1.3 The Sigma Notation and Partial Sums

In the previous sections, you learned about the individual terms of a sequence. In this section, you will learn how to add the terms of a sequence, i.e. find the sum of the terms.

Partial sums

Given the sequence a_n

$S_1 = a_1$, S_1 is the first term of the sequence.

$S_2 = a_1 + a_2$, S_2 is the sum of the first two terms of the sequence.

$S_3 = a_1 + a_2 + a_3$, S_3 is the sum of the first three terms of the sequence.

$S_4 = a_1 + a_2 + a_3 + a_4$, S_4 is the sum of the first four terms of the sequence,

and so on.

$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$, S_n is the sum of the first n terms of the sequence called the partial sum.

Definition 1.4

Let $\{a_n\}_{n=1}^{n=\infty}$ be a sequence. The sum of the first n terms of the sequence, denoted by S_n is called the partial sum of the sequence. Such summation is denoted as follows.

$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$, where k is the index of the summation, 1 is the lower limit of summation, n is the upper limit of the summation and $\sum$ is the sigma notation or the summation notation.

Example 1

Find the sum of the first five even natural numbers.

Solution

$a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8,\ a_5 = 10$. Then,

$$\begin{aligned} S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= 2 + 4 + 6 + 8 + 10 \\ &= 30 \end{aligned}$$

Example 2

Let $a_n = 3n + 1$, find S_6

Solution

$$a_n = 3n + 1$$

$$a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_5 = 16, a_6 = 19$$

$$\begin{aligned} S_6 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= 4 + 7 + 10 + 13 + 16 + 19 \\ &= 69 \end{aligned}$$

Example 3

Given the general term $a_n = \dfrac{1}{n(n+1)}$, find the sum of the first

a. 99 terms		b. n terms

Solution

By using partial fraction decomposition:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

Solving for A and B gives $A = 1$ and $B = -1$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$(a)\quad S_{99}=(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots+(\frac{1}{99}-\frac{1}{100})=1-\frac{1}{100}=0.99$$

$$(b)\quad S_n=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\ldots+\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}=\frac{n}{n+1}$$

Note: Such a sequence is said to be telescoping sequence.

Exercise 1.10

1. Find the sum of:
   a. the first five odd natural numbers.
   b. the first ten odd natural numbers.
2. Find the sums of the following sequences to the term given.
   a. $a_n=4n-3,\ S_5$. b. $a_n=3-5n,\ S_8$. c. $a_n=n^2+1,\ S_6$.
3. Given the general term $a_n=\dfrac{2}{n^2+5n+6}$, find the sum of the first n^{th} terms.

Sigma notation

Sigma notation is a method used to write out a long sum in a concise way. We use sigma notation for writing finite and infinite numbers of terms in a sequence. The sum is denoted by the sigma notation using the Greek letter $\sum$ (sigma).

Example 1

Express the following sigma notation in the form of the sum

a. $\sum_{k=1}^{8} 3k$ b. $\sum_{k=2}^{6} k^2$

Solution

a. $\sum_{k=1}^{8} 3k=3+6+9+12+15+18+21+24=108$. b. $\sum_{k=2}^{6} k^2=4+9+16+25+36=90$.

Example 2

Which one of the following express the sum, $3^2+4^2+5^2+6^2+7^2$?

a. $\sum_{k=3}^{7} k^2$ b. $\sum_{i=3}^{7} i^2$ c. $\sum_{k=2}^{6} (k+1)^2$

Solution

All of them express the given sum.

Exercise 1.11

1. Express the following sigma notations in the form of a sum.

   a. $\sum_{k=1}^{6} 2k$ b. $\sum_{k=3}^{5} k^2$ c. $\sum_{k=1}^{n} 3^k$ d. $\sum_{k=3}^{5} k^3$

2. Express the following using the sigma notation

   $2^2+4^2+6^2+8^2+10^2+12^2.$

1.3.1 Sigma Notation

Properties of sigma notation

The sequence $\{a_n\}$, where all the terms are c, the sum of the first n$^{\text{th}}$ term is

$$\sum_{k=1}^{n} a_n = c+c+c+\ldots+c = nc$$

That is, $\sum_{k=1}^{n} a_n = nc$

In particular, $\sum_{k=1}^{n} 1 = n(1) = n.$

Properties of Sigma Notation

(1) $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$, c is a constant

(2) $\sum_{k=1}^{n} (a_k+b_k) = \sum_{k=1}^{n} a_i + \sum_{k=1}^{n} b_k$

(3) $\sum_{k=1}^{n} (a_k-b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(4) $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$,$1\le m\le n$

