UNIT 1

SEQUENCES AND SERIES

Unit Outcomes

By the end of this unit, you will be able to:

- Understand sequence and series.
- Compute terms of a sequence from a given rule.
- Use given terms to develop a formula that represent the sequence.
- Identify different types of sequences and series.
- Compute the partial and infinite sum of some sequences.
- Apply your understanding the knowledge of sequences and series to real-life problems.

Unit Contents

KEY Words

- arithmetic sequence
- finite sequence
- partial sums
- common difference
- general term
- sequence
- common ratio
- convergent series
- divergent series
- geometric sequence
- infinite sequence
- infinite series
- series
- terms of a sequence
- recursion formula

Introduction

Mathematics has an enormous number of uses in our daily life. There is, in fact, no area of life that is not affected by mathematics. For instance, civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programs useful. Mathematics is present everywhere from distance, time and money to art, design and music.

A sequence is an arrangement of numbers in a definite order according to some rule. Sequences and Series play an important role in various aspects of our lives. They help us to predict, evaluate and monitor the outcome of a situation or event and help us in decision making.

1.1 Sequences

Activity 1.1

1. People wants to plant trees in a certain pattern in the green area of a community like 20 plants in the first row, 34 plants in the second row and 48 plants in the third row, and so on. How many trees people will plant in the 5^{th} and 6^{th} row?
2. Find the next two terms of the following sequence.

   a. $\{50, 47, 44, 41, \ldots\}$ b. $\{2, 12, 22, 32, \ldots\}$

Definition 1.1

A sequence is a function whose domain is the collection of all integers greater than or equal to a given integer m (usually 0 or 1). A sequence is usually denoted by a_n. The functional values: $a_1, a_2, a_3, ..., a_n, ...$ are called the terms of a sequence, and a_n is called the general term, or the n^{th} term of the sequence. There are two types of sequences depending on its last term.

Finite Sequence: A sequence that has a last term. The domain of a finite sequence is $1, 2, 3, ..., n$.

Infinite sequence: A sequence that does not have a last term. The domain of an infinite sequence is the set of natural numbers $(\mathbb{N})$.

Example 1

List the first five terms of each of the sequences whose general terms are given below where n is a positive integer.

$(a)\ a_n = 2n - 1 \qquad (b)\ a_n = \left(-\frac{1}{3}\right)^{n-1} \qquad (c)\ a_n = \frac{1}{n}$

Solution

$(a)\ a_n = 2n - 1:$

$$a_1 = 2(1) - 1 = 1, \quad a_2 = 2(2) - 1 = 3, \quad a_3 = 2(3) - 1 = 5,$$
$$a_4 = 2(4) - 1 = 7, \quad a_5 = 2(5) - 1 = 9.$$

Therefore, the first five terms are $1, 3, 5, 7$ and 9.

$(b)\ a_n = \left(-\frac{1}{3}\right)^{n-1}$

$$a_1 = \left(-\frac{1}{3}\right)^{1-1} = 1, \quad a_2 = \left(-\frac{1}{3}\right)^{2-1} = -\frac{1}{3}, \quad a_3 = \left(-\frac{1}{3}\right)^{3-1} = \frac{1}{9},$$
$$a_4 = \left(-\frac{1}{3}\right)^{4-1} = -\frac{1}{27}, \quad a_5 = \left(-\frac{1}{3}\right)^{5-1} = \frac{1}{81}.$$

Therefore, the first five terms are 1, $-\frac{1}{3}$, $\frac{1}{9}$, $-\frac{1}{27}$ and $\frac{1}{81}$.

(c) $a_n = \frac{1}{n}$:

$$a_1 = \frac{1}{1} = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3},\ a_4 = \frac{1}{4},\ a_5 = \frac{1}{5}.$$

Therefore, the first five terms are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$.

Example 2

Draw the graph of sequence $a_n = 2n + 1$.

Solution

Make a table with n and a_n, as follows. Then plot each ordered pair (n, a_n).

n	1	2	3	4	5
a_n	3	5	7	9	11

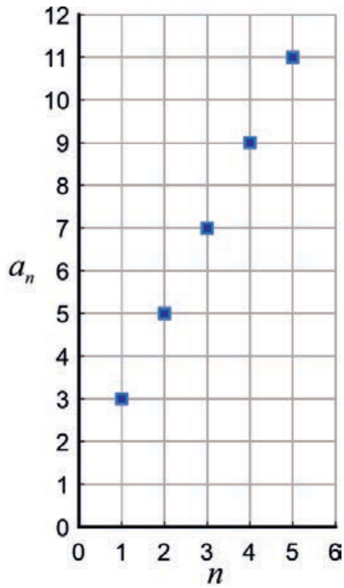

Figure 1.1

Note

From the Figure 1.1, we can observe that the graph of the sequence follows the pattern of a linear equation.

Exercise 1.1

1. List the first five terms of each of the sequences whose general terms are given below where n is a positive integer:

   a. $a_n = 2n$ b. $a_n = \left(\frac{1}{2}\right)^{n-1}$ c. $a_n = \frac{n}{n+1}$ d. $a_n = n^3$

2. Draw the graph of the following sequences and observe the pattern of the sequence.

   a. $a_n = 3n - 1$, b. $a_n = \frac{1}{n}$, c. $a_n = (-1)^n$.

3. Bontu's uncle gave 130 Ethiopian birr to her in January, in the next month she saves money and has 210 Ethiopian birr and in the third month she has 290 Ethiopian birr. How much money will she have in the fourth, fifth, sixth and seventh month respectively.

