# Regneregler

Wednesday, 26 November 2025 19.25

| Rules of Exponents or Laws of Exponents | |
|---|---|
| Multiplication Rule | $a^x \times a^y = a^{x+y}$ |
| Division Rule | $a^x \div a^y = a^{x-y}$ |
| Power of a Power Rule | $\left(a^x\right)^y = a^{xy}$ |
| Power of a Product Rule | $\left(ab\right)^x = a^x b^x$ |
| Power of a Fraction Rule | $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ |
| Zero Exponent | $a^0 = 1$ |
| Negative Exponent | $a^{-x} = \frac{1}{a^x}$ |
| Fractional Exponent | $a^{\frac{x}{y}} = \sqrt[y]{a^x}$ |

## Summation Shortcuts

$$\sum_{n=1}^{k} n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

$$\sum_{n=1}^{k} n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$\sum_{n=1}^{k} n^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$$

$$\sum_{n=1}^{k} n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{6n^5+15n^4+10n^3-n}{30}$$

$$\sum_{n=1}^{k} n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12} = \frac{2n^6+6n^5+5n^4-n^2}{12}$$

**Brøker**

Brøker er tal på formen

$$\frac{a}{b},$$

hvor $a, b$ er tal samt $b \neq 0$. $a$ er *tælleren* og $b$ er *nævneren*.

**Regneregler**

Der gælder

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}, \quad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}, \quad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc},$$

$$a\frac{b}{c} = \frac{ab}{c}, \quad \frac{\frac{a}{b}}{c} = \frac{a}{bc}, \quad \frac{a}{\frac{b}{c}} = \frac{ac}{b}.$$

**Forkorte/Forlænge Brøker**

Fælles faktorer kan forkortes:

$$\frac{a}{b} = \frac{ac}{bc}$$

**Potenser**

Potenser er tal på formen $x^a$, $x$ er *grundtallet* og $a$ er *eksponenten*.

**Regneregler**

Der gælder

$$x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad (xy)^a = x^a y^a,$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}, \quad (x^a)^b = x^{ab}, \quad x^{-a} = \frac{1}{x^a}.$$

**Rødder**

Hvis $x \geq 0$ og $n \in \mathbf{Z}_+$ så findes et tal $\sqrt[n]{x} > 0$ så

$$(\sqrt[n]{x})^n = x.$$

Bemærk at $\sqrt[n]{x} = x^{\frac{1}{n}}$.

**Regneregler**

Der gælder

$$\sqrt[n]{x} = x^{\frac{1}{n}}, \quad \sqrt[n]{x^m} = x^{\frac{m}{n}} = (\sqrt[n]{x})^m,$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}, \quad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}.$$

**Kvadratsætninger**

Der gælder

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$(a-b)^2 = a^2 + b^2 - 2ab$$

$$(a+b)(a-b) = a^2 - b^2.$$

**Ligninger**

Ligninger kan reduceres med følgende regler:

1. Man må lægge til/trække fra med det samme tal på begge sider af et lighedstegn.
2. Man må gange/dividere med det samme tal (undtagen 0) på begge sider af et lighedstegn.

**Andengradsligninger**

Andengradsligninger er på formen

$$ax^2 + bx + c = 0, \qquad (1)$$

Løsningerne til (1) er

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

**Faktorisering**

Hvis $ax^2 + bx + c = 0$ har rødder $r_1$ og $r_2$ så gælder.

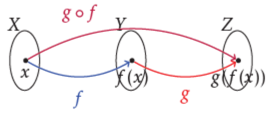
$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

**Funktioner**

En funktion $f: X \to Y$ tildeler alle $x \in X$ *præcis ét* element $f(x) \in Y$.

**Sammensatte funktioner**

Hvis $f: X \to Y$ og $g: Y \to Z$ defineres sammensætningen $g \circ f: X \to Z$ ved $(g \circ f)(x) = g(f(x))$. $f$ er den *indre funktion*, $g$ er den *ydre funktion*




**Inverse funktioner**

To funktioner $f: X \to Y$ og $g: Y \to X$ er hinandens *inverse* hvis

$$f(g(y)) = y, \quad \text{og} \quad g(f(x)) = x$$

for alle $x$ i $X$ og $y$ i $Y$.

**Polynomier**

Et førstegradspolynomium har forskrift:

$$f(x) = ax + b.$$

Et andengradspolynomium har forskrift:

$$f(x) = ax^2 + bx + c.$$

**Logaritmer og eksponentialfunktioner**

*Logaritmen med grundtal* $a$, $\log_a: ]0, \infty[ \to \mathbf{R}$ er invers til eksponentialfunktionen $f_a(x) = a^x$ ($a > 0$, $a \neq 1$). Der gælder at

$$\log_a(a^x) = x \quad \text{og} \quad a^{\log_a(y)} = y$$

og vi har

$$\ln x = \log_e x, \qquad \log x = \log_{10} x$$

**Regneregler**

Der gælder

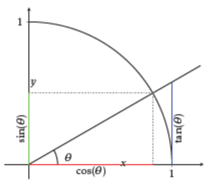
$$\log_a(xy) = \log_a(x) + \log_a(y),$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y),$$

$$\log_a(x^r) = r\log_a(x).$$

**Trigonometriske funktioner**

De trigonometriske funktioner er defineret ud fra enhedscirklen:




Der gælder at $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ samt

| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | - |

**Differentialregning**

Den afledede af $f$ skrives som $f' = \frac{d}{dx}f = \frac{df}{dx}$.

**Regneregler**

Der gælder at

| $f(x)$ | $f'(x)$ |
|---|---|
| $c$ | 0 |
| $x$ | 1 |
| $x^n$ | $nx^{n-1}$ |
| $e^x$ | $e^x$ |
| $e^{cx}$ | $ce^{cx}$ |
| $a^x$ | $a^x \ln a$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\cos x$ | $-\sin x$ |
| $\sin x$ | $\cos x$ |
| $\tan x$ | $1 + \tan^2(x)$ |

**Generelle regneregler**

Der gælder at

$$(cf)'(x) = cf'(x)$$

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Den sidste regneregel kaldes *kædereglen*.

**Ubestemte integraler**

En funktion $f$ har *stamfunktion* $F$ hvis

$$F'(x) = f(x).$$

Det ubestemte integral af $f$ er

$$\int f(x)\,dx = F(x) + k,$$

hvor $F'(x) = f(x)$ og $k \in \mathbf{R}$.

**Generelle regneregler**

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

$$\int f(g(x))g'(x)\,dx = F(g(x)) + k.$$

Den 3. regel kaldes *delvis integration* og den sidste kaldes *integration ved substitution*.

**Regneregler**

Der gælder at

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $c$ | $cx + k$ |
| $x$ | $\frac{1}{2}x^2 + k$ |
| $x^n$ | $\frac{1}{n+1}x^{n+1} + k$ |
| $e^x$ | $e^x + k$ |
| $e^{cx}$ | $\frac{1}{c}e^{cx} + k$ |
| $\frac{1}{x}$ | $\ln(\lvert x\rvert) + k$ |
| $\ln x$ | $x\ln(x) - x + k$ |
| $\cos x$ | $\sin x + k$ |
| $\sin x$ | $-\cos x + k$ |
| $\tan x$ | $-\ln(\lvert\cos(x)\rvert) + k$ |

**Integration ved substitution**

Givet et integral på formen $\int f(g(x))g'(x)\,dx$ anvendes metoden:

1. Lad $u = g(x)$.
2. Udregn $\frac{du}{dx}$ og isoler $dx$.
3. Substituer $g(x)$ og $dx$.
4. Udregn integralet mht. $u$.
5. Substituer tilbage.

**Besemte integraler**

Det bestemte integral af $f$ i intervallet $[a,b]$ til

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

hvor $F$ er en stamfunktion til $f$.

**Generelle regneregler**

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)g(x)\,dx = [f(x)G(x)]_a^b - \int_a^b f'(x)G(x)\,dx$$

$$\int_a^b f(g(x))g'(x)\,dx = [F(x)]_{g(a)}^{g(b)}.$$

**Integration ved substitution**

Givet et integral på formen $\int_a^b f(g(x))g'(x)\,dx$ anvendes metoden

1. Lad $u = g(x)$.
2. Udregn $\frac{du}{dx}$ og isoler $dx$.
3. Substituer $g(x)$, $dx$ samt grænser.
4. Udregn integralet mht. $u$.

**Differentialligninger**

**Løsningsformler**

| Differentiallign. | Fuldstændig løsn. |
|---|---|
| $f'(x) = k$ | $f(x) = kx + c$ |
| $f'(x) = h(x)$ | $f(x) = \int h(x)\,dx$ |
| $f'(x) = kf(x)$ | $f(x) = ce^{kx}$ |
| $f'(x) + af(x) = b$ | $f(x) = \frac{b}{a} + ce^{-ax}$ |

**Panserformlen**

Differentialligningen

$$f'(x) + a(x)f(x) = b(x)$$

har fuldstændig løsning

$$f(x) = e^{-A(x)}\int b(x)e^{A(x)}\,dx + ce^{-A(x)},$$

hvor $A'(x) = a(x)$.

**Vektorer i planen**

En vektor $\vec{u}$ i planen skrives som $\vec{u} = [x, y]$ hvor $x, y \in \mathbf{R}$.

**Regneregler**

For $\vec{u} = [x_1, y_1]$, $\vec{v} = [x_2, y_2]$, $c \in \mathbf{R}$ er

$$\vec{u} \pm \vec{v} = \begin{bmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \end{bmatrix}, \qquad \vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2,$$

$$c\vec{u} = \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}, \quad \det(\vec{u}, \vec{v}) = x_1y_2 - x_2y_1$$

Længden af $\vec{u}$ er $\|\vec{u}\| = \sqrt{x_1^2 + y_1^2}$.

**Vinklen mellem to vektorer**

For vinklen $\theta$ mellem $\vec{u}, \vec{v}$ er

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad \sin\theta = \frac{\det(\vec{u}, \vec{v})}{\|\vec{u}\|\,\|\vec{v}\|}$$

Yderligere gælder

1. $\vec{u}$ og $\vec{v}$ er ortogonale $\Leftrightarrow \vec{u}\cdot\vec{v} = 0$.
2. $\vec{u}$ og $\vec{v}$ er parallelle $\Leftrightarrow \det(\vec{u}, \vec{v}) = 0$.

**Vektorer i rummet**

En vektor $\vec{u}$ i rummet skrives som $\vec{u} = [x, y, z]$ hvor $x, y, z \in \mathbf{R}$.

**Regneregler**

For $\vec{u} = [x_1, y_1, z_1]$, $\vec{v} = [x_2, y_2, z_2]$ og $c \in \mathbf{R}$ gælder

$$\vec{u} \pm \vec{v} = \begin{bmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \\ z_1 \pm z_2 \end{bmatrix}, \qquad c\vec{u} = \begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix},$$

$$\vec{u}\cdot\vec{v} = x_1x_2 + y_1y_2 + z_1z_2.$$

Længden af $\vec{u}$ er $\|\vec{u}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

Krydsproduktet er givet ved

$$\vec{u} \times \vec{v} = \begin{bmatrix} y_1z_2 - z_1y_2 \\ z_1x_2 - x_1z_2 \\ x_1y_2 - y_1x_2 \end{bmatrix}$$

**Vinklen mellem to vektorer**

For vinklen $\theta$ mellem $\vec{u}$ og $\vec{v}$ gælder

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad \sin\theta = \frac{\|\vec{u}\times\vec{v}\|}{\|\vec{u}\|\,\|\vec{v}\|}$$

Yderligere gælder

1. $\vec{u}$ og $\vec{v}$ er ortogonale $\Leftrightarrow \vec{u}\cdot\vec{v} = 0$.
2. $\vec{u}$ og $\vec{v}$ er parallelle $\Leftrightarrow \vec{u}\times\vec{v} = 0$.

