Notice that when converting a recurring decimal that is less than one to a fraction, we write the repeating digits to the numerator and in the denominator of the equivalent fraction write as many 9's as is the number of digits in the repeating pattern.

Exercise 1.21

Convert the following recurring decimals to fractions.

a. $0.\dot{4}$ b. $3.\dot{7}$ c. $0.\dot{5}\dot{6}$

1.5 Applications of Sequence and Series in Daily Life

This section is devoted to the application of arithmetic and geometric progressions or geometric series that are associated with real-life situations.

Activity 1.9

1. A man accepts a position with an initial salary of 5200 ETB per month. It is understood that he will receive an automatic increase of 320 ETB in the very next month and each month thereafter.
   a. Find his salary for the tenth month.
   b. What are his total earnings during the first year?
2. A carpenter was hired to build 192 window frames. The first day he/she made five frames and each day, thereafter he/she made two more frames than he /she made the day before. How many days did it take him/her to finish the job?

Here are some examples followed by exercises. They illustrate some useful applications.

Example 1

Observe the pattern in Figure 1.5:

a. If the pattern continues, find the number of letters in the column containing the letter M.

b. If the total number of letters in the pattern is 361, which letter will be in the last column?

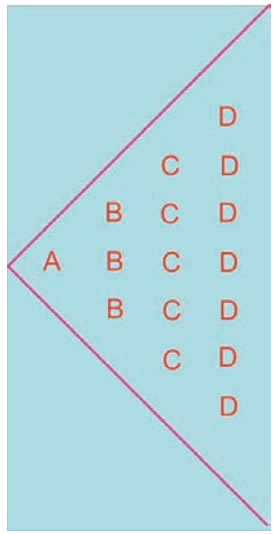


Figure 1.5

Solution

a. If you observe the structure of the letters in figure 1.5, it is of the form: $1, 3, 5, 7, 9, \ldots$

This is an arithmetic series with first term $A_1 = 1$ and common difference $d = 2$. Therefore, the n term of the arithmetic sequence is given by

$$A_n = A_1 + (n-1)d$$

For the letter M: $n = 13$;

$A_{13} = A_1 + (n-1)d = 1 + (13-1)(2) = 1 + (12)(2) = 25.$

b.

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$361 = \frac{n}{2}[2(1) + (n-1)(2)]$$

$$361 = n[1 + n - 1]$$

$$361 = n^2$$

$$n = \pm 19 \text{ (as } n > 0)$$

$$\therefore n = 19$$

So, the letter "S" will be in the last column.

Example 2

A theatre is filling up at a rate of 4 people in the first minute, 6 people in the second minute and 8 people in the third minute and so on. After 6 minutes, the theatre is half full. After how many minutes will the theatre be full?

Solution

The structure of the problem has an infinite arithmetic series of the form: $4+6+8+\ldots$

So, the common difference of the problem is calculated as

$$d = A_2 - A_1 = A_3 - A_2 = 2$$

Therefore, the sum of arithmetic series whose first term $A_1 = 4$ and common difference $d = 2$, is written as

$$S_n = \frac{n}{2}[2A_1 + (n-1)d] \Rightarrow S_6 = \frac{6}{2}[2(4) + (6-1)(2)] = 3(18) = 54 \text{ (theatre half full).}$$

Therefore, the capacity of the theatre is $2 \times 54 = 108$

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$108 = \frac{n}{2}[2(4) + (n-1)(2)]$$

$$216 = n[8 + 2n - 2]$$

$$2n^2 + 6n - 216 = 0$$

$$n^2 + 3n - 108 = 0$$

$$(n+12)(n-9) = 0$$

$\therefore n = -12$ or $n = 9$, where n must be a positive integer.

Therefore, $n = 9$. It takes 9 minutes for the theatre to be full.

Example 3

A job applicant finds that a firm offers a starting annual salary of Birr 32,500 with a guaranteed raise of Birr 1,400 per year.

a. What would the annual salary be in the tenth year?

b. How much would be earned at the firm over the first 10 years ?

