# CONTROL SYSTEM

A handbook guide to the basics of
Control systems engineering

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# Control System: A handbook guide to the basics of Control Systems Engineering


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*"We can't control systems or figure them out.*
*But we can dance with them!"— Donella Meadows*

# Contents

# Chapter 1
# Introduction to Control Systems

## 1.1 Objective

- To define a control system and describe some applications.
- To discuss the advantages of studying control systems.
- To know the historical developments that resulted in modern control theory.
- To identify the fundamental characteristics and setups of control systems.
- To describe the objectives of control system analysis and design.
- To understand the process of designing a control system.

## 1.2 Theory

### Introduction

Control systems are integrated in modern day life. Numerous applications are visible all around us: rockets ignite and the space shuttle lifts off into earth orbit; a metallic part is automatically machined in splashing cooling water; and a self-guided vehicle delivering material to workstations in an aerospace assembly plant glides along the floor in search of its destination.

Naturally, we are not the only sole inventors of self-regulating systems; they also exist in nature. Numerous control systems exist within our bodies, such as the pancreas, which regulates our blood sugar. When we are in a state of "fight or flight," our adrenaline levels rise along with our heart rate, increasing the amount of oxygen given to our cells. Our eyes track a moving object to maintain visual contact; our hands seize the object and precisely position it in a specified area.

Even the nonphysical realm appears to be self-regulating. Models demonstrating automatic regulation of student performance have been proposed. The model's input is the student's available study time, while the model's output is the grade. The model can be used to forecast how long it will take for a grade to climb if an unexpected increase in study time is available. Using this technique, you can assess whether more study during the final week of the term is worthwhile. These are just a few instances of the kind of automated systems that we can design and encounter in this world of technology and modernization.

### Control System Definition

A **control system** consists of subsystems and processes (or plants) assembled for the purpose of obtaining a desired output with desired performance, given a specified input. Figure 1.1 shows a control system in its simplest form, where the input represents a desired output.

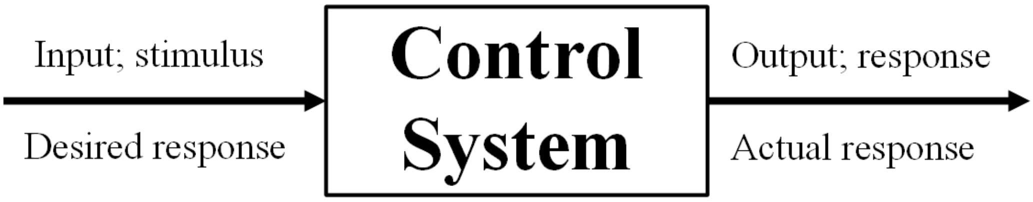


**Figure 1.1:** A control system's simplified description

**Two major measures of performance of a system:**

- Transient Response

> **Definition 1.1**
> *The **Transient Response** is that part of the response curve due to the system and the way the system acquires or dissipates energy. In stable systems it is the part of the response plot prior to the steady-state response.* ♣

- Steady-state error

> **Definition 1.2**
> *The **Steady-state error** is the difference between the input and the output of a system after the natural response has decayed to zero.* ♣

## Advantages of Control Systems

We build control systems for four primary reasons:

- Power amplification
- Remote control
- Convenience of input form
- Compensation for disturbances

## Open-loop vs Closed-loop Control System

An **open-loop control system** uses a controller and an actuator to obtain the desired response, as shown in 1.2.

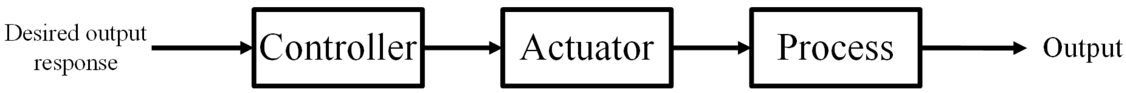


**Figure 1.2:** An Open-loop control system (without feedback)

> **Definition 1.3**
> *An **open-loop control system** utilizes an actuating device to control the process directly without using feedback.* ♣

**Examples of Open-loop system:**

- Toaster (input: time, output: color)
- Irrigation sprinkler (input: time, output: soil moisture)
- Stepper motors in inkjet printers (input: steps, output: position)
- Motor voltage speed control (input: voltage, output: speed)

In contrast to an open-loop control system, a **closed-loop control** system utilizes an additional measure of the actual output to compare the actual output with the desired output response. The measure of the output is called the **feedback signal**. A simple closed-loop feedback control system is shown in Figure 1.3. A feedback control system is a control system that tends to maintain a prescribed relationship of one system variable to another by comparing functions of these variables and using the difference as a means of control.

A feedback control system often uses a function of a prescribed relationship between the output and reference input to control the process. Often the difference between the output of the process under control and the reference input is amplified and used to control the process so that the difference is continually reduced. In general, the difference between the desired output and the actual output is equal to the error, which is then adjusted by the controller. The output of the controller causes the actuator to modulate the process in order to reduce the error. The system shown in Figure 1.3 is a negative feedback control system, because the output is subtracted from the input and the difference is used as the input signal to the controller.

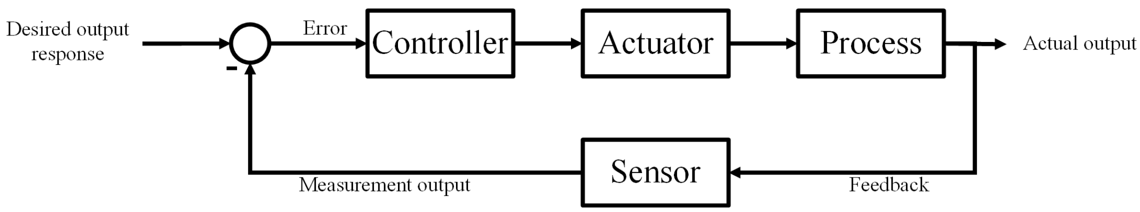


**Figure 1.3:** Closed-loop feedback control system (with feedback)

**Definition 1.4**

*A **closed-loop control system** uses a measurement of the output and feedback of this signal to compare it with the desired output (reference or command).* ♣

One of the many advantages of a closed-loop control system over open-loop control is the ability to reject external disturbances and improve measurement noise attenuation. External disturbances and measurement noise are inevitable in real-world applications and must be addressed in practical control system designs. As indicated in Figure 1.4, disturbances and measurement noise are added as external inputs into the block diagram. Table 1.1 shows the summary table of the advantages and disadvantages of Open-loop and closed-loop systems.

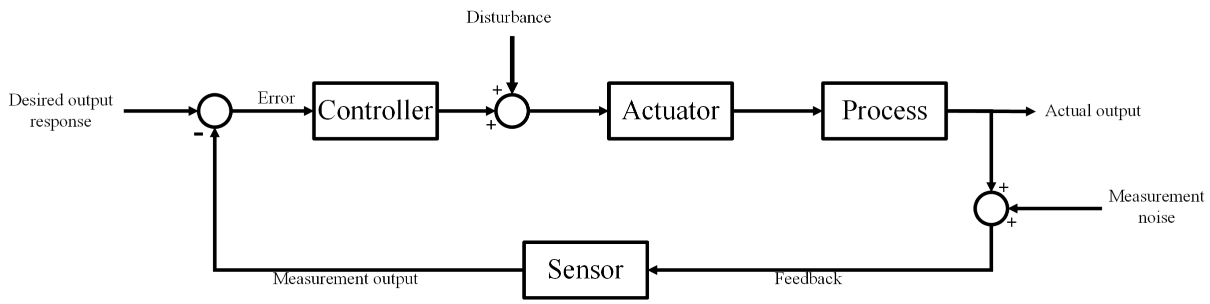


**Figure 1.4:** Closed-loop feedback system with external disturbances and measurement noise

| System | Advantages | Disadvantages |
|---|---|---|
| Open-loop | Simple<br>Inexpensive | Lower Accuracy<br>Higher sensitivity to noise |
| Closed-loop | Higher Accuracy<br>Less sensitivity to noise | Complex<br>Expensive<br>Stability |

**Table 1.1:** Advantages and Disadvantages of Open- and Closed-loop systems

**Examples of Closed-loop system:**

- Cruise control (input: throttle, measurement: speed, output: speed)
- Automated Vehicle. A simple block diagram of an automobile steering control system is shown in Figure 1.5. The desired course is compared with a measurement of the actual course in order to generate a measure of the error. This measurement is obtained by visual and tactile (body movement) feedback, as provided by the feel of the steering wheel by the hand (sensor). This feedback system is a familiar version of the steering control system in an ocean liner or the flight controls in a large airplane.