**Linjer og Planer**

Planen/linjen gennem punktet med stedvektor $\vec{x}_0$ med normalvektor $\vec{n}$ beskrives ved alle vektorer $\vec{x}$ der løser ligningen

$$\vec{n}\cdot(\vec{x} - \vec{x}_0) = 0$$

En linje i rummet/planen gennem punktet med stedvektor $\vec{x}_0$ og retning $\vec{r}$ har parameterfremstilling

$$\vec{x}_0 + t\vec{r}, \quad t \in \mathbf{R}.$$

| $f'(x)$ | $f(x)$ | $F(x) = \int f(x)dx$ | |
|---|---|---|---|
| $0$ | $a$ | $a\cdot x$ | (58) |
| $n\cdot x^{n-1}$ | $x^n$ | $\frac{1}{n+1}\cdot x^{n+1}$ | (59) |
| $\frac{1}{2\sqrt{x}} = \frac{1}{2}\cdot x^{-\frac{1}{2}}$ | $\sqrt{x} = x^{\frac{1}{2}}$ | $\frac{2}{3}\cdot x^{\frac{3}{2}}$ | (60) |
| $-\frac{1}{x^2} = -x^{-2}$ | $\frac{1}{x} = x^{-1}$ | $\ln\lvert x\rvert$ | (61) |
| $\frac{1}{x}$ | $\ln(x)$ | $x\cdot\ln(x) - x$ | (62) |
| $e^x$ | $e^x$ | $e^x$ | (63) |
| $k\cdot e^{k\cdot x}$ | $e^{k\cdot x}$ | $\frac{1}{k}\cdot e^{k\cdot x}$ | (64) |
| $\ln(a)\cdot a^x$ | $a^x$ | $\frac{1}{\ln(a)}\cdot a^x$ | (65) |
| $-\sin(x)$ | $\cos(x)$ | $\sin(x)$ | (66) |
| $\cos(x)$ | $\sin(x)$ | $-\cos(x)$ | (67) |

Tabel over differentialkvotienter og stamfunktioner

| Funktion f(x) | Differentialkvotienten f'(x) | Stamfunktionerne F(x) |
|---|---|---|
| 0 | 0 | k |
| a | 0 | a·x+k |
| x | 1 | $\frac{1}{2}\cdot x^2+k$ |
| $x^2$ | 2x | $\frac{1}{3}\cdot x^3+k$ |
| $x^3$ | $3x^2$ | $\frac{1}{4}\cdot x^4+k$ |
| $x^4$ | $4x^3$ | $\frac{1}{5}\cdot x^5+k$ |
| $x^n$ | $n\cdot x^{n-1}$ | $\frac{1}{n+1}\cdot x^{n+1}+k$ |
| $\sqrt{x}$ | $\frac{1}{2\cdot\sqrt{x}}=\frac{1}{2}x^{-\frac{1}{2}}$ | $\frac{2}{3}\cdot x^{\frac{3}{2}}+k$ |
| $\sqrt[n]{x}$ | $\frac{\sqrt[n]{x}}{n\cdot x}$ | $\frac{n}{n+1}\cdot x^{\frac{n+1}{n}}+k$ |
| $\frac{1}{x}$ | $-\frac{1}{x^2}$ | ln(x)+k |
| $\frac{1}{x^2}$ | $-\frac{2}{x^3}$ | $-\frac{1}{x}+k$ |
| $e^x$ | $e^x$ | $e^x+k$ |
| $e^{b\cdot x}$ | $b\cdot e^{b\cdot x}$ | $\frac{e^{bx}}{b}+k$ |
| $a^x$ | $a^x\ln(a)$ | $\frac{a^x}{\ln(a)}+k$ |
| a·x+b | a | $\frac{1}{2}\cdot a\cdot x^2+b\cdot x+k$ |
| $a\cdot x^2$ | 2·a·x | $\frac{1}{3}\cdot a\cdot x^3+k$ |
| $b\cdot a^x$ | $b\cdot a^x\ln(a)$ | $\frac{b}{\ln(a)}\cdot a^x+k$ |
| $b\cdot x^a$ | $b\cdot a\cdot x^{a-1}$ | $\frac{b}{a+1}\cdot x^{a+1}+k$ |
| ln(x) | $\frac{1}{x}$ | x·ln(x)-x+k |
| log(x) | $\frac{\log(e)}{x}$ | Log(e)·(x·ln(x)-x)+k |
| sin(x) | cos(x) | -cos(x)+k |
| cos(x) | -sin(x) | sin(x)+k |

# ExamSet2023

Monday, November 17, 2025 2:20 PM

ExamStochasticSyste...

# Exam: Stochastic Processes

Date and Time: Tuesday January 3, 2023, 13:00 16:00.

This entire problem set contains **4 pages**. Please make sure that you have received all pages.

The exam is graded according to your answer as a whole: both quantity and quality count. We value concise arguments showing your command of the topics. Simply answering "Yes." or "No." will not do that!

We recommend that you read through each problem thoroughly before starting to solve it. Should you happen to get stuck at some point, we recommend that you continue and anyway try to solve the rest. You always have the opportunity to sketch or explain how you would have continued if you hadn't got stuck.

It is allowed to use books, lecture notes, your own notes, calculators and computers during the exam. Communication to others during the exam is not allowed therefore, the use of internet is strictly forbidden.

1

**Problem 1:**

The process $U(n)$ is an i.i.d. random process with samples drawn identically and independently from the following probability mass function (pmf):

$$p_U(u) = \begin{cases} 1/3 & \text{for } u = 1 \\ 1/3 & \text{for } u = 0 \\ 1/3 & \text{for } u = -1 \\ 0 & \text{otherwise} \end{cases}.$$

**1.1** Calculate the mean function and the autocorrelation function (ACF) of $U(n)$.

**1.2** Is $U(n)$ a wide-sense stationary (WSS) process? Is it strict-sense stationary (SSS)? Justify your answers.

Next, consider the following process built form the samples of $U(n)$:

$$X(n) = \sum_{i=0}^{n} U(i), \qquad n = 0, 1, \ldots.$$

**1.3** Draw a sketch of a 10-sample long realization of $U(n)$ and of the corresponding realization of $X(n)$.

**1.4** Calculate the mean and variance functions of $X(n)$.

**1.5** Based on the results of the previous item, is $X(n)$ WSS or SSS? Why?

Finally, consider a WSS continuous-time process $Y(t)$ with an ACF reading $R_Y(\tau) = \cos(\pi\tau)$.

**1.6** Calculate and sketch the power spectral density (PSD) of the process.

**1.7** Calculate the covariance matrix of the random vector $\boldsymbol{Y} = [Y(0), Y(1), Y(5/2)]^T$.

2

$$U(n) \overset{iid}{\sim} \mathcal{P}_U(u) = \begin{cases} 1/3 & \text{for } u=1 \\ 1/3 & \text{for } u=0 \\ 1/3 & \text{for } u=-1 \\ 0 & \text{otherwise} \end{cases}$$

1.1)

$\mathbb{E}[U(n)] = 1/3 \cdot 1 + 1/3 \cdot 0 + 1/3 \cdot (-1) + 0$

$\mathbb{E}[U(n_1)U(n_2)] = \mathbb{E}[U(n_1)]\,\mathbb{E}[U(n_2)] \Big|_{n_1 \neq n_2}$

$R(\tau \neq 0) = 0$

$\mathbb{E}[U(n)^2] = \mathrm{Var}(U(n))$

$= 1/3 \cdot 1^2 + 1/3 \cdot 0^2 + 1/3 \cdot (-1)^2$

$= 2/3$

$R(\tau) = \frac{2}{3}\delta[\tau]$

For this process $U(n)$:

- Each sample $U(n)$ is drawn **independently**.
- All samples have the **same PMF**
  $P(U=1) = P(U=0) = P(U=-1) = 1/3.$

We already know:

- Mean: $E[U(n)] = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + (-1) \cdot \frac{1}{3} = 0$, same for all $n$.
- Autocorrelation:

$$R_U(k) = E[U(n)U(n+k)] = \begin{cases} E[U^2] = \frac{1^2}{3} + \frac{0^2}{3} + \frac{(-1)^2}{3} = \frac{2}{3}, & k = 0, \\ E[U(n)]E[U(n+k)] = 0, & k \neq 0, \end{cases}$$

which depends only on the lag $k$, not on $n$.

So it is wide-sense stationary (WSS).

Now, is it **strict-sense stationary (SSS)**?

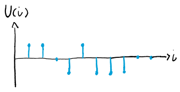
For SSS, *all* finite-dimensional distributions must be invariant to a time shift. Because the samples are:

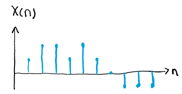
- independent, and
- identically distributed for every time index,

the joint distribution of $(U(n_1), \ldots, U(n_k))$ is just the product of the same 1-D PMF, no matter what times $n_1, \ldots, n_k$ you choose. Shifting all indices by some $m$ doesn't change that joint distribution at all.

Therefore the process is actually **strict-sense stationary** (and hence automatically WSS as well), not "only WSS".

1.3) $X(n) = \sum_{i=0}^{n} U(i)$




1.4)

$\mathbb{E}[X(n)] = \mathbb{E}[\sum_{i=0}^{n} U(i)] = \sum_{i=0}^{n} \mathbb{E}[U(i)]$

$\mathbb{E}[X(n)] = 0$

$\mathrm{Var}[X(n)] = \mathrm{Var}[\sum_{i=0}^{n} U(i)]$

$= \sum_{i=0}^{n} \mathrm{Var}[U(i)]$

$= (n+1)\mathrm{Var}[U(i)]$

$= (n+1)\frac{2}{3}$

Since $U(i)$ is iid $\mathrm{cov}(U(i_1), U(i_2)) = 0$ for all $i_1 \neq i_2$, this must be $\mathrm{Var}[X(n)] = \sum_{i=0}^{n} \mathrm{Var}[U(i)] = (n+1)\mathrm{Var}[U(i)]$

$\mathrm{Var}(X(n)) = \mathbb{E}[(x - \mu_x)^2]$

$= \mathbb{E}[(\sum_{i=0}^{n} U(i))^2]$

$= \mathbb{E}[U(i_n)^2] + \mathbb{E}[U(i_1)U(i_2)]$

$= \sum_{i=0}^{n} \mathbb{E}[U(i)^2]$

$= \sum_{i=0}^{n} \mathrm{Var}(U(i))$

$= (n+1)\mathrm{Var}(U(i))$

1.5)

Not WSS or SSS

$R(\tau = 0) = \mathbb{E}[x(n)^2] = \mathrm{Var}(x(n))$

1.6)

$Y(t)$ — $\cos(2\pi f_0 t)$

$R_Y(\tau) = \cos(\pi\tau)$ ← freq.

