1.2 Arithmetic and Geometric Sequences

1.2.1 Arithmetic Sequences

Activity 1.3

Find the difference between consecutive terms for each of the following sequences.

a. 3, 7,11,15, ... b. $3, -1, -5, -9, \ldots$ c. $2, \frac{7}{3}, \frac{8}{3}, 3, \ldots$

Definition 1.2

Arithmetic sequence or arithmetic progression is a sequence in which each term except the first is obtained by adding a fixed number (positive or negative) to the preceding term. The fixed number is called common difference of the sequence.

Example 1

For the following arithmetic sequence 2, 5, 8, 11, 14,…, what is the first term, third term and common difference? Find the 6^{th} term.

Solution

$$2 \xrightarrow{+3} 5 \xrightarrow{+3} 8 \xrightarrow{+3} 11 \xrightarrow{+3} 14$$

1^{st} term: 2, 3^{rd} term: 8

The common difference is 3.

6^{th} term: 5^{th} term $+ 3 = 14 + 3 = 17$.

Example 2

Given that the 1^{st} term of an arithmetic sequence is 10, and the common difference is -4, find the terms from 2^{nd} to 5^{th} term.

Solution

2^{nd} term: 1^{st} term + (-4) = 10 + (-4) = 6.

3^{rd} term: 2^{nd} term + (-4) = 6 + (-4) = 2.

4^{th} term: 3^{rd} term + (-4) = 2 + (-4) = -2.

5^{th} term: 4^{th} term + (-4) = -2+ (-4) = -6.

Therefore, the terms from 2^{nd} to 5^{th} term are 6, 2, -2, -6.

Exercise 1.3

1. For the following arithmetic sequences, what are the first term, third term and common difference? Find the 6^{th} term.

   a. {3, 5, 7, 9, 11...} b. {9, 6, 3, 0, -3…}

2. Find the terms from 2^{nd} to 5^{th} of an arithmetic sequence when the

   a. 1st term is 1, and the common difference is 5.

   b. 1st term is 2, and the common difference is $-\frac{1}{2}$.

General term of arithmetic sequence

From the Activity 1.3, (a), 3, 7, 11, 15, 19,… the first term is 3, and the common difference is 4. Let the common difference be d, and let A_n be the n^{th} term of the sequence, then,

$$A_1 = 3$$
$$A_2 = 3+4 = A_1 + d$$
$$A_3 = 3+4+4 = A_1 + 2d$$
$$A_4 = 3+4+4+4 = A_1 + 3d$$
$$\cdots$$
$$A_n = 3+4+4+4+...+4 = A_1 + (n-1)d$$

Suppose $A_n : A_1, A_2, A_3, A_4, A_5, A_6, ...$ is an arithmetic sequence.

$$A_2 = A_1 + d$$
$$A_3 = A_2 + d = A_1 + d + d = A_1 + 2d$$
$$A_4 = A_3 + d = A_1 + 2d + d = A_1 + 3d$$

$$
\begin{aligned}
A_5 &= A_4 + d = A_1 + 3d + d = A_1 + 4d \\
A_6 &= A_5 + d = A_1 + 4d + d = A_1 + 5d \\
&\vdots \\
A_n &= A_{n-1} + d = A_1 + (n-2)d + d = A_1 + (n-1)d
\end{aligned}
$$

Thus, the following formula is derived:

Theorem 1.1

If A_n is an arithmetic sequence with the first term A_1 and a common difference d, then the n^{th} term of the sequence is given by $A_n = A_1 + (n-1)d$.

Example 1

Find the general term of the sequence A_n, when the first term is 5, and the common difference is 4.

Solution

The given sequence is arithmetic sequence with $A_1 = 5$ and $d = 4$.

From the theorem 1.1, we have the formula $A_n = A_1 + (n-1)d$. So,

$A_n = A_1 + (n-1)d = 5 + (n-1)(4) = 5 + 4n - 4 = 4n + 1$.