Solution

a. The annual salary at the firm forms the arithmetic sequence;

32,500, 33,900, 35,300, … with first term $A_1 = 32,500$ and common difference $d = 1,400$. Thus, $A_n = A_1 + (n-1)d$, substituting the values we obtain;

$A_{10} = 32,500 + (10-1)(1,400) =$ Birr 45,100

b. To determine the amount that would be earned over the first 10 years, we need to add the first 10 annual salaries;

$S_{10} = A_1 + A_2 + A_3 + \ldots + A_{10} = 10\left(\frac{A_1 + A_{10}}{2}\right)$ (It is 10 times the average of the first and the last term.)

$$S_{10} = \frac{10}{2}(32,500 + 45,100) = \text{ Birr } 388,000$$

Therefore, a total of Birr 388,000 would be earned at the firm over the first 10 years.

Exercise 1.22

1. A person is scheduled to get a raise of Birr 250 every 6 months during his/her first 5 years on the job. If their starting salary is Birr 25,250 per year, what will his/her annual salary be at the end of the 3rd year?
2. Bontu begins a saving program in which she will save Birr 1,000 the first year. Each subsequent year, she will save 200 more than she did the previous year. How much will she save during the eighth year?

Example 4

A woman deposits Birr 3,500 in a bank account paying an annual interest at a rate of 6%. Show that whether the amounts she has in the account at the end of each year form a geometric sequence.

Solution

Let $G_1 = 3,500$. Then,

$$G_2 = G_1 + \frac{6}{100}G_1 = G_1(1+0.06) = G_1(1.06)$$

$$G_3 = G_2 + \frac{6}{100}G_2 = G_2(1+0.06) = G_1(1.06)(1.06) = G_1(1.06)^2$$

$$G_4 = G_3 + \frac{6}{100}G_3 = G_3(1+0.06) = G_1(1.06)^2(1.06) = G_1(1.06)^3$$

Continuing in this way, you get $G_n = G_1(1.06)^{n-1}$

Since the ratio of any two consecutive terms is a constant, which is 1.06, this sequence is a geometric sequence.

Example 5

Suppose a radioactive substance loses half of its mass per year. If we start with 100 grams of a radioactive substance, how much will be left after 10 years?

Solution

Let us record the amount of the radioactive substance left after each year starting with $G_0 = 100$. Note that each term is half of the previous term and hence,

$G_1 = \frac{1}{2}G_0 = 100\left(\frac{1}{2}\right)$ is the amount left at the end of year 1.

$G_2 = \frac{1}{2}G_1 = 100\left(\frac{1}{2}\right)^2$ is the amount left at the end of year 2.

$G_3 = \frac{1}{2}G_2 == 100\left(\frac{1}{2}\right)^3$ is the amount left at the end of year 3.

If you continue in this way, you see that the ratio of any two consecutive terms is a constant, which is $\frac{1}{2}$, and hence this sequence is a geometric sequence.

Therefore, after ten years, the amount of the substance left is given by:

$$G_{10} = G_0\left(\frac{1}{2}\right)^{10} = 100\left(\frac{1}{2}\right)^{10} = \frac{100}{1,024} = 0.09765625 \text{ g.}$$

Exercise 1.23

1. A certain item loses one-tenth of its value each year. If the item is worth Birr 28,000 today, how much will it worth 4 years from now?
2. A boat is now worth Birr 34,000 and loses 12% of its value each year. What will it worth after 5 years?
3. The population of a certain town is increasing at a rate of 2.5% per year. If the population is currently 100,000, what will the population be 10 years from now?
4. Sofia deposits Birr 3,500 in a bank account paying an annual interest rate of 6%.Find the amount she has at the end of the
   a. first year
   b. second year
   c. third year
   d. fourth year
   e. n^{th} year
   f. Do the amounts she has at the end of each year form a geometric sequence? Explain.

