Does the rank-correlation get rid of inequality? Group inequality and positional mobility

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Abstract

Rank-based measures of intergenerational mobility are generally justified by their invariance to changes in inequality. However, I show that whenever the source of inequality is uncorrelated to parent ranks, such as in the cases of gender and birth order, increasing equality leads to a fall in rank mobility as measured by the rank correlation. I develop a method to ex-post quantify the importance of inequality for mobility measurement using cross-sectional income distributions and show that US income mobility could have fallen by as much as 24 percent since 1970 due to increased gender equality. Without specifying a policy objective of interest, it is therefore unclear which conclusions to draw from differences in rank-correlations across societies or changes over time.

Keywords: Economic inequality, gender inequality, intergenerational mobility, mobility measurement, rank-correlation, wealth inequality

JEL Codes: C63,D31,E24,J16,J62

1 Introduction

Rank-based measures have become popular in investigations of intergenerational mobility in outcomes such as income, wealth, education, and health (Chetty et al., 2014a,b; Adermon et al., 2018; Wong et al., 2019; Fletcher and Jajtner, 2019; Fagereng et al., 2021). An often-touted advantage of such measures is the ability to disentangle mobility from inequality. This, in principle, allows researchers to compare mobility across societies with different levels of inequality.

I show in this paper that the separation of inequality and mobility is only valid when we are not interested in the actual mechanisms creating or inhibiting mobility. Specifically, I find that intergenerational rank correlation is increased by gender inequality. In other words, a society with high gender inequality will have *higher* rank-mobility (i.e. a lower rank correlation between parent and child) than a society with low gender inequality. The intuition is as follows: If a society has no gender inequality, the gender of a child will not matter for the measure of mobility, ie. the association between the positions of parent and child in society. On the other hand, if a society has high gender inequality, the gender of the child becomes an important determinant of income. As the gender of a child is essentially random, and thus uncorrelated with parental rank, the sorting into genders will register as mobility from one generation to the next. Consequently, increasing gender equality leads to less mobility. Therefore, it becomes unclear what to conclude from differences in rank-based mobility when underlying inequality between genders differs between societies.[1]

Other definitions of groups, such as race, ethnicity, and education also affect the rank correlation but in ambiguous directions. Mobility might seem lower or higher due to inequality between groups. In that view, rank-based measures of mobility are uninformative on the mechanisms that create the barriers to mobility.

[1]The use of rank mobility is often criticized for ignoring that it is easier to change position in a more compressed distribution and positional mobility may therefore still reflect inequality. The issue in the present paper, however, goes further than that. Not only may the shape of the aggregate distribution matter, but even when the aggregate distribution remains unchanged over time, the *composition* of the distribution is a central factor.

To assess the quantitative implications of changes in gender inequality on mobility, I develop an adjustment factor that can be applied ex-post to existing studies of rank mobility using only cross-sectional income distributions. I calibrate the adjustment factor to US income data from 1970 to 2019 and show that, solely due to the narrowing gender income gap, the intergenerational rank correlation based on individual incomes should have risen by 24 percent. In other words, without any change in intergenerational mobility *within* genders, the aggregate mobility would appear to have fallen substantially due to increased gender equality. Based on this finding, a policy-maker favoring higher rank mobility should favor an *increase* in gender discrimination. The implied absurd policy recommendations reflect the issue posed by inequality for mobility measurement. There are two fundamental questions for policy evaluation and social welfare in the context of intergenerational mobility; Why should a policymaker care about mobility and what is the relevant measure of mobility? In the mobility literature, there is, unfortunately, no general agreement as to what kind of mobility should be valued, and whether mobility is desired as a goal in and of itself or as a means to lessen overall inequality (Fields and Ok, 1999).

An important distinction in the literature is between absolute and positional, or rank, mobility. *Absolute mobility* is computed using outcomes in levels and analyzing cardinal changes. Though not uncontroversial, one can generalize many of the insights from the inequality literature and link different measures to social welfare (Kolm, 1977; Atkinson and Bourguignon, 1982; Maasoumi, 1986; Tsui, 2002; Decancq and Lugo, 2013; Bourguignon and Chakravarty, 2019). Approaches based on *positional mobility* abstract from levels of outcomes and instead focus on the position, or rank, in the distribution and include fractional transition matrices and rank correlations (Fields and Ok, 1999). By focusing on positions, such approaches deliberately ignore questions of intensity (eg. how much does income change) and monotonic growth (e.g. how much more can a child in the next generation consume on average)

The justification for positional mobility in terms of social welfare is less well-understood than measures based on absolute mobility. D'Agostino and Dardanoni (2009) and Cowell and Flachaire (2018) develop axiomatic properties of mobility measures assuming social welfare is a function of positional mobility alone. D'Agostino and Dardanoni (2009) show that the rank correlation has desirable properties as a measure of positional mobility. Such approaches, however, contradict the often stated motivation for using positional mobility in the applied literature; the ability to decompose intergenerational outcome distributions into inequality and mobility. There is no reason to think that a policy-maker interested in mobility does not care about inequality or vice-versa. King (1980) and Fields and Ok (1999) develop frameworks to decouple positional mobility (or exchange mobility) and absolute mobility where "full exchange mobility" means maximizing positional movement (corresponding to a rank correlation of minus one). However, these frameworks have not taken hold in the applied literature.[2]

The missing link between theory and the empirical literature causes problems for the interpretation and proper application of the rank correlation. In the present case of gender, there is no theoretical guidance as to whether one should first rank within gender or rank all individuals jointly. I show that researchers depending on this choice may reach contradicting conclusions when comparing societies over time or space.

If society thinks gender is an important mechanism, one would ideally be able to account for gender in the analysis of mobility changes across time or between societies. In this sense, the issue that gender inequality poses for the interpretation of the rank correlation can be interpreted in the tradition of the equality-of-opportunity literature, surveyed by Roemer and Trannoy (2015). In this literature, the goal is to isolate the inequality caused by circumstances outside the control of the individual including parental outcomes. This literature explicitly views mobility as an ex-ante concept and parental outcomes as causal determinants of children's outcomes. In the case where the set of circumstances consists of a single parental outcome, such an approach simplifies to an inequality index over the residual distribution of the child outcomes, and the rank

[2]This may be because these frameworks conflict with the principle of origin independence which is the main justification for mobility in the equality of opportunity literature (corresponding to a rank correlation of zero).

correlation can be framed as a measure of equality of opportunity.

However, as soon as the set of circumstances one wishes to account for grows beyond a single explanatory variable, such as adding gender, the rank correlation as a measure of equality of opportunity breaks down. The ranked outcome of the child as a left-hand side variable is by definition not independently distributed and one cannot linearly decompose the rank mobility as one would with the intergenerational elasticity (IGE), for example in a multiple regression framework. [3]

In conclusion, the rank correlation and other positional mobility measures may be an answer to a question that is ill-poised. It does not readily inform policymakers on generally accepted social objectives and possible mobility-enhancing policies. Unfortunately, such considerations are generally missing from applied work on positional intergenerational mobility where rank correlations are generally accepted as important measures. A consequence is that measurement of (changes to) positional mobility may merely reflect statistical artifacts rather than policy-relevant information.

