CSCI-SHU 360 Homework1

Ninghao Lu nl2752

2024 Fall

Exercise 1 Simpson's Paradox

1.1 Solution:

For machine 1:

Let A_1 denote the event that I win on machine 1, my winning probability is

$$P(A_1) = \frac{40}{40+60} = \frac{40}{100} = 0.4$$

Let B_1 denote the event that my friend wins on machine 1, the winning probability is

$$P(B_1) = \frac{30}{30+70} = \frac{30}{100} = 0.3$$

Since $0.4 > 0.3$, we have $P(A_1) > P(B_1)$, I am more likely to win on machine 1.

For machine 2:

Let A_2 denote the event that I win on machine 2, my winning probability is

$$P(A_2) = \frac{210}{210+830} = \frac{210}{1040} = \frac{21}{104}$$

Let B_2 denote the event that my friend wins on machine 2, the winning probability is

$$P(B_2) = \frac{14}{14+70} = \frac{14}{84} = \frac{1}{6}$$

Since $\frac{21}{104} > \frac{1}{6}$, we have $P(A_2) > P(B_2)$, I am more likely to win on machine 2.
In conclusion, I am more likely to win on both machines.

1.2 Solution:

Let A denote the event that I win on both machines and let B denote the event that my friend wins on both machines.

$$P(A) = \frac{\text{Number of wins}}{\text{Number of trials}} = \frac{40+210}{40+60+210+830} = \frac{250}{1140} = \frac{25}{114}$$
$$P(B) = \frac{\text{Number of wins}}{\text{Number of trials}} = \frac{30+14}{30+70+14+70} = \frac{44}{184} = \frac{11}{46}$$

Since $\frac{25}{114} > \frac{11}{46}$, we have $P(A) < P(B)$, my friend is more likely to win.

1.3 Solution:

Let M_1 denote the event that I play on machine 1, M_2 denote the event that I play on the second machine. Let N_1 denote the event that my friend plays on machine 1, N_2 denote the event that my friend plays on the second machine.
My winning probability is

$$\begin{aligned}
P(A) &= P(A \cap M_1) + P(A \cap M_2) = P(A|M_1)P(M_1) + P(A|M_2)P(M_2) \\
&= \frac{2}{5} \cdot \frac{100}{100 + 1040} + \frac{21}{104} \cdot \frac{1040}{100 + 1040} = \frac{2}{5} \cdot \frac{100}{1140} + \frac{21}{104} \cdot \frac{1040}{1140} \\
&= \frac{4}{114} + \frac{21}{114} \\
&= \frac{25}{114}
\end{aligned}$$

My friend's winning probability is

$$\begin{aligned}
P(B) &= P(B \cap N_1) + P(A \cap N_2) = P(B|N_1)P(N_1) + P(B|N_2)P(N_2) \\
&= \frac{3}{7} \cdot \frac{100}{100 + 84} + \frac{14}{84} \cdot \frac{84}{100 + 84} = \frac{3}{10} \cdot \frac{100}{184} + \frac{1}{6} \cdot \frac{84}{184} \\
&= \frac{30}{184} + \frac{14}{184} \\
&= \frac{11}{46}
\end{aligned}$$

From the above computing process we can see the the overall probability is affected by both the probability on each machine and the fraction of the number of trails on each machine. Although my winning probability is higher on each machine, I conduct more much trials on machine 2 and my winning probability on is lower than that on machine 1, which lowers my overall probability.

Exercise 2 Matrix as operations

2.1 Solution:

We want to find a matrix W that transforms a_1 to b_1 and a_2 to b_2. Let

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.8 & 1.6 \end{bmatrix}$$

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.6 & -0.2 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} w_{11} + w_{12} \\ w_{21} + w_{22} \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1.6 \end{bmatrix}$$

$$\begin{bmatrix} w_{11} - w_{12} \\ w_{21} - w_{22} \end{bmatrix} = \begin{bmatrix} 2.6 \\ -0.2 \end{bmatrix}$$