Example 6

A man was injured in an accident at work. He receives a disability grant of 4800 ETB in the first year. This grant increases by a fixed amount each year.

a. What is the annual rate of increase if he received a total of 143,500 ETB over 20 years?

b. His initial annual expenses are 2600 ETB, which increases at a rate of 400 ETB per year. After how many years will his expenses exceed his income?

Solution

a. Since the grant increases with a fixed amount each year, the problem is infinite arithmetic series and the sum is given as

$$4800+(4800+d)+(4800+2d)+\ldots$$

This is an arithmetic series with first term is $A_1=4800$ and the common difference d. So, the sum of the arithmetic series is given by

$$S_n=\frac{n}{2}[2A_1+(n-1)d]$$

Therefore, the sum over 20 years is given by

$$S_{20}=\frac{20}{2}[2(4800)+(20-1)d]=143,500$$
$$143,500=10[9600+19d]$$
$$14,350=9600+19d$$
$$19d=4750$$
$$d=250.$$

Therefore, the annual increase if he received a total of 143,500 ETB over 20 years is 250 ETB.

b. The structure of the series is given by

$$2600+3000+3400+\ldots$$

This is an arithmetic series whose first term $A_1=2600$ and common difference $d=400.$

So, the sum of the arithmetic series is given by

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

$$S_{expenses} = \frac{n}{2}[2(2600) + (n-1)(400)]$$

$$= \frac{n}{2}[5200 + 400n - 400] = \frac{n}{2}[4800 + 400n]$$

$$S_{income} = \frac{n}{2}[2(4800) + (n-1)(250)]$$

$$= \frac{n}{2}[9600 + 250n - 250] = \frac{n}{2}[9350 + 250n]$$

Let $S_{\text{expenses}} = S_{\text{income}}$

$$\frac{n}{2}[4800 + 400n] = \frac{n}{2}[9350 + 250n]$$

$$4800 + 400n = 9350 + 250n$$

$$150n = 4550$$

$$n = \frac{4550}{150} = 30.333...$$

Therefore, his expenses will exceed his income after 30 years.

Exercise 1.24

1. A job applicant finds that a firm A offers a starting salary of Birr 31,100 with a guaranteed raise of Birr 1,200 per year, whereas firm B offers a higher starting salary of Birr 35,100 but will guarantee a yearly raise of only Birr 900.
   a. What would the annual salary be in the 11^{th} year at firm A?
   b. What would the annual salary be in the 11^{th} year at firm B?
   c. Over the first 11 years, how much would be earned at firm A?
   d. Over the first 11 years, how much would be earned at firm B?
   e. Compare the amount earned in 11 years in firms A and B.
2. A contest offers a total of 18 prizes. The first prize is worth Birr 10,000, and each consecutive prize is worth Birr 500 less than the next higher prize. Find the value of the eighteenth prize and the total value of the prizes.

3. A contest offers 10 prizes with a total value of Birr 13,250. If the difference in value between consecutive prizes is Birr 250, what is the value of the first prize?

Example 7

A flower 110 cm high is planted in a garden. At the end of the first year, the flower is 120 cm tall. Thereafter the growth of the flower each year is half of its growth in the previous year. Show that the height of the flower will never exceed 130 cm. Draw a graph of the relationship between time and growth.

Solution

The annual growth of the flower: $10, 5, \frac{5}{2}, \frac{5}{4}, \ldots$ and the sum of the growth:

$$10 + 5 + \frac{5}{2} + \frac{5}{4} + \ldots$$

This is a geometric series with the first term $G_1 = 10$ and common ratio $r = \frac{1}{2}$.

$$S_\infty = \frac{G_1}{1-r} = \frac{10}{1-\frac{1}{2}} = 20.$$

Note: we may join the points on the graph because the growth is continuous.