$S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau) \cdot e^{-j2\pi f\tau}\, d\tau$

$= \int_{-\infty}^{\infty} \cos(\pi\tau) \cdot e^{-j2\pi f\tau}\, d\tau$

$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

$= \int \frac{e^{\pi\tau} + e^{-j\pi\tau}}{2} e^{-j2\pi f\tau}\, d\tau$

$= \int \frac{e^{i\pi\tau(1-2f)}}{2} + \frac{e^{-i\pi\tau(1-2f)}}{2}\, d\tau$

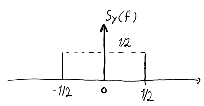
$= \int \frac{\cos((\pi - 2\pi f)\tau) + i\sin((\pi - 2\pi f)\tau)}{2} + \frac{\cos((\pi - 2\pi f)\tau) - i\sin((\pi - 2\pi f)\tau)}{2}$

$= \int \cos((\pi - 2\pi f)\tau)\, d\tau$

$= \int \cos(\pi(1-2f)\tau)\, d\tau$

Siden PSD er en FFT af ACF som trækker vores freq. componenter ud af signalet, så må det resulterer:

$S_Y(f) = (\delta[f - 1/2] + \delta[f + 1/2])\, 1/2$




Vores PSD er en Fourier transform af vores ACF. Vi har en frekvens på 1/2

1.7)

$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[(X_1 - \mu_x)(X_2 - \mu_x)]$

$= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1 \mu_x] - \mathbb{E}[\mu_x X_2] - \mu_x^2$

(Hvis $X_1$ og $X_2$ er ens, så er det det samme som at tage var)

$\mathrm{Var}(X_1) = \mathbb{E}[(X_1 - \mu_x)^2]$

Bliver givet vector $\bar{Y} = [Y(0), Y(1), Y(2{,}5)]^T$ og skal finde cov. matrix.

$$Y \cdot Y^T = [Y(0), Y(1), Y(2{,}5)] \cdot \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2.5) \end{bmatrix}$$

2

**Problem 2:**

A white Gaussian noise (WGN) process with variance $\sigma^2$ is input to a linear time invariant (LTI) system with impulse response

$$h(n) = \delta(n) + a\delta(n-2)$$

where $a$ is an unknown constant. The output process, $X(n)$, is known to have the following PSD and ACF properties:

$$S_X(1/8) = 4, \qquad R_X(2) = -2. \tag{1}$$

**2.1** Find the values of the noise variance $\sigma^2$ and the coefficient $a$ so that the conditions in (1) are fulfilled.

**2.2** What type of ARMA process is $X(n)$?

**2.3** Calculate and sketch the full ACF and PSD of $X(n)$.

Next, let $Y(n)$ be an autorregressive process given by

$$Y(n) = Z(n) + \frac{1}{3}Y(n-1) + \frac{2}{9}Y(n-2)$$

where $Z(n)$ is WGN with unit variance.

**2.4** Use the Yule-Walker equations to pose a system of equations to calculate the ACF of $Y(n)$ for time lags $k = 0, 1, 2$. (Just pose the system of equations, you don't need to solve it).

**2.5** Assuming that a 100-sample long realization of $Y(n)$ is available, explain with pseudocode how it can be used to **estimate** the ACF of the process for time lags $k = -99, -98, \ldots, 0, \ldots, 98, 99$.

3

**Problem 3:**

The position of sensor nodes in an Internet of Things (IoT) network is modeled using a two-dimensional Poisson point process $X \sim \texttt{PoissonPP}(S, \varrho)$ defined in a rectangular area $S = [-2, 2] \times [-3, 3]$ and with intensity function

$$\varrho(x, y) = \begin{cases} \frac{1}{4}x^2 & \text{if } (x, y) \in S \\ 0 & \text{otherwise.} \end{cases}$$

**3.1** Calculate the expected number of sensor nodes within the area $S$.

**3.2** Sketch a possible realization of the point process $X$ in $S$. Annotate your sketch to explain in which areas you expect the density of nodes to be highest.

Next, we define two equally sized subregions of $S$ as:

$$B_1 = (0, 2] \times (0, 3]$$
$$B_2 = [-2, 0) \times [-3, 0)$$

**3.3** What is the probability that region $B_1$ contains more than 1 sensor?

**3.4** Assume we know that region $B_2$ contains 7 sensors. Under this condition, what is now the probability that $B_1$ contains more than 1 sensor?

To conclude, each of the sensors transmits an amount of data packets per unit time that depends on its coordinates according to the function $f(x, y) = \frac{3y}{x}$. Thus, the total number of data packets transmitted per unit time in the whole network is

$$N_p = \sum_{(x,y) \in X} \frac{3y}{x}$$

**3.5** Calculate the expected value of the random variable $N_p$. *(Hint: use Campbell's theorem).*

4

og skal finde cov matrix.

$$Y \cdot Y^T = [Y(0), Y(1), Y(2,5)] \cdot \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2.5) \end{bmatrix}$$

$$\text{cov}(Y) = \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2.5) \end{bmatrix} [Y(0), \; Y(1), \; Y(2.5)] \begin{bmatrix} \mathbb{E}[Y(0), Y(0)] & \mathbb{E}[Y(0), Y(1)] & \mathbb{E}[Y(0), Y(2.5)] \\ \mathbb{E}[Y(1), Y(0)] & \mathbb{E}[Y(1), Y(1)] & \mathbb{E}[Y(1), Y(2.5)] \\ \mathbb{E}[Y(2.5), Y(0)] & \mathbb{E}[Y(2.5), Y(1)] & \mathbb{E}[Y(2.5), Y(2.5)] \end{bmatrix}$$

Antager at mean, $\mu_x$, er 0

$$= \begin{bmatrix} R(0) & R(1) & R(2.5) \\ R(1) & R(0) & R(1.5) \\ R(2.5) & R(1.5) & R(0) \end{bmatrix}$$

Nok mere rigtig:

Covariance funktion:

$$R_x(t_1, t_2) = C_x(t_1, t_2) + \mu_x(t_1)\mu_x(t_2)$$
$$\Updownarrow$$
$$C_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$$
$$= \boxed{R_x(t_1, t_2) - \mu_x^2(t_1, t_2)}$$

Dette er hvad der skal stå i hver input af matrix

$$\text{cov}(Y) = \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2.5) \end{bmatrix} [Y(0), \; Y(1), \; Y(2.5)] \begin{bmatrix} \mathbb{E}[Y(0), Y(0)] - \mu_y^2 & \mathbb{E}[Y(0), Y(1)] - \mu_y^2 & \mathbb{E}[Y(0), Y(2.5)] - \mu_y^2 \\ \mathbb{E}[Y(1), Y(0)] - \mu_y^2 & \mathbb{E}[Y(1), Y(1)] - \mu_y^2 & \mathbb{E}[Y(1), Y(2.5)] - \mu_y^2 \\ \mathbb{E}[Y(2.5), Y(0)] - \mu_y^2 & \mathbb{E}[Y(2.5), Y(1)] - \mu_y^2 & \mathbb{E}[Y(2.5), Y(2.5)] - \mu_y^2 \end{bmatrix}$$

$$= \begin{bmatrix} R_y(0) - \mu_y^2 & R_y(1) - \mu_y^2 & R_y(2.5) - \mu_y^2 \\ R_y(1) - \mu_y^2 & R_y(0) - \mu_y^2 & R_y(1.5) - \mu_y^2 \\ R_y(2.5) - \mu_y^2 & R_y(1.5) - \mu_y^2 & R_y(0) - \mu_y^2 \end{bmatrix}$$

## 2:

WGN process with variance $\sigma^2$ input to LTI System.

Impulse response:

$$h(n) = \delta(n) + a\,\delta(n-2)$$

$$U(n) \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$$

$$S_x(1/8) = 4$$
$$R_x(2) = -2 \qquad (1)$$

$X(n) \to$ LTI $h(n)$ $\to Y(n)$

$Z(n) \to$ LTI $h(n)$ $\to X(n)$

2.1) Find $\sigma^2$ and 'a' so that (1) is fulfilled.

$Y(n) = h(n) * X(n)$

$$X(n) = h(n) * Z(n) = \sum_{K=-\infty}^{\infty} h(K) \cdot Z(n-K)$$

Kan se at $h(n)$ er defineret ved $n = 0$ & 2

$$\underbrace{\delta(n) = 0 \quad og \quad \delta(n-2) = 2}_{\text{Vores } n \text{ vardier}}$$

Kan satte det ind i $X(n)$ hvor nu satter vi vores $n$ vardier bliver $K$ vardierne istedet $K = 0, 2$

$$X(n) = \sum_{K=0,2} h(K) \cdot Z(n-K) = \Big(\underbrace{\delta(0)}_{1} + a\underbrace{\delta(0-2)}_{0}\Big) Z(n-0)\Big) + \Big(\underbrace{\delta(2)}_{0} + a\underbrace{\delta(2-2)}_{1}\Big) Z(n-2)\Big)$$

$$= (1 + a \cdot 0) Z(n-0) + (0 + a \cdot 1) Z(n-2)$$

$$= (1 \cdot Z(n-0) + a \cdot 0 \cdot Z(n-0)) + (0 \cdot Z(n-2) + a \cdot 1 \cdot Z(n-2))$$

$$= \underline{\underline{Z(n) + aZ(n-2)}}$$

ACF:
$R_x(\tau) = \mathbb{E}[X(n) + X(n+\tau)]$

$$R_x(\tau) = \mathbb{E}[X(n) + X(n+\tau)]$$

$$= \mathbb{E}[(Z(n) + aZ(n-2))(Z(n+\tau) + aZ(n-2+\tau))]$$

$$= \mathbb{E}[Z(n)Z(n+\tau)] + \mathbb{E}[Z(n) \cdot aZ(n-2+\tau)] + \mathbb{E}[aZ(n-2) \cdot Z(n+\tau)] + \mathbb{E}[a^2 Z(n-2) \cdot Z(n-2+\tau)]$$

$$= R_z(\tau) + aR_z(\tau-2) + aR_z(\tau+2) + a^2 R_z(\overbrace{+2-2\,\tau}^{\tau})$$

$(R_x(t, t+\tau) = R_x(\tau)$

$$= \sigma^2\delta(\tau) + a\sigma^2\delta(\tau-2) + a\sigma^2\delta(\tau+2) + a^2\sigma^2\delta(\tau)$$

$(R_x(n_1, n_2) = \mu_x^2 + \sigma_x^2\,\delta(n_2 - n_1)$ i dette tilfælde er vores mean 0

Det opg. beder om

$$= \underbrace{(\sigma^2 + a^2\sigma^2)\,\delta(\tau)}_{0} + \underbrace{a\sigma^2\delta(\tau-2)}_{-2} + \underbrace{a\sigma^2\delta(\tau+2)}_{+2}$$

$$R(2) = \cancel{(\sigma^2 + a^2\sigma^2)\,\delta(2)} + a\sigma^2\underbrace{\delta(2-2)}_{1} + \cancel{a\sigma^2\delta(2+2)}$$

$$= a\sigma^2 = -2$$ hvad opgaven beder om

$$\sigma^2 = -\frac{2}{a}$$

$$S_x(f) = \sum_{\tau=-\infty}^{\infty} R_x(\tau) \cdot e^{-2\pi j f \tau}$$

only defined for $\tau = 0, -2, +2$

$$= \Big((\sigma^2 + a^2\sigma^2)\underbrace{\delta(0)}_{1} + a\sigma^2\underbrace{\delta(0-2)}_{0} + a\sigma^2\underbrace{\delta(0+2)}_{0}\Big)\, \overbrace{e^{-2\pi j f 0}}^{1}$$
$$+ \Big((\sigma^2 + a^2\sigma^2)\underbrace{\delta(-2)}_{0} + a\sigma^2\underbrace{\delta(-2-2)}_{0} + a\sigma^2\underbrace{\delta(-2+2)}_{1}\Big)\, e^{-2\pi j f -2}$$
$$+ \Big((\sigma^2 + a^2\sigma^2)\underbrace{\delta(2)}_{0} + a\sigma^2\underbrace{\delta(2+2)}_{1} + a\sigma^2\underbrace{\delta(2+2)}_{0}\Big)\, e^{-2\pi j f 2}$$

$$= (\sigma^2 + a^2\sigma^2 \cdot 1) + (a\sigma^2 \cdot e^{4\pi j f}) + (a\sigma^2 \cdot e^{-4\pi j f})$$