Solving the equations, we have

$$w_{11} = 0.9 \quad w_{12} = -1.7 \quad w_{21} = 0.7 \quad w_{22} = 0.9$$

In conclusion,

$$W = \begin{bmatrix} 0.9 & -1.7 \\ 0.7 & 0.9 \end{bmatrix}$$

2.2 Solution

For a matrix V which rotates clockwise by α degrees, we assume it to be

$$\begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix}$$

Since tan$\alpha = 3$, suppose $\alpha \in [0, 180°]$, we have sin$\alpha = \frac{3}{\sqrt{10}}$, cos$\alpha = \frac{1}{\sqrt{10}}$. As a result,

$$V = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

For a matrix Σ which scales the y-axis by 2, while keeps the x-axis unchanged, let $x = \begin{bmatrix} a \\ b \end{bmatrix}$, with a, b $\in \mathbb{R}$

$$\Sigma x = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} x = \begin{bmatrix} a\Sigma_{11} + b\Sigma_{12} \\ a\Sigma_{21} + b\Sigma_{22} \end{bmatrix} = \begin{bmatrix} a \\ 2b \end{bmatrix}$$

Hence, we can take $\Sigma_{11} = 1, \Sigma_{12} = 0, \Sigma_{21} = 0, \Sigma_{22} = 2$ and

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

For a matrix U which rotates counter-clockwise by β degrees, we assume it to be

$$\begin{bmatrix} cos\beta & -sin\beta \\ sin\beta & cos\beta \end{bmatrix}$$

Since tan$\beta = \frac{1}{3}$, suppose $\beta \in [0, 180°]$, we have sin$\beta = \frac{1}{\sqrt{10}}$, cos$\beta = \frac{3}{\sqrt{10}}$. As a result,

$$U = \begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Therefore,

$$\begin{aligned} U\Sigma V &= \begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{6}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{10} & \frac{7}{10} \\ -\frac{17}{10} & \frac{9}{10} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.7 \\ -1.7 & 0.9 \end{bmatrix} \end{aligned}$$

We could discover that $U\Sigma V = W^T$

2.3 Solution

Compute eigenvalues and eigenvectors

$$W^TW = \begin{bmatrix} \frac{9}{10} & \frac{7}{10} \\ -\frac{17}{10} & \frac{9}{10} \end{bmatrix} \begin{bmatrix} \frac{9}{10} & -\frac{17}{10} \\ \frac{7}{10} & \frac{9}{10} \end{bmatrix} = \begin{bmatrix} \frac{13}{10} & -\frac{9}{10} \\ -\frac{9}{10} & \frac{37}{10} \end{bmatrix}$$

To compute the eigenvalues and eigenvectors, the characteristic equation is

$$\begin{aligned} det(W^TW - \lambda I) &= 0 \\ (1.3 - \lambda)(3.7 - \lambda) - 0.81 &= 0 \\ \lambda^2 - 5\lambda + 4 &= 0 \\ (\lambda - 4)(\lambda - 1) &= 0 \end{aligned}$$

Therefore, the eigenvalues are $\lambda = 1$ and $\lambda = 4$.
For $\lambda = 1$,

$$W^T W - \lambda I = W^T W - I = \begin{bmatrix} 0.3 & -0.9 \\ -0.9 & 2.7 \end{bmatrix}$$

We have, $(W^T W - I)x = 0$, which is

$$\begin{bmatrix} 0.3 & -0.9 \\ -0.9 & 2.7 \end{bmatrix} x = 0$$

So x can be $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

For $\lambda = 4$,

$$W^T W - \lambda I = W^T W - 4I = \begin{bmatrix} -2.7 & -0.9 \\ -0.9 & -0.3 \end{bmatrix}$$

We have, $(W^T W - 4I)x = 0$, which is

$$\begin{bmatrix} -2.7 & -0.9 \\ -0.9 & -0.3 \end{bmatrix} x = 0$$

So x can be $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

For $\lambda = 1$, its corresponding eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$; for $\lambda = 4$, its corresponding eigenvector is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Shape after transformation

Suppose every point on the unit circle gets transformed by W, we will get an ellipse after the transformation. This is because transforming by W is equivalent to transforming by $U\Sigma V$ according to previous question. This is a process that first rotates each point (still get a circle), then scaling (resulting in an ellipse), and finally rotating the ellipse.