Example 2

What is the 31^{st} term of the sequence $1, 4, 7, 10, \ldots$?

Solution

The given sequence is arithmetic sequence with $A_1 = 1$ and $d = 3$

$A_n = A_1 + (n-1)d = 1 + 3(n-1) = 3n - 2$.

Then,

$A_{31} = 3(31) - 2 = 93 - 2 = 91$.

Exercise 1.4

1. Find the general term of the sequence A_n, when

   a. $A_1 = 2, \ d = 3.$ b. $A_1 = 10, \ d = -5.$

2. What is the 10^{th} term of the sequence, 10, 6, 2, -2, …?

Further on arithmetic sequences

Example 1

When the third term is 10 and the sixth term is 1,

a. Find the general term of sequence A_n.

b. Find A_8.

Solution

a. Applying arithmetic sequence formula and substituting existing values yields:

$$A_3 = A_1 + (3-1)d, \qquad 10 = A_1 + 2d$$
$$A_6 = A_1 + (6-1)d, \qquad 1 = A_1 + 5d$$
$$\begin{cases} A_1 + 2d = 10 \\ A_1 + 5d = 1 \end{cases}$$

Subtracting these two equations,

$$3d = -9 \Rightarrow d = -3 \ , \ A_1 + 2d = 10, \ A_1 + 2(-3) = 10, \ A_1 = 10 + 6 = 16.$$

Therefore, the general term becomes

$$A_n = A_1 + (n-1)d = 16 + (n-1)(-3) = 16 - 3n + 3 = 19 - 3n.$$

b. $A_n = 19 - 3n \Rightarrow A_8 = 19 - 3(8) = 19 - 24 = -5.$

Example 2

Determine whether or not the sequences with the following general terms are arithmetic. *a.* $a_n = 3n - 2$ *b.* $a_n = 3n^2 - 2$

Solution

a. To solve such types of problem, we have to show the difference between successive terms is constant.

$$a_n = 3n - 2$$
$$a_{n+1} = 3(n+1) - 2 = 3n + 1$$
$$a_{n+1} - a_n = 3n + 1 - (3n - 2) = 3.$$

Since, the difference between successive term is constant, it arithmetic sequence

b. $a_n = 3n^2 - 2$

$$a_n = 3n^2 - 2$$
$$a_{n+1} = 3(n+1)^2 - 2 = 3(n^2 + 2n + 1) - 2 = 3n^2 + 6n + 1$$
$$a_{n+1} - a_n = 3n^2 + 6n + 1 - (3n^2 - 2) = 6n + 3.$$

Since, the difference between successive terms is not constant, it is not arithmetic sequence.

Exercise 1.5

1. Find the general term of the arithmetic sequence A_n, when

   a. $A_4 = 15,\ \ A_8 = 27.$ b. $A_5 = 20,\ \ A_{10} = 0.$

2. Given arithmetic sequence with $A_2 = 3$ and $A_5 = 24$. Find A_n and A_{11}.

3. Determine whether or not the sequences with the following general terms are arithmetic.

   a. $a_n = 7n - 3$ b. $a_n = 5n - 3$ c. $a_n = n^2 + n + 1$ d. $a_n = 3^n$

Arithmetic mean between two numbers

The term(s) of arithmetic sequence that lie between two given terms are called the arithmetic mean.

Example 1

Given that $1, x, 8$ is an arithmetic sequence, find x.

Solution

Since it is arithmetic sequence, the difference between two consecutive terms is constant.

$$x-1=8-x$$
$$2x=9$$
$$x=\frac{9}{2}$$

Example 2

The first and sixth terms of an arithmetic sequence are 4 and 29. Find the values of terms 2, 3, 4 and 5.

Solution

Let the four terms be A_2, A_3, A_4, A_5.