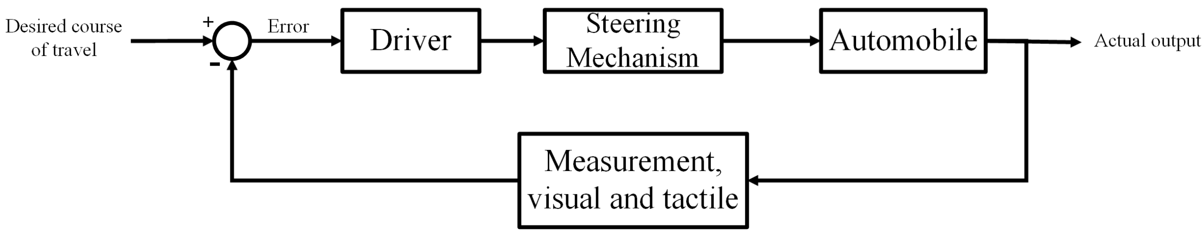


**Figure 1.5:** Automobile steering control system

## A History of Control Systems

Feedback control systems have a rich history and are known to be older than humanity. The first inhabitants of our planet have a variety of biological control systems built into them. Let us now tackle the evolution of human-designed control systems.

- **Liquid-Level Control.** Around 300 B.C., the Greeks began developing feedback systems. Ktesibios designed a water clock that worked by allowing water to trickle into a measuring container at a steady rate. The water level in the measuring container could be used to tell the time. The supply tank has to be kept at a constant level in order for water to trickle at a steady rate. This was accomplished with the use of a float valve, similar to the water-level control used in today's flush toilets. Philon of Byzantium used the idea of liquid-level control to an oil lamp shortly afterwards Ktesibios. The lamp was made up of two vertically stacked oil containers wherein the lower pan served as the flame's fuel supply and the lidded upper bowl served as a fuel storage for the pan beneath. The containers were linked by two capillary tubes and another tube called a vertical riser that was inserted into the oil in the lower pan to a level slightly below the surface. The base of the vertical riser was exposed to air as the oil burnt, forcing oil from the reservoir above to flow through the capillary tubes and into the pan. When the former oil level in the pan was restored, the transfer of fuel from the higher reservoir to the pan ceased, preventing air from entering the vertical riser. As a result, the system maintained a constant liquid level in the lower container.

- **Steam Pressure and Temperature Control.** Denis Papin's creation of the safety valve in 1681 marked the beginning of steam pressure regulation. The notion was expanded upon by weighing the valve top. If the boiler's upward pressure surpassed the weight, steam was discharged and the pressure dropped. If it did not exceed the weight, the valve did not open, and the pressure inside the boiler increased. As a result, the weight on the valve top determined the internal pressure of the boiler. Cornelis Drebbel of Holland developed a purely mechanical temperature control device for hatching eggs in the seventeenth century as well. A vial of alcohol and mercury with a floater was employed in the gadget. The floater was linked to a damper, which controlled the flame. A part of the vial was placed in the incubator to detect the heat produced by the fire. The alcohol and mercury expanded as the temperature rose, raising the floater, closing the damper, and decreasing the flame. Lower temperatures allowed the float to drop, allowing the damper to open and the flame to flare up.
- **Speed Control.** Edmund Lee applied speed control to a windmill in 1745. As the wind blew harder, the blades were pushed farther back, leaving less space. As the wind decreased, more blade area became available. In 1809, William Cubitt improved on the concept by separating the windmill sail into movable louvers. Moreover, James Watt devised the flyball speed governor in the eighteenth century to regulate the speed of steam engines. As the rotational speed of this device increases, two spinning fly balls rise. A steam valve attached to the flyball mechanism closes with ascending fly balls and opens with descending fly balls, allowing the speed to be controlled.
- **Stability, Stabilization, and Steering.** In the later part of the nineteenth century, control systems theory as we know it now began to form. In 1868, James Clerk Maxwell established the stability criterion for a third-order system based on differential equation coefficients. In 1874, Edward John Routh was able to extend the stability criterion to fifth-order systems by applying a suggestion from William Kingdon Clifford that Maxwell had previously overlooked. The topic for the Adams Prize in 1877 was "The Criterion of Dynamical Stability," and Routh responded with a work titled A Treatise on the Stability of a Given State of Motion, which received the

prize. The development of control systems was focused on ship steering and stabilization during the second half of the nineteenth century. In 1874, Henry Bessemer relocated the ship's salon to keep it steady by utilizing a gyro to detect motion and power supplied by the ship's hydraulic system. Other efforts were made to steady gun platforms as well as entire ships, with pendulums used to detect motion.

- **Twentieth-Century Developments.** In 1922, the Sperry Gyroscope Company built an automatic steering system that improved performance by including components of compensating and adaptive control. However, Nicholas Minorsky is credited with much of the basic theory utilized today to improve the performance of automatic control systems, specifically the proportional-plus-integral-plus-derivative (PID), or three-mode, controllers. During the late 1920s and early 1930s, the analysis of feedback amplifiers was developed by H. W. Bode and H. Nyquist at Bell Telephone Laboratories. On the other hand, Walter R. Evans created a graphical technique for plotting the roots of a characteristic equation of a feedback system whose parameters fluctuated across a specific range of values while working in the aerospace industry in 1948.

## Analysis and Objective

**Analysis** is the process by which a system's performance is determined. For example, we evaluate its transient response and steady-state error to determine if they meet the desired specifications. **Design** is the process by which a system's performance is created or changed. For example, if a system's transient response and steady-state error are analyzed and found not to meet the specifications, then we change parameters or add additional components to meet the specifications.

Three major objectives of systems analysis and design:

- Producing the desired **transient response**;
- Reducing **steady-state error**, and
- Achieving **stability**.

**Definition 1.5**

***Transient Response*** *is that part of the response curve due to the system and the way the system acquires or dissipates energy.* ♣

**Definition 1.6**

***Steady-State Response*** *is that part of the total response function due to the input and is typically of the same form as the input and its derivatives.* ♣

**Definition 1.7**

***Stability*** *is the characteristic of a system defined by a natural response that decays to zero as time approaches infinity.* ♣

## The Design Process

1. Transform requirements into a physical system
    - System concept
    - Qualitative description
    - Determine inputs and outputs
    - Description of the physical system
2. Draw a functional block diagram
    - Detailed layout
    - Describes the component parts of the system (function and hardware) and shows their interconnections
3. Create a schematic
    - Transform the physical system into a schematic diagram
    - Make approximations and neglect certain phenomena
    - Start simple, check assumptions later through analysis and simulation, if too simple, i.e., does not adequately account for observed behavior, add phenomena
    - Use knowledge of the physical system, physical laws, and practical experience
4. Develop a mathematical model (block diagram)
    - Use physical laws
    - Relationship between the inputs and outputs of the dynamic system
    - Linear, time-invariant (LTI) differential equations (DEs)
    - High order, nonlinear, time-varying, or partial DEs
    - Transfer functions (alternate representations of LTI DEs transformed using the Laplace transform)
    - State-space representation (alternate representation of nth-order DEs as $n$ simultaneous first-order DEs
    - Knowledge of parameter values
5. Reduce the block diagram
    - Interconnect subsystem models to form block diagrams of larger systems
    - Each block represents a mathematical description with dynamics, relations, inputs, outputs, and parameters
6. Analyze and design
    - Compare time response specifications and performance requirements
        - Test input waveform signals
        - Sensitivity analysis
    - Improve time response and performance
        - Adjusting system parameters
        - Design additional hardware
        - Minimize sensitivity over an expected range of environmental changes

## 1.3 Laboratory Experiment

### Module Exercises

1. In your own words, what is a control system and how does it contribute to the modern day world?
2. Provide at least three examples of Open-loop and Closed-loop systems you encounter in your daily lives.
3. Give three reasons for using feedback control systems and at least one reason for not using them.
4. Explain a typical control system analysis and system design task.
5. Name and distinguish the three major design criteria for control systems.
6. A temperature control system works by detecting the difference between the thermostat setting and the actual temperature and then opening a fuel valve proportionally to this difference. Create a functional closed-loop block diagram that includes the input and output transducers, the controller, and the plant. Identify the input and output signals of all previously described subsystems.