Relationship between transformed shape and eigenvalues and eigenvectors

The eigenvalues corresponds to squared length of the semi-major and semi-minor of the ellipse. And the eigenvectors indicate the directions of the axes.
Explanation:
Since U and V are orthonormal matrix, and Σ is symmetric, for a vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ which is on the unit circle, we have

$$x^T x = 1$$

because for any point on the unit cricle its coordinates satisfies $x_1^2 + x_2^2 = 1$. Since $W^T = U\Sigma V$, after transformation by W, we have

$$\begin{aligned} (Wx)^T Wx = x^T W^T W x &= x^T (U\Sigma V)(U\Sigma V)^T x \\ &= x^T U\Sigma V V^T \Sigma U^T x \\ &= x^T U \Sigma^2 U^T x \end{aligned}$$

Here we let y $= U^T x$, where $y^T y = (U^T x)^T (U^T x) = x^T x = 1$, and the equation above becomes

$$\begin{aligned} x^T U \Sigma^2 U^T x &= y^T \Sigma^2 y \\ &= \sum_{i=1}^{2} \sigma_i^2 y_i^2 \\ &= \sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 \end{aligned}$$

where σ_1, σ_2 are elements on the diagonal of matrix Σ. And we assume

$$\sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 = k, \ k \in \mathbb{R}$$

From this equation, we can see that the original equation of a unit circle $x_1^2 + x_2^2 = 1$ is transformed into an equation for an ellipse. The semi-major and semi-minor of the ellipse is σ_1^2, σ_2^2. **It is quite obvious that eigenvalues corresponds to squared length of the semi-major and semi-minor of the ellipse since the eigenvalues of $W^T W$ is σ_1^2, σ_2^2.**

For the unit circle, the direction of the axes can be indicated by $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For the current ellipse, the direction of the axes after transformation can be indicated by $c_1 = U^T v_1$ and $c_2 = U^T v_2$. Since U is a 2×2 rotation matrix, the eigenvector of U^T and U are the same. **It is also obvious that the eigenvectors indicate the directions of the axes since the eigenvectors of $W^T W$ are columns of U.**

2.4 Solution

The determinant of W is

$$det(W) = det \begin{bmatrix} 0.9 & -1.7 \\ 0.7 & 0.9 \end{bmatrix} = 0.81 + 1.19 = 2$$

The area of the ellipse is $S_{ellipse} = 2 \cdot 1 \cdot \pi = 2\pi$, which is equal to the determinant of W $\cdot$ {Area of the unit circle}. So the relationship between the determinant and the area of a transformed shape is that:
the absolute value of determinant is the fraction between the area of the transformed shape and the area of the original shape.
Based on that, one simple explanation for $det(AB) = det(A)det(B)$ is transforming a shape by AB is equivalent to conduct the transformation with A first and then B, and the scaling effect of AB is equivalent to the cumulative scaling effect of A and B.

Exercise 3 Some practices

3.1 Solution

Let a discrete random variable Y = X^e, its variance can be written as

$$Var(Y) = E[Y^2] - (E[Y])^2 = E[(X^3)^2] - (E[X^3])^2$$

Since Var(Y) ≥ 0, we have $E[(X^3)^2] - (E[X^3])^2 \geq 0$. Therefore, $(E[X^3])^2 \leq E[X^6]$.

3.2 Solution

We can proceed by Cauchy-Schwarz Inequality, which states that for any two random variables A and B,

$$(E[AB])^2 \leq E[A^2]E[B^2]$$

Applying this inequality here, we let A = X^3, B = 1 and we have

$$(E[X^3])^2 \leq E[(X^3)^2]E[1] = E[X^6]E[1] = E[X^6]$$

Therefore, we can conclude that $(E[X^3])^2 \leq E[X^6]$.

