Mandatory1 MAT-MEK4270

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1.2.3

$$u(t,x,y) = e^{i(k_x x + x_y y + \omega t)}$$

to show that this satisfies the wave equation we put it into PDE:

$$u_{tt} = c^2 \nabla^2 u$$

$$u_{tt} = i^2 \omega^2 u = -\omega^2 u, c^2 \nabla^2 u = c^2 i^2 (k_x^2 + k_y^2) = -c^2 (k_x^2 + k_y^2)$$

yields:

$$\omega^2 = c^2 (k_x^2 + k_y^2)$$

$$\omega = c\sqrt{k_x^2 + k_y^2}$$

that was defined to be the dispersion coefficient, hence u satisfoies the PDE.

1.2.4

$$u_{ij}^n = e^{i(kh(i+j) - \hat{\omega} n \Delta t)}$$

for $k_x = k_y = k$ we get $\omega = ck\sqrt{2}$. Putting u_{ij}^n into the discretized eq. yields:

$$\frac{e^{i\hat{\omega}\Delta t} - 2 + e^{-i\hat{\omega}\Delta t}}{\Delta t^2} e^{i(kh(i+j) + \hat{\omega} n \Delta t} = 2c^2 (\frac{e^{ikh} - 2 + e^{-ikh}}{h^2} e^{i(kh(i+j) + \hat{\omega} n \Delta t})$$

now we get

$$\frac{e^{i\hat{\omega}\Delta t} - 2 + e^{-i\hat{\omega}\Delta t}}{\Delta t^2} = 2c^2 (\frac{e^{ikh} - 2 + e^{-ikh}}{h^2})$$

this can from trigonometric properties be written as:

$$\frac{4}{\Delta t^2} sin^2(\frac{\hat{\omega}\Delta t}{2}) = c^2 \frac{8}{h^2} sin^2(\frac{kh}{2})$$

and applying that $cfl = \frac{c\Delta t}{h} = 1/\sqrt{2}$ we get

$$sin^2(\frac{\hat{\omega}\Delta t}{2}) = sin^2(\frac{kh}{2})$$

which implies that

$$\frac{\hat{\omega}\Delta t}{2} = \frac{kh}{2}$$

$$\hat{\omega} = \frac{kh}{\Delta t} = ck\sqrt{2} = \omega$$

as we wanted to show

1.2.5

see code in Wave2d.py