$e^{jx} = \cos(x) + j\sin(x)$

$$S_x(1/8) = (\sigma^2 + a^2\sigma^2) + a\sigma^2 \cdot \cos(4 \cdot (1/8) \cdot \pi) + \cancel{a\sigma^2 j \sin(4 \cdot (1/8) \cdot \pi)}$$
$$+ a\sigma^2 \cdot \cos(4 \cdot (1/8) \cdot \pi) - \cancel{a\sigma^2 j \sin(4 \cdot (1/8) \cdot \pi)}$$

$$S_x(1/8) = (\sigma^2 + a^2\sigma^2) + a\sigma^2 \cdot 2\cos(4 \cdot (1/8) \cdot \pi)$$

$$= (\sigma^2 + a^2\sigma^2) + a\sigma^2 \cancel{2\cos(4\cdot(1/8)\cdot\pi)}$$

$\cos(4\cdot(1/8)\pi) = \cos(1/2\,\pi) = 0$

$90°$

$$= \sigma^2 + a^2\sigma^2 = 4$$

Sætter det lig 4
Pga det stod
i opg. beskrivelse

henter lige ind

$$\sigma^2 = -\frac{2}{a}$$

$$\Updownarrow$$

$$-\frac{2}{a} - a^{\cancel{2}}\frac{2}{\cancel{a}} = 4$$

$$\Updownarrow$$

$$-\frac{2}{a} - 2a = 4$$

$$\Updownarrow$$

$$-2 - 2a^2 = 4a$$

$$\Updownarrow$$

$$-2a^2 - 4a - 2 = 0$$

$$a = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\cdot(-2)(-2)}}{-4}$$

$$a = \underline{\underline{-1}}$$

$$\sigma^2 = \underline{\underline{2}}$$

You have:

$$h(n) = \delta(n) + a\delta(n-2)$$

A WGN process $W(n)$ with variance $\sigma^2$ is fed into an LTI system.
The output is:

$$X(n) = W(n) + aW(n-2)$$

This is the definition of a **moving-average process of order 2**, although note that the lag is 2, not 1.

Formally:

$$X(n) = W(n) + aW(n-2)$$

That is a **MA(2)** process *but with the middle coefficient equal to 0:*

$$X(n) = 1\cdot W(n) + 0\cdot W(n-1) + a\cdot W(n-2)$$

✓ **So what the question wants:**

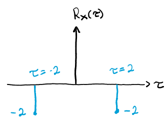
**Answer:**
The process $X(n)$ is a **moving-average process of order 2**, i.e. an **MA(2)** process.

But some instructors want you to be explicit:

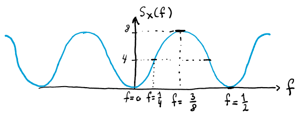
- Is it ARMA? → Yes, trivially ARMA(0,2)
  (because MA(q) is ARMA(0,q))
- MA order? → 2
- AR order? → 0

✓ **Final phrasing for an exam answer:**
**"The process $X(n) = W(n) + aW(n-2)$ is an MA(2) process (or ARMA(0,2)).**
**It contains no autoregressive component and a 2-lag moving-average term."**

2.3)




$$S_X(f) = \sigma^2 a^2 \sigma^2 + 2a\sigma^2\cos(4\pi f) = 4 - 4\cos(4\pi f)$$




2.4) $Y(n) = Z(n) + \frac{1}{3}Y(n-1) + \frac{2}{9}Y(n-2)$

$k = 0,1,2$

$$\bar{\Phi} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{9} \end{bmatrix}$$ coefficienterne for $Y(n)$ $\qquad Z \sim \mathcal{N}(0,1)$

$$X(n) = \sum_{i=1}^{p} \phi_i X(n-i) + Z(n)$$ (Lecture notes s. 30)

$$Y(n) = \sum_{i=1}^{2} \phi_i Y(n-i) + Z(n)$$

$$R_Y(\tau) = \sum_{i=1}^{2} \phi_i R_Y(i-\tau) + \sigma_Z^2\,\delta(\tau)$$

- - - -

$$R_Y(0) = \sum_{i=1}^{2} \frac{1}{3} R_Y(1-0) + \frac{2}{9} R_Y(2-0) + \sigma^2\cdot 1$$

$$R_Y(1) = \sum_{i=1}^{2} \frac{1}{3} R_Y(1-1) + \frac{2}{9} R_Y(2-1) + \sigma^2\cdot 0$$

$$R_Y(2) = \sum_{i=1}^{2} \frac{1}{3} R_Y(1-2) + \frac{2}{9} R_Y(2-2) + \sigma^2\cdot 0$$

- - - - -

$$R_Y(0) = \sum_{i=1}^{2} \frac{1}{3} R_Y(1) + \frac{2}{9} R_Y(2) + \sigma^2$$

$$R_Y(1) = \sum_{i=1}^{2} \frac{1}{3} R_Y(0) + \frac{2}{9} R_Y(1)$$

$$R_Y(2) = \sum_{i=1}^{2} \frac{1}{3} R_Y(-1) + \frac{2}{9} R_Y(0)$$

2.5)

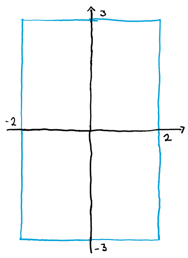
3:

$X \sim \text{PoissonPP}(S, \varrho)$

$S = [-2,2] \times [-3,3]$

$$\varrho(x,y) = \begin{cases} \frac{1}{4}x^2 & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

3.1)




$$\mu_X(B) = \int_B \varrho(x)\,dx$$

$$= \int_{-2}^{2}\int_{-3}^{3} 1/4\, x^2\,dx\,dy$$

løser integralet for y
(Der er ikke nogen y
Så derfor sker der intet)

$$\int_A^B f(x)\,dy = (B-A)\cdot f(x)$$

$$= \int_{-2}^{2} 1/4\,x^2\cdot 3 - 1/4\,x^2\cdot(-3)\,dx$$

$$= \int_{-2}^{2} 3/4\,x^2 - (-3/4)\,x^2\,dx$$

$$= \int_{-2}^{2} \frac{6}{4}x^2\,dx$$

$$= \frac{6}{4}\cdot\frac{1}{2+1}\cdot x^{2+1}$$

$$\int x^n = \frac{1}{n+1}x^{n+1}$$

$$= \frac{6}{12}x^3$$

upper bound
lower bound

$$F(B) - F(A) = f(x)\,B - f(x)\,A$$

$$= \frac{6}{12}2^3 - \frac{6}{12}(-2)^3$$

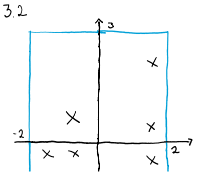
$$= \frac{6}{12}8 - \frac{6}{12}(-8)$$

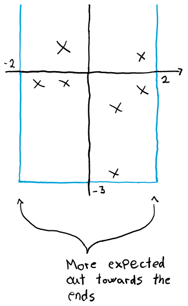
$$= \frac{48}{12} - \left(-\frac{48}{12}\right)$$

$$= \frac{96}{12}$$

$$\mu = \underline{\underline{8}}$$

3.2






3.3)

$B_1 = (0,2] \times (0,3]$

$B_2 = [-2,0) \times [-3,0]$

Antager dette, men kan ikke altid.

Kan antage det fordi begge regions er symmetrisk

$\mu_{B_1} = \mu_{B_2} = 1/4\,\mu = 2$

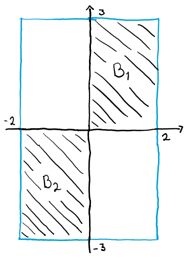
$\mathcal{P}_{B_1}(>1) = 1 - \mathcal{P}_{B_1}(0) - \mathcal{P}_{B_1}(1)$

$P(k) = \frac{\mu^k \cdot e^{-\mu}}{k!}$

$\mathcal{P}_{B_1}(>1) = 1 - \frac{2^0 \cdot e^{-2}}{0!} - \frac{2^1 \cdot e^{-2}}{1!}$

$= 1 - e^{-2} - 2 \cdot e^{-2}$

$= 0.59$




3.4) Given the regions $B_1$ and $B_2$ are mutually independent random variables, then the probability of region $B_1$ is unchanged.

"An important property of The Poisson point process is that if $B_1, B_2, \ldots, B_n \subset S$ are fixed disjoint regions, then the corresponding region counts $N_X(B_1), N_X(B_2) \ldots, N_X(B_n)$ are mutally independent random variables."

-Poisson lecture notes

3.5) $f(x,y) = \frac{3y}{x}$, $N_p = \sum_{(x,y)\in X} \frac{3y}{x}$

$\mathbb{E}\left[\sum_{\vec{x}\in\lambda} g(\vec{x})\right] = \int_S g(\vec{x})\, \varrho_X(\vec{x})\, d(\vec{x})$

$\mathbb{E}\left[\sum_{(x,y)\in X} f(x,y)\right] = \int_{-2}^{2}\int_{-3}^{3} f(x,y)\, \varrho(x,y)\, dy\,dx$

$= \iint \frac{3y}{x} \cdot \frac{1}{4}x^2 \; dy\,dx$

$= \iint \frac{3y}{x} \cdot \frac{x^2}{4} \; dy\,dx$

$= \iint \frac{3yx^{\cancel{2}}}{4\cancel{x}}$

$= \iint \frac{3yx}{4}$

$= 0$

# ExamSet2024

Sunday, November 2, 2025 4:08 PM

Problem 1

1.1)

PSD:

$$P_X(f) = \sum_{k=-\infty}^{\infty} r_X[k]\, e^{(-i2\pi f k)}$$

Real Part:

$$P_X(f) = \sum_{k=-\infty}^{\infty} r_X[k] \cos^{(-i2\pi f k)}$$

$$-\frac{2}{3}\cdot\left(\frac{2}{3}\right) = \frac{4}{9}$$

ExamStochasticSyste...

## Exam: Stochastic Systems

Date: Thursday January 11, 2024.
Troels Pedersen, AAU

This entire problem set contains **5 pages**. Please make sure that you have received all pages.

The exam is graded according to your answer as a whole: both quantity and quality count. We value concise arguments showing your command of the topics. Simply answering "Yes." or "No." will not do that!

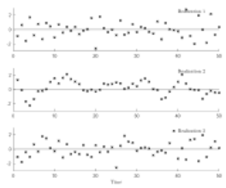
We recommend that you read through each problem thoroughly before starting to solve it. Should you happen to get stuck at some point, we recommend that you continue and anyway try to solve the rest. You always have the opportunity to sketch or explain how you would have continued if you hadn't got stuck.

It is allowed to use books, lecture notes, your own notes, calculators and computers during the exam. Communication to others during the exam is not allowed.

1

**Problem 1:**

Realizations of three different discrete-time random processes are displayed in the figure below. All three processes are wide-sense stationary (WSS) with zero mean and unit variance.




Power spectral densities (PSD's) corresponding to three different discrete-time WSS processes are shown in the figure below.

**1.1** Match each of the three realizations to one of the three PSD's. For each pair, explain why you have grouped together that particular realization with that particular PSD.

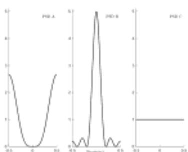
See chat notes

**1.2** It can be identified directly from PSD A, PSD B and PSD C, that all three of them correspond to processes with zero mean. Explain how.