3.3 Solution

Since A, B are both PSD matrices of shape n $\times$ n, we have

$$x^T Ax \geq 0, x^T Bx \geq 0 \quad \text{for every vector x.}$$

To show that $\lambda A + (1-\lambda)B$ is also PSD for $0 \leq \lambda \leq 1$, we want to show $x^T(\lambda A + (1-\lambda)B)x \geq 0$ for every vector x.

$$\begin{aligned} x^T(\lambda A + (1-\lambda)B)x &= x^T \lambda Ax + x^T(1-\lambda)Bx \\ &= \lambda x^T Ax + (1-\lambda)x^T Bx \end{aligned}$$

Since $x^T Ax \geq 0, x^T Bx \geq 0$ for every vector x and $\lambda \in [0,1]$, we have

$$\lambda x^T Ax + (1-\lambda)x^T Bx \geq 0$$

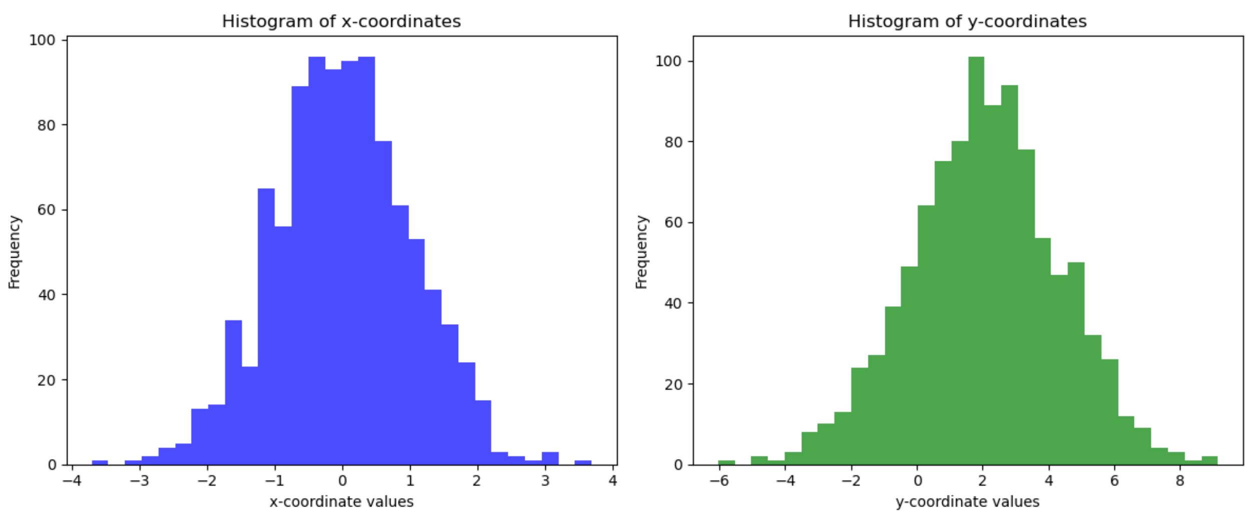
Therefore, $x^T \lambda A + (1-\lambda)Bx \geq 0$ for for every vector x and $\lambda \in [0,1]$. We can conclude that $\lambda A + (1-\lambda)B$ is also PSD for $\lambda \in [0,1]$.

Exercise 4

4.1 Solution:

The mean of X is $\begin{bmatrix} 0.01909265 & 2.06052385 \end{bmatrix}$, the covariance of X is $\begin{bmatrix} 1.03589238 & 2.06244165 \\ 2.06244165 & 5.08839176 \end{bmatrix}$

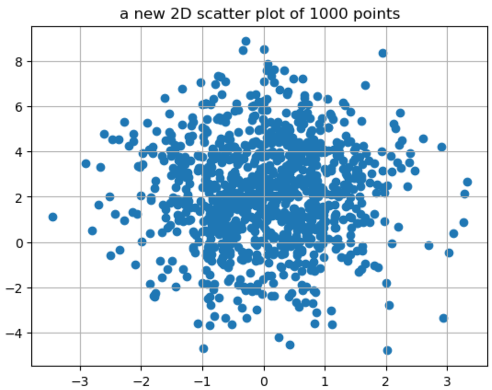
4.2 Solution:



4.3 Solution:

From the histogam, we can tell that the x-coordinates of X samples are from some Gaussian distribution. The mean of x-coordinates is 0.019092651459051646. The variance of x-coordinates is 1.0348564864741474. We can also tell that the y-coordinates of X samples are from some Gaussian distribution.
The mean of x-coordinates is 2.0605238541899022. The variance of x-coordinates is 2.0605238541899022.