### Simulation Activity I

### Constructing Open- and Closed-loop System in Simulink®

### Objective:

- To be familiarized with the concepts of Open and Closed-loop System
- To know the purpose of controllers to the systems

### Procedures

A. **Constructing the Open-loop system**
   (a). Start the **MATLAB software**, open **Simulink®** and create a **blank model**.
   (b). Go to the **Library Browser** and click **Sources** and drag the **Step-block** to the canvas.
   (c). Double click the canvas and type **Transfer Function** to insert a transfer function block.
   (d). Double click the canvas and type **Scope** to insert a scope block.
   (e). For the Open-loop System in Simulink®, create a block diagram consisting of a **Step-response**, **Transfer Function**, and **Scope blocks** connected to each other.
   (f). Double click on the Transfer function block to change the parameters.
   (g). The given transfer function is $F(s) = \frac{2}{2s^2+5s+2}$. Modify the parameters then click **Apply** then **Ok**.
   (h). Double click the canvas and type **Mux**. Connect the **step input** and the **transfer function block** to **Mux**, then connect it to the **Scope block**.
   (i). Click **Run** to simulate the system.

(j). Double click the **scope block** to see the output plot. Record the result.

B. **Converting the open-loop system into closed-loop system**

(a). Double click the canvas then type **Sum block** then insert it to the canvas.

(b). Modify the sum block by double clicking the block and change it to **+-**.

(c). Insert two **gain blocks** by double clicking the canvas and typing **Gain block**. One will serve as the proportional gain block, and the other is the sensor or feedback of the system.

(d). Connect the **step block** to the positive input of the **sum block**. Then connect the **sum block** to the **proportional control gain block.**

(e). Connect the **proportional control gain block** to the **Transfer function block.**

(f). Create an additional connection between the transfer function block and the sensor block.

(g). Connect the sensor to the negative input of the sum block.

(h). Modify the controller gain by double clicking the block to have a steady-state error closer to the reference input which is 1.

(i). Click **Run** to simulate the system.

(j). Double click the **scope block** to see the output plot. Record the result.

## Simulation Activity II

## Understanding system with and without control

## Objective:

- To apply PID control in a second order system
- To differentiate the response of system with and without control

## Procedures

1. Download the **“MassSpringDamper_Simulink.slx”** file from the e-Learning Google Drive.
2. Open the Simulink file in the MATLAB Software.
3. After opening the file, label the model as **“Without Control”** by adding text.
4. Likewise, download the **“FeedbackModel_Multibody.slx**” file from the e-Learning Google Drive and open it in Simulink.
5. Merge the Feedback Model and Mass Spring Damper Simulink by copying and pasting it in one file. Add a **“With Control”** label on the Feedback Model. This model is the one we are going to modify with the PID controller block.
6. You can notice that some of the blocks are highlighted with red. This is because their values m, b, and k are not yet defined in the base workspace. Thus, define the value of the mass (m), damping coefficient (b) and spring constant (k) in MATLAB. In the command window, type

```
m= 2; %Mass value
b=128; %Damping Coefficient Value
k=3; %Spring Constant
```

7. 7. To modify the controller, double click the PID Controller Block to set the values of $k_p$, $k_i$, and $k_d$. Set **Proportional (P)= 450**, **Integral (I)=350**, and **Derivative (D)=10**. Also, set the **Filter Coefficient (N)=100**.
8. Connect the two models in a scope with 3 inputs: The Step Input, the position without control and the position with control as illustrated below.

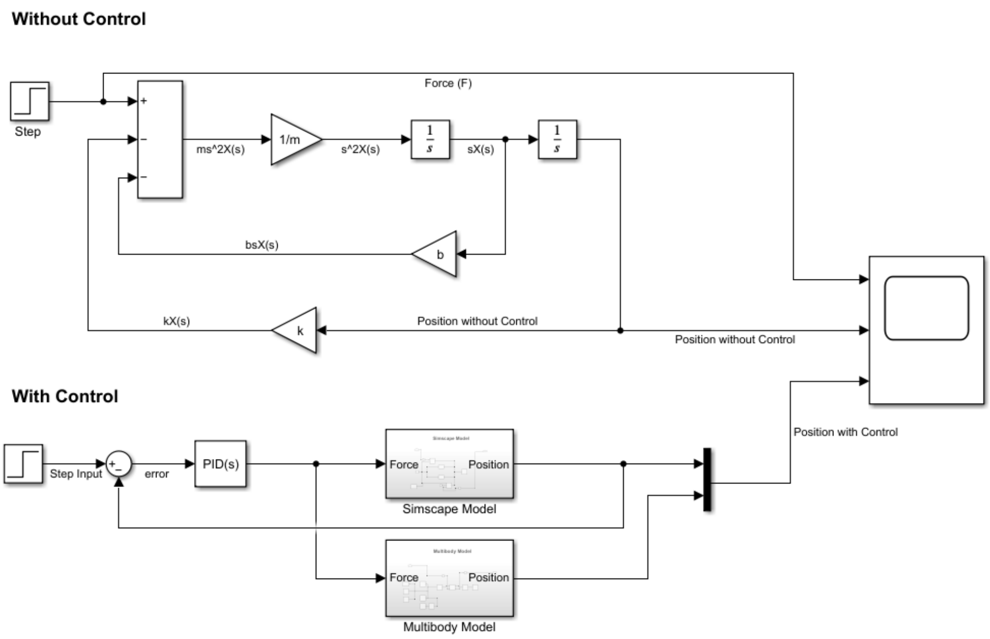


Figure 1.6

9. **Run** the simulation, visualize the simulation using Mechanics explorer and compare the results of the two models in the scope block.

## 1.4 Questions to Ponder

1. What is the effect of the proportional gain to the closed loop system?
2. What can you notice on the response of the system with and without control?
3. How do you think controllers improve system responses?
4. What is the difference between P-, PI-, and PID control? Explain briefly.

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# Chapter 2
# Modeling in the Frequency Domain

## 2.1 Objectives

- To find a time function's Laplace transform and inverse Laplace transform
- To obtain the transfer function from a differential equation and solve the differential equation using the transfer function
- To find the transfer function for linear, time-invariant systems

## 2.2 Theory

The design and analysis of control systems rely primarily on mathematical models of physical processes. Differential equations are used to characterize the dynamic behavior. We will discuss linearization approximations, which allow us to apply Laplace transform methods because most physical systems are nonlinear. To graphically represent the interconnections, the transfer function blocks can be grouped into block diagrams or signal-flow graphs. For building and analyzing complex control systems, block diagrams (and signal-flow graphs) are extremely useful and simple tools.