*This problem continues on the following page.*

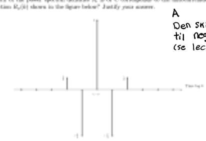
- Trend is not moving anywhere just moving around 0

2




**1.3** Which of the power spectral densities A, B or C corresponds to the autocorrelation function $R_x(k)$ shown in the figure below? *Justify your answer.*

A
Den skifter over til negativ correl. (se lec 5 quiz)



**1.4** Assume now that $\{X(n)\}$ is a zero mean Gaussian process with autocorrelation function $R_x(k)$ as given in the above figure. State the pdf of the vector $\mathbf{X}$ defined as

$$\mathbf{X} = \begin{bmatrix} X(23) \\ X(24) \end{bmatrix}.$$

**1.5** Calculate the mean and covariance of the vector $\mathbf{Y}$ defined as

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{X}.$$

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**Problem 2:**

Consider the linear time-invariant (LTIV) system


where $X(n)$ is a discrete-time WSS process with zero mean and autocorrelation function $R_x(k)$. The transfer function of the system is

$$H(f) = 1 - \frac{3}{4}\exp(-j4\pi f), \qquad -\tfrac{1}{2} \le f < \tfrac{1}{2}.$$

**2.1** Compute the mean of the output process $\{Y(n)\}$.

**2.2** *Justify* without calculations that $\{Y(n)\}$ is WSS.

**2.3** Express the autocorrelation function $R_Y(k)$ of the output process $\{Y(n)\}$ in terms of $R_X(k)$.

**2.4** Let $\delta$ be the Kronecker delta. State at least two reasons why $R_Y(k)$ cannot be of the form:

$$R_Y(k) = -2\delta(k+1) - \delta(k) + \delta(k-2)$$

Notice that the above equation is not true.

1.1) Realization 2 = PSD B
Realization 1 = PSD C
Realization 3 = PSD A

1.4) 1.5)

$$\mathbb{E}[X] = \mathbb{E}\left[\begin{bmatrix} X(23) \\ X(24) \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathrm{Cov}(X) = \begin{bmatrix} \mathrm{Var}(X(23)) & \mathrm{Cov}(X(23), X(24)) \\ \mathrm{Cov}(X(23), X(24)) & \mathrm{Var}(X(24)) \end{bmatrix} = \begin{bmatrix} R(0) & R(1) \\ R(1) & R(0) \end{bmatrix}$$

$$f_X(x) = \frac{1}{2\pi\sqrt{\det(\mathrm{cov}(x))}}\, e^{\left(-\frac{1}{2}x^T \mathrm{cov}(x)^{-1} x\right)}$$

$$Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X(23) \\ X(24 \end{bmatrix} = \begin{bmatrix} 1\cdot X(23) + 0\cdot X(24) \\ 1\cdot X(23) + 1\cdot X(24) \end{bmatrix} = \begin{bmatrix} X(23) \\ X(23) + X(24) \end{bmatrix}$$

$$\mathbb{E}[Y] = \begin{bmatrix} \mathbb{E}[X(23)] \\ \mathbb{E}[X(23)] + \mathbb{E}[X(24)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathrm{cov}(Y) = \begin{bmatrix} Y_1^2 & Y_1 Y_2 \\ Y_2 Y_1 & Y_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} R_x(0) & \mathbb{E}[X(23)\cdot(X(23) + X(24))] \\ \mathbb{E}[(X(23) + X(24))\cdot X(23)] & \mathbb{E}\left[(X(23) + X(24))^2\right] \end{bmatrix}$$

$$\mathbb{E}[X(23)(X(23) + X(24))] = \mathbb{E}[X(23)] + \mathbb{E}[X(23) + X(24)]$$
$$= R_x(0) + R_x(1)$$

$$\mathbb{E}[(X(23) + X(24))\cdot X(23)] = \mathbb{E}[X(23) + X(24)] + \mathbb{E}[X(23)]$$
$$= R_x(1) + R_x(0)$$

$$\mathbb{E}[(X(23) + X(24))^2] = \mathbb{E}[X(23)^2 + X(24)^2 + 2X(23)X(24)]$$
$$= R_x(0) + R_x(0) + 2R_x(1)$$
$$= 2R_x(0) + 2R_x(1)$$

$$\mathrm{cov}(Y) = \begin{bmatrix} R_x(0) & R_x(0) + R(1) \\ R_x(1) + R_x(0) & 2R_x(0) + 2R_x(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 + (-\frac{2}{3}) \\ -\frac{2}{3} + 1 & 2\cdot 1 + 2\cdot(-\frac{2}{3}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

We're told:

- $\{X(n)\}$ is a **zero-mean Gaussian process** with autocorrelation $R_x(k)$ (and we're implicitly assuming it's WSS, so $R_x(k)$ depends only on the lag $k$).
- Define the 2-D random vector

$$\mathbf{X} = \begin{bmatrix} X(23) \\ X(24) \end{bmatrix}.$$

### 1.4 – PDF of $\mathbf{X}$

Because the process is Gaussian, **any finite collection of samples is jointly Gaussian**. So $\mathbf{X}$ is a 2-D (bivariate) Gaussian vector.

#### Mean of $\mathbf{X}$

The process is zero mean, so

$$E[X(23)] = E[X(24)] = 0$$

and therefore

$$\mu_X = E[\mathbf{X}] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

#### Covariance matrix of $\mathbf{X}$

For a WSS process,

$$\mathrm{cov}(X(n), X(m)) = R_x(m-n).$$

So:

- $\mathrm{var}(X(23)) = R_x(0)$
- $\mathrm{var}(X(24)) = R_x(0)$
- $\mathrm{cov}(X(23), X(24)) = R_x(1)$

Thus

$$C_X = \mathrm{Cov}(\mathbf{X}) = \begin{bmatrix} R_x(0) & R_x(1) \\ R_x(1) & R_x(0) \end{bmatrix}.$$

Call $R_0 = R_x(0)$, $R_1 = R_x(1)$ for short.

#### Bivariate Gaussian pdf

The pdf of a 2-D Gaussian with mean $\mu_X$ and covariance $C_X$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det C_X}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu_X)^T C_X^{-1}(\mathbf{x} - \mu_X)\right),$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

If you want it fully expanded:

- $\det C_X = R_0^2 - R_1^2$,
- $C_X^{-1} = \frac{1}{R_0^2 - R_1^2}\begin{bmatrix} R_0 & -R_1 \\ -R_1 & R_0 \end{bmatrix}$.

So

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{R_0^2 - R_1^2}} \exp\left(-\frac{1}{2(R_0^2 - R_1^2)}\left[R_0(x_1^2 + x_2^2) - 2R_1 x_1 x_2\right]\right).$$

That's the pdf of $\mathbf{X}$.

### 1.5 – Mean and covariance of $\mathbf{Y}$

We define

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{X} = A\mathbf{X}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Let's use linear expectation and covariance rules.

#### Mean of $\mathbf{Y}$

$$E[\mathbf{Y}] = A\,E[\mathbf{X}] = A\,\mu_X = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $\mathbf{Y}$ is also zero mean.

#### Covariance of $\mathbf{Y}$

$$C_Y = \mathrm{Cov}(\mathbf{Y}) = AC_X A^T.$$

We already have

$$C_X = \begin{bmatrix} R_0 & R_1 \\ R_1 & R_0 \end{bmatrix}.$$

First compute $AC_X$:

$$AC_X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} R_0 & R_1 \\ R_1 & R_0 \end{bmatrix} = \begin{bmatrix} R_0 & R_1 \\ R_0 + R_1 & R_0 + R_1 \end{bmatrix}.$$

Then multiply by $A^T$:

$$C_Y = (AC_X)A^T = \begin{bmatrix} R_0 & R_1 \\ R_0 + R_1 & R_0 + R_1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_0 & R_0 + R_1 \\ R_0 + R_1 & 2(R_0 + R_1) \end{bmatrix}.$$

So:

- $E[\mathbf{Y}] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $C_Y = \begin{bmatrix} R_x(0) & R_x(0) + R_x(1) \\ R_x(0) + R_x(1) & 2(R_x(0) + R_x(1)) \end{bmatrix}$.

Since $X$ is Gaussian and $\mathbf{Y}$ is a linear transform of $\mathbf{X}$, $\mathbf{Y}$ is also jointly Gaussian with that mean and covariance.

**Problem 2:**

Consider the linear time-invariant (LTIV) system:

Input process $[X(n)]$ → LTIV system $h(n)$ → Output process $[Y(n)]$

where $X(n)$ is a discrete-time WSS process with zero mean and autocorrelation function $R_X(k)$. The transfer function of the system is

$$H(f) = 1 - \frac{3}{4}\exp(-j4\pi f), \quad -\frac{1}{2} \le f < \frac{1}{2}.$$

**2.1** Compute the mean of the output process $[Y(n)]$.

**2.2** Justify without calculations that $[Y(n)]$ is WSS.

**2.3** Express the autocorrelation function $R_Y(k)$ of the output process $[Y(n)]$ in terms of $R_X(k)$.

**2.4** Let $\delta$ be the Kronecker delta. State at least two reasons why $R_Y(k)$ cannot be of the form:

$$R_Y(k) = -3\delta(k+1) - \delta(k) + \delta(k-2)$$

Notice that the above equation is not true.

**2.5** Suppose that the input $[X(n)]$ is a Gaussian process with

$$R_X(k) = 2\delta(k).$$

Compute and sketch the power spectral density (PSD) $S_Y(f)$ for the output process $[Y(n)]$ in this case.

**2.6** Identify wether the output is an AR(p), MA(q) or ARMA(p,q) process. Give the values for $p$ and $q$.

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**Problem 3:**

Consider the discrete time, discrete valued process $[X(n)]$ defined as

$$X(0) = \begin{cases} 1 & \text{with probability } b, \ 0 \le b \le 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n = 0$ and for $n = 1, 2, 3, \ldots$, conditioned on $X(n-1)$, as

$$X(n) = \begin{cases} 1 & \text{with probability } a + (1-2a)X(n-1) \text{ with } 0 \le a \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

**3.1** Explain using pseudo-code how 3 independent realization of $X(n)$ of each 100 samples can be simulated.

**3.2** Verify that $[X(n)]$ is a 2-state, discrete time Markov process.

**3.3** Draw the state-diagram for $[X(n)]$ and give its state transition probability matrix.

**3.4** Show that the steady state probability vector $\pi$ equals

$$\pi = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

**3.5** Now, assume $b = \frac{1}{2}$ and $a = 0.9$. Compute the mean of $X(n)$. Furthermore, evaluate the autocorrelation function at time-lags $k = 0, 1$, and $2$.

**3.6** Let $Y(n)$ be defined as $X(n)$:

$$Y(n) = 1 - 2X(n).$$

Which type of process is $Y(n)$ and what can you say about it?

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$$X(0) = \begin{cases} 1 & \text{with Prob. } 0 \le b \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(n) = \begin{cases} 1 & \text{Prob. } a + (1-2a)X(n-1) \quad 0 \le a \le 1 \end{cases}$$

3.1)

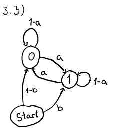
```
for i in 1...3
    b = b
    a = a
    if(U(0,1) ≤ b)
        x[0] = 1
    else
        x[0] = 0
    for n in 1...100
        if(U(0,1) ≤ a
            x[n] = not(x[n-1])
        else
            x[n] = x[n-1]

    if(U(0,1) ≤ (a+(1-2a)·x[n-1]
        x[n] = 1
    else
        x[n] = 0
Stem Plot (x[n], n)
```

3.2)

Scenario 1)
$X(n-1) = 0$
$\text{Prob}(X(n)=1) = a$
$\text{Prob}(X(n)=0) = 1-a$

Scenario 2)
$X(n-1) = 1$
$\text{Prob}(X(n)=1) = a + (1-2a)\cdot 1$
$= 1-a$
$\text{Prob}(X(n)=0) = a$

3.3




$$P_x = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-a & a \\ a & 1-a \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \end{array}$$

3.4 $\bar{\pi} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$P(n-1) = P(n) = \pi$

$\pi^T = \pi^T \bar{\bar{P}}$

$\bar{\pi} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix}$

$\pi^T P = \pi^T$

$\pi_0 + \pi_1 = 1$

$\pi_0 = \pi_0(1-a) + \pi_1 a$

$\pi_1 = 1 - \pi_0$

$\Updownarrow$

$\pi_0 = \pi_0(1-a) + (1-\pi_0)a = \pi_0 - a\pi_0 + a - a\pi_0$

$\pi_0 - \pi_0 + 2a\pi_0 = \pi_0 - \pi_0 - 2a\pi_0 + 2a\pi_0 + a$

$2a\pi_0 = a$

$\pi_0 = \frac{2}{a}$

$\pi_0 = \frac{1}{2} \iff \pi_1 = 1 - \pi_0 = 1 - \frac{1}{2} = \frac{1}{2}$

$\bar{\pi} = \left[\frac{1}{2}\right]$

If you want it fully expanded:

- $\det C_X = R_0^2 - R_1^2$,
- $C_X^{-1} = \frac{1}{R_0^2 - R_1^2}\begin{bmatrix} R_0 & -R_1 \\ -R_1 & R_0 \end{bmatrix}$.

