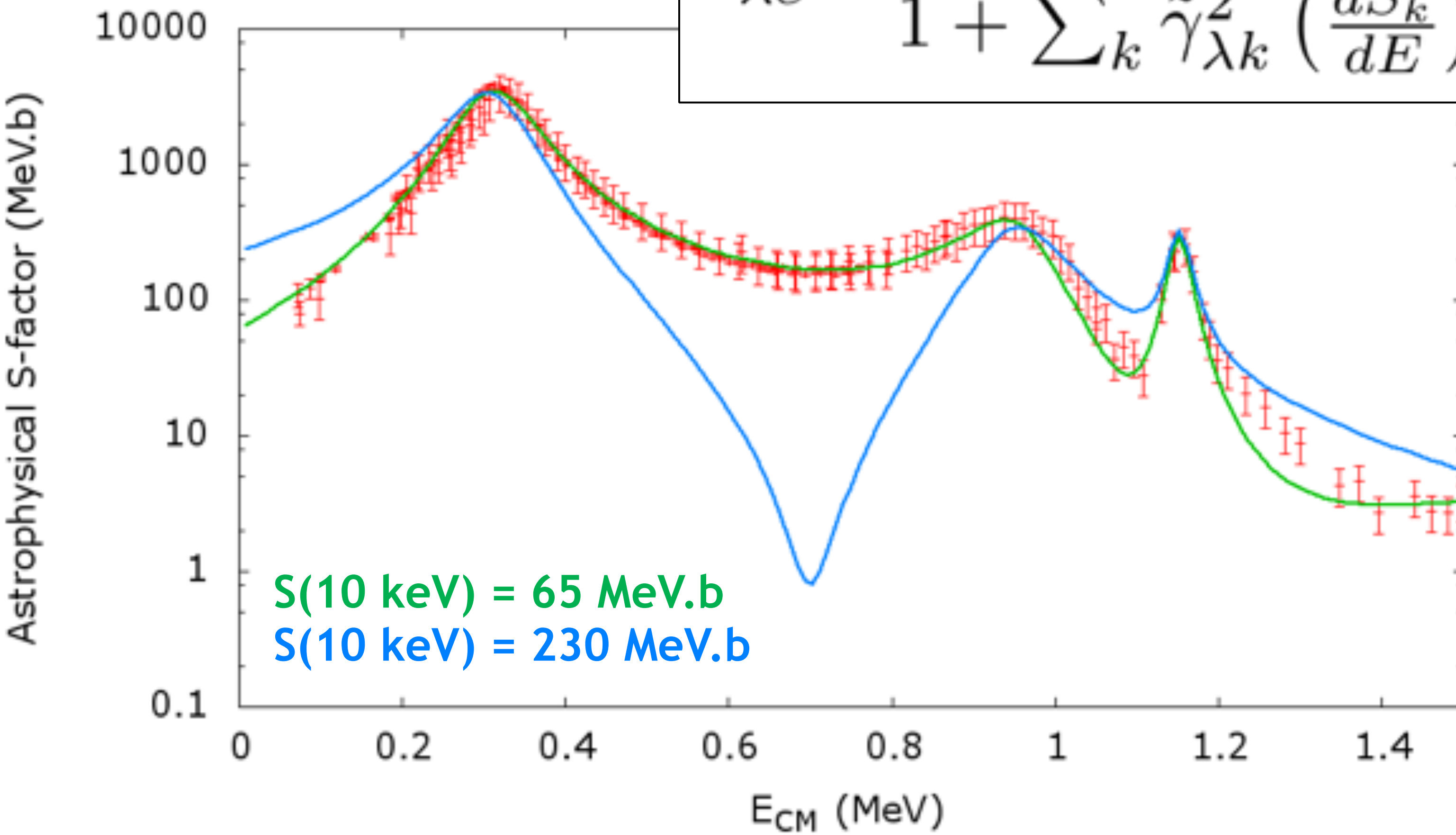


$^{15}N(p,\alpha)^{12}C$

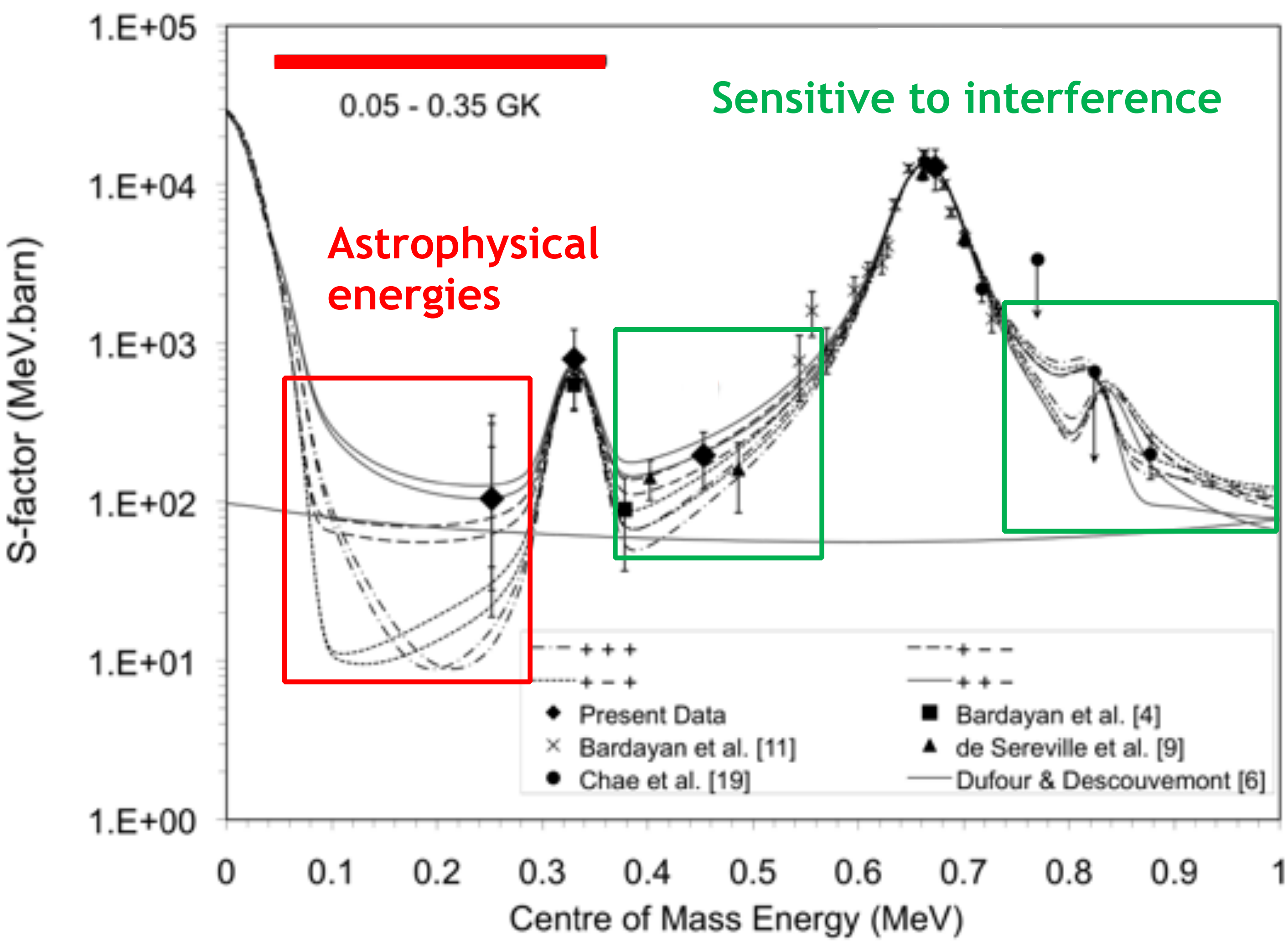


$^{15}N(p,\alpha)^{12}C$

$$\Gamma^{o}_{\lambda c} = \frac{2P_c\tilde{\gamma}^2_{\lambda c}}{1+\sum_k \tilde{\gamma}^2_{\lambda k}\left(\frac{dS_k}{dE}\right)_{\tilde{E}_\lambda}}$$



$^{18}F(p,\alpha)^{15}O$ - interferences



C. Beer et al., Phys. Rev. C 83, 042801(R) (2011)
via Alison Laird

Summary

R-matrix may be used to parameterize cross sections, allowing you to calculate the cross section where you do not have data (e.g. in astrophysics, materials analysis) and extract resonance properties.

Alternative parameterizations of the R-matrix exist to relate formal reduced width amplitudes arising from integrals over the compound nucleus surface to “observed”-type widths Γ.

R-matrix can be used to understand the essential structure and elements determining the cross section at astrophysical energies. This can help guide future experiments.