2 Mobility and inequality

The basic framework to conceptualize inequality and mobility follows Jäntti and Jenkins (2015) closely. Let a family be defined by a parent and a child outcome, X_i and Y_i respectively. Throughout, I assume that outcomes are continuously distributed. Most measures of intergenerational mobility can be thought of as describing the joint distribution of (X_i, Y_i). Denote this bivariate joint distribution $H(x, y)$ with the corresponding marginal distributions, $F(x)$ and $G(y)$ where x and y are values of X_i and Y_i. The marginal distributions are the outcome distributions of the two generations and all the regular measures of inequality and social welfare within generations can be calculated

[3]A related but distinct issue is that of estimating causal effects. The ranking of the child outcome variable breaks the *Stable Unit Treatment Values Assumption* usually invoked in causal analysis (Rubin, 1980). This implies that if ranks are based on realized outcomes, causal estimands of effects on ranks cannot be estimated by invoking identifying assumption of (quasi-) random assignment. As an example, assume that a subset of individuals receives a treatment that affects outcomes. When ranking the outcomes ex-post, the untreated will be affected by the treatment of others through changing the outcome distribution.

from F and G.[4]

It is well known that changes in inequality within generations directly affect measures of mobility. This makes interpretation difficult; If two societies have different measures of mobility, how much of the difference can be attributed to different levels of inequality, and how much to the transfer of privilege between generations? As an alternative, a branch of the mobility literature has moved towards a focus on *positions* in distributions, i.e. the ranks of the individuals (e.g. Chetty et al. (2014a)). Denote the ranks of parent and child as U_i and R_i respectively. Assuming continuous outcome distributions, a rank is defined as the value of the cumulative distribution function evaluated at the level of outcome, $U_i = F(X_i)$, and $R_i = G(Y_i)$. By construction, ranks are uniformly distributed on the unit interval. In other words, regardless of the shape of the actual outcome distribution, the corresponding rank distribution remains unchanged. The invariance of the rank distribution to positive monotonic transformations in principle makes it possible to separate out mobility from inequality by estimating the dependence structure between ranks rather than the actual outcomes.[5]

There exist several measures of mobility based on ranks, the most popular being transition matrices and the rank correlation (also known as the Spearman correlation coefficient). The latter is simply the linear correlation coefficient of the parent and child rank in the population:[6]

$$\rho = \frac{Cov(U_i, R_i)}{Var(U_i)} = \frac{Cov(U_i, R_i)}{\sigma_U \sigma_R} = 12 \times Cov(U_i, R_i) \tag{1}$$

A rank correlation of one means that all children inherit the position of their parents perfectly and can therefore be interpreted as a perfectly immobile society. While the definition of a perfectly immobile society is uncontroversial, the perfectly mobile society in terms of the rank correlation could either be when the correlation equals zero,

[4]For example, the IGE is given as $\beta_{IGE} = Cov(\log Y_i, \log X_i)/Var(\log X_i)$

[5]Note, however, that this decomposition presumes that society assigns value to positional mobility across generations separate from actual outcomes of individuals. The implied properties of such a social welfare function are unclear as discussed in the Section 1.

[6]Note that ranks of parents and children are both uniformly on the unit interval and that this distribution has a variance of $1/12$. The rank-correlation is thus simply the covariance multiplied by 12.

and parental outcomes yield no information as is typical in the equality of opportunity literature, or minus one which maximizes the movement in status, but is completely predicted by parental position (Prais, 1955; Shorrocks, 1978).[7] Thus, there is no sense in which the rank correlation identifies a perfect mobile society without normative judgment on why mobility matters. However, I describe a society as more mobile when it has a lower rank correlation and note that I have yet to see a negative rank correlation in applied research on intergenerational mobility.[8]

Because of the invariance of the distribution of ranks with respect to differences in inequality over time or between countries, the rank correlation has been touted as superior over linear correlation measures based on outcomes for measuring mobility in incomes, such as the intergenerational income elasticity (Chetty et al., 2014a,b) and has increasingly been applied to the analysis of wealth mobility (Fagereng et al., 2021; Adermon et al., 2018) and health mobility (Wong et al., 2019; Fletcher and Jajtner, 2019).

The framework above concerns a bivariate distribution, i.e. the link between one child outcome and one parental outcome. However, we might think of the child outcome as being determined by characteristics, which might in turn be a function of the characteristics of parents. To model such a characteristic in a simple and tractable way, I assume that families may belong to one of two groups, $j \in \{a, b\}$ and construct an indicator variable to denote group affiliation, $S_i = \mathbf{1}(i \in a)$ with $E[S_i] = \mu$. Due to the law of total covariance, the covariance between parent and child rank can be decomposed into between- and within-group components:

$$Cov(U_i, R_i) = \underbrace{Cov\left(E\left[U|S\right], E\left[R|S_i\right]\right)}_{\text{Between group covariance}} + \underbrace{E\left[Cov(U_i, R_i|S_i)\right]}_{\text{Within group covariance}}, \tag{2}$$

[7] Prais (1955) and Shorrocks (1978) develop these terms in the context of transition matrices. The rank correlation however is a dependence measure on the limit of the transition matrix when the number of categories equals the number of observations, i.e. the empirical copula.

[8] Note, that the same rank correlation can correspond to dramatically different changes in levels of income as income distributions are decidedly non-uniform in practice. In other words, the “intensity” of movement across generations is ignored (Shorrocks, 1978).

The within- and between-group components can be calculated as:

$$\text{Between group covariance} = (1-\mu)\mu\left[(\overline{U}^a - \overline{U}^b)(\overline{R}^a - \overline{R}^b)\right]$$
$$\text{Within group covariance} = \mu Cov(U_i, R_i|S_i = a) + (1-\mu)Cov(U_i, R_i|S_i = b),$$

where $\overline{U}^j$ and $\overline{R}^j$ denote the mean rank in the group j of parents and children respectively.

This decomposition highlights the presence of the marginal distributions of the two groups in the rank correlation.[9] If societies differ in group inequality this will be reflected in differences in the rank correlation. The fact that between-group inequality still matters may or may not be a problem depending on the question asked. However, only in the case where we find that the group affiliation is irrelevant for policy questions can we claim that we can abstract from inequality with measures based on ranks.

Different kinds of group partitions will have different implications for the decomposition. Two salient dimensions are race and gender, the latter being the subject of this paper. Race is generally inherited, and S_i will therefore correlate both with the parental outcome and the child outcome (and corresponding ranks). This implies that changes in racial inequality will affect the rank correlation both through the between- and within-components.[10] The gender of the child, however, is approximately random and thus uncorrelated with the rank of the parent, which implies that $\overline{U}^a = \overline{U}^b$. This leaves only the within-group covariance in Equation (2).[11]

Adjustment factor derivation with gender inequality I now proceed to investigate the importance of gender inequality for rank mobility. I assume a data-generating process where families draw a parent rank, U_i, and a child's *within-gender* rank, V_i, and the gender of the child. The gender of the child is described by $S_i = \mathbf{1}(\textit{Child is male})$ and

[9]The rank correlation is obtained by multiplying with 12.

[10]I discuss the case of race at the end of this section.

