1.4 Infinite Series

Activity 1.7

Supposing that a tree grows by half its height in a year. It then grows half of the amount of the previous year. What would be the height of the tree be if it continues to grow at the same rate?

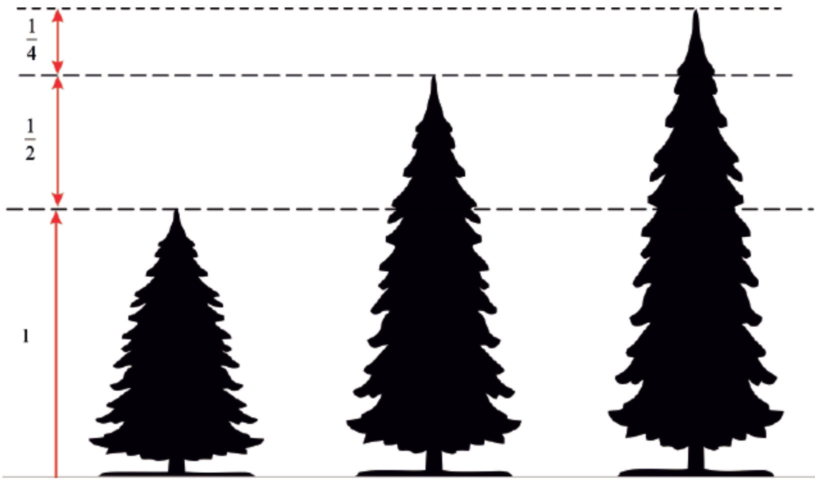

Figure 1.5

Suppose we have the sequence: $a_1, a_2, a_3, ..., a_n, ...$

An infinite sum of the form: $a_1 + a_2 + a_3 + ... + a_n + ...$ is called an infinite series and using summation, we can write as: $a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n$

1.4.1. Divergence and Convergence of Infinite Sequence

Activity 1.8

For each of the sequences i – v:

a. Write the formula for the n^{th} term.

b. Find the sum where the sequence converges.

i. 1, 2, 3, 4,... ii. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$

iii. −1, 1, −1, 1,... iv. 1, 1.5, 2.25, 3.375,...

v. 0.12, 0.0012, 0.000012, 0.00000012,...

For an infinite sequence, there are cases of convergence and divergence, when n approaches to infinity, as follows:

Convergence

a. Sequence a_n converges to α: $n \to \infty,\ a_n \to \alpha$

Divergence

b. Sequence a_n diverges to positive infinity: $n \to \infty,\ a_n \to \infty$.

c. Sequence a_n diverges to negative infinity: $n \to \infty,\ a_n \to -\infty$.

d. Sequence a_n vibrates: a_n has no limit. i.e. the value oscillate or vibrate back and forth between numbers.

Divergence/convergence of infinite geometric sequence

Considering infinite geometric sequence $G_n = r^n$, where $G_1 = r$, and the common ratio is r, there are cases of convergence and divergence, when n approaches to infinity, as follows:

i) $r > 1$, When $n \to \infty$, then $G_n \to \infty$ (diverge)

ii) $r = 1$, When $n \to \infty$, then $G_n \to 1$ (converge)

iii) $|r| < 1$, When $n \to \infty$, then $G_n \to 0$ (converge)

iv) $r \leq -1$, When $n \to \infty$, then G_n vibrates (no limit, diverge).

Example

Find whether the given geometric sequences diverge, converge or vibrate as n approaches to infinity.

a. $G_n = (\sqrt{2})^n$ *b.* $G_n = \left(\frac{2}{3}\right)^n$ *c.* $G_n = (-3)^n$

Solution

a. $G_n = (\sqrt{2})^n$

As $\sqrt{2} > 1$, when $n \to \infty$, then $G_n \to \infty$, it diverges.

b. $G_n = \left(\frac{2}{3}\right)^n$

As $\left|\frac{2}{3}\right| < 1$, when $n \to \infty$, then $G_n \to 0$, it diverges.

c. $G_n = (-3)^n$

As $-3 \leq -1$, when $n \to \infty$, then G_n vibrates

Exercise 1.18

Find whether the given geometric sequences diverge, converge or vibrates as n approaches infinity.

a. $G_n = (\sqrt{3})^n$ b. $G_n = \left(\frac{3}{4}\right)^n$ c. $G_n = \left(-\frac{1}{2}\right)^n$

Infinite series

Definition 1.5

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and S_n be the n^{th} partial sum such that, as n gets larger and larger, S_n tends to s, where s is a real number, then we say the infinite series $\sum_{n=1}^{\infty} a_n$ converges and is written as $\sum_{n=1}^{\infty} a_n = s$.

