Incidences, tilings, and fields: auxiliary computations

M. Skopenkov

In Sections 1, 2, and 3, we apply the same algorithm to the matrices arising in Cases 1,2, and 3 respectively of the proof of Example 3.13

Section 1. Checking of Case 1

```
In[•]:= niterations = 3;
m = 12; n = 9;
matrix =
  {{0, 0, 0, 1, 0, 0, 0, 0, 1}, {1, 1, -1, 0, 1, -1, 0, 0, 0}, {0, 1, -1, 0, -1, 1, 1, 0, 0},
   {0, 1, -1, 0, -1, -1, 0, 1, 1}, {0, -1, 1, 0, -1, -1, 1, 0, 1}, {0, -1, 1, 0, 1, -1, 0, 1, 0},
   {1, -1, 1, 0, -1, 1, 0, 0, 0}, {0, 0, 0, 1, 1, 0, 1, 0, 0}, {0, 0, 0, 1, 0, 1, 0, 1, 0},
   {0, 1, 1, -1, 0, 0, 0, 0, 0}, {0, 0, 1, 1, 0, 0, 0, 0, 0}, {0, 1, 0, 1, 0, 0, 0, 0, 0}};
matrix // MatrixForm
(* Small tests: m=3; n=3;
matrix={{1,1,-1},{1,1,1},{-1,1,1}};*)
(* m=4; n=4;
matrix={{0,1,1,-1},{0,1,-1,1},{1,1,1,1},{-1,1,0,-1}};*)
IncidenceAxiomContradictionQ[submatrix_] :=
  If[submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, 1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, 0}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, -1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, 1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, 0}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, -1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, 1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, 0}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, -1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, 1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, 0}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, -1}}, True, False];
For[iteration = 1, iteration ≤ niterations, iteration++,
 For[i1 = 1, i1 ≤ m, i1++,
  For[j1 = 1, j1 ≤ n, j1++,
   If[matrix[[i1, j1]] == 0,
    For[i2 = 1, i2 ≤ m, i2++,
     For[i3 = 1, i3 ≤ m, i3++,
      For[j2 = 1, j2 ≤ n, j2++,
       For[j3 = 1, j3 ≤ n, j3++,
        (*Print[i2,i3,j2,j3]*)
        If[i2 ≠ i1 && i3 ≠ i1 && i2 ≠ i3 &&
          j2 ≠ j1 && j3 ≠ j1 && j2 ≠ j3 && IncidenceAxiomContradictionQ[
           {{1, matrix[[i1, j2]], matrix[[i1, j3]]}, {matrix[[i2, j1]], matrix[[i2, j2]],
             matrix[[i2, j3]]}, {matrix[[i3, j1]], matrix[[i3, j2]], matrix[[i3, j3]]}}],
         (*Print[i2,i3,j2,j3];
         Print[{{1,matrix[[i1,j2]],matrix[[i1,j3]]},{matrix[[i2,j1]],matrix[[i2,j2]],
            matrix[[i2,j3]]},{matrix[[i3,j1]],matrix[[i3,j2]],matrix[[i3,j3]]}}];*)
         matrix[[i1, j1]] = -1
        ]]]]]]]]]
matrix // MatrixForm
```

Out[$\circ$]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[$\circ$]//MatrixForm=