[11]I am purposefully vague about the definition of the parent. I think of the parent as either a couple or either father or mother.

gender is assumed to be orthogonal to parental outcomes.[12] The gender of a child maps to separate outcome distributions through the quantile functions, ${G^m}^{-1}(v)$ and ${G^f}^{-1}(v)$ for males and females respectively:

$$y(v, s) = (1 - s){G^f}^{-1}(v) + s{G^m}^{-1}(v), \tag{3}$$

where s and v are values of S_i and V_i respectively and y is the outcome. In this simplified setup, gender is only important for mobility insofar as the gender-specific outcome distributions differ, $G^f \neq G^m$. The share of males in a cohort is given by $E[S_i] = \mu$ which I assume is stable over time. The aggregate outcome distribution of the child generation is a mixture distribution of the income distributions of the two genders:

$$G(y) = (1 - \mu)G^f(y) + \mu G^m(y). \tag{4}$$

Again, let R_i be the *aggregate* rank of the child, i.e. the outcome rank of the child when compared to every other child regardless of gender. Following Equations (3) and (4), the *aggregate* rank can be expressed as a function of the *within-gender* rank and the gender indicator, $R_i = r(V_i, S_i)$. Using the *probability integral transform*:

$$r(v, s) = G(y(v, s)) = \mu G^m(y(v, s)) + (1 - \mu)G^f(y(v, s)) \tag{5}$$

Equation (5) can be simplified by conditioning on gender and using Equation (3):

$$r(v, 0) = (1 - \mu)v + \mu G^m({G^f}^{-1}(v)) \tag{6}$$

$$r(v, 1) = (1 - \mu)G^f({G^m}^{-1}(v)) + \mu v. \tag{7}$$

[12]There is evidence that the gender of a child might not be completely uncorrelated to parent outcomes, such as income, through the linkage between mortality rates of male and female fetuses and the mother's circumstances and lifestyle (Orzack et al., 2015) and through parental behavioral responses to the gender of a child (Lundberg et al., 2007). Nonetheless, the assumption of orthogonality between gender and parent outcome is maintained throughout the rest of the analysis.

These two expressions involve a transformation of the within-gender rank, v, to the other-gender rank. This is achieved by first applying the own-gender quantile function to retrieve an outcome in level and then mapping to the corresponding rank for the other using the cumulative distribution function. These transformations can be approximated by a first-order Taylor polynomial around a value $\hat{v}$:

$$G^m(G^{f^{-1}}(v)) \approx G^m\left(G^{f^{-1}}(\hat{v})\right) + \underbrace{\left.\frac{g^m\left(G^{f^{-1}}(v)\right)}{g^f\left(G^{f^{-1}}(v)\right)}\right|_{\hat{v}}}_{\lambda^f(\hat{v})}(v-\hat{v}) \tag{8}$$

$$G^f(G^{m^{-1}}(v)) \approx G^f\left(G^{m-1}(\hat{v})\right) + \underbrace{\left.\frac{g^f\left(G^{m-1}(v)\right)}{g^m\left(G^{m-1}(v)\right)}\right|_{\hat{v}}}_{\lambda^m(\hat{v})}(v-\hat{v}), \tag{9}$$

where $\lambda^m(v)$ is a likelihood ratio evaluated at the outcome corresponding to a rank, v, in the male outcome distribution. Conversely, $\lambda^f(v)$ is the reciprocal likelihood ratio evaluated at the outcome corresponding to the rank in the female distribution.

To focus on the role of inequality, I assume that the covariance of parent rank and within-gender child rank is equal across genders. Simplifying notation for conditioning I therefore assume that $Cov(U_i, V_i|0) = Cov(U_i, V_i|1)$.[13] Using Equations (8) and (9) a

[13]This is not required for my results to hold but simplifies exposition considerably. I return to this assumption in the end of this section.

first-order Taylor expansion of within-group component in Equation (2) yields:

$$
\begin{aligned}
E[Cov(U_i, R_i|S_i)] =& (1-\mu)Cov(U_i, R_i|0) + \mu Cov(U_i, R_i|1) \\
=& (1-\mu)Cov(U_i, r(V_i, 1)|0) + \mu Cov(U_i, r(V_i, 0)|1) \\
\approx& (1-\mu)Cov(U_i, \left\{\mu\lambda^f(\hat{v}) + (1-\mu)\right\} V_i|0) \\
& + \mu Cov(U_i, \{(1-\mu)\lambda^m(\hat{v}) + \mu\} V_i|1) \\
=& (1-\mu)\left\{\mu\lambda^f(\hat{v}) + (1-\mu)\right\} Cov(U_i, V_i|0) \\
& + \mu\{(1-\mu)\lambda^m(\hat{v}) + \mu\} Cov(U_i, V_i|1) \\
=& \left[(1-\mu)\left\{\mu\lambda^f(\hat{v}) + (1-\mu)\right\} + \mu\{(1-\mu)\lambda^m(\hat{v}) + \mu\}\right] Cov(U_i, V_i) \\
=& A(\hat{v})Cov(U_i, V_i), \qquad (10)
\end{aligned}
$$

where

$$
A(\hat{v}) = \left[\mu^2 + (1-\mu)^2 + \mu(1-\mu)\left\{\lambda^m(\hat{v}) + \lambda^f(\hat{v})\right\}\right],
$$

where the fifth equality follows from assuming equal covariance for males and females.

The aggregate covariance is thus, *to an approximation*, a linear function of the within-gender covariance, where the relationship is governed by an adjustment term, $A(\hat{v})$. This adjustment term is, in turn, a function of two likelihood ratios, λ^m, and λ^f evaluated at different outcomes and can be obtained from cross-sectional data on child outcomes. The formulation of the adjustment factor in (10) allows researchers to assess ex-post the importance of gender inequality for the rank correlation in a given study without access to the joint distribution of parent and child outcomes.[14]

Comparative statics I now proceed to investigate the properties of the adjustment term and the implications of gender inequality for the magnitude of the wedge between within-gender and aggregate rank correlations. As densities can only take positive values, it follows that the adjustment term is non-negative and the total covariance will

[14]As seen in Equation (10), the assumption of equal within-gender covariance can easily be relaxed by stopping derivations at the 4th row. However, without equal covariance, one cannot establish a one-to-one mapping between within-gender covariance and the aggregate covariance.

always have the same sign as the within-gender correlation. Under full gender equality it follows that $\lambda^m(\cdot) = \lambda^f(\cdot) = 1 \Rightarrow \quad A(\cdot) \quad = \quad 1$. In other words, when there is no gender inequality, the within-gender covariance equals the total covariance. What remains to be shown is whether the aggregate correlation is larger than the within-gender correlation, i.e. whether the adjustment term is larger or smaller than one.

The likelihood ratios reflect gender inequality. For instance, if the likelihood ratio g^m/g^f is monotonically rising in the income level, this implies that the male income distribution stochastically dominates the female distribution. To gain additional insights, assume that this likelihood ratio is indeed monotonically increasing. It follows that densities cross only once.[15] An example of such a situation can be seen in Figure 1 with two artificial distributions: one for women (in red) and one for men (in blue). The top plot displays the two cumulative distribution functions (CDF). The distributions are chosen such that they have monotonic likelihood ratios. I assume that the populations of men and women are of equal sizes, i.e. $\mu = 0.5$. In this case, the adjustment factor simplifies to $A(\hat{v}) = \left[\frac{3}{4} + \frac{1}{4}\left\{\lambda^m(\hat{v}) + \lambda^f(\hat{v})\right\}\right]$.