4.4 Solution:



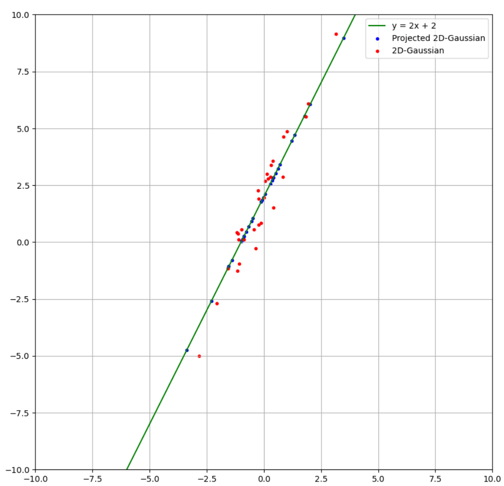
Difference:

The original plot shows that there's a linear correlation between x-coordinates and y-coordinates, which shows that x and y are dependent and this coincides with the previous code. The new graph's circular pattern shows that there's no apparent relationship between x and y coordinates (They are independent).

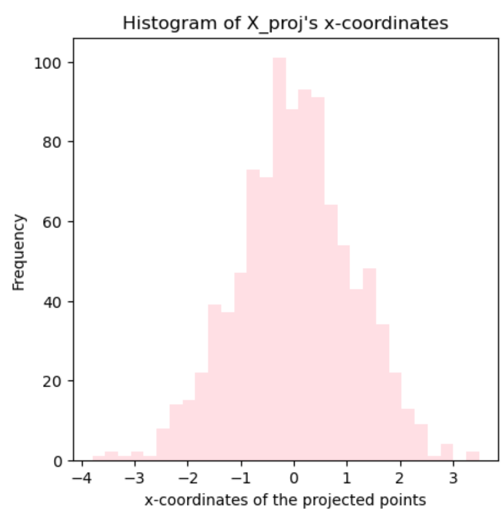
What causes such difference?

From the code in the previous section in the jupyter notebook we can see that the first plot is generated from some multivariate Gaussian distribution with a covariance between x and y coordinates so the graph they form shows a linear pattern, indicating they are correlated. While the second plot is generated from 2 independent 1D Gaussian distributions, so their graph shows a circular pattern.

4.5 Solution:



4.6 Solution:



4.6 Solution: (Additional Text)

The histogram shows that the x-coordinates of the projected points are from some Gaussian distribution. The mean of x-coordinates: 0.028028071967771222 and the variance of x-coordinates: 1.1843834728271614.

Exercise 5

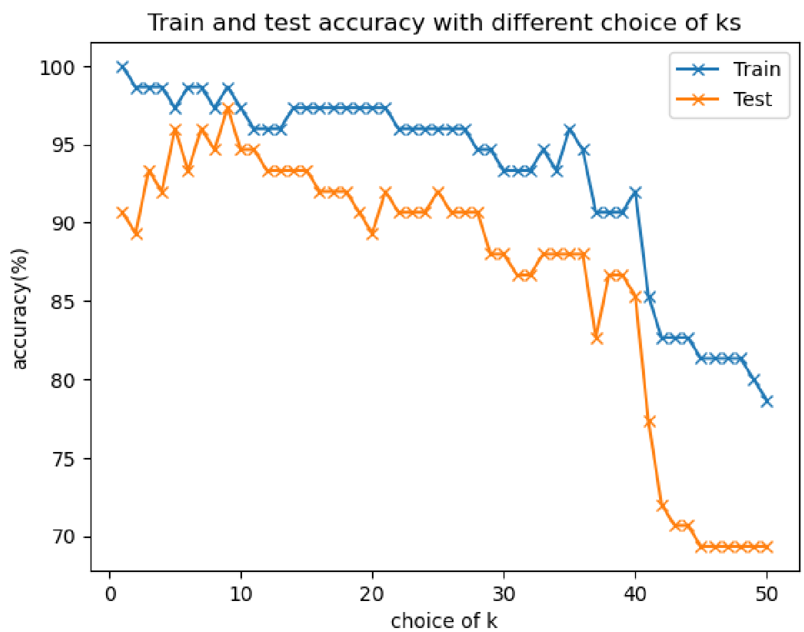
5.1 Solution:

- Elements in class setosa: 50.
- Elements in class versicolor: 50.
- Elements in class virginica: 50.