Example

Evaluate each of the following sigma notations:

(a) $\sum_{k=1}^{3} 4k$ (b) $\sum_{k=1}^{5} (3k\text{-}2)$ (c) $\sum_{k=1}^{6} 2^{k-1}$

Solution

(a) $\sum_{k=1}^{3} 4k = 4+8+12 = 24$

Using the above property (1), you can also calculate,

$$\sum_{k=1}^{3} 4k = 4\sum_{k=1}^{3} k = 4(1+2+3) = 4(6) = 24$$

(b) $\sum_{k=1}^{5} (3k-2) = 1+4+7+10+13 = 35$

Using property (1) and (3), you can also calculate,

$$\begin{aligned}\sum_{k=1}^{5} (3k-2) &= 3\sum_{k=1}^{5} k - \sum_{k=1}^{5} 2 = 3(1+2+3+4+5)-(2+2+2+2+2)\\ &= 3(1+2+3+4+5)-(2+2+2+2+2)\\ &= 3(15)-2(5)\\ &= 35\end{aligned}$$

(c) $\sum_{k=1}^{6} 2^{k-1} = 2^0+2^1+2^2+2^3+2^4+2^5$

$$\begin{aligned}&= 1+2+4+8+16+32\\ &= 63\end{aligned}$$

Using the above property (4), you can also calculate,

$$\begin{aligned}\sum_{k=1}^{6} 2^{k-1} &= \sum_{k=1}^{3} 2^{k-1} + \sum_{k=4}^{6} 2^{k-1}\\ &= (2^0+2^1+2^2)+(2^3+2^4+2^5)\\ &= 63\end{aligned}$$

Exercise 1.12

Evaluate each of the following sigma notations.

a. $\sum_{k=1}^{4} 5k$ b. $\sum_{k=1}^{5} (4k-1)$ c. $\sum_{k=3}^{6} (k^2-4)$

d. $\sum_{k=2}^{5} 3$ e. $\sum_{k=1}^{8} (k^3+2k^2-3k+5)$

1.3.2 Sum of Arithmetic Sequences

Activity 1.5

Find the sum of the first ten terms of the sequence $5, 15, 25, 35, \ldots$

HISTORICAL NOTE

Carl Friedrich Gauss (1777-1855)

A teacher of Gauss, at his elementary school, asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. This is what Gauss did:

$$\begin{array}{r} 1+\ \ 2+\ 3+\ldots+100 \\ 100+99+98+\ldots+\ \ \ 1 \\ \hline 101+101+101+\ldots+101 \end{array}$$

$$\frac{100\times 101}{2}=5050.$$

To find the sum of the first 100 natural numbers, Gauss worked as follows. Writing the sum forward and backward then adding them together yields:

$$S_{100}=1+2+3+\ldots+98+99+100$$
$$S_{100}=100+99+98+\ldots+3+2+1$$
$$2S_{100}=101+101+101+\ldots+101+101+101$$
$$2S_{100}=100\times 101$$

Therefore, $S_{100} = \frac{1}{2} \times 100 \times 101 = 5050$

The sum of the first n natural numbers can also be calculated as follows:

$$S_n = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$
$$S_n = n + (n-1) + (n-2) + \ldots 3 + 2 + 1$$
$$2S_n = (n+1) + (n+1) + \ldots + (n+1) + (n+1)$$

Therefore, $2S_n = n(n+1)$

> The sum of the first n consecutive natural numbers is $S_n = \frac{n}{2}(n+1)$.

Example

Find the sum of the first a. 40 natural numbers b. 150 natural numbers

Solution

a) Using the formula $S_n = \frac{n}{2}(n+1)$, $S_{40} = \frac{40}{2}(40+1) = 20(41) = 820$

b) Using the formula $S_n = \frac{n}{2}(n+1)$, $S_{150} = \frac{150}{2}(150+1) = 75(151) = 11,325$

Exercise 1.13

1. Find the sum of the first
   a. 30 natural numbers.
   b. 99 natural numbers.
   c. 200 natural numbers.
2. If the sum of the first n natural numbers is 3240, what is the value of n ?