However, if such an s does not exist or is infinite, we say the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Let us consider the geometric sequence $1, \frac{2}{3}, \frac{4}{9}, \ldots$ where $G_1 = 1$ and $r = \frac{2}{3}$. We have partial sum,

$$S_n = \frac{G_1(1-r^n)}{1-r} = \frac{1\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

Let us study the behavior of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger, approaching to infinity.

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes increasingly large, $\left(\frac{2}{3}\right)^n$ becomes increasingly small. In other words, as $n \to \infty$, $\left(\frac{2}{3}\right)^n \to 0$. Consequently, we find set on the line below the sum of infinitely many terms of the sequence above is $S=3$.
Thus, for an infinite geometric sequence $G_1, G_1r, G_1r^2, \ldots,$ if numerical value of the common ratio r is between -1 and 1, then $S_n = \frac{G_1(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1r^n}{1-r}$.

In this case, $r^n \to 0$ as $n \to \infty$ since $|r|<1$ and then $\frac{G_1r^n}{1-r} \to 0$. Therefore, $S_n \to \frac{G_1}{1-r}$ as $n \to \infty$. Symbolically, the sum to infinity of an infinite geometric series is denoted by S_∞. Thus, we have $S_\infty = \frac{G_1}{1-r}$.

Note

Recall that if $\sum_{n=1}^{\infty} G_1 r^{n-1} = G_1 + G_1 r + G_1 r^2 + \ldots$ is a geometric series with first term G_1 and common ratio r, then $S_n = \frac{G_1(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r}$.

(i) If $|r|<1$, as $n \to \infty$, $r^n \to 0$ so that $S_n = \frac{G_1}{1-r} - \frac{G_1 r^n}{1-r} \to \frac{G_1}{1-r}, S_\infty = \frac{G_1}{1-r}$.

(ii) If $|r| \geq 1$, then the series diverges or vibrates as follows.

a. If $r=1$, then $\sum_{k=1}^{n} G_k = nG_1$, when $n \to \infty$, $\sum_{k=1}^{\infty} G_k = \infty \ (G_1 > 0)$

$$\sum_{k=1}^{\infty} G_k = -\infty \ (G_1 < 0)$$

b. If $r \neq 1$, then $S_n = \sum_{k=1}^{\infty} G_k = \frac{G_1(1-r^n)}{1-r}$

Then, If $r \leq -1$, when $n \to \infty$, then $\sum_{k=1}^{\infty} G_k$ vibrates,

If $r > 1$, when $n \to \infty$, then $\sum_{k=1}^{\infty} G_k = \infty \ (G_1 > 0)$

$$\sum_{k=1}^{\infty} G_k = -\infty \ (G_1 < 0)$$

Example 1

Determine whether the series $\sum_{n=1}^{\infty} 3^n$ converges or diverges.

Solution

The series $\sum_{n=1}^{\infty} 3^n = 3 + 9 + 27 + \ldots + \ldots$ is a geometric series with first term $G_1 = 3$ and common ratio $r = 3$.

Hence, the partial sum is given by $S_n = \frac{G_1(1-r^n)}{1-r}$

Substituting the values, we obtain,

$$S_n = \frac{G_1(1-r^n)}{1-r} = \frac{3[1-3^n]}{1-3} = -\frac{3}{2}[1-3^n] = -\frac{3}{2} + \frac{3}{2}3^n.$$

Thus, as $n \to \infty$, $S_n \to \infty$. Therefore, the series diverges.

Example 2

Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$

Solution

The geometric series has the first term $G_1 = 5$ and the common ratio $r = \frac{-\frac{10}{3}}{5} = -\frac{2}{3}$.

Since

$|r| = \left|\frac{-2}{3}\right| = \frac{2}{3} < 1,$

$$S_\infty = 5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots = \frac{G_1}{1-r} = \frac{5}{1-\left(\frac{-2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3.$$

then the series converges and its sum is given by

Exercise 1.19

Find the sum for each of the following, if it exists, assuming the patterns continue as in the first few terms.

a. $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

b. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots$

c. $\frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \ldots$

d. $\frac{1}{5} + \frac{-1}{10} + \frac{1}{20} + \frac{-1}{40} + \ldots$

e. $7 + \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$

Further on infinite series

Example 1

Find the sum $\sum_{k=1}^{\infty} 5^{3-k}$

Solution

$$5^{3-k} = 5^3 \times 5^{-k} = 5^3\left(\frac{1}{5}\right)^k = 5^2\left(\frac{1}{5}\right)^{k-1}$$

It is a geometric series whose first term $G_1 = 25$ and the common ratio $r = \frac{1}{5}$.