### Introduction

Mathematical Models of systems are qualitative ways to understand complex systems. It is necessary therefore to analyze the relationships between the system variables and to obtain a mathematical model. Differential Equations are usually used because the systems under consideration are dynamic in nature. Furthermore, if these equations can be linearized, then the Laplace transform can be used to simplify the method of solution. In practice, the complexity of systems and our ignorance of all the relevant factors necessitate the introduction of assumptions concerning the system operation. The approach to dynamic system modeling can be listed as follows:

1. Define the system and its components.
2. Formulate the mathematical model and fundamental necessary assumptions based on basic principles.
3. Obtain the differential equations representing the mathematical model.
4. Solve the equations for the desired output variables.
5. Examine the solutions and the assumptions.
6. If necessary, reanalyze or redesign the system.

## Laplace Transform Review

A system represented by a differential equation is difficult to model as a block diagram. Thus, we now lay the groundwork for the Laplace transform, with which we can represent the input, output, and system as separate entities. Further, their interrelationship will be simply algebraic.

### *Definition of Laplace Transform*

As $f$ is a function defined on the interval $t \geq 0$, then the integral is called the Laplace Transform of $f$, provided that the integral converges.

$$\mathscr{L}[f(t)] = F(s) \int_{0-}^{\infty} f(t)e^{-st}dt$$

When the given integral converges, the result is a function of $s$.

Furthermore, the inverse Laplace transform denoted by $\mathscr{L}^{-1}\{F(s)\}$, which allows us to find $f(t)$ given $F(s)$, is:

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi}j \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{-st}ds = f(t)u(t)$$

| **Item No.** | ***f(t)*** | **F(s)** |
|---|---|---|
| 1. | $\delta(t)$ | 1 |
| 2. | $u(t)$ | $\frac{1}{s}$ |
| 3. | $tu(t)$ | $\frac{1}{s^2}$ |
| 4. | $t^n u(t)$ | $\frac{n!}{s^{n+1}}$ |
| 5. | $e^{-at}u(t)$ | $\frac{1}{s+a}$ |
| 6. | $\sin \omega t u(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| 7. | $\cos \omega t u(t)$ | $\frac{s}{s^2+\omega^2}$ |

**Table 2.1:** Laplace Transform Table

### *Inverse Laplace Transform*

*f(t)* represents the inverse Laplace transform of *F(s)*. It is defined by the integral

$$\mathscr{L}^{-1}\{F(s)\} = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\, e^{st} ds = f(t)\, u(t)$$

where,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

is the unit step function. Multiplying *f(t)* to *u(t)* yields a time function that is zero for $t > 0$.

**Example 2.1** Evaluate $\mathscr{L}^{-1}\left(\dfrac{4}{s-5}\right)$.

**Solution**

$$\mathscr{L}^{-1}\left(\frac{4}{s-5}\right) = \frac{1}{4}\mathscr{L}^{-1}\left(\frac{1}{s-5}\right)$$
$$\mathscr{L}^{-1}\left(\frac{4}{s-5}\right) = \frac{1}{4}e^{5t}u(t)$$

### *Linearity of Inverse Laplace Transform*

The inverse Laplace Transform is a linear operator for any function *F(s)* and *G(s)* whose inverse transform exists as *f(t)* and *g(t)*, respectively.

$$\mathscr{L}^{-1}\{aF(s) + bG(s)\} = a\mathscr{L}^{-1}\{F(s)\} + b\mathscr{L}^{-1}\{G(s)\}$$
$$\mathscr{L}^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t)$$

**Example 2.2** Evaluate $\mathscr{L}^{-1}\left(\dfrac{6-2s}{s^2+4}\right)$.

**Solution**

$$\mathscr{L}^{-1}\left(\frac{6-2s}{s^2+4}\right) = \mathscr{L}^{-1}\left(\frac{6-2s}{s^2+4} - \frac{2s}{s^2+4}\right)$$
$$\mathscr{L}^{-1}\left(\frac{6-2s}{s^2+4}\right) = 3\mathscr{L}^{-1}\left(\frac{2}{s^2+2^2}\right) - 2\left(\frac{s}{s^2+2^2}\right)$$
$$\mathscr{L}^{-1}\left(\frac{6-2s}{s^2+4}\right) = 3\sin 2tu(t) - 2\cos 2tu(t)$$

## Laplace Transform Solution of a Differential Equation

In this section, we will use Laplace Transform in solving differential equations in evaluating quantities such as $\mathscr{L} = \left\{\dfrac{dy}{dt}\right\}$ and $\mathscr{L} = \left\{\dfrac{d^2y}{dt^2}\right\}$.

**Example 2.3** Evaluate: $\mathscr{L}\{f'(t)\}$ with $f(0) = 1$ where $F(s) = \dfrac{1}{s-1}$

**Solution**

$$\mathscr{L}\{f'(t)\} = sF(s) - 1$$
$$\mathscr{L}\{f'(t)\} = \frac{s}{s-1} - 1$$
$$\mathscr{L}\{f'(t)\} = \frac{1}{s-1}$$

## The Transfer Function

**Definition 2.1**

*The transfer function of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.* ♣

The transfer function of a system (or element) represents the relationship describing the dynamics of the system under consideration This function will allow separation of the input, system, and output into three separate and distinct parts, unlike the differential equation. The function will also allow us to algebraically combine mathematical representations of subsystems to yield a total system representation.

The transfer function is the ratio of output transform, $C(s)$ to the input $R(s)$,

$$\frac{C(s)}{R(s)} = G(s) = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \cdot + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \cdot + a_0\right)}$$

Notice that the denominator of the transfer function is identical to the characteristic polynomial of the differential equation. Also, we can find the output, $C(s)$ by using:

$$C(s) = R(s)G(s)$$

### *Electrical Network Transfer Functions*

In this section, we will apply the transfer function to the mathematical modeling of electric circuits including passive networks and operational amplifier circuits. Table 2.2 summarizes the components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

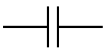
| Component | Voltage-Current | Current-Voltage | Voltage-charge | Impedance $Z(s) = \frac{V(s)}{I(s)}$ | Admittance $Y(s) = \frac{I(s)}{V(s)}$ |
|---|---|---|---|---|---|
|  Capacitor | $v(t) = \frac{1}{C}\int_0^1 i(\tau)\,d\tau$ | $i(t) = C\frac{dv(t)}{dt}$ | $v(t) = \frac{1}{C}q(t)$ | $\frac{1}{Cs}$ | $Cs$ |
|  Resistor | $v(t) = Ri(t)$ | $i(t) = \frac{1}{R}V(t)$ | $v(t) = R\frac{dq(t)}{dt}$ | R | $\frac{1}{R} = G$ |
|  Inductor | $v(t) = L\frac{di(t)}{dt}$ | $i(t) = \frac{1}{L}\int_0^1 v(\tau)\,d\tau$ | $v(t) = L\frac{d^2q(t)}{dt^2}$ | $Ls$ | $\frac{1}{Ls}$ |

**Table 2.2:** Summary for Electrical Network

**Example 2.4** Find the transfer function of Fig. 2.1

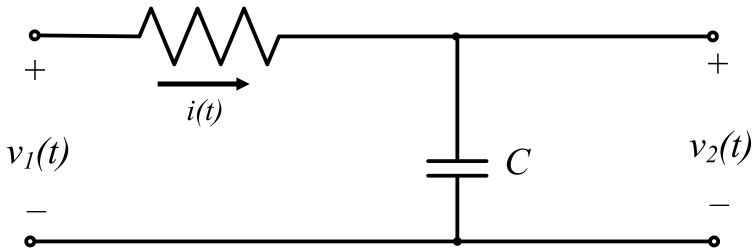



**Figure 2.1:** RC Circuit

**Solution**

*Using Kirchhoff's Law, we have the relation of the input voltage and the current*

$$V_1\,(s) = \left(1 + \frac{1}{Cs}\right) I(s),$$

*The output voltage is*

$$V_2\,(s) = I\,(s)\left(\frac{1}{Cs}\right)$$

$$G(s)\frac{V_2(s)}{V_1(s)} = \frac{1}{RCs+1} = \frac{1}{\tau RC+1}$$

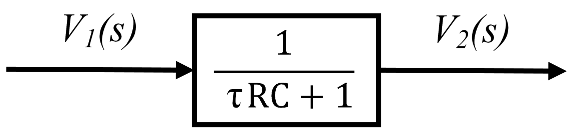



**Figure 2.2:** Block Diagram of RC electrical network

**Techniques for simplifying Electrical Network Problems:**

1. Redraw the original network showing all time variables, such as *v(t), i(t)*, and *vc(t)*, as Laplace transforms *V(s), I(s)*, and *VC(s)*, respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of DC circuits, where we represent resistors with their resistance value