So

$$f_{\mathbf{X}}(x_1, x_2) = \frac{1}{2\pi\sqrt{R_0^2 - R_1^2}} \exp\left(-\frac{1}{2(R_0^2 - R_1^2)}\left[R_0(x_1^2 + x_2^2) - 2R_1 x_1 x_2\right]\right).$$

That's the pdf of $\mathbf{X}$.

2:

$X(n) \rightarrow \boxed{\text{LTI } h(n)} \xrightarrow{Y(n)}$

$H(f) = 1 - \frac{3}{4}e^{(-j4\pi f)} \qquad -\frac{1}{2} \le f < \frac{1}{2}$

$X(n)$ = WSS (zero mean)

2.1) $\mu_Y = \mu_X \sum_n h(n)$ (Lecture 9 Slide 13 tabel)

$= H(0)\cdot\mu_X$

$= H(0)\cdot\mu_X$

$= \left(1 - \frac{3}{4}e^{-j4\pi 0}\right)\cdot 0$

$= \underline{\underline{0}}$

2.2) $\mu_Y(n) := \mathbb{E}[Y(n)] = \mathbb{E}[h(n) * X(n)]$

$= \mathbb{E}\left[\sum_{j=-\infty}^{\infty} h(j)X(n-j)\right]$

$= \sum_j h(j)\mathbb{E}[X(n-j)]$

$\mathbb{E}[Y(n)] = \sum_j \overbrace{h(j)}^{\text{ikke afhængig af tidspunkt } n} \mathbb{E}[X(n-j)]$ Er givet $X(n)$ er WSS. en WWS's ACF er kun afhængig af time lag $\tau$. Derfor må $Y(n)$ være WSS.

2.3) (Tjek Lec 9 Slides 8-10)

$R_Y(n, n+k) = \mathbb{E}[Y(n)\cdot Y(n+k)]$

$= \mathbb{E}\left[Y(n)\sum_j h(j)X(n+k-j)\right]$

$= \sum_j h(j)\underbrace{\mathbb{E}[Y(n)X(n+k-j)]}_{:= R_{YX}(n,n+k-j)}$

$R_{YX}(n, n+k) = \mathbb{E}[Y(n)X(n+k)]$

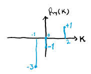
$= \mathbb{E}\left[\sum_j h(j)X(n-j)X(n+k)\right]$

$= \sum_j h(j)\underbrace{\mathbb{E}[X(n-j)X(n+k)]}_{:= R_X(k+j)}$

$R_{YX}(k) = h(-k) * R_X(k)$

$\boxed{R_Y(k) = h(k) * h(-k) * R_X(k)}$

2.4 $R_Y(k) = -3\delta(k+1) - \delta(k) + \delta(k-2)$




can't of the form because:

1) it is not symmetrical there are no $-3\delta(k-1)$ or $\delta(k+2)$

2) The largest amplitude is not in 0

3) The amplitude in 0 is negative

2.5

$X(n)$ = Gaussian Process

$R_X(k) = 2\delta(k)$

$S_Y(f) = |H(f)|^2 S_X(f)$

$S_X(f) = \sum_k R_X(k)\cdot e^{-j2\pi f k}$

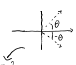
$R_X(k) = 2\delta(k)$

$S_X(f) = 2\cdot R_X(0)\cdot e^{-j2\pi f 0}$

$= 2\, R_X(0)\cdot e^0$

$= 2\cdot 1\cdot 1$

$= 2$



$|H(f)|^2 = H(f)\cdot H(f)^*$

$= \left(1 - \frac{3}{4}e^{-j4\pi f}\right)\left(1 - \frac{3}{4}e^{j4\pi f}\right)$

$= 1 - \frac{3}{4}e^{j4\pi f} - \frac{3}{4}e^{-j4\pi f} + \left(\frac{3}{4}\right)^2$

$= -\frac{6}{4}\cos(4\pi f) + 1 + \frac{9}{16}$ $\quad e^{-j\varphi} + e^{j\varphi} = 2\cos(\varphi)$

$= -\frac{6}{4}\cos(4\pi f) + \frac{25}{16}$

$= \frac{25}{16} - \frac{6}{4}\cos(4\pi f)$

$S_Y(f) = |H(f)|^2 S_X(f)$

$= \left(\frac{25}{16} - \frac{6}{4}\cos(4\pi f)\right) 2$

$= \underline{\frac{50}{16} - 3\cos(4\pi f)}$

2.5

$S_Y(f) = -3\cos(4\pi f) + \frac{50}{16}$


(No negative frequencies exist in PSD's)

2.6) $Y(n) = h(n) * X(n)$

$= (\delta(n) - \frac{3}{4}\delta(n-2)) * X(n)$

$= \sum_j h(j)\cdot X(n-j) = X(n) - \frac{3}{4}X(n-2)$

$$\pi_0 - \pi_0 + 2a\pi_0 = \pi_0 - \pi_0 - 2a\pi_0 + 2a\pi_0 + a$$

$$2a\pi_0 = a$$
$$\pi_0 = \frac{2}{a}$$

$$\pi_0 = \frac{1}{2} \iff \pi_1 = 1 - \pi_0 = 1 - \frac{1}{2} = \frac{1}{2}$$
$$\bar{\pi} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

2.6)

$$Y(n) = h(n) * X(n)$$
$$= (\delta(n) - \tfrac{3}{4}\delta(n-2)) * X(n)$$
$$= \sum_j h(j) \cdot x(n-j) = X(n) - \tfrac{3}{4} x(n-2)$$

MA(2) 2. orden (Kigger efter Størst tids lag)

$$Y(n) \overset{MA}{\propto} X(n-k)$$
$$Y(n) \overset{AR}{\propto} Y(n-k)$$
$$Y(n) \overset{ARMA}{\propto} X(n-k)\, Y(n-k)$$

# LTI exercises

Friday, November 21, 2025 1:32 PM

## The z-transform

TABLE 3.1 SOME COMMON $z$-TRANSFORM PAIRS

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $a^n u[n]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7. $na^n u[n]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 8. $-na^n u[-n-1]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 9. $\cos(\omega_0 n)u[n]$ | $\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $\sin(\omega_0 n)u[n]$ | $\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $r^n\cos(\omega_0 n)u[n]$ | $\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $r^n\sin(\omega_0 n)u[n]$ | $\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\dfrac{1-a^N z^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

**18.1** (☺) **(f)** An LSI system with system function $\mathcal{H}(z) = 1 - z^{-1} - z^{-2}$ is used to filter a discrete-time white noise random process with variance $\sigma_U^2 = 1$. Determine the ACS and PSD of the output random process.

$$H(z) = 1 - z^{-1} - z^{-2} = \frac{Y(z)}{X(z)}$$

$\downarrow z$

$$h(n) = \delta[n] - \delta[n-1] - \delta[n-2]$$

$$Y[n] = h(n) * X(n) = \sum_j h(j)\cdot X(n-j)$$

$$= X(n) - X(n-1) - X(n-2)$$

ACF:

$$R_y(k) = \mathbb{E}[y(n)\cdot y(n+k)]$$

$$= \mathbb{E}[(X(n) - X(n-1) - X(n-2))\cdot(X(n+k) - X(n+k-1) - X(n+k-2))]$$

$$= \mathbb{E}[(X(n)X(n+k)) - (X(n)X(n+k-1)) - (X(n)X(n+k-2))$$
$$- (X(n-1)X(n+k)) + (X(n-1)X(n+k-1)) + (X(n-1)X(n+k-2))$$
$$- (X(n-2)X(n+k)) + (X(n-2)X(n+k-1)) + (X(n-2)X(n+k-2))]$$

$$= \mathbb{E}[(X(n)X(n+K)) - (X(n)X(n+K-1)) - (X(n)X(n+K-2))$$
$$- (X(n-1)X(n+K)) + (X(n-1)X(n+K-1)) + (X(n-1)X(n+K-2))$$
$$- (X(n-2)X(n+K)) + (X(n-2)X(n+K-1)) + (X(n-2)X(n+k-2))]$$

$$= \mathbb{E}[X(n)X(n+K)] - \mathbb{E}[X(n)X(n+k-1)] - \mathbb{E}[X(n)X(n+k-2)]$$
$$- \mathbb{E}[X(n-1)X(n+K)] + \mathbb{E}[X(n-1)X(n+K-1)] + \mathbb{E}[X(n-1)X(n+K-2)]$$
$$- \mathbb{E}[X(n-2)X(n+K)] + \mathbb{E}[X(n-2)X(n+k-1)] + \mathbb{E}[X(n-2)X(n+k-2)]$$

$$= R_x(K) - R_x(K-1) - R_x(k-2)$$
$$- R_x(k+1) + R_x(k+1-1) + R_x(K+1-2)$$
$$- R_x(K+2) + R_x(K+2-1) + R_x(K+2-2)$$

$$= R_x(K) - \cancel{R_x(k-1)} - R_x(k-2)$$
$$- \cancel{R_x(K+1)} + R_x(k) + \cancel{R_x(K-1)}$$
$$- R_x(K+2) + \cancel{R_x(K+1)} + R_x(K)$$

$$R_y(k) = 3R_x(k) - R_x(k-2) - R_x(k+2)$$

$$R_x(k) = \mu_x^2 + \sigma^2 \delta(k)$$
$$= \delta(k)\big|_{\mathcal{N}(0,1)}$$
ACF for iid

$$R_y(k) = 3\delta(k) - \delta(k-2) - \delta(k+2)$$

PSD:

$$S_y(f) = \sum_k R_y(k) e^{-j2\pi kf}$$
$$= 3\cdot 1 - \underbrace{e^{-j4\pi f} - e^{j4\pi f}}_{-2\cos(4\pi f)}$$

$$e^{j\varphi} - e^{-j\varphi} = 2\cos(\varphi)$$

$$= 3 - 2\cos(4\pi f)$$

**18.2 (f)** A discrete-time WSS random process with mean $\mu_U = 2$ is input to an LSI system with impulse response $h[n] = (1/2)^n$ for $n \geq 0$ and $h[n] = 0$ for $n < 0$. Find the mean sequence at the system output.

$$U[n] \sim WSS$$

$$\mu_U = \mathbb{E}[U[n]] = 2$$

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\mu_Y(n) := \mathbb{E}[Y[n]] = h(n) * X(n)$$

$$\mu_Y(n) := \mathbb{E}[Y[n]] = h(n) * X(n)$$

$$= \mu_x \sum_j h(j)$$

$$Y[n] = h[n] * U[n]$$

$$\underline{\underline{\mathbb{E}[Y[n]] = \mu_U \cdot \sum_j^n h(j)}}$$

# Markov exercises

Saturday, November 22, 2025 10:45 AM

**22.1 (w)** A Markov chain has the states "A" and "B" or equivalently 0 and 1. If the conditional probabilities are $P[A|B] = 0.1$ and $P[B|A] = 0.4$, draw the



state probability diagram. Also, find the transition probability matrix.