Denote the point of crossing of the two densities by y^* where $g^m(y^*) = g^f(y^*)$. The single crossing implies the following:

$$\frac{g^m(y)}{g^f(y)} < 1 \text{ if } y < y^* \tag{11}$$

$$\frac{g^m(y)}{g^f(y)} > 1 \text{ if } y > y^* \tag{12}$$

In other words, the male density is lower than the female density when the outcome is below the crossing point and higher when the outcome is above. Using this observation

[15]This is a fairly realistic assumption when the outcome is income. In the empirical analysis in section 3 I show that the assumption of monotonic likelihood ratios in incomes is reasonable in an American context.

we can bound the interval, where $\lambda^m(v)$ and $\lambda^f(v)$ are *both* smaller than one:

$$\lambda^m(v) < 1 \Leftrightarrow \frac{g^f\left(G^{m-1}(v)\right)}{g^m\left(G^{m-1}(v)\right)} < 1 \Leftrightarrow G^{m-1}(v) < y^* \Leftrightarrow v < G^m(y^*) \tag{13}$$

$$\lambda^f(v) < 1 \Leftrightarrow \frac{g^m\left(G^{f^{-1}}(v)\right)}{g^f\left(G^{f^{-1}}(v)\right)} < 1 \Leftrightarrow G^{f^{-1}}(v) > y^* \Leftrightarrow v > G^f(y^*), \tag{14}$$

where the second inequality follows from the single crossing property. Therefore, the adjustment term is less than one for $\hat{v} \in \left[G^m(y^*), G^f(y^*)\right]$.[16]

The approximated relationship in Equation (10) is evaluated at a *rank* level. However, if we want to approximate the adjustment factor at an *outcome* level, this may correspond to two different ranks: one for women and one for men. The two ranks give rise to two different adjustment factors. In order to collapse the factors into a simple evaluation, I suggest taking the density-weighted mean, where the densities are evaluated at an outcome level in the aggregate distribution. Denote $\bar{y}$ as the outcome of evaluation and $\omega = \frac{g^m(\bar{y})}{g^m(\bar{y})+g^f(\bar{y})}$ as the share of males at $\bar{y}$. Now define the two ranks corresponding to $\bar{y}$ for men as $\bar{v}^m$ and for women $\bar{v}^f$. We can now calculate the weighted mean of the adjustment term:

$$\begin{aligned}&\omega A(\bar{v}^m) + (1-\omega)A(\bar{v}^f)\\=&\frac{3}{4} + \frac{1}{4}\left(\omega\lambda^f(\bar{v}^m) + (1-\omega)\lambda^m(\bar{v}^f)\right),\end{aligned} \tag{15}$$

where I have assumed equal overall shares of men and women.[17] Taking the density-

[16] The interval can seen in Figure 1 in the following way: Find the crossing point, y^*, in the middle figure. Evaluate this value in the CDFs in the top figure to find the corresponding bounding ranks, $\bar{v}^m = G^m(y^*)$ and $\bar{v}^f = G^f(y^*)$. These bounds can, in turn, be transferred back into outcome levels, denoted by y^- and y^+. The adjustment term is less than one for every rank chosen within these bounds. Note that these bounds are sufficient but not necessary conditions for an adjustment factor smaller than one. The adjustment term can be smaller in cases where either λ^m or λ^f are larger than one, as long as the other ratio is sufficiently small.

[17] The more general case is given by:

$$\begin{aligned}&\omega A(\bar{v}^m) + (1-\omega)A(\bar{v}^f)\\=&\mu^2 + (1-\mu)^2 + \mu(1-\mu)\left(\omega\left[\lambda^m(\bar{v}^m) + \lambda^f(\bar{v}^m)\right] + (1-\omega)\left[\lambda^m(\bar{v}^f) + \lambda^f(v^f)\right]\right)\\=&\mu^2 + (1-\mu)^2 + \mu(1-\mu)\left(1 + \omega\lambda^f(\bar{v}^m) + (1-\omega)\lambda^m(\bar{v}^f)\right).\end{aligned}$$

When I assume overall equal shares of males and females in the economy, $\mu = 1/2$ this term collapses to weighted mean is a convenient choice as it simplifies the terms considerably. Other approaches are also possible. As long as the outcome level of evaluation is such that the corresponding male rank translates into an outcome for females below the crossing point for the densities, then $\lambda^f(v^m) < 1$. Likewise, if the corresponding female rank corresponds to an outcome for men above the crossing point, then $\lambda^m(v^f) < 1$. This implies that other weighted means where these conditions are fulfilled also will evaluate to a mean adjustment factor less than one.

An elaborate example of such an evaluation at the mean of the outcome distributions is displayed in Appendix Figure 5. In the example, the adjustment factor is less than one for all outcome levels but for the ones in the extreme tails of the distribution where there is little common distributional support among the two genders empirically. Thus for any reasonable income level, the adjustment factor is less than one.

Extensions Throughout this section, I have assumed that the covariance in within-gender ranks is equal for sons and daughters which may not be realistic. If the adjustment factor is different from 1 the convex combination of (possibly unequal) covariances for separate groups will not provide researchers with the aggregate covariance. This can be seen from the 4th line in Equation (10):

$$
\begin{aligned}
&E[Cov(U_i, R_i|S_i)] = \\
&\left\{\mu(1-\mu)\lambda^f(\hat{v}) + (1-\mu)^2\right\} Cov(U_i, V_i|0) + \left\{\mu(1-\mu)\lambda^m(\hat{v}) + \mu^2\right\} Cov(U_i, V_i|1) \\
&< (1-\mu)Cov(U_i, V_i|0) + \mu\, Cov(U_i, V_i|1) = E[Cov(U_i, V_i|S_i)] \quad \text{for } \hat{v} \in \left[G^m(y^*), G^f(y^*)\right].
\end{aligned}
$$

Equation 15.

For $\mu = 1/2$ this simplifies further:

$$\begin{aligned}E[Cov(U_i, R_i|S_i)] =& \frac{1}{4}\left\{\lambda^f(\hat{v}) + 1\right\} Cov(U_i, V_i|0) + \frac{1}{4}\left\{\lambda^m(\hat{v}) + 1\right\} Cov(U_i, V_i|1) \\ <& \frac{1}{2} Cov(U_i, V_i|0) + \frac{1}{2}\, Cov(U_i, V_i|1) = E[Cov(U_i, V_i|S_i)] \\ & \text{for } \hat{v} \in \left[G^m(y^*), G^f(y^*)\right].\end{aligned}$$

In other words, as λ^f and λ^m are generally less than 1, the rank correlation is not linearly separable with weights adding up to 1. Consequently, the rank correlation cannot linearly decomposed into within-group rank correlations. This is in stark contrast to absolute measures of mobility such as the IGE, where the aggregate covariance can be decomposed by interacting the parental outcome with other control variables.

Throughout I have discussed the problem of inequality for position-based mobility measures in terms of gender inequality. Under the strong assumption that fertility is uncorrelated with parent income, birth order is a random process much akin to gender. As such, the rank correlation will tend to be lower, the larger is the systematic difference in earnings or wealth between older and younger siblings. This is relevant in the context of wealth inequality and inheritance where societies differ in the extent to which the oldest child inherits the estate of deceased parents. These institutional features have long been recognized to affect inequality, see Menchik (1980) and my results indicate that they also directly affect rank mobility. In a context where each estate has two descendants, the rank correlation will be maximized when they each receive an equal share of the bequests. For any other sharing rule, the rank correlation will be smaller. Thus legislative or normative changes towards a more equal sharing of estates among siblings will tend to mechanically lower mobility in society.

The results for gender above assume zero between-group covariance which is a result of orthogonality between parental outcomes and group affiliation of the child. This is a natural assumption for gender, but for most partitions of society, one cannot expect orthogonality. An important case is racial or ethnic inequality. Assignment to race will be highly correlated with parent outcomes. In this case, the aggregate rank correlation can be *higher* than the within-race rank correlation. This can be seen directly from the decomposition of the parent-child covariance into its within- and between-group components in Equation (2). With race, the between-race covariance will be positive if race inequality is persistent from one generation to the next. In other words, nor in the case of race inequality, does the rank correlation allow us to ignore group inequality, but whether aggregate mobility is larger than within-group mobility is not determined a priori.