5.2 Solution:

The accuracy is 100 %. This result doesn't give me useful information because this model is trained on the whole dataset, which means it actually memorizes this dataset and this leads to overfitting. The model will assign each data point to itself when we use the same data set for training and testing when k = 1, resulting in accuracy = 100 %. But this doesn't mean this model will perform well on new data. As a result, when we train the model, we should split the dataset so that we can test its generalization ability.

5.3 Solution:



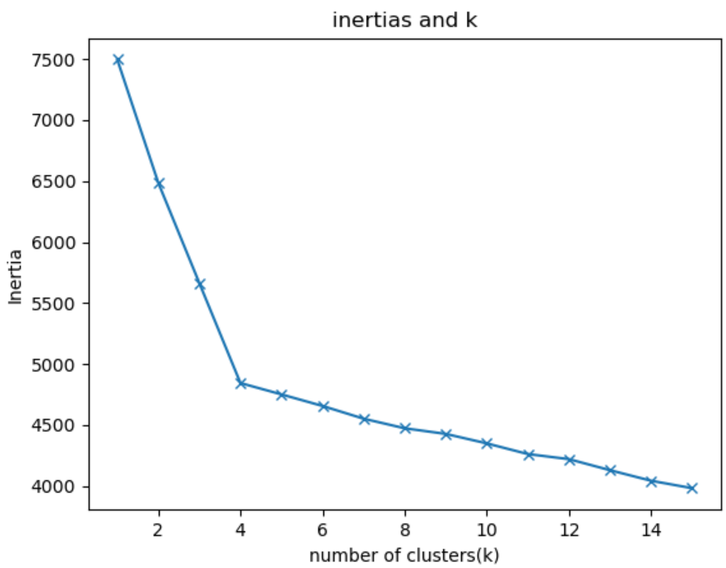
According to the graph above, when k = 9, the training accuracy and testing accuracy become most closest and this means the model has the best ability to generalize when k = 9, indicating the best possible performance. So 9 is the optimal k.

5.4 Solution:

The predicted class is: versicolor.

Exercise 6

6.1 Solution:

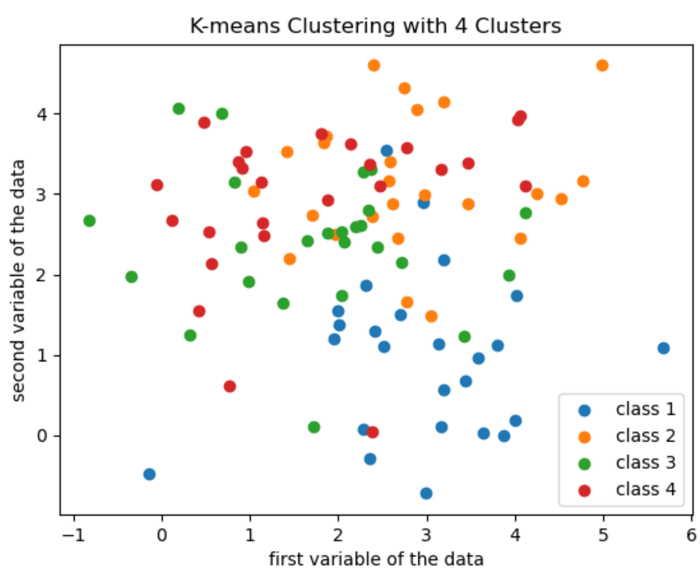


Based on this plot, the elbow point indicates that 4 clusters should be used for this data.

6.2 Solution:

25 observations are placed in each cluster, the value of inertia is 4844.925817623823.

6.3 Solution:



Based on this plot, I cannot conclude that 4 is good for the number of centers since graph based on the first and second variable does not separate the data points. The data points on the graph mixed together. Maybe we need to choose other variables to better visualize.