So, $4, A_2, A_3, A_4, A_5, 29$ form an arithmetic sequence. Then, $A_n = A_1 + (n-1)d$.

$$A_1 = 4$$
$$A_6 = A_1 + (6-1)d$$
$$29 = 4 + 5d$$
$$d = 5$$

Thus, $A_n = 4 + (n-1)5 = 5n-1$

$A_2 = 4 + 5 \times 1 = 9$

$A_3 = 4 + 5 \times 2 = 14$

$A_4 = 4 + 5 \times 3 = 19$

$A_5 = 4 + 5 \times 4 = 24.$

Exercise 1.6

1. Given that the sequence $3, x, 7$ is an arithmetic sequence, find x.
2. Given that the sequence $\frac{1}{12}, \frac{1}{x}, \frac{1}{6}$ is an arithmetic sequence, find x.
3. Find the arithmetic mean of 4 and 14.
4. Insert four arithmetic means between 4 and 14 to create an arithmetic sequence.

1.2.2 Geometric Sequences

Activity 1.4

Find the ratio between the consecutive terms of each of the following sequences.

a. 2 ,6, 18, 54,... *b.* $2, -2, 2, -2, \ldots$ *c.* $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$

Definition 1.3

A geometric sequence or geometric progression is one in which the ratio between consecutive terms is a non-zero constant. This constant is called the common ratio.

$\{G_n\}$ is geometric sequence if and only if $r = \frac{G_{n+1}}{G_n}$, $r \in \mathbb{R}$ & $r \neq 0$ where r is the common ratio.

Example 1

For the following geometric sequence 3, 6, 12, 24, 48,…, find the common ratio, r, and the 6^{th} term.

Solution

The common ratio:

$$\frac{G_2}{G_1}=\frac{6}{3}=2,\ \frac{G_3}{G_2}=\frac{12}{6}=2,\ \frac{G_4}{G_3}=\frac{24}{12}=2,\ldots$$

$$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \cdots$$

Thus, $r=2$.

The 6^{th} term: $G_6=G_5r=48\times 2=96$.

Example 2

The 1^{st} term of a geometric sequence is $-\frac{1}{2}$, and its common ratio is $\frac{1}{2}$, find the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} term.

Solution

We are given $G_1=-\frac{1}{2}$ and $r=\frac{1}{2}$, then

$$G_2=G_1r=-\frac{1}{2}\times\frac{1}{2}=-\frac{1}{4}$$

$$G_3=G_2r=-\frac{1}{4}\times\frac{1}{2}=-\frac{1}{8}$$

$$G_4=G_3r=-\frac{1}{8}\times\frac{1}{2}=-\frac{1}{16}$$

$$G_5=G_4r=-\frac{1}{16}\times\frac{1}{2}=-\frac{1}{32}.$$

Exercise 1.7

1. For the geometric sequence 1, 2, 4, 8, 16…, find the common ratio, r, and the 6^{th} term.
2. Given that the 1^{st} term of a geometric sequence is 1, and its common ratio is 3, find the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} term.

Determining the nth term of geometric sequence

From the activity 1.4, (a), we have the geometric sequence 2, 6, 18, 54,… which has a common ratio of 3.

$$G_1 = 2$$
$$G_2 = 6 = 2\times 3 = G_1 \times 3^1$$
$$G_3 = 18 = 6\times 3 = G_2\times 3 = G_1\times 3\times 3 = G_1\times 3^2$$
$$G_4 = 54 = 18\times 3 = G_3\times 3 = G_1\times 3\times 3\times 3 = G_1\times 3^3$$
$$\cdots$$
$$G_n = G_1\times 3\times 3\times 3\times \ldots\times 3 = G_1\times 3^{n-1}$$

From Definition 1.3,

$$G_2 = G_1\times r$$
$$G_3 = G_2\times r = G_1\times r\times r = G_1\times r^2$$
$$G_4 = G_3\times r = G_1\times r^2\times r = G_1\times r^3$$
$$G_5 = G_4\times r = G_1\times r^3\times r = G_1\times r^4$$
$$\cdots \qquad \cdots$$
$$G_n = G_1\times r^{n-1}$$

Thus, the following theorem is deduced:

Theorem 1.2

If $\{G_n\}$ is a geometric sequence with the first term G_1 and common ratio r, then the n^{th} term of the sequence is given by $G_n = G_1 r^{n-1}$.