**Transfer Function on Electrical Problems can be solved using:**

1. Simple Circuits via Mesh Analysis
   a. Transfer Function— Single Loop via the Differential Equation
   b. Transfer Function— Single Loop via Transform Methods
2. Simple Circuits via Nodal Analysis
   a. Transfer Function—Single Node via Transform Methods
3. Simple Circuits via Voltage Division
   a. Transfer Function—Single Loop via Voltage Division
4. Complex Circuits via Mesh Analysis
   a. Transfer Function— Multiple Loops
5. Complex Circuits via Nodal Analysis

   a. Transfer Function—Multiple Nodes
   b. Transfer Function—Multiple Nodes with Current Sources
6. Mesh Equations via Inspection

## *Translational Mechanical System Transfer Function*

Mechanical systems parallel electrical networks to such an extent that there are analogies between electrical and mechanical components and variables. Mechanical systems, like electrical networks, have three passive, linear components. Two of them, the spring and the mass, are energy-storage elements: one of them, the viscous damper, dissipates energy. The two energy-storage elements are analogous to the two electrical energy-storage elements, the inductor and capacitor. The energy dissipator is analogous to electrical resistance. Let us take a look at these mechanical elements, which are shown in Fig. 2.3. In the table, *K*, *fv*, and *M* are called spring constant, coefficient of viscous friction, and mass, respectively.

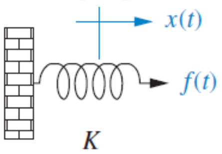
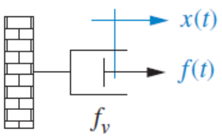
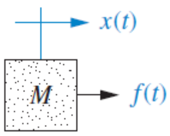
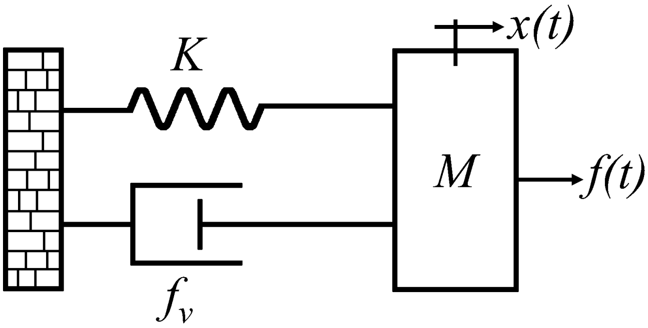
| Component | Force-velocity | Force-displacement | Impedence $Z_M(s) = F(s)/X(s)$ |
|---|---|---|---|
| Spring   | $f(t) = K\int_0^t v(\tau)d\tau$ | $f(t) = Kx(t)$ | $K$ |
| Viscous damper   | $f(t) = f_v v(t)$ | $f(t) = f_v \frac{dx(t)}{dt}$ | $f_v s$ |
| Mass   | $f(t) = M\frac{dv(t)}{dt}$ | $f(t) = M\frac{d^2x(t)}{dt^2}$ | $Ms^2$ |

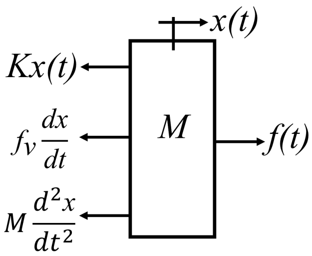
**Figure 2.3:** Summary for Translational Mechanical System

### Example 2.5 Transfer Function—One Equation of Motion

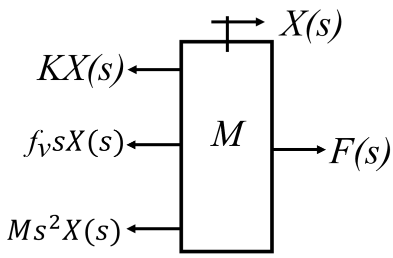
Find the transfer function, $\dfrac{X(s)}{F(s)}$, for the system below.




### Solution




**(a)** Free-body diagram of mass, spring, and damper system




**(b)** Transformed free-body diagram

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

$$\left(Ms^2 + f_v s + K\right)X(s) = F(s)$$

*Transfer function:*

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

## *Rotational Mechanical Transfer Function*

Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement. The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation. Fig. 2.5 shows the components along with the relationships between torque and angular velocity, as well as angular displacement. Notice that the symbols for the components look the same as translational symbols, but they are undergoing rotation and not translation. Also notice that the term associated with the mass is replaced by inertia.

The values of $K$, $D$, and $J$ are called *spring constant*, *coefficient of viscous friction*, and *moment of inertia*, respectively.

The concept of degrees of freedom carries over to rotational systems, except that we test a point of motion by rotating it while holding still all other points of motion.

**Note** *The number of points of motion that can be rotated while all others are held still equals the number of equations of motion required to describe the system.*

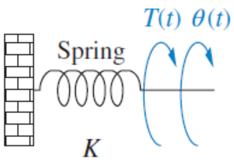
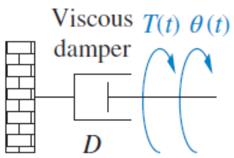
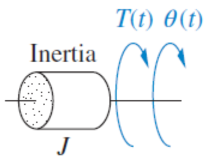
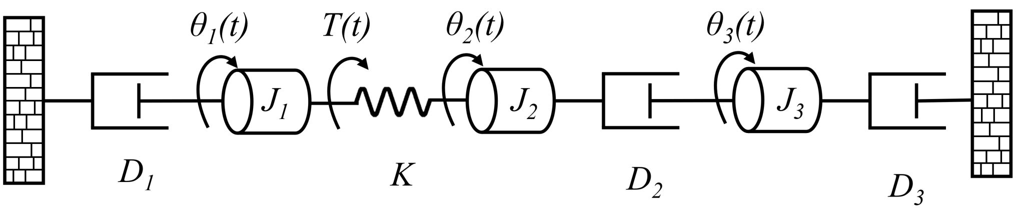
| Component | Torque-angular velocity | Torque-angular displacement | Impedence $Z_M(s) = T(s)/\theta(s)$ |
|---|---|---|---|
|   | $T(t) = K\int_0^t \omega(\tau)d\tau$ | $T(t) = K\theta(t)$ | $K$ |
|   | $T(t) = D\omega(t)$ | $T(t) = D\dfrac{d\theta(t)}{dt}$ | $Ds$ |
|   | $T(t) = J\dfrac{d\omega(t)}{dt}$ | $T(t) = J\dfrac{d^2\theta(t)}{dt^2}$ | $Js^2$ |

**Figure 2.5:** Summary for Rotational Mechanical System

**Example 2.6 Transfer Function of a three-degree of-freedom rotational system**



### Solution

*The equations will take on the following form:*

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_1\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_2(s)$$
$$- \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_1\end{bmatrix}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_2\end{bmatrix}\theta_2(s)$$
$$- \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_2\end{bmatrix}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_2(s)$$
$$+ \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_3\end{bmatrix}$$

*Hence,*

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) - 0\theta_3(s) = T(s)$$
$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) - D_2 s\theta_3(s) = 0$$
$$-0\theta_1(s) - D_2 s\theta_2(s) + (J_3 s^2 + D_3 s + D_2 s)\theta_3(s) = 0$$

## *Transfer Function for Systems with Gears*

Gears provide mechanical advantage to rotational systems. It allows us to match the drive system and the load—a trade-off between speed and torque. For many applications, gears exhibit backlash, which occurs because of the loose fit between two meshed gears. The drive gear rotates through a small angle before making contact with the meshed gear. The result is that the angular rotation of the output gear does not occur until a small angular rotation of the input gear has occurred. In this section, we idealize the behavior of gears and assume that there is no backlash.