22.1)

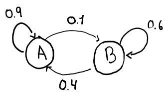
$P(A|B) = 0.1$

$P(B|A) = 0.4$

$$\bar{\bar{P}} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\bar{\bar{P}} = \begin{bmatrix} 1-0.1 & 0.1 \\ 0.4 & 1-0.4 \end{bmatrix}$$

$$\begin{array}{c|cc} & A & B \\ \hline A & 0.9 & 0.1 \\ B & 0.4 & 0.6 \end{array}$$




**22.2** (⌣) **(f)** For the state probability diagram shown in Figure 22.2 find the probability of obtaining the outcomes $X[n] = 0, 1, 0, 1, 1$ for $n = 0, 1, 2, 3, 4$, respectively.

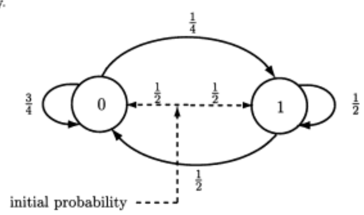



Figure 22.2: Markov state probability diagram for putting example.

$X[n] = 0, 1, 0, 1, 1$

$n = 0, 1, 2, 3, 4$

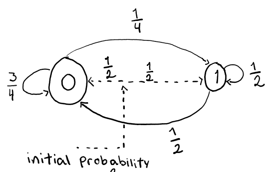



fig. 22.2

$$\bar{\bar{P}} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 3/4 & 1/4 \\ 1 & 1/2 & 1/2 \end{array}$$

$$P(X[n]) = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2}$$

$$= \frac{1}{128}$$

**22.3 (f)** For the state probability diagram shown in Figure 22.3 find the probabilities of the outcomes $X[n] = 0, 1, 0, 1, 1, 1$ for $n = 0, 1, 2, 3, 4, 5$, respectively and also for $X[n] = 1, 1, 0, 1, 1, 1$ for $n = 0, 1, 2, 3, 4, 5$, respectively. Compare the two and explain the difference.

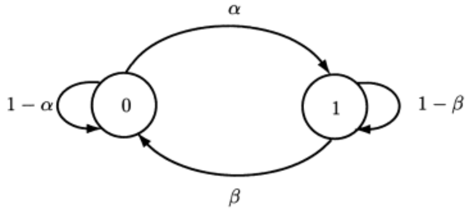
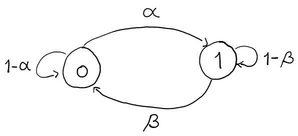



Figure 22.3: General two-state probability diagram.

$X_1[n] = 0,1,0,1,1,1 \quad$ for $n = 0,1,2,3,4,5$

$X_2[n] = 1,1,0,1,1,1 \quad$ for $n = 0,1,2,3,4,5$




$$\overline{\overline{P}} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \end{array}$$

Assuming : $P(0) = \begin{bmatrix} Q \\ 1-Q \end{bmatrix}$ ← Initial Probabilities

$P(X_1[n]) = Q \cdot \alpha \cdot \beta \cdot \alpha \cdot (1-\beta) \cdot (1-\beta)$

$P(X_2[n]) = (Q-1) \cdot (1-\beta) \cdot \beta \cdot \alpha \cdot (1-\beta) \cdot (1-\beta)$

Explain:

- They are two different sequences
- $X_1$ begins in '0' while $X_2$ begins in '1'

**22.5** (⌣) **(t)** In this problem we give an example of a random process that does *not* have the Markov property. The random process is defined as an *exclusive OR* logical function. This is $Y[n] = X[n] \oplus X[n-1]$ for $n \geq 0$, where $X[n]$ for $n \geq 0$ takes on values 0 and 1 with probabilities $1-p$ and $p$, respectively. The $X[n]$'s are IID. Also, for $n = 0$ we define $Y[0] = X[0]$. The definition of this operation is that $Y[n] = 0$ only if $X[n]$ and $X[n-1]$ are the same (both equal to 0 or both equal to 1), and otherwise $Y[n] = 1$. Determine $P[Y[2] = 1 | Y[1] = 1, Y[0] = 0]$ and $P[Y[2] = 1 | Y[1] = 1]$ to show that they are not equal in general.

Skal vise at $Y[n]$ ikke er Markov.

Det kan gøres ved at vise at:

$P(Y[2] = 1 \mid Y[1] = 1, Y[0] = 0)$

ikke er lig:

$P(Y[2] = 1 \mid Y[1] = 1)$

Hvis deres sandsynligheder ikke er ens så betyder det at de afhænger af noget mere end den tidligere state, og bryder Markov

Får fortalt $Y[0] = X[0]$.
Ud fra $P(Y[2] = 1 \mid Y[1] = 1, Y[0] = 0)$
kan vi se at $Y[0] = 0$

Ud fra XOR operationen: $Y[n] = X[n] \oplus X[n-1]$

| X(n) | X(n-1) | Y(n) |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

når vi så kigger på $Y[1]$ så kan vi refferere til XOR tabellen

index

| X(1) | X(1-1) | Y(1) |
|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ |
| ~~0~~ | ~~1~~ | ~~1~~ |
| 1 | 0 | 1 |
| ~~1~~ | ~~1~~ | ~~0~~ |

Det betyder $X(1) = 1$

Det samme nu for $Y(2)$

index $X(1)=1$ — index $Y(2)$ (vi ved den er = 1)

| X(2) | X(2-1) | Y(2) |
|---|---|---|
| ~~0~~ | ~~0~~ | ~~0~~ |
| 0 | 1 | 1 |
| ~~1~~ | ~~0~~ | ~~1~~ |
| ~~1~~ | ~~1~~ | ~~0~~ |

$X(2) = 0$

$P(X(2)=0) = \underline{1-P}$

Nu for $P[Y[2]=1 \mid Y[1]=1]$

Vil gerne finde sandsynligheden for de to sekvenser:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Starter med at finde $P(B)$

opstilles de mulige sekvenser for $Y(1) = 1$

$P(Y(1)=1) = P(X(0)=0) \cdot P(X(0)=1)$
$\quad + P(X(0)=1) \cdot P(X(0)=0)$

opstiller de mulige sekvenser for $Y(1)=1$

$$P(Y(1)=1) = P(X(0)=0)\cdot P(X(0)=1) + P(X(0)=1)\cdot P(X(0)=0)$$

$$= (1-p)\cdot p + p\cdot(1-p)$$

$$= 2\cdot(p(1-p))$$

Nu for $P(A\cap B)$
opstiller sekvenserne for $Y(2)=1 \cap X(1)=1$
hvor de to første
skal være anderledes
for hinanden

De to første skal være anderledes

$$P(X(2)=1\cap X(1)=1) = P(X(0)=0)\cdot P(X(1)=1)\cdot P(X(2)=0) + P(X(0)=1)\cdot P(X(1)=0)\cdot P(X(2)=1)$$

$$=(1-p)\cdot p\cdot(1-p) + p\cdot(1-p)\cdot p$$

$$= \underline{\underline{p(1-p)^2 + p^2(1-p)}}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$P(Y[2]=1|Y[1]=1) = \frac{\cancel{p}\cancel{(1-p)}^2 + p^{\cancel{2}}\cancel{(1-p)}}{2\cancel{p}\cancel{(1-p)}}$$

$$= \frac{(1-p)+p}{2}$$

$$= \frac{1-\cancel{p}+\cancel{p}}{2}$$

$$= \underline{\underline{\frac{1}{2}}}$$

$$P[Y[2]=1|Y[1]=1, Y[0]=0] = 1-p$$

$$P[Y[2]=1|Y[1]=1] = \frac{1}{2}$$

$$\underline{\underline{P[Y[2]=1|Y[1]=1, Y[0]=0] \neq P[Y[2]=1|Y[1]=1]}}$$

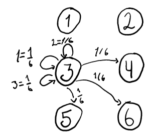
**22.7 (w)** A fair die is tossed many times in succession. The tosses are independent of each other. Let $X[n]$ denote the maximum of the first $n+1$ tosses. Determine the transition probability matrix. Hint: The maximum value cannot decrease as $n$ increases.

$$\bar{p} = [1/6 \;\; 1/6 \;\; 1/6 \;\; 1/6 \;\; 1/6 \;\; 1/6]$$

Det næste højeste slag

$$\bar{\bar{P}} = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 2 & 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 3 & 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 4 & 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 5 & 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 6 & 0 & 0 & 0 & 0 & 0 & 6/6 \end{array}$$

Det nuværende højeste slag



22.9

$$\bar{\bar{P}} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

**22.9** (⌣) **(w,c)** A digital communication system transmits a 0 or a 1. After 10 miles of cable a repeater decodes the bit and declares it either a 0 or a 1. The probability of a decoding error is 0.1 as shown schematically in Figure 22.11. It is then retransmitted to the next repeater located 10 miles away. If the repeaters are all located 10 miles apart and the communication system is 50 miles in length, find the probability of an error if a 0 is initially transmitted. Hint: You will need a computer to work this problem.

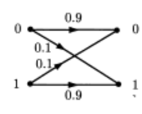



Figure 22.11: One section of a communication link.

22.9

**22.9** (☺) **(w,c)** A digital communication system transmits a 0 or a 1. After 10 miles of cable a repeater decodes the bit and declares it either a 0 or a 1. The probability of a decoding error is 0.1 as shown schematically in Figure 22.11. It is then retransmitted to the next repeater located 10 miles away. If the repeaters are all located 10 miles apart and the communication system is 50 miles in length, find the probability of an error if a 0 is initially transmitted. Hint: You will need a computer to work this problem.

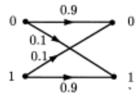
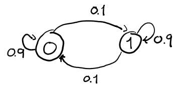



Figure 22.11: One section of a communication link.

$\bar{\bar{P}} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$




$X(0) = 0$

$P(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ← 100% den er 0 initially ← 0% den er 1 initially

$P(X(5)=0) = P(X(5)=0 \mid X(4)=1 \cdot X(4)=0)$
$\quad + P(X(5)=1 \mid X(4)=0 \cdot X(4)=1)$

$\bar{P}(1) = \begin{bmatrix} P(X(1)=0) \\ P(X(0)=1) \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} = \bar{P}(0) \cdot \bar{\bar{P}}$

$P(X(2)=0 \mid X(1)) = P(X(2)=0 \mid X(1)=0) \cdot P(X(1)=0)$
$\quad + P(X(2)=1 \mid X(1)=1) \cdot P(X(1)=1)$

$\quad = \bar{\bar{P}}_{0,0} \cdot \bar{P}_0(1) + \bar{\bar{P}}_{1,0} \cdot \bar{P}_1(1)$

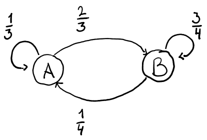
$\bar{P}(2) = \bar{P}(1) \cdot \bar{\bar{P}} = \bar{P}(0) \cdot \bar{\bar{P}}^2$

$\bar{P}(5) = \bar{P}(0) \cdot \bar{\bar{P}}^5$

Generalt set: $\bar{P}(n) = \bar{P}(0) \cdot \bar{\bar{P}}^n$

22.16

$\bar{\bar{P}} \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$



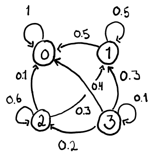

$\pi^T = [\pi_0, \pi_1] = \left[\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right]$

$\pi^T = \left[\frac{\frac{1}{4}}{\frac{2}{3}+\frac{1}{4}}, \frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{4}}\right]$