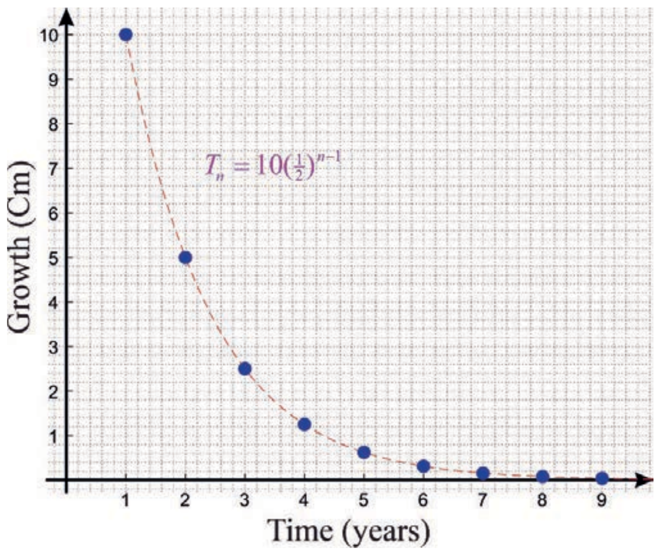


Figure 1.6

Therefore the growth of the flower is limited to 20 cm, and the maximum height of the shrub is therefore 110 cm + 20 cm = 130 cm.

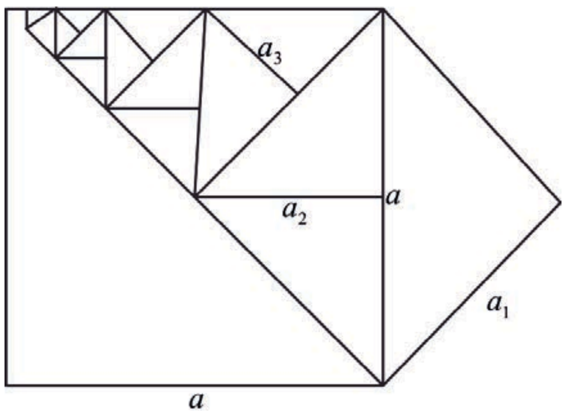
Example 8

Given a square with side length a. The side of the second square is half of its diagonal. The side of the third square is half of the diagonal of the second square and so on as shown in the figure below. Find the sum of the areas of all these squares.

Solution

Let the side of square be a_n and diagonal d_n.

Then,



$$a_0 = a$$

$$a_1 = \frac{1}{2}d_0 = \frac{1}{2}\sqrt{2}a = \frac{\sqrt{2}a}{2}$$

$$a_2 = \frac{1}{2}d_1 = \frac{1}{2}\sqrt{2}a_1 = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}a}{2}$$

$$a_3 = \frac{1}{2}d_2 = \frac{1}{2}\sqrt{2}a_2 = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}a}{2}$$

$$\dots\dots\dots\dots$$

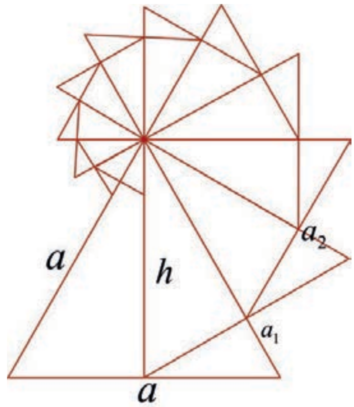
$$S_n = a_0^{\,2} + a_1^2 + a_2^2 + a_3^2 + \dots$$

$$= a^2 + \frac{1}{2}a^2 + \frac{1}{4}a^2 + \frac{1}{8}a^2 + \dots$$

Since, $r = \dfrac{G_n}{G_{n-1}} = \dfrac{1}{2}$, $|r| < 1$, then $S_\infty = \dfrac{G_1}{1-r} = \dfrac{a^2}{1-\frac{1}{2}} = 2a^2$.

Exercise 1.25

1. Suppose a ball is dropped from a height of h m and always rebounds to r % of the height from which it falls. Show that the total vertical distance that could be covered by the ball is $h\left(\frac{r+1}{1-r}\right)$ m. Assume that the ball will never stop bouncing.



2. Given an equilateral triangle with side length a, its height is the side of another equilateral triangle. The height of this triangle is then the side of the third equilateral triangle and so on, as shown in the diagram. Find the sum of the areas of all these triangles.