The linearized interaction between two gears is depicted in Fig. 2.6. An input gear with radius $r_1$ and $N_1$ teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius $r_2$ and $N_2$ teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$.

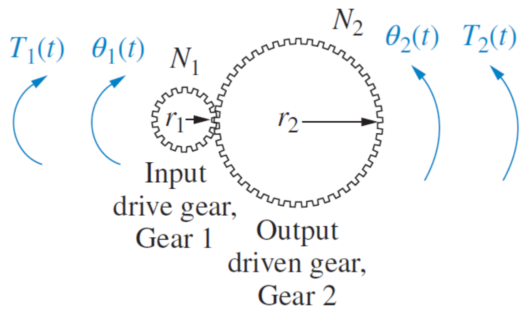



**Figure 2.6:** Gear system

As the gears turn on Fig. 2.6, the distance traveled along each gear's circumference is the same. Thus,

$$r_1\theta_1 = r_2\theta_2$$
$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

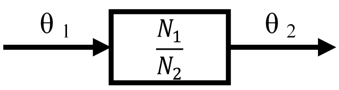
since the ratio of the number of teeth along the circumference is in the same proportion as the ratio of the radii.

**Note** *The ratio of the angular displacement of the gears is* ***inversely proportional*** *to the ratio of the number of teeth.*

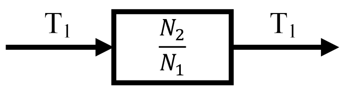
**Relationship of input torque and delivered torque:**

$$T_1\theta_1 = T_2\theta_2$$
$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

**Note** *Torques are* ***directly proportional*** *to the ratio of the number of teeth*




**(a)** Transfer function for angular displacement in lossless gears



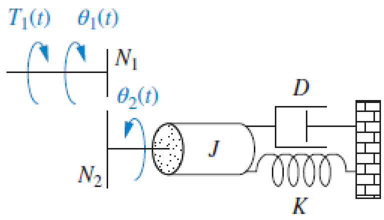

**(b)** Transfer function for torque in lossless gears

**Figure 2.7**

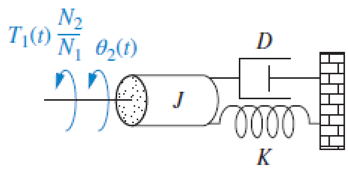
### Mechanical Impedances driven by Gears:

Fig. 2.7b can be reflected to the output by multiplying by $\frac{N_2}{N_1}$. The equation of motion can be written as:

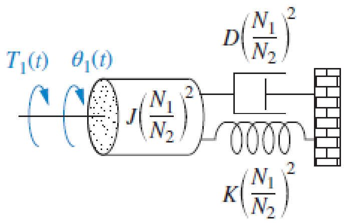
$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$




**(a)** Rotational system driven by gears




**(b)** Equivalent system at the output after reflection of input torque




**(c)** Equivalent system at the input after reflection of impedances

**Figure 2.8**

Using Fig. 2.8a to obtain $\theta_2(s)$ in terms of $\theta_1(s)$, we get

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

After simplification,

$$\left[J\left(\frac{N_1}{N_2}\right)^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

which suggests the equivalent system at the input and without gears shown in 2.8c. Thus, the load can be thought of as having been reflected from the output to the input.

**Note** *Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio where the impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft.*

$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2$$

## *Electromechanical System Transfer Functions*

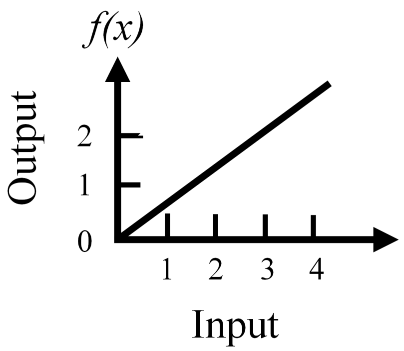
Electromechanical systems are hybrids of electrical and mechanical variables. Applications for systems with electromechanical components are robot controls, sun and star trackers, and computer tape and disk-drive position controls. In this section, we will apply the transfer function for an armature-controlled dc servomotor.

### *Electric Circuit Analogs*

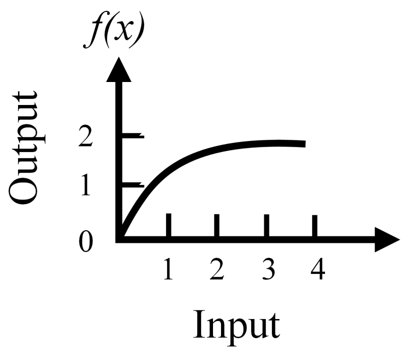
In this section, we show the commonality of systems from the various disciplines by demonstrating that the mechanical systems with which we worked can be represented by equivalent electric circuits. An electric circuit that is analogous to a system from another discipline is called an electric circuit analog. Analogs can be obtained by comparing the describing equations, such as the equations of motion of a mechanical system, with either electrical mesh or nodal equations. When compared with mesh equations, the resulting electrical circuit is called a **series analog**. When compared with nodal equations, the resulting electrical circuit is called a **parallel analog**.

## Nonlinearities

In this section, we will define linear and nonlinear and show how to distinguish the two.




**(a)** Linear system




**(b)** Nonlinear system

**Figure 2.9**

#### Properties of Linear System

1. Homogeneity
2. Superposition

**Definition 2.2**

***Saturation*** *is a non-linearity condition wherein the response does not change over an increasing input.* ♣

**Definition 2.3**

***Dead zone*** *is a non-linearity caused by a system not responding to a very low input signal.* ♣

**Definition 2.4**

***Backlash*** *is when the system does not respond to a small input range.* ♣

A designer can often make a linear approximation to a nonlinear system. Linear approximations simplify the analysis and design of a system and are used as long as the results yield a good approximation to reality. For example, a linear relationship can be established at a point on the nonlinear curve if the range of input values about that point is small and the origin is translated to that point. Electronic amplifiers are an example of physical devices that perform linear amplification with small excursions about a point.

## Linearization

In this section, we show how to obtain linear approximations to nonlinear systems in order to obtain transfer function.