$$\begin{pmatrix} 0 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

Section 2. Checking of Case 2

In[ ]:=
```
niterations = 3;
m = 16; n = 10;
matrix = {{-1, 0, 0, 1, 0, 0, 0, 0, 1, 0}, {1, 1, -1, 0, 1, -1, 0, 0, 0, 0},
   {0, 1, -1, 0, -1, 1, 1, 0, 0, 0}, {0, 1, -1, 0, -1, -1, 0, 1, 1, 0},
   {0, -1, 1, 0, -1, -1, 1, 0, 1, 0}, {0, -1, 1, 0, 1, -1, 0, 1, 0, 0},
   {1, -1, 1, 0, -1, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 1, 0, 1, 0, 0, 1},
   {0, 0, 0, 1, 0, 1, 0, 1, 0, 0}, {0, 1, 1, 0, 0, 0, 0, 0, 0, -1}, {0, 0, 1, 1, 0, 0, 0, 0, 0, 0},
   {0, 1, 0, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0, 1, 1},
   {0, 0, 0, 0, 0, 1, 0, 0, 0, 1}, {0, 0, 1, 0, 0, 0, 0, 0, 0, 1}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 1}};
matrix // MatrixForm
IncidenceAxiomContradictionQ[submatrix_] :=
  If[submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, 1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, 0}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, -1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, 1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, 0}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, -1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, 1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, 0}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, -1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, 1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, 0}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, -1}}, True, False];
For[iteration = 1, iteration ≤ niterations, iteration++,
 For[i1 = 1, i1 ≤ m, i1++,
  For[j1 = 1, j1 ≤ n, j1++,
   If[matrix[[i1, j1]] == 0,
    For[i2 = 1, i2 ≤ m, i2++,
     For[i3 = 1, i3 ≤ m, i3++,
      For[j2 = 1, j2 ≤ n, j2++,
       For[j3 = 1, j3 ≤ n, j3++,
        If[i2 ≠ i1 && i3 ≠ i1 && i2 ≠ i3 &&
          j2 ≠ j1 && j3 ≠ j1 && j2 ≠ j3 && IncidenceAxiomContradictionQ[
           {{1, matrix[[i1, j2]], matrix[[i1, j3]]}, {matrix[[i2, j1]], matrix[[i2, j2]],
             matrix[[i2, j3]]}, {matrix[[i3, j1]], matrix[[i3, j2]], matrix[[i3, j3]]}}],
         matrix[[i1, j1]] = -1
        ]]]]]]]]]
matrix // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix}
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix}
-1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1
\end{pmatrix}$$

Section 3. Checking of Case 3

In[ ]:=
```
niterations = 3;
m = 16; n = 10;
matrix = {{-1, 0, 0, 1, 0, 0, 0, 0, 1, 0}, {1, 1, -1, 0, 1, -1, 0, 0, 0, 0},
   {0, 1, -1, 0, -1, 1, 1, 0, 0, 0}, {0, 1, -1, 0, -1, -1, 0, 1, 1, 0},
   {0, -1, 1, 0, -1, -1, 1, 0, 1, 0}, {0, -1, 1, 0, 1, -1, 0, 1, 0, 0},
   {1, -1, 1, 0, -1, 1, 0, 0, 0, 0}, {0, 0, 0, 1, 1, 0, 1, 0, 0, 1},
   {0, 0, 0, 1, 0, 1, 0, 1, 0, 0}, {0, 1, 1, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 1, 1, 0, 0, 0, 0, 0, 0},
   {0, 1, 0, 1, 0, 0, 0, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0, 1, 1},
   {0, 0, 0, 0, 0, 1, 0, 0, 0, 1}, {0, 0, 1, 0, 0, 0, 0, 0, 0, 1}, {0, 1, 0, 0, 0, 0, 0, 0, 0, 1}};
matrix // MatrixForm
IncidenceAxiomContradictionQ[submatrix_] :=
  If[submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, 1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, 0}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {-1, 1, -1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, 1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, 0}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {-1, 1, -1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, 1}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, 0}} ||
    submatrix == {{1, 1, -1}, {1, 1, 1}, {1, -1, -1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, 1}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, 0}} ||
    submatrix == {{1, 1, 1}, {1, 1, -1}, {1, -1, -1}}, True, False];
For[iteration = 1, iteration ≤ niterations, iteration++,
 For[i1 = 1, i1 ≤ m, i1++,
  For[j1 = 1, j1 ≤ n, j1++,
   If[matrix[[i1, j1]] == 0,
    For[i2 = 1, i2 ≤ m, i2++,
     For[i3 = 1, i3 ≤ m, i3++,
      For[j2 = 1, j2 ≤ n, j2++,
       For[j3 = 1, j3 ≤ n, j3++,
        If[i2 ≠ i1 && i3 ≠ i1 && i2 ≠ i3 &&
          j2 ≠ j1 && j3 ≠ j1 && j2 ≠ j3 && IncidenceAxiomContradictionQ[
           {{1, matrix[[i1, j2]], matrix[[i1, j3]]}, {matrix[[i2, j1]], matrix[[i2, j2]],
             matrix[[i2, j3]]}, {matrix[[i3, j1]], matrix[[i3, j2]], matrix[[i3, j3]]}}],
         matrix[[i1, j1]] = -1
        ]]]]]]]]]
matrix // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix}
-1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
-1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 1
\end{pmatrix}$$