3 Gender equality and income mobility in the US

The adjustment term can be used to ex-post quantify the importance of gender inequality for the rank correlation. To assess the empirical relevance, I calibrate the adjustment term with empirical cross-sectional income distributions and investigate the importance of gender inequality in the United States.

I estimate the income distributions on data provided by IPUMS-USA (Ruggles et al., 2021) which has harmonized cross-sectional surveys to ensure comparability. I restrict attention to samples between 1970 and 2019 and keep individuals between the ages of 30 to 39. The variable of interest is *total individual income* (*inctot*) and only individuals with positive earnings are kept in the data. This is not an innocuous restriction as it affects individuals along the extensive margin. However including zero-incomes also breaks with the assumption of smooth distributions. As the purpose of this section is to serve as an illustration, I find the drawbacks of exclusion of zero acceptable and it can be shown that the exclusion of zero incomes likely underestimates the magnitude of the change in the rank correlation over the sample period.[18] The sample is weighted and I draw artificial samples from the original sample according to these weights. I then apply Kernel-Density-Estimation (KDE) to obtain densities, cumulative distributions, and

[18]If there are more women than men outside the labor force, this implies that the quantile function for women is biased downwards. As $\lambda^m(v)$ is generally decreasing in v, this would imply that the adjustment factor is biased upwards towards one. Consequently, an increase in labor market participation by women would lead to a more dramatic increase in the adjustment term over time.

Year	N	Share men	Average income, 1000 USD				Sample
			All	Men	Women	Men/Women-1	
1970	173,312	0.62	48.1	62.6	24.4	1.57	Form 1 Metro
1980	1,355,511	0.56	43.7	57.4	26.4	1.18	5%
1990	1,850,708	0.53	46.7	58.5	33.4	0.75	5%
2000	1,884,950	0.53	51.7	62.7	39.3	0.59	5%
2005	330,079	0.53	51.7	62.2	39.9	0.56	ACS
2010	322,104	0.53	48.5	55.9	40.3	0.39	ACS
2015	322,255	0.53	49.8	57.3	41.4	0.38	ACS
2019	346,996	0.53	54.5	62.2	45.8	0.36	ACS

Table 1: Descriptive statistics for samples

The table presents descriptive statistics for the constructed samples used to estimate distributions. All means are weighted. Data are retrieved from IPUMS-USA (Ruggles et al., 2021). Incomes are converted to 2019-prices using the CPI-U index. The right-most column lists the sample used in a given year within the IPUMS data.

quantile functions by gender and year. I smooth the estimated functions to interpolate values for incomes not present in the data.[19]

Descriptive statistics on income by gender are provided in Table 1. All income averages are weighted according to sampling weights. Beginning in 1970 conditional on having a positive income, men on average earned 157 percent of what women while in 2019 that number decreased to 36 percent. Thus a substantial decrease in gender inequality has occurred since 1970 measured by differences in averages across genders.

The estimated densities are shown in Figure 2. From 1970 to 2019 the incomes rose but the shapes of the distributions also changed. While the single-crossing property is preserved, the densities for men and women are much more alike today than in 1970 implying greater gender equality.

Figure 3 shows the likelihood ratios and quantile ratios of the estimated income distribution of men over women. Disregarding small deviations, the likelihood ratios in

[19]When estimating empirical distributions, there are several issues such as top coding, weighting, and sampling error. As the calculations are made for illustrative purposes, I ignore most of these things. That means that I regard my quantitative results as suggestive and not definitive, The full code used to generate the results is available on my website. I use income in a single year. It is recognized across the literature that this is problematic as it may be a poor proxy for lifetime income, see Chen et al. (2017) and Guvenen et al. (2017) for recent discussions. However, Chetty et al. (2014b) find that the rank correlation in the US is insensitive to the number of years over which to average income levels. Where relevant I convert incomes into 2019-prices using the consumer-price index.

each year appear to be increasing monotonically as seen in Figure 3b. This implies that the income distributions of males stochastically dominate the distributions of females in each year. This is reflected in Figure 3a where the quantile ratios are consistently above 1. However, the ratios have approached one from above throughout the period since 1970. This implies that while stochastic dominance is maintained, men and women are considerably less unequal today than in the seventies.

Using the estimated densities, I calibrate the adjustment factor, developed in section 2. The density-weighted mean adjustment factor evaluated at the aggregate mean and median income in each year is shown in Figure 4. The general trend confirms the intuition that the increased gender equality should in and of itself increase the total rank correlation. The adjustment factor rises from 0.79 to 0.98 between 1970 and 2019 as seen in Figure 4. Assuming constant within-gender intergenerational mobility, the aggregate mobility would have *fallen* by 24 percent solely due to greater gender equality.[20] In other words, the greater gender equality achieved over the last half of the twentieth century has decreased mobility considerably if measured by the aggregate rank correlation.

To gauge the importance of the choice of income level in which to approximate the income factor, I calibrate the adjustment factor for incomes corresponding to male ranks between the first and the 99th percentile. The results are shown in Appendix Figure 6. In practice, as long as there is support, it matters little at which point in the income distribution the adjustment factor is calculated.[21]

4 Conclusion

The potential for confusing gender equality with decreased mobility shows how the interpretation of rank-based mobility measures is less obvious than it might initially appear. Positional mobility measures are an important part of the toolbox for investigating

[20] Calculated as 0.98/0.79-1=0.24.

[21] In the extreme tails likelihood ratios become extreme as lack of empirical support implies dividing with zero and the adjustment factor therefore explodes.

mobility and inequality but too little focus has been afforded to the importance of group inequality for these measures. Only under a very specific understanding of mobility is it correct to claim that the use of rank mobility allows the researcher to disregard inequality. No one correct mobility measure exists, nor should there, but this paper has illustrated the perils of comparing rank-mobility in societies over time and across space without theoretical guidance on who to compare with whom and why ranks are appropriate in the first place. Thus, the conclusion of Fields and Ok (1999) that the *"income mobility literature is still distressingly far from being unified on how to measure mobility and make mobility comparisons"* still holds two decades later.

5 Declarations

5.1 Funding

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5.2 Ethics

The author has no relevant financial or non-financial interests to disclose. The author has no competing interests to declare that are relevant to the content of this article.

5.3 Data availability

Data for estimating income distributions are available from `https://usa.ipums.org/usa` (Ruggles et al., 2021). Code for reproducing empirical results are available from the following repository: `https://github.com/MikkelGandil/papers/tree/master/rank_correlations`.