Therefore, the sum becomes:

$$S_\infty = \sum_{k=1}^{\infty} 5^{3-k} = 5^2 + 5^1 + 5^0 + 5^{-1} + \ldots = \frac{G_1}{1-r} = \frac{25}{1-\frac{1}{5}} = \frac{125}{4}$$

Example 2

Find the sum $\sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{2}{3}\right)^k$

Solution

$$(-1)^{k+1}\left(\frac{2}{3}\right)^k = (-1)^k(-1)\left(\frac{2}{3}\right)^k = (-1)\left(-\frac{2}{3}\right)^k = -\left(-\frac{2}{3}\right)^k$$

It is a geometric series whose first term $G_1 = \frac{2}{3}$ and the common ratio $r = -\frac{2}{3}$.

Therefore, the sum becomes

$$S_\infty = \sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{2}{3}\right)^k = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 + \ldots = \frac{G_1}{1-r} = \frac{\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{2}{5}$$

Exercise 1.20

Find the sums for each of these geometric series

a. $\sum_{k=1}^{\infty} 5\left(\frac{1}{3}\right)^{k-1}$ b. $\sum_{k=1}^{\infty} 2^{1-k}$ c. $\sum_{k=1}^{\infty} 5^{k-3}$ d. $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k+3}\left(\frac{2}{3}\right)^{k-2}$

1.4.2. Recurring Decimals

Recurring or repeating decimals are rational numbers (fractions) whose representations as a decimal contain a pattern of digit that repeats indefinitely after decimal point. The decimal that start their recurring cycle immediately after the decimal point are called **purely recurring decimals.**

Purely recurring decimals are converted to an irreducible fraction whose prime factors in the denominator can only be the prime numbers other than 2 or 5, i.e. the prime numbers from the sequence $\{3,7,11,13,17,19,...\}$. The decimals that have some extra digits before the repeating the sequence of digits are called the **mixed recurring decimals**.

The repeating sequence may consist of just one digit or of any finite number of digits. The number of digits in the repeating pattern is called **the period.**

Mixed requiring decimals are converted to an irreducible fraction whose denominator is a product of 2's and/or 5's besides the prime numbers from the sequence $\{3,7,11,13,17,19,...\}$.

All recurring decimals are infinite decimals.

Converting purely recurring decimals to fraction

Example 1

Convert the recurring decimal $0.\dot{3}$ to a fraction.

Solution

We can write the given decimal as the sum of the infinite converging geometric series as follows:

$$0.\dot{3} = 0.333333\ldots = 0.3 + 0.03 + 0.003 + \ldots$$
$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$$

This is the infinite geometric series, where $G_1 = \frac{3}{10}$ and $r = \frac{\frac{3}{100}}{\frac{3}{10}} = \frac{1}{10}$

Since, $|r| < 1$ then $S_\infty = \frac{G_1}{1-r} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{3}$

Example 2

Convert the recurring decimal $0.\dot{4}\dot{7}$ to a fraction.

Solution

We can write the given recurring decimal as the sum of the infinite converging geometric series as follows:

$$0.\dot{4}\dot{7} = 0.474747\ldots = 0.47 + 0.0047 + 0.000047 + \ldots$$
$$= \frac{47}{100} + \frac{47}{10000} + \frac{47}{1000000} + \ldots$$

Since, $r = \frac{G_n}{G_{n-1}} = \frac{1}{100}$, $|r| < 1$ then $S_\infty = \frac{G_1}{1-r}$,

Therefore, by substituting $G_1 = \frac{47}{100}$ and $r = \frac{1}{100}$ in to the formula:

$$0.\dot{4}\dot{7} = \frac{G_1}{1-r} = \frac{\frac{47}{100}}{1-\frac{1}{100}} = \frac{\frac{47}{100}}{\frac{99}{100}} = \frac{47}{99}$$

Notice that when converting a recurring decimal that is less than one to a fraction, we write the repeating digits to the numerator and in the denominator of the equivalent fraction write as many 9's as is the number of digits in the repeating pattern.

Exercise 1.21

Convert the following recurring decimals to fractions.

a. $0.\dot{4}$ b. $3.\dot{7}$ c. $0.\dot{5}\dot{6}$

1.5 Applications of Sequence and Series in Daily Life

This section is devoted to the application of arithmetic and geometric progressions or geometric series that are associated with real-life situations.

Activity 1.9

1. A man accepts a position with an initial salary of 5200 ETB per month. It is understood that he will receive an automatic increase of 320 ETB in the very next month and each month thereafter.
   a. Find his salary for the tenth month.
   b. What are his total earnings during the first year?
2. A carpenter was hired to build 192 window frames. The first day he/she made five frames and each day, thereafter he/she made two more frames than he /she made the day before. How many days did it take him/her to finish the job?

Here are some examples followed by exercises. They illustrate some useful applications.