1. Recognize the nonlinear component and write the nonlinear differential equation.
2. Linearize the nonlinear differential equation, and then we take the Laplace transform of the linearized differential equation, assuming zero initial conditions.
3. Separate input and output variables and form the transfer function

## 2.3 Laboratory Experiment

### Module Exercises

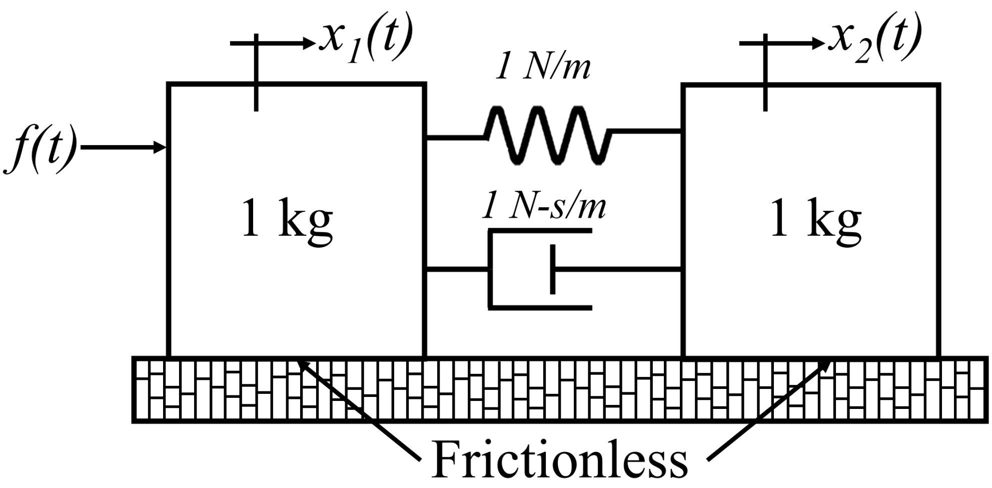
1. Derive the Laplace transform of the following:

   a. $\mathscr{L}(3t^4 - 2t^{\frac{3}{2}} + 6)$
   b. $\mathscr{L}(\cos^2 at)$

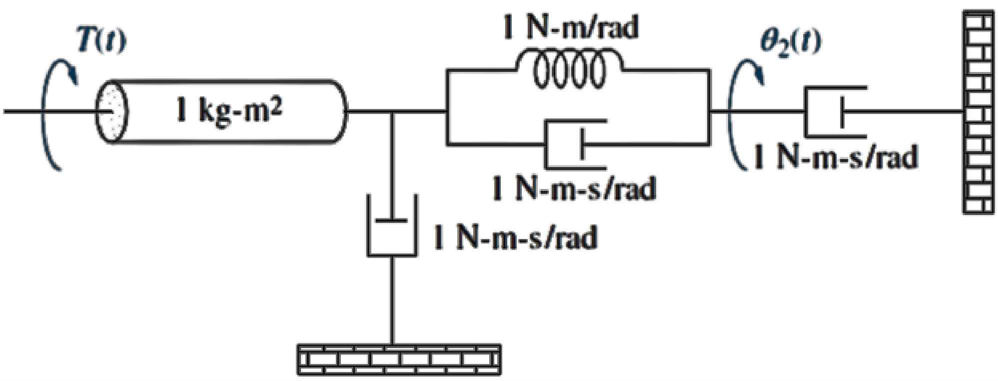
2. Derive the inverse Laplace transform of the following:

   a. $\mathscr{L}^{-1}\left(\dfrac{1}{s-3}\right)$
   b. $\mathscr{L}^{-1}\left(\dfrac{2s+3}{s^2+2s+5}\right)$

3. Find the transfer function, $G(s) = \dfrac{X_2(s)}{F(s)}$, for the translational mechanical network below:




4. For the rotational mechanical systems below, find the transfer function $G(s) = \dfrac{\theta_2(s)}{T(s)}$



## Simulation Activity

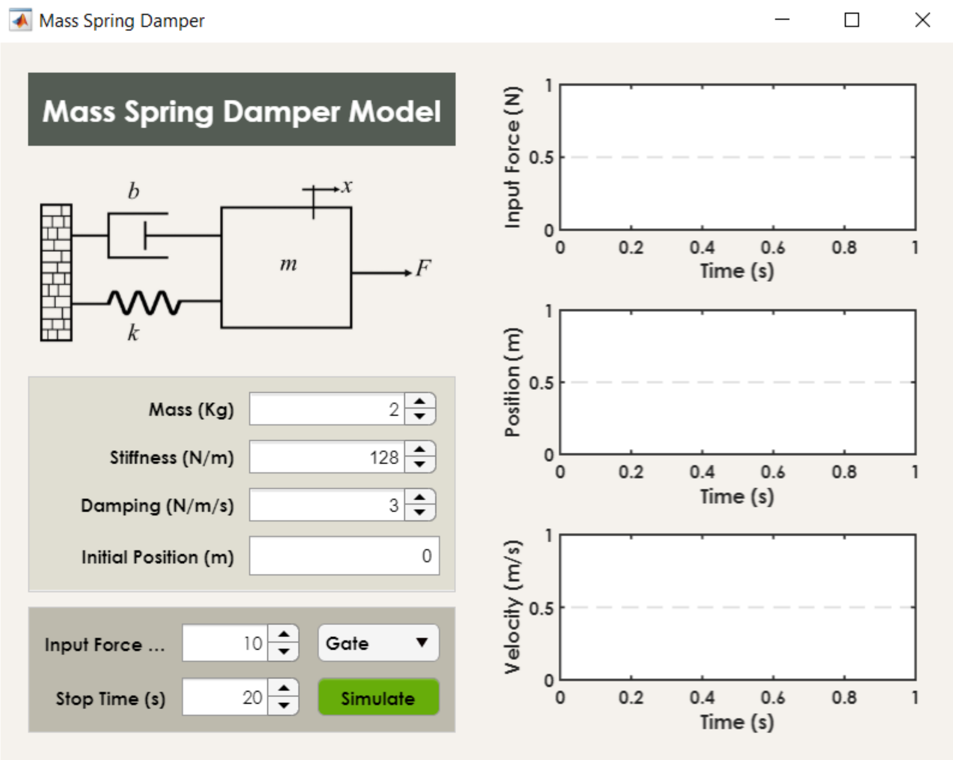
## The Mass-Spring-Damper System: Analysis of a Second-Order System

## Objective:

- To understand the relationship of various parameters of a mass spring damper system

## Procedures

1. Download the **"MassSpringDamper_MATLABApp"** file from the e-Learning Google Drive.The MATLAB Application is illustrated below:




2. With the help of the MATLAB App of a Mass Spring Damper System, simulate various set of values of the app's parameters, and record the result in the table below:

| Set | Mass | Stiffness | Damping | Initial Position | Input Force | Type of Input | Stop Time |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | | | | | | | |
| 3 | | | | | | | |
| 4 | | | | | | | |
| 5 | | | | | | | |
| 6 | | | | | | | |
| 7 | | | | | | | |
| 8 | | | | | | | |
| 9 | | | | | | | |
| 10 | | | | | | | |

## 2.4 Questions to Ponder

1. How can type of input affect the output of the system?
2. What relationship among parameters of the mass spring damper have you observed?

# Bibliography

Dorf, Richard C. and Robert H. Bishop (2017). Modern Control Systems. 13th ed. Hoboken, NJ: Pearson Education, Inc. ISBN: 9780134407623.

Nise, Norman (2015). Control Systems Engineering. 7th ed. Hoboken, NJ: John Wiley & Sons, Inc. ISBN: 9781118170519.

# Chapter 3
# Modeling in Time Domain

## 3.1 Objective

- To recognize the concepts of state variables, output equations, and state differential equations.
- To find the state-space representation of a linear, time-invariant system.
- To obtain a state-space representation from a transfer function.
- To obtain a transfer function from a state-space representation.

## 3.2 Theory

### Frequency Domain vs. Time Domain

- **Classical approach, or fequency domain technique**. This method is based on converting a system's differential equation to a transfer function, resulting in a mathematical model of the system that algebraically connects an output representation to an input representation. The primary disadvantage of this classical approach is its limited applicability, as it can only be applied to linear, time-invariant, or approximated systems.
- **Modern, or state-space, time-domain approach**. This is a unified approach to modeling, analyzing, and designing a wide variety of systems. The state-space approach, for example, can be used to represent nonlinear systems with backlash, saturation, and dead zone.

### The State-Space Representation

A state-space representation, therefore, consists of:

- **State Equation**. A set of n simultaneous, first-order DEs that expresses the time derivatives of the n states of a system as linear combinations of the states and inputs.

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{3.1}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}; A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}; B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{bmatrix}$$

- **Output Equation**. An equation that expresses the measured output variables of a system as linear combinations of the states and inputs.