Derivation of sum of arithmetic sequence

Let $\{A_n\}_{n=1}^{\infty}$ be an arithmetic sequence.

$$S_n = A_1 + A_2 + A_3 + \ldots + A_n \text{ , where } A_n = A_1 + (n-1)d$$
$$S_n = A_1 + (A_1 + d) + (A_1 + 2d) + (A_1 + 3d) + \ldots + (A_1 + (n-1)d)$$

By collecting all the A_1 terms (there are n of them) we get,

$$S_n = nA_1 + [d + 2d + 3d + ... + (n-1)d]$$

Now factoring out d from within the brackets,

$$S_n = nA_1 + d[1 + 2 + 3 + ... + (n-1)]$$

Inside the brackets you have the sum of the first $(n-1)$ positive integers. Thus by using the formula, $S_n = \frac{n}{2}(n+1)$,

$$S_n = nA_1 + d\left(\frac{n-1}{2}\right)n = \frac{2nA_1 + n(n-1)d}{2} = \frac{n}{2}[2A_1 + (n-1)d].$$

This formula can be written as:

$$S_n = \frac{n}{2}[A_1 + \{A_1 + (n-1)d\}]$$

$$= \frac{n}{2}(A_1 + A_n)$$

Hence, the following theorem is derived:

Theorem 1.3

The sum S_n of the first n terms of an arithmetic sequence with first term A_1 and common difference d is $S_n = \sum_{k=1}^{n} A_k = \frac{n}{2}[2A_1 + (n-1)d]$.

This formula can also be written as:

$$S_n = n\left(\frac{A_1 + A_n}{2}\right),$$ where A_n is the n^{th} term.

This alternative formula is useful when the first and the last terms are known.

Example 1

Find the sum of the first 35 terms of the sequence whose general term is $A_n = 5n$.

Solution

From the general term, we get $A_1 = 5, A_2 = 10, A_3 = 15, \ldots,$ this shows that the given sequence is an arithmetic sequence. So, $A_{35} = 5(35) = 175.$

Since we can easily identify the first and the 35^{th} terms, we use the formula

$$S_n = \frac{n}{2}(A_1 + A_n) = n\left(\frac{A_1 + A_n}{2}\right)$$

Thus, substituting $A_1 = 5$, and $A_{35} = 175$

$$S_{35} = \frac{35}{2}(5+175) = 35\left(\frac{5+175}{2}\right) = 35(90) = 3{,}150$$

Example 2

If the 1^{st} term of arithmetic sequence is 4, common difference is 5, then find the sum of the first 40 terms.

Solution

Given $A_1 = 4, \quad d = 5,$

$$S_n = \frac{n}{2}\left[2A_1 + (n-1)d\right] \Rightarrow S_{40} = \frac{40}{2}\left[2(4) + (40-1)(5)\right] = 20(8+195) = 20(203) = 4060.$$

Exercise 1.14

1. Find the partial sum of the following arithmetic sequences:

   a. $A_1 = 2,$ and last term $A_{10} = 21.$

   b. $A_1 = 40,$ and last term $A_{26} = 0.$

2. Find the sum of the following arithmetic sequences:

   a. $A_1 = 2,\ d = 3,\ n = 10$ b. $A_1 = 30,\ d = -5,\ n = 12$

Further on sum of arithmetic sequence

Example 1

Find the sum S_7 of the arithmetic sequence whose 4^{th} term is 2 and 7^{th} term is 17.

Solution

Applying the formula $A_n = A_1 + (n-1)d$

$$\begin{cases} A_4 = A_1 + 3d = 2 \\ A_7 = A_1 + 6d = 17 \end{cases}$$

Subtracting the first equation from the second equation,

$$3d = 17 - 2$$
$$d = 5$$

Thus,

$$A_1 = A_4 - 3d$$
$$A_1 = 2 - 3 \times 5 = -13$$
$$S_n = n\left(\frac{A_1 + A_n}{2}\right)$$
$$S_7 = 7\left(\frac{-13+17}{2}\right) = 7(2) = 14.$$

Example 2

For a given arithmetic sequence the sum $S_{10} = 165$ and $A_1 = 3$, find A_{10}.

Solution

Since the first term and the sum are given, applying the formula gives:

$$S_n = n\left(\frac{A_1 + A_n}{2}\right)$$

$$S_{10} = 165, A_1 = 3, A_{10} = ?$$

$$S_{10} = 10\left(\frac{A_1 + A_{10}}{2}\right)$$

$$165 = \frac{10}{2}(3 + A_{10})$$

$$\frac{165}{5} = 3 + A_{10}$$

$$A_{10} = 33 - 3 = 30.$$

Example 3

Find the sum of integers from 1 to 100 that are divisible by 10.