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A Additional figures

List of Figures

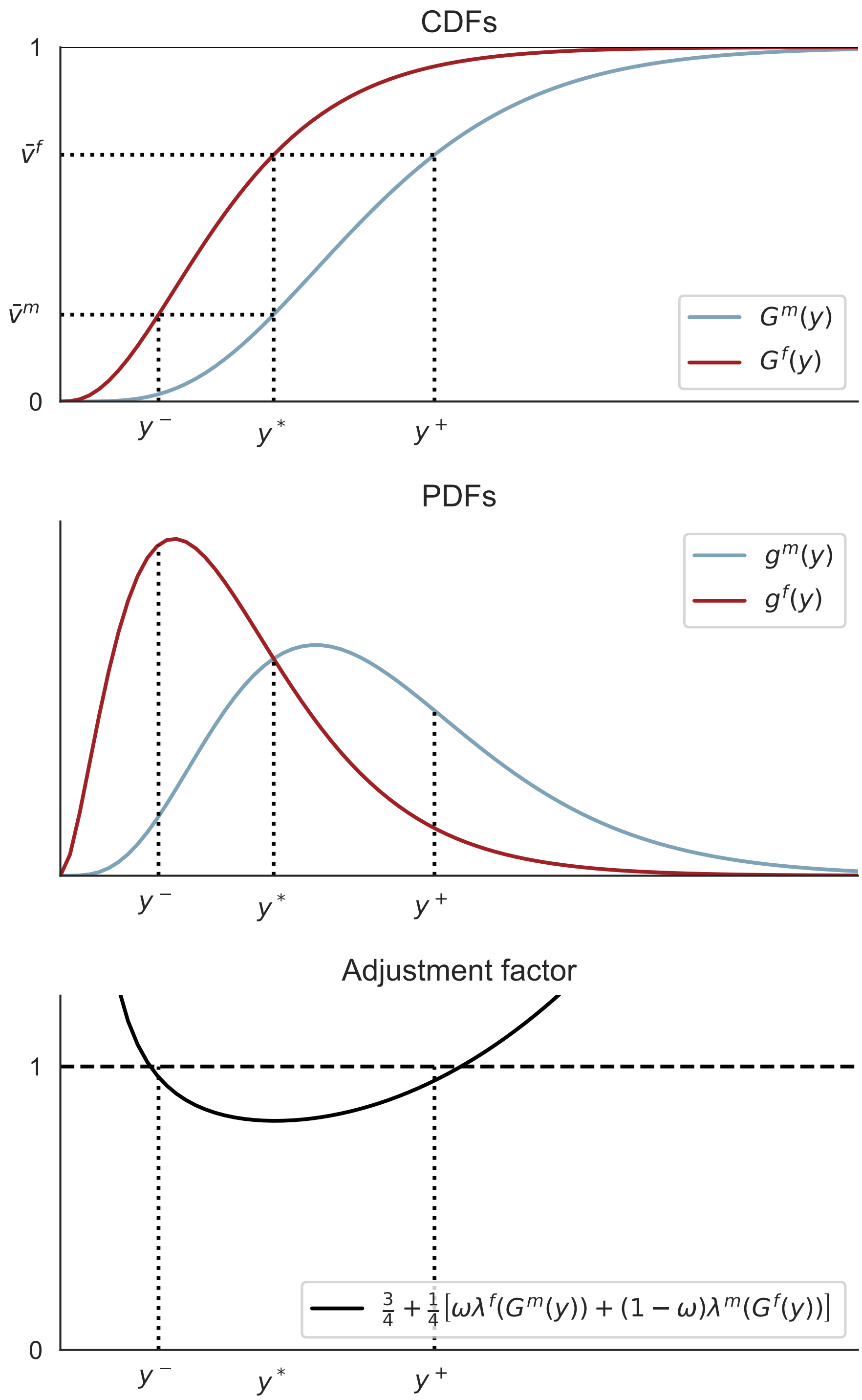


Figure 1: The range where aggregate covariance < within-gender covariance

The figure depicts artificial distributions which exhibit monotonic likelihood ratios and $\frac{g^f(v)}{g^m(v)}$ is decreasing. The top figure displays the cumulative distribution functions (CDFs). From these functions one can identify $\bar{v}^m = G^m(y^*)$ and $\bar{v}^f = G^f(y^*)$ where the likelihood ratios are evaluated, where y^* is the crossing of densities. The ranks are transformed back to outcomes, $y^- = G^{f^{-1}}(\bar{v}^m)$ and $y^+ = G^{m^{-1}}(\bar{v}^f)$. Evaluating the covariance at any outcome level in the interval (y^-, y^+) the aggregate rank correlation is smaller than the within-gender correlation. With the distributions specified in this example, y^- and y^+ correspond to the 13th and the 82nd percentile in the aggregate outcome distribution.

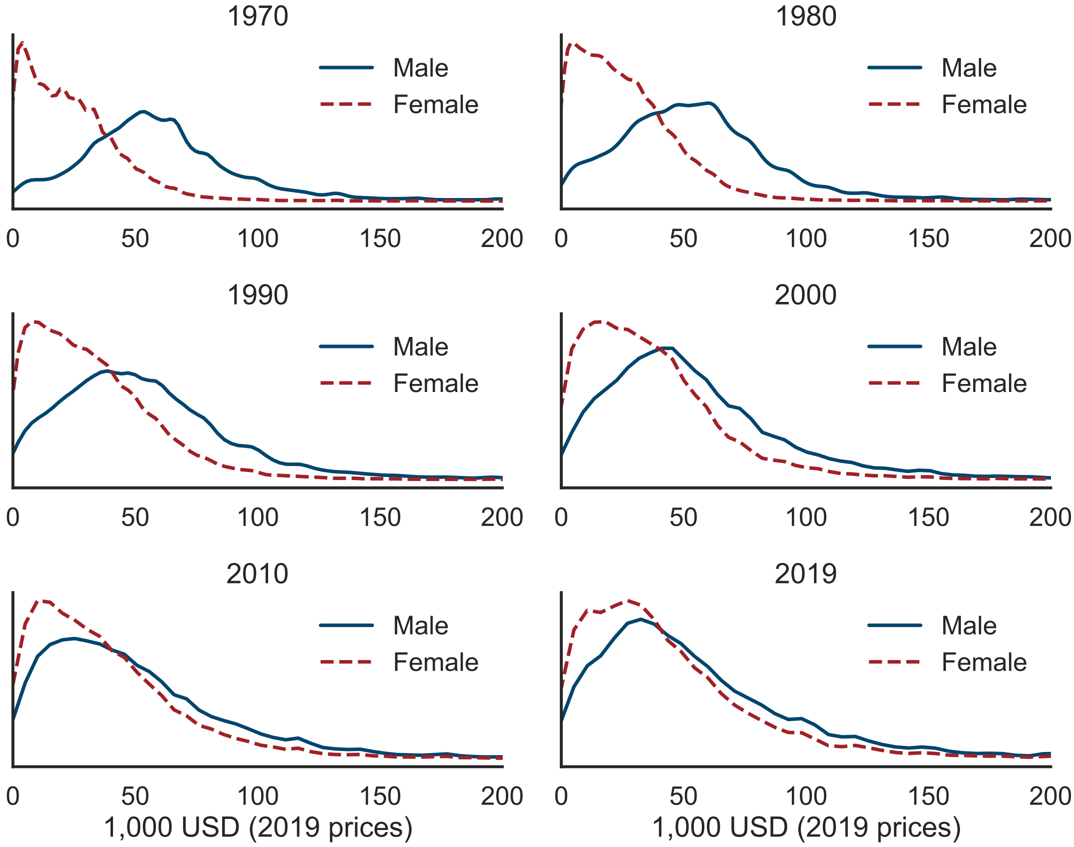
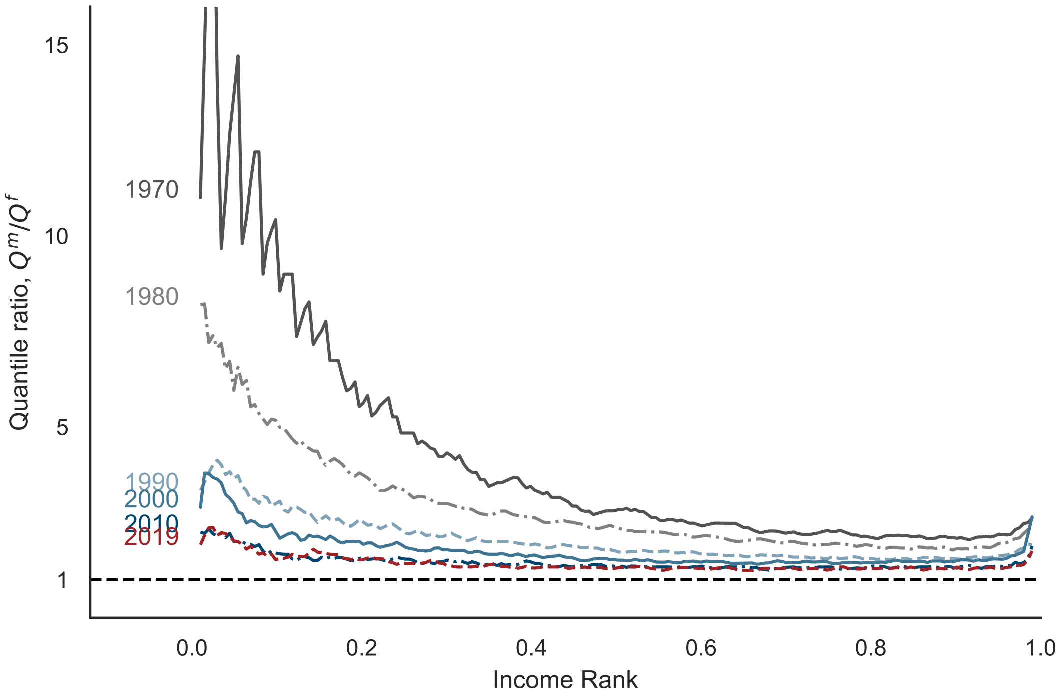