Fibonacci and Mulatu sequences

Activity 1.2

Two squares with a side length of 1 are arranged side by side. The rectangle made has a vertical length of 1 and a horizontal length of 2. Place a square with a side length of 2 next to it. Then, the rectangle made has vertical length is 3, horizontal length is 2. After this, arrange a square with a side length of 3, a square with a length of 5, and a square with a length of 8 to make a large rectangle.

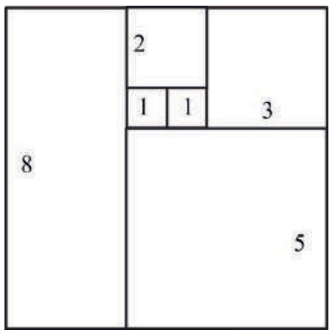

Figure 1.2

Based on Figure 1.2 above, answer the following questions:

a. What is the sum of the area of all the inner squares?

b. What is the area of the outer rectangle?

c. What is the relation between your answer to parts a and b?

Fibonacci's sequence

HISTORICAL NOTE

Leonardo Fibonacci (circa 1170, 1240)

Italian mathematician Leonardo Fibonacci made advances in number theory and algebra. Fibonacci, also called Leonardo of Pisa, produced numbers that have many interesting properties such as the birth rates of rabbits and the spiral growth of leaves on some trees.

He is especially known for his work on series of numbers, including the Fibonacci series. Each number in a Fibonacci series is equal to the sum of the two numbers that came before it. Fibonacci sequence arose when he was trying to solve a problem of the following kind concerning the breeding of rabbits.

"Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at the age of two months. If we start with one new born pair, how many pairs of rabbits will we have in the n^{th} month?"

Fibonacci sequence is defined as: $F_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$

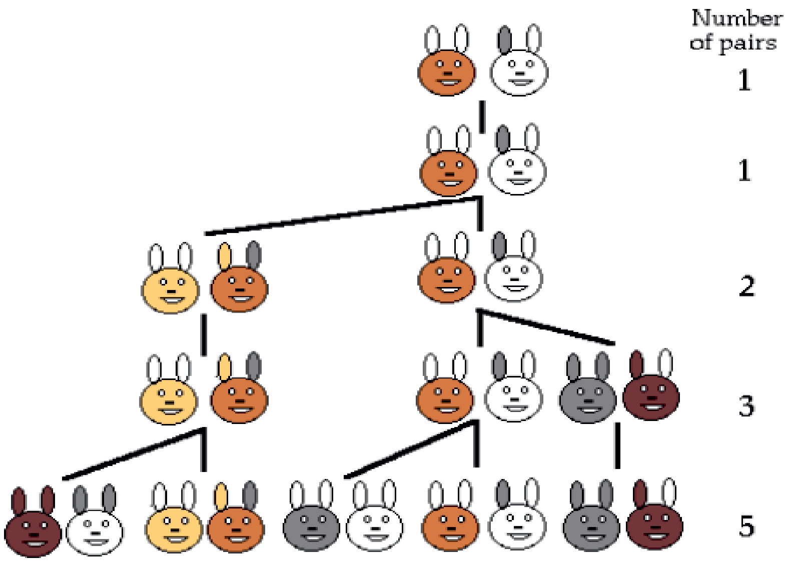

Figure 1.3

Example

List the first eight terms of the Fibonacci sequence and draw its graph.

Solution

Make a table with n and a_n, as follows. Then plot each ordered pair (n, F_n).

n	0	1	2	3	4	5	6	7	8	9
F_n	1	1	2	3	5	8	13	21	34	55

So, the graph becomes

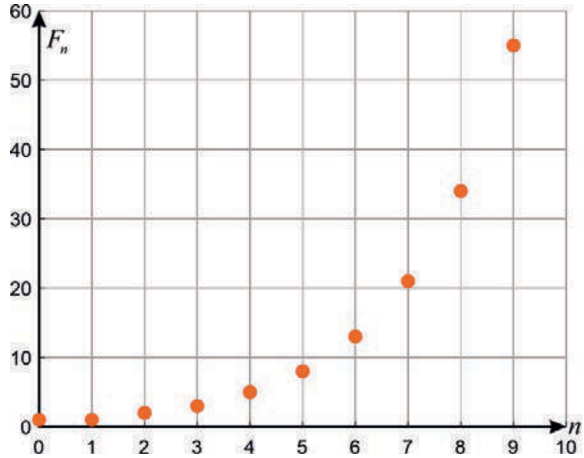

Figure 1.4

Mulatu sequence

HISTORICAL NOTE

Mulatu Lemma

Professor Mulatu Lemma is an Ethiopian Mathematician.

He pursued an education through a university in Ethiopia, earning a Bachelor of Science in 1977 from Addis Ababa University.

He continued his studies at the aforementioned university and obtained a Master of Science in applied mathematics in 1982. Following these accomplishments, he was enrolled at Kent State University, where he pursued a Master of Arts in pure mathematics in 1993. Prof. Mulatu completed his academic journey with a Doctor of Philosophy from Kent State University in 1994. In 2011, he introduced the Mulatu's Number (named after him) to the mathematical community and to the world.

Professor Mulatu introduced a sequence of the form:

$$M_n = \begin{cases} 4 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ M_{n-1} + M_{n-2} & \text{if } n \geq 2. \end{cases}$$

Recursive Sequence

A sequence that relates to the general term a_n of a sequence where one or more of the terms that comes before it is said to be defined recursively. The domain of recursive sequence can be the set of whole numbers. For Example, Mulatu and Fibonacci sequences are some of the examples of recursion formula.

Exercise 1.2

1. Find the 12^{th} term of the Fibonacci's sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$.
2. List the first eight terms of Mulatu's sequences and draw its graph.