Solution

The number of such integer is $\frac{100}{10} = 10$

The 1^{st} term is 10, and the last term is 100. Then, $S_n = 10\left(\frac{10+100}{2}\right) = 550.$

Exercise 1.15

1. Find S_5 of the arithmetic sequence whose 3^{rd} term is 5 and 5^{th} term is 11.
2. Given the sum of an arithmetic sequence is $S_8 = 120$ and $A_1 = 1$, find A_8 and A_n.
3. Find the sum of the integers from 1 to 100 that are divisible by 2 or 5.
4. Find the sum of odd integers from 1 to 2001.

1.3.2 Sum of Geometric Sequences

Activity 1.6

1. Find the sum of the following geometric sequences:

   a. $\{2, 4, 8, 16, 32\}$ b. $\left\{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}\right\}$

2. Find the sum of the first 5 terms of the sequence $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

In order to answer such types of problems, consider the following:

Let $\{G_n\}_{n=1}^{\infty}$ be a geometric sequence, then its associated geometric sum, S_n is:

$$S_n = G_1 + G_2 + G_3 + \ldots + G_{n-1} + G_n, \text{ where } G_n = r^{n-1}G_1$$
$$S_n = G_1 + rG_1 + r^2G_1 + \ldots + r^{n-2}G_1 + r^{n-1}G_1$$

Factorizing out G_1,

$$S_n = G_1(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1}) \qquad (1)$$

Multiplying both sides of equation (1) by r

$$rS_n = G_1(r + r^2 + r^3 \ldots + r^{n-1} + r^n) \qquad (2)$$

Subtracting the equation (2) from equation (1),

$$S_n - rS_n = G_1(1 + r + r^2 + r^3 + \ldots + r^{n-2} + r^{n-1}) - G_1(r + r^2 + r^3 + \ldots + r^{n-1} + r^n)$$
$$(1-r)s_n = G_1(1-r^n),$$
$$S_n = \frac{G_1(1-r^n)}{1-r} \text{ for } r \neq 1$$

Thus, the following theorem is inferred:

Theorem 1.4

Let $\{G_n\}_{n=1}^{\infty}$ be a geometric sequence with common ratio r. Then the sum of the first n terms S_n is given by, $S_n = \begin{cases} nG_1, & \text{if } r = 1 \\ G_1\dfrac{(1-r^n)}{1-r} = G_1\dfrac{(r^n-1)}{r-1}, & \text{if } r \neq 1. \end{cases}$

Example

Find the sum of the sequences in activity 1.6 using this theorem, and confirm the results.

Solution

a. $r = 2$, $G_1 = 2$ and $n = 5$

$$S_5 = 2\left(\frac{2^5 - 1}{2 - 1}\right) = 62$$

b. $\left\{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}\right\}$

$G_1 = 1$, $r = \frac{2}{3}$ and $n = 4$

$$S_4 = 1\left(\frac{\left(\frac{2}{3}\right)^4 - 1}{\frac{2}{3} - 1}\right) = \frac{65}{27}$$

Exercise 1.16

1. Find the sum of the from 1^{st} to n^{th} of the following geometric sequences:

   a. $G_1 = 3,\ \ r = 2$ b. $G_1 = 1,\ \ r = \frac{1}{2}$

2. Given the geometric sequence: 1, 3, 9, 27,.... find S_n.

Further on the sum of geometric sequences

Example

The sum of the first three terms of a geometric sequence is 7, and the sum from 4^{th} to 6^{th} terms is 56. Find the first term and the common ratio.

Solution

Let the first term be G_1 and common ratio r. Then,

$$G_1 + G_1 r + G_1 r^2 = 7 \qquad (1)$$

$$G_1 r^3 + G_1 r^4 + G_1 r^5 = 56 \qquad (2)$$

From (2),

$$r^3(G_1 + G_1 r + G_1 r^2) = 56$$

Substituting (1),

$$7r^3 = 56$$
$$r^3 = 8$$
$$r = 2$$

Then,

$$G_1 + 2G_1 + 4G_1 = 7$$
$$7G_1 = 7$$
$$G_1 = 1$$

Therefore, the common ratio is 2, and the 1^{st} term is 1.

Exercise 1.17

1. The sum of the first three terms of a geometric sequence is 9, and the sum from the 4^{th} to 6^{th} term is -18. Find the first term and common ratio.
2. How many terms of the sequence: $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$ are needed to give the sum $\frac{3069}{512}$?
3. Find the sum to indicated number of terms in each of the geometric sequence in questions *a* to *d*:

   a. $0.15, 0.015, 0.0015, \ldots$ *n* terms.

   b. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ *n* terms.

   b. $1, -a, a^2, -a^3, \ldots$ *n* terms (if $a \neq -1$).

   d. $x^3, x^5, x^7, \ldots$ *n* terms (if $x \neq \pm 1$).