Example 1

Find G_n, when the first term is 3 and the common ratio is 2.

Solution

Given $G_1 = 3$ and $r = 2$, then

$$G_n = G_1 r^{n-1} = 3\times 2^{n-1}.$$

Example 2

Find the n^{th} term, G_n of the sequence $1, \ -2, \ 4, \ -8, \ 16, \ldots$.

Solution

$G_1 = 1 \ , r = -2$

Applying the formula for the n^{th} term of a geometric sequence, $G_n = G_1 r^{n-1}$.

$$G_n = (-2)^{n-1}.$$

Example 3

Find the 6^{th} term of the geometric sequence whose first term is 1 and common ratio is 2.

Solution

Given $G_1 = 1$ and $r = 2$, then $G_n = G_1 r^{n-1} = 1 \times 2^{n-1} = 2^{n-1}$.

Therefore, $G_6 = 2^{6-1} = 2^5$ or 32.

Exercise 1.8

1. For each of the following, find the n^{th} term of the geometric sequence.
   a. $G_1 = 2, \ r = 5$
   b. $G_1 = 1, \ r = -3$
   c. $G_1 = 2, \ r = -2$
   d. $G_1 = -3, \ r = \frac{1}{2}$
2. Find the n^{th} term G_n of the following sequences
   a. $3, 6, 12, 24, \ldots$
   b. $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots$
   c. $27, 9, 3, 1, \ldots$
3. Find the 5^{th} term of the geometric sequence whose first term is -3 and common ratio is $-\frac{1}{2}$.

Geometric mean between two numbers

When a, m, and b are terms in a geometric sequence, then m is called the geometric mean between a and b. $(a \neq 0,\ b \neq 0,\ m \neq 0)$. In a geometric sequence, the ratio between consecutive terms is constant.

$$r = \frac{m}{a} = \frac{b}{m}$$
$$m^2 = ab$$
$$m = \pm\sqrt{ab}$$

Example 1

When $2, x, 5, \ldots$ is a geometric sequence, find $x (x \neq 0)$.

Solution

As the ratio between the consecutive terms is the same,

$$\frac{x}{2} = \frac{5}{x},\ \ x^2 = 10,\ \ x = \pm\sqrt{10}.$$

Example 2

Find geometric mean between 2 and 8.

Solution

Let the geometric mean be $m (m \neq 0)$, then

$$r = \frac{m}{2} = \frac{8}{m}$$
$$m^2 = 2 \times 8$$
$$m = \pm\sqrt{16} = \pm 4.$$

Therefore, the geometric mean between 2 and 8 is -4 or 4.

Example 3

Find the 8^{th} term of the geometric sequence whose 1^{st} term is 5 and 4^{th} term is $\frac{1}{25}$.

(Express the answer in the form of exponent).

Solution

$G_1 = 5$ and $G_4 = \frac{1}{25}$,

$G_4 = G_1r^3 = 5r^3 = \frac{1}{25}$.

Then, $r^3 = \left(\frac{1}{5}\right)^3$, therefore $r = \frac{1}{5}$.

$$G_8 = G_1r^7 = 5\left(\frac{1}{5}\right)^7 = \frac{1}{5^6}.$$

Exercise 1.9

1. Find the geometric mean between 3 and 12.
2. In a geometric sequence, the 2^{nd} term is 12 and the 6^{th} term is 192. Find the 11^{th} term.
3. If $x, 4x+3, 7x+6$ are consecutive terms of a geometric sequence, find the value(s) of x, $x \neq 0$.
4. Find three consecutive terms of a geometric sequence, such that their sum is 35 and their product is 1000. Let the terms be $\frac{a}{r}$, a and ar. $(a \neq 0, r \neq 0)$