PROBLEM 1

By definition:

$$a_k = \frac{1}{T} \int_T f(t) e^{2\pi i \frac{kt}{T}} dt$$

> Consider the coefficient for $k=0$

$$a_0 = \frac{1}{T} \int_T f(t)\, dt \leftarrow \text{It is the actuall mean}$$

- We observe that the term a_0 of the Fourier series correspond to the mean value of the signal.

PROBLEM 2

$$\mathcal{F}\{e^{-a|x|}\} = \int_{-\infty}^{+\infty} e^{-a|x|} e^{-2\pi i k x} dx =$$

$$= \int_{-\infty}^{0} e^{ax} e^{-2\pi i k x} dx + \int_{0}^{+\infty} e^{-ax} e^{-2\pi i k x} dx =$$

$$= \int_{-\infty}^{0} e^{(-2\pi i k + a)x} dx + \int_{0}^{+\infty} e^{-(2\pi i k + a)} dx =$$

$$= -\int_{+\infty}^{0} e^{(-2\pi i k + a)(-t)} dt + \int_{0}^{+\infty} e^{-(2\pi i k + a)} dx =$$

$$= \int_{0}^{+\infty} e^{-(-2\pi i k + a)t} dt + \int_{0}^{+\infty} e^{-(2\pi i k + a)x} dx = (\text{see note}) =$$

$$= \frac{1}{-2\pi i k + a} + \frac{1}{2\pi i k + a} = \frac{1}{2\pi i k + a} - \frac{1}{2\pi i k - a}$$

NOTE: $\int_{0}^{+\infty} e^{-\alpha x} dx = \frac{1}{\alpha}$ if $\mathrm{Real}(\alpha) > 0$ $(\alpha \in \mathbb{C})$

PROBLEM 3

We can write $f(x)$ as follows:

$$f(x) = 2\Lambda(x/2) - \Lambda(x)$$

Therefore:

$$\mathcal{F}\{f(x)\}(k) = \mathcal{F}\{2\Lambda(x/2)\} - \mathcal{F}\{\Lambda(x)\} =$$

$$= 2\mathcal{F}\{\Lambda(x/2)\} - \mathcal{F}\{\Lambda\}(k) = 2 \cdot \frac{1}{(1/2)} \mathcal{F}\{\Lambda\}(k/(1/2)) - \mathrm{sinc}^2(k) =$$

Similarity theorem

$$= 4\,\mathrm{sinc}^2(2k) - \mathrm{sinc}^2(k)$$

$$\boxed{F(k) = 4\,\mathrm{sinc}^2(2k) - \mathrm{sinc}^2(k)}$$

PROBLEM 5

Let consider a simplier case:

$$g(x) = \mathrm{SINC}(x) * \mathrm{SINC}(x)$$

> Using the result of the convolution theorem:

$$\mathcal{F}\{g\} = \mathcal{F}\{\mathrm{SINC} * \mathrm{SINC}\} = \Pi \cdot \Pi = \Pi$$

However: $\mathcal{F}^{-1}\{\Pi\} = \mathrm{SINC} \Rightarrow g(x) = \mathrm{SINC}(x)$

→ Consequently, by induction, we have:

$$\boxed{f(x) = \mathrm{SINC}(x)}$$

PROBLEM 6

$$f(t) \longrightarrow \boxed{\mathcal{F}} \longrightarrow F(k)$$

- LINEARITY: YES

$$\mathcal{F}\{a f(x) + b g(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\} \quad \checkmark$$

- SHIFT-INVARIANT: No

$$\mathcal{F}\{f(x-\delta)\} = e^{-2\pi i \delta v} \cdot \mathcal{F}\{f\}(k) \neq \mathcal{F}\{f\}(k-\delta)$$

↳ No shift-invariant

- No CAUSAL AND HAS MEMORY:

→ Because the Fourier transform depends on both past and future events!

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i k t}\, dt$$

It goes over all time points

PROBLEM 7

→ Recall that the output of a Linear Shift-Invariant system CAN be described as follows:

$$y(t) = x(t) * h(t)$$

where $h(t)$ is the impulse response of the system.

> If we consider the input $x(t) = e^{2\pi i k t}$ then:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(x)\, e^{2\pi i k (t-x)}\, dx =$$

$$= e^{2\pi i k t} \int_{-\infty}^{+\infty} h(x)\, e^{-2\pi i x}\, dx = e^{2\pi i k t} H(k)$$

$$\Rightarrow \quad y(t) = H(k)\, e^{2\pi i k t}$$

Therefore, we observe that $x(t) = e^{2\pi i k t}$ IS an eigenfunction of any LSI system, with eigenvalue $H(k)$.

Problem 8

Let first apply the Fourier transform to the intensity:

Note that:

— WE USE THE DERIVATIVE THEOREM —

$$I(k) = \int_{-\infty}^{+\infty} i(t)\, e^{-2\pi k t}\, dt \Rightarrow \begin{cases} \dfrac{di}{dt} = 2\pi i k\, I(k) \\[2ex] \dfrac{d^2 i}{dt^2} = -4\pi^2 k^2\, I(k) \end{cases}$$

Then, applying the Fourier transform to the equation:

$$-4\pi^2 k^2 L\, I(k) + 2\pi i k R\, I(k) + \frac{1}{C} I(k) = 2\pi i k\, V(k)$$

This is also known as the complex Ohm's law !!

Solving now for $I(k)$:

$$I(k) = \frac{2\pi i k\, V(k)}{C^{-1} + 2\pi i k R - 4\pi^2 k^2 L}$$

Therefore:

$$i(t) = \int_{-\infty}^{+\infty} \frac{2\pi i k\, V(k)}{C^{-1} + 2\pi i k R - 4\pi^2 k^2 L}\, e^{2\pi i k t}\, dt$$