$= \left[\frac{3}{11}, \frac{8}{11}\right]$

22.25

$$\bar{\bar{P}} = \begin{array}{cc} & \text{Machine fail} \\ & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \text{active state} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.1 & 0.3 & 0.6 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{bmatrix} \end{array}$$




$P(x(n)=0) \geq 0.8$

$$\begin{aligned} P(x(n)=0) = \ & P(x(n)=0 \mid X(n-1)=3) \cdot P(x(n-1)=3) \\ & + P(x(n)=0 \mid X(n-1)=2) \cdot P(x(n-1)=2) \\ & + P(x(n)=0 \mid X(n-1)=1) \cdot P(x(n-1)=1) \\ & + P(x(n)=0 \mid x(n-1)=0) \cdot P(x(n-1)=0) \end{aligned}$$

$\bar{P}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ("Assume that initially all 3 machines are working")

$\bar{P}(x(1)=0) = \bar{P}(0) \cdot \bar{\bar{P}}^1$

$\bar{P}(x(n)=\ ) = \bar{P}(0) \cdot \bar{\bar{P}}^n$

$$\bar{\bar{D}}_p = \begin{bmatrix} \lambda_1^n & & & 0 \\ & \lambda_2^n & & \\ & & \lambda_3^n & \\ 0 & & & \lambda_4^n \end{bmatrix}$$

eigenvector

$\bar{\bar{P}}^n = \bar{V} \cdot \bar{\bar{D}}_p^n \cdot \bar{V}^{-1}$

# CDF exercises

Monday, November 24, 2025 9:56 AM

**13.2** (:) **(w)** Determine if the proposed conditional PDF

$$p_{Y|X}(y|x) = \begin{cases} c\exp(-y/x) & y \geq 0, x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is a valid conditional PDF for some $c$. If so, find the required value of $c$.

$$P_{Y|X}(x|y) = \begin{cases} ce^{-\frac{y}{x}}, & y \geq 0,\ x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$P(y(x))$ is valid if $\int P(y|x)\,dy = 1$ for all $x$

$$\int_0^\infty ce^{-y/x} = \left[-cx\,e^{-y/x}\right]_0^\infty$$

$$-cx \cdot e^{-\infty/x} - (-cx \cdot e^{-0/x}) = cx$$

$$\Downarrow$$

$$c = \frac{1}{x}$$

**13.4** (:) **(f)** If

$$p_{X,Y}(x,y) = \begin{cases} 2\exp[-(x+y)] & 0 \leq y \leq x, x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

find $p_{Y|X}(y|x)$.

$$P_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)}, & 0 \leq y \leq x,\ x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find $P_{Y|X}(y|x)$:

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{\int_{-\infty}^{\infty} P_{X,Y}(x,y)\,dx}$$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{\int_{-\infty}^{\infty} P_{X,Y}(x,y)\,dy}$$

$$e^a \cdot e^b = e^{a+b}$$

$$P_{X,Y}(x,y) = \begin{cases} 2e^{-x} \cdot e^{-y}, & \text{if } 0 \leq y \leq x,\ 0 \leq x \\ 0, & \text{otherwise} \end{cases}$$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{\int_{-\infty}^{\infty} P_{X,Y}(x,y)\,dx}$$

$$p_{Y|X}(y|x) = \frac{\quad}{\int_{-\infty}^{\infty} P_{X,Y}(x,y)\, dx}$$

$$= \frac{2e^{-x}\cdot e^{-y}}{\int_0^x 2e^{-x}\cdot e^{-y}\, dy}$$

$$= \frac{2e^{-x}\cdot e^{-y}}{\left[2e^{-x}\cdot e^{-y}\right]_0^x}$$

$$= \frac{2e^{-x}\cdot e^{-y}}{-2e^{-x}\cdot e^{-x} - (-2e^{-x}e^{-0})}$$

$$= \frac{2e^{-x}\cdot e^{-y}}{-2e^{-x}e^{-x} + 2e^{-x}}$$

$$= \frac{\cancel{2e^{-x}}\cdot e^{-y}}{\cancel{2e^{-x}}(1-e^{-x})}$$

$$= \underline{\underline{\frac{e^{-y}}{(1-e^{-x})}}}$$

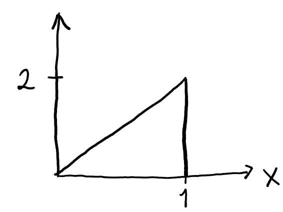
**13.5 (w)** Plot the joint PDF

$$p_{X,Y}(x,y) = \begin{cases} 2x & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

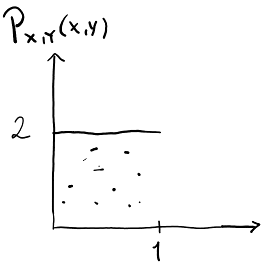


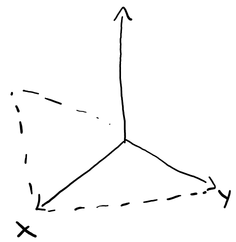
Next determine by inspection the conditional PDF $p_{Y|X}(y|x)$. Recall that the conditional PDF is just the normalized cross-section of the joint PDF.

$P_{x,y}(x,y)$




$P_{x,y}(x,y)$


$P_{X,Y}(x,y)$
2
1

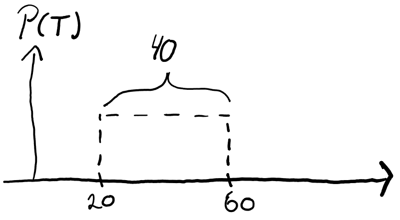

x
y

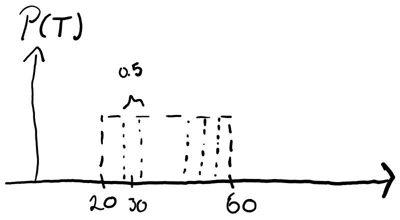
# PDF, CDF exercises

Monday, November 24, 2025 11:14 AM

**10.2** (☺) **(w)** The temperature in degrees Fahrenheit is modeled as a uniform random variable with $T \sim \mathcal{U}(20, 60)$. If $T$ is rounded off to the nearest $1/2°$ to form $\hat{T}$, what is $P[\hat{T} = 30°]$? What can you say about the use of a PDF versus a PMF to describe the probabilistic outcome of a physical experiment?

$T \sim U(20,60)$




$P(\hat{T} = 30°) = \frac{0.5}{40} = \frac{1}{80}$ (Sandsynlighed for at vores 0.5 slot rammer én af de 80 mulige slots i vores 40 span)

PDF:

Sandsynligheden for at man rammer et specifik punkt er 0, da der er uendelig mange punkter i en PDF, (PDF = Cont.)

PMF:

Sandsynligheden for at man rammer et specifik punkt er uniformly distrubueret da du kan talle antal slots, (PMF = disc.)

**10.4** (☺) **(w)** Which of the functions shown in Figure 10.33 are valid PDFs? If a function is not a PDF, why not?

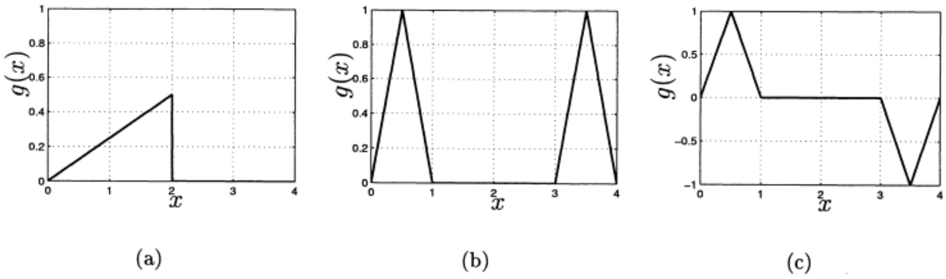



Figure 10.33: Possible PDFs for Problem 10.4.

(a) Den er ikke en PDF.
Den integreres ikke til 1

$A_{tre.} = \frac{1}{2} \cdot h \cdot b$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot 2$

$= \frac{1}{4} \cdot \frac{2}{1}$

$= \frac{2}{4}$

$= \frac{1}{2}$

(B) Er en PDF

$A_{trek.} = \frac{1}{2} \cdot h \cdot b$

$= \frac{1}{2} \cdot 1 \cdot 1$

$= \frac{1}{2}$

De begge har et areal som 0.5 og summen af deres arealer giver 1

(C) Den har negative Probability.
Kan ikke være PDF.

Troels: DER KAN IKKE
VÆRE NEGATIVE
SANDSYNLIGHEDDER.
DET ER DUMPE GRUNLAG
HVIS MAN TROR DET

**10.12** (◡̈) **(w)** A constant or DC current source that outputs 1 amp is connected to a resistor of nominal resistance of 1 ohm. If the resistance value can vary according to $R \sim \mathcal{N}(1, 0.1)$, what is the probability that the voltage across the resistor will be between 0.99 and 1.01 volts?

Vi vil integrere fra 0.99 → 1.01

$\mu = 1$, $a = 0.99$
$\sigma^2 = 0.1$, $b = 1.01$

$\sigma = \sqrt{\sigma^2} = 0.316$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F(x) = \int_a^b f(x)\,dx$$

$$f(x) = \frac{1}{\sqrt{0.1}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{\sqrt{0.1}}\right)^2}$$

$$F(x) = \int_{0.99}^{1.01} \frac{1}{\sqrt{0.1}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{\sqrt{0.1}}\right)^2} dx$$

$$= 0.025$$
$$= \underline{\underline{2.5\%}}$$

**10.21 (f,c)** If $X \sim \mathcal{N}(0, 1)$, determine the number of outcomes out of 1000 that you would expect to occur within the interval $[1, 2]$. Next conduct a computer simulation to carry out this experiment. How many outcomes actually occur within this interval?

Pre-discussion med johan:

Jeg forstår opgaven som at finde
hvad sansynligheden for at der er et outkome
og så ganger sandsynligheden med 1000

Post-discussion med johan:

Når man læser noget ligne
"hvor mange i dette sted"
så skal man tænke binomial

binom mean:

$$\mu = n \cdot p$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F(x) = \int_a^b f(x)\,dx$$

$\mu = 0$ $\quad$ $a = 1$

$\sigma^2 = 1$ $\quad$ $b = 2$

$\sigma = \sqrt{\sigma^2} = 1$

$$f(x) = \frac{1}{\sqrt{1}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-0}{\sqrt{1}}\right)^2}$$

$$F(x) = \int^{2} \frac{1}{\sqrt{1}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-0}{\sqrt{1}}\right)^2}$$

$$P(f(x)) = \int_1^2 \frac{1}{\sqrt{1}\ \sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-0}{\sqrt{1}}\right)^2}$$

$$= 0.1359$$

$$\mu = 1000 \cdot 0.1359$$

$$= \underline{\underline{135.9}}$$

**10.22** (◡̈) **(w)** If $X \sim \mathcal{N}(\mu, \sigma^2)$, find the CDF of $X$ in terms of $\Phi(x)$.

# Expectations formulas

Tuesday, November 25, 2025 10:52 AM

$$\mathbb{E}[x] = \sum_i x_i P(x_i)$$

$$Cov(x) = \mathbb{E}[(x_1 - \mathbb{E}[x_1]) \cdot (x_2 - \mathbb{E}[x_2])]$$

$$\mu(n) = \mathbb{E}[X(n)]$$

$$R_x(\tau) = \mathbb{E}[X(n) \cdot X(n+\tau)]$$

Sample autocorrelation function:

$$\hat{R}_x(k) = \frac{1}{N} \sum_{n=1}^{N-k} X(n) X(n+k) \qquad \text{for } k = 0, 1, \dots, N-1$$

Sample covariance:

$$\gamma_k = \frac{1}{N} \sum_{t}^{N-k} (X_t - \hat{X}) \cdot (X_{t+k} - \hat{X})$$