Multidimensional Fourier Transform

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Fourier Transform

Problem 1. The average value of a function $f : [0, T] \to \mathbb{R}$ is given by:

$$\text{mean}(f) = \frac{1}{T}\int_0^T f(t)dt.$$

Assuming we know all the coefficients of its Fourier series, a_k, of an arbitrary function, say f, what is the mean of the function? Do we need to reconstruct the original function to calculate it?

Problem 2. Find the Fourier transform of $f(x) = e^{-a|x|}$ with $a > 0$.

Problem 3. Find the Fourier transform of the following function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} x+2 & -2 < x \leqslant -1 \\ 1 & -1 < x \leqslant 1 \\ 2-x & 1 < x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}.$$

Hint: *Is there any simpler way to write this function?*

Problem 4. Prove the following properties of the Fourier transform:

- If a function, f, is real and even its Fourier transform, F, is real and even.
- If a function, f, is real and odd its Fourier transform, F, is imaginary and odd.
- If a function, f, is real the magnitude of its Fourier transform, $||F||$, is even.
- If a function, f, is real the phase of its Fourier transform, $\phi(F)$, is odd.

Problem 5. Find a simple expression for the function

$$f(x) = \underbrace{\text{sinc}(x) * \text{sinc}(x) * \cdots \text{sinc}(x)}_{N\text{times}}.$$

Systems

Problem 6. The Fourier transform is an operator that accepts a function as an input and returns a function as an output. Therefore it is also a system! Determine if the properties of the *Fourier system* is linear, shift-invariant, causal or has memory.

Problem 7. Sinusoidal functions are of great interest when studying shift invariant systems. Show that the functions $f(x) = e^{i2\pi kx}$ are eigenfunctions of *any* shift invariant system. Determine their corresponding eigenvalues.

Problem 8. A classical RLC circuit consists of an inductance L, a capacitor of capacitance C and a resistor of resistance R. The equation of this simple system is given by the differential equation:

$$L\frac{di^2}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}.$$

If we now the voltage applied to the system, $v(t)$, use the Fourier transform to determine the intensity $i(t)$

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Problem 9. Prove the similarity property of the multidimensional Fourier transform

Problem 10. Plot the step two-dimensional square function $\Pi(x_1, x_2)$. Determine and plot its corresponding Fourier Transform.

Problem 11. Prove that any rotation in the spatial domain corresponds to an identical rotation in the frequency domain.