Figure 2: Estimated densities by year and gender

The figure shows the estimated densities based on data retrieved from IPUMS-USA (Ruggles et al., 2021). Incomes are converted to 2019-prices using the CPI-U index. Densities are estimated by repeatedly drawing from weighted samples and performing kernel-density estimates which are smoothed to interpolate.

(a) Quantile ratios



(b) Likelihood ratios

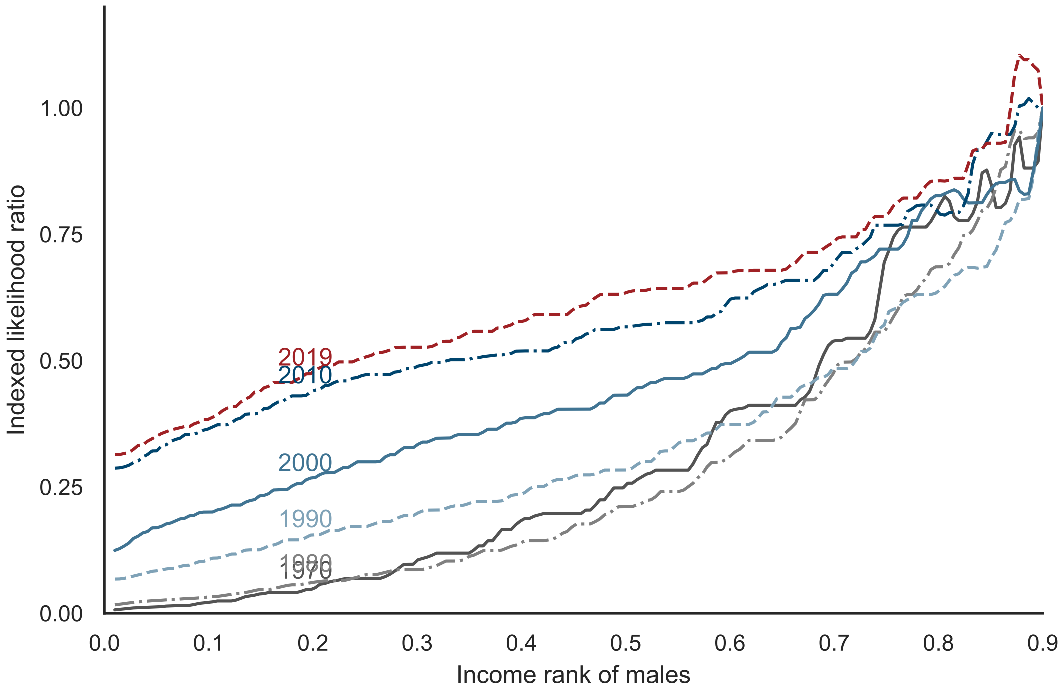


Figure 3: Estimated distributions

Figure 3a shows the male quantile distribution divided by the female, and will thus take the value 1 when the quantiles are equal. I omit ratios at the very bottom, as these are very volatile due to division by values close to zero. First-order stochastic dominance is maintained for all years. Figure 3b shows the density of males divided by the density of women, i.e. likelihood ratios. For comparison, the ratios are indexed at the 90th percentile of the male income distribution in a given year and evaluated at incomes corresponding to male ranks.

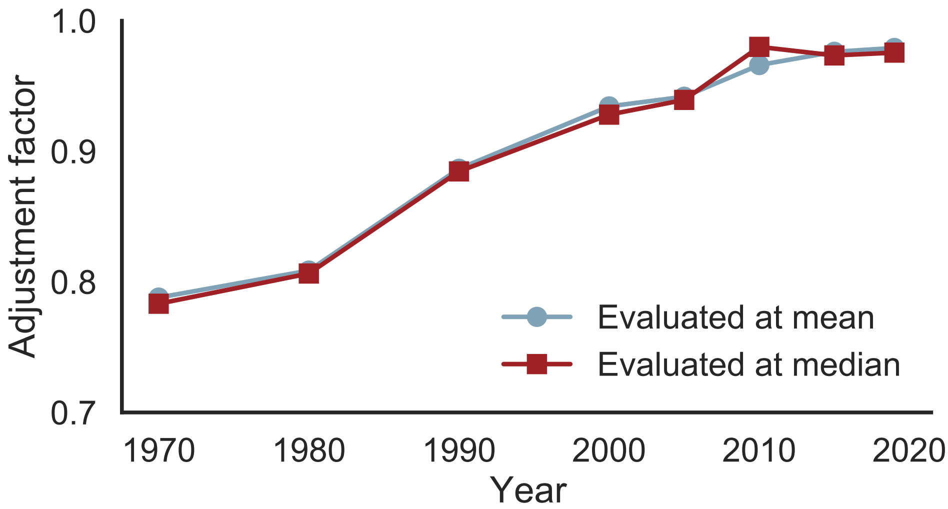


Figure 4: Calibrated adjustment factor

The figure shows the calibrated density-weighted adjustment factors according to Equation (15) evaluated at the aggregate mean and median. The weights are not sensitive to the choice of the income level at which to evaluate the adjustment factors. For the value of the adjustment factor for other income levels, I refer to Appendix Figure 6.

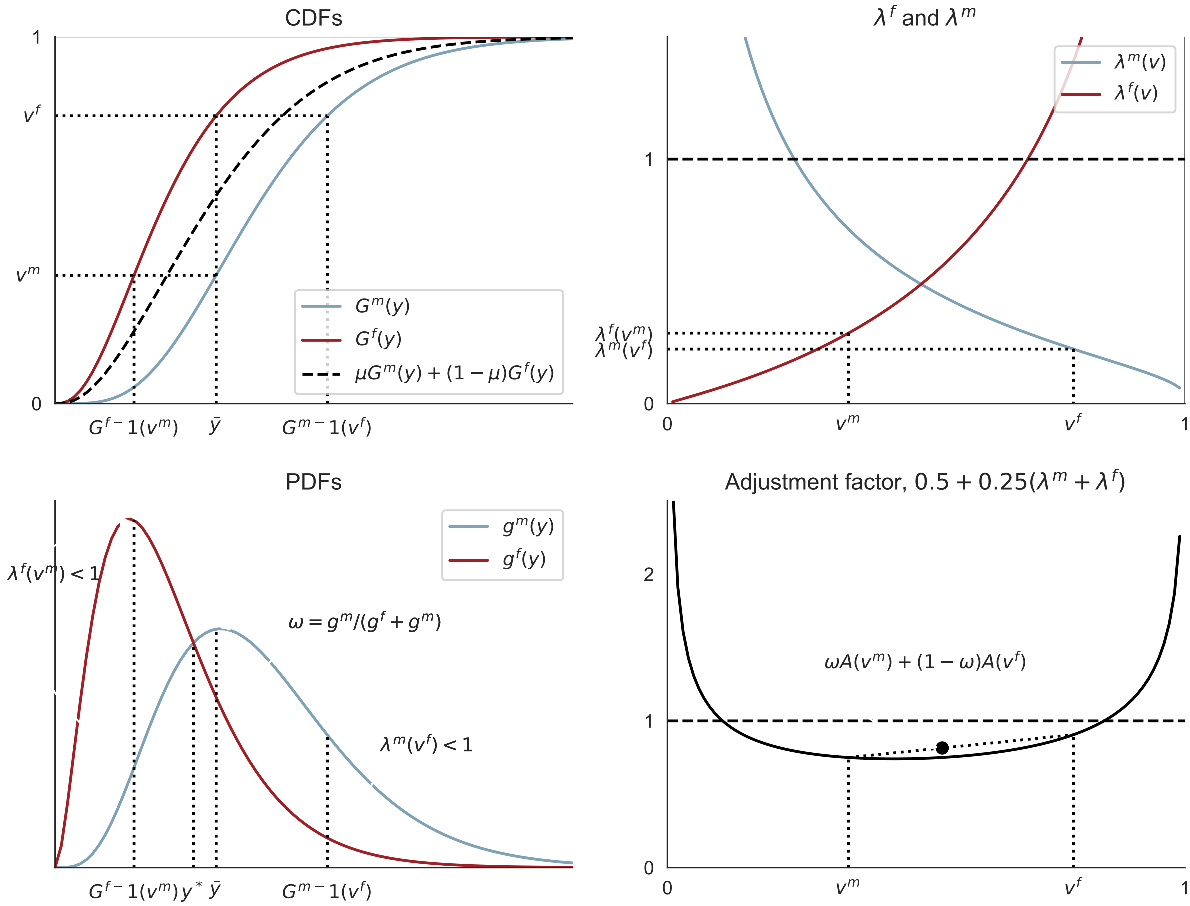
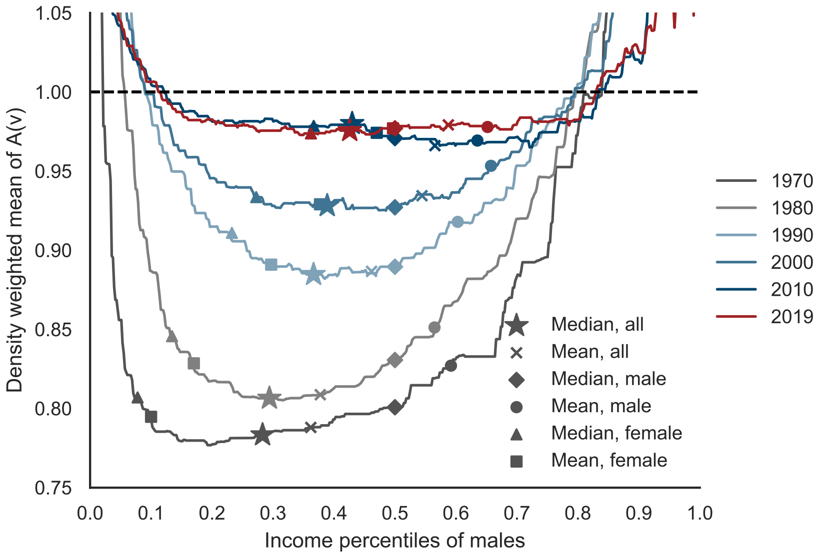


Figure 5: Example of evaluation of the adjustment factor at the mean income

The figure depicts artificial distributions which exhibit monotonic likelihood ratios. In this example the adjustment factor is evaluated at the mean of the aggregate distribution, $\bar{y}$. The top left figure displays the cumulative distribution functions (CDFs). From these functions one can identify $v^m = G^m(\bar{y})$ and $v^f = G^f(E(y))$ where the likelihood ratios are evaluated. In this example $G^{f^{-1}}(v^m) < y^* < G^{f^{-1}}(v^f)$, where y^* is the crossing of densities. This implies that both $\lambda^f(v^m)$ and $\lambda^m(v^f)$ are below one. This can be read off the top right figure, which plots the values for λ^f and λ^m for all values of the rank, v. The adjustment factor for all values of v is shown in the bottom right figure, where the density weighted mean between the two adjustment factors are shown as a black circle.

(a) Adjustment factor as function of income level



(b) Adjustment factor over time for different income levels

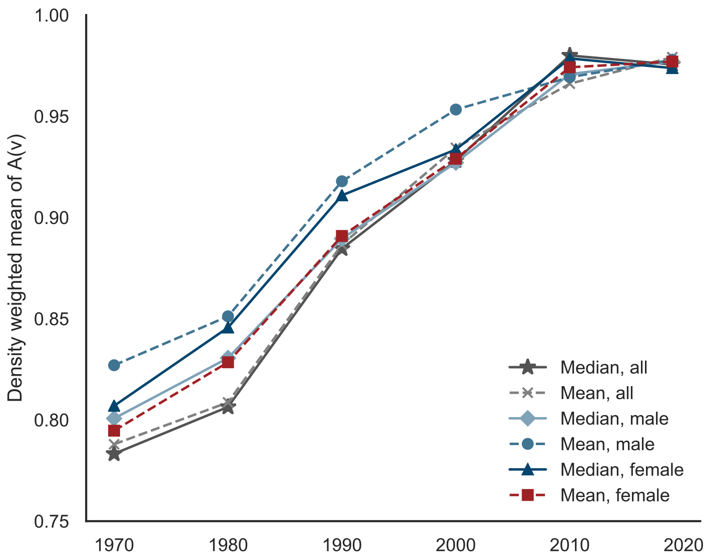


Figure 6: Evaluation of the adjustment factor at different levels of income

The figure shows the importance of the choice of evaluation point. For each year I calculate the density weighted adjustment factor at an income corresponding to the rank on the x-axis for males. The ranks serve as a normalisation of the income distributions, such that one can compare across years. The very flat lines indicate that it matters little where the adjustment factor is approximated, less it not be in the tails. The figure also displays the adjustment factors for means and medians for the two genders and for the total income distribution. These are represented by the symbols on the line placed at the corresponding ranks in the male income distribution. By definition, the median of males are always located at 0.5 on the x-axis.