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \tag{3.2}$$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}; C = \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{p,1} & \cdots & c_{p,n} \end{bmatrix}; D = \begin{bmatrix} d_{1,1} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots \\ d_{p,1} & \cdots & b_{p,m} \end{bmatrix}$$

where

x = state vector

ẋ = derivative of the state vector with respect to time

y = output vector

u = input vector

A = system matrix

B = input matrix

C = output matrix

D = feedforward matrix

- **System variable**. Any variable that responds to an input or initial conditions in a system.
- **State variable**.The smallest set of variables that determine or represent the variables in a system. It is the particular subset of all possible system variables.
- **State vector**. A vector whose elements are the state variables.

## Finding the State-space Representation of LTI Systems

Steps in solving the State-space representation of a linear, time-invariant system.

1. Form the equation of motion (i.e electrical, translational mechanical, rotational mechanical systems, or systems with gears)
2. Define the set/s of state variables which can represent the variables in a system and can be a first differential equation.
3. Rename all the system variables including state variables, input variable and output variable (given) for easy manipulation and identification.
4. Differentiate and manipulate the system variables to form the state vector.
5. Determine state equation/s and output equation/s using state vectors through the matrix form.

**Example 3.1** Find the state equations for the translational mechanical system shown in figure below:

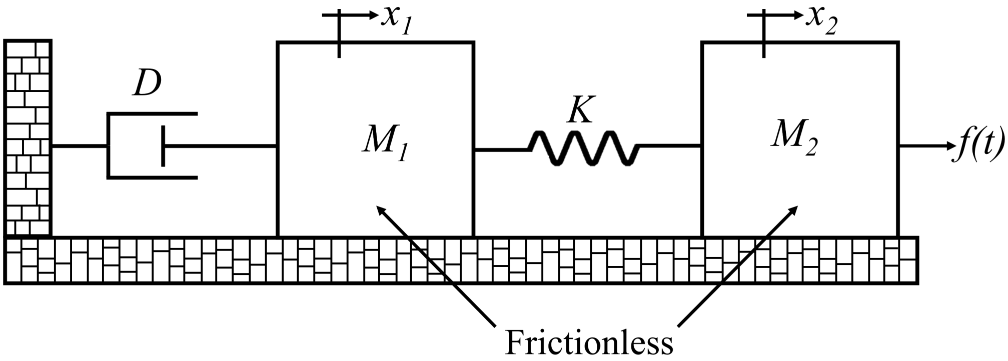



**Figure 3.1:** Translational mechanical system

## Solution

1. *Form the equation of motion.*

   *In this case there will be two equations since it has 2 degrees of freedom.*

$$M_1\ddot{x}_1(t) + D_1\dot{x}_1(t) + K_1x_1(t) - K_2x_2(t) = 0$$
$$M_2\ddot{x}_2(t) - K_1x_1(t) + K_2x_2(t) = f(t)$$

2. *Define the sets of state variables which can represent the variables in a system and can be a first differential equation.*

   *State variables:*

$$x_1(t) \rightarrow \dot{x}_1(t)$$
$$x_2(t) \rightarrow \dot{x}_2(t)$$

3. *Rename all the system variables including state variables, input variable and output variable (given) for easy manipulation and identification.*

   *Because the output equation is not required in this problem, we will only rename the state variables and input variables.*

*State variables:*

$$x_1(t) = Z_1$$
$$\dot{x}_1(t) = Z_2$$
$$x_2(t) = Z_3$$
$$\dot{x}_2(t) = Z_4$$

*Input variable:*

$$f(t) = u_1(t)$$

4. *Differentiate and manipulate the system variables to form the state vector.*

$$\begin{aligned}
\dot{x}_1(t) &= \dot{Z}_1 = Z_2 \\
\ddot{x}_1(t) &= \dot{Z}_2 = -\frac{K}{M_1}Z_1 - \frac{D}{M_1}Z_2 + \frac{K}{M_1}Z_3 \\
\dot{x}_2(t) &= \dot{Z}_3 = Z_4 \\
\ddot{x}_2(t) &= \dot{Z}_4 = \frac{K}{M_2}Z_1 - \frac{K}{M_2}Z_3 + \frac{1}{M_2}u_1
\end{aligned}$$

5. *Determine state equation/s by using the matrix form.*

*State equation:*

$$\dot{x} = Ax + Bu \tag{3.3}$$

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \\ \dot{Z}_3 \\ \dot{Z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M_1} & -\frac{B}{M_1} & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & -\frac{K}{M_2} & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} u_1$$

## Transfer Function to State-Space Representation

Steps in converting a Transfer Function to State-Space Representation with a constant term in numerator.

Consider the given transfer function:

$$\frac{C(s)}{R(s)} = \frac{100}{s^3 + 10s^2 + 7s + 100}$$

1. Convert the given transfer function to its differential equation by cross multiplication.

$$C(s)[s^3 + 10s^2 + 7s + 100] = 100R(s)$$

2. Take the inverse Laplace transform of both sides, assuming zero initial conditions.

$$\mathcal{L}^{-1}\{[C(s)[s^3 + 10s^2 + 7s + 100] = 100R(s)\}$$
$$\dddot{c}(t) + 10\ddot{c}(t) + 7\dot{c}(t) + 100c(t) = 100r(t)$$

3. Select the output as the state variable and get the successive derivatives.

State variables:

$$x_1 = c$$
$$x_2 = \dot{c}$$
$$x_3 = \ddot{c}$$

Input variable:

$$y(t) = c$$

4. Differentiate the set of state variables on both sides, and obtain the state and output equations by substituting the values from the differentiation. Represent the equations in state-space in phase-variable form.

$$\begin{aligned}
\dot{x}_1 &= \dot{c} = x_2 \\
\dot{x}_2 &= \ddot{c} = x_3 \\
\dot{x}_3 &= \dddot{c} = -100x_1 - 7x_2 - 10x_3 + 100r \\
y &= c = x_1
\end{aligned}$$

In vector matrix form,

State equation:

$$\dot{x} = Ax + Bu \tag{3.4}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -100 & 7 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} r$$

**Steps in converting a Transfer Function to State-Space Representation with a polynomial in the numerator**.

Consider the given transfer function:

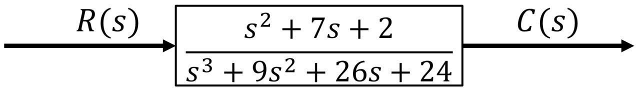



**Figure 3.2:** Transfer function

1. Separate the numerator and denominator as two cascaded transfer functions.

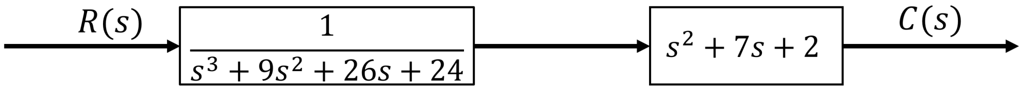



**Figure 3.3:** Decomposed transfer function

2. Convert the first transfer function with the denominator only to phase-variable representation in state-space that yields to a state equation.

$$\frac{C(s)}{R(s)} = \frac{1}{s^3+9s^2+26s+24}$$
$$C(s)(s^3+9s^2+26s+240 = R(s)$$
$$\mathcal{L}^{-1}\{[C(s)(s^3+9s^2+26s+240 = R(s)\}$$
$$\dddot{c}(t)+9\ddot{c}(t)+26\dot{c}(t)+24c(t) = r(t)$$

State variables:

$$x_1 = c = y$$
$$x_2 = \dot{c}$$
$$x_3 = \ddot{c}$$

Differentiate both sides,

$$\dot{x}_1 = \dot{c} = x_2$$
$$\dot{x}_2 = \ddot{c} = x_3$$
$$\dot{x}_3 = \dddot{c} = -24x_1 - 26x_2 - 9x_3 + r$$

State equation:

$$\dot{x} = Ax + Bu \tag{3.5}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

3. The second transfer function with the numerator only yields to:

$$Y(s) = C(s) = (b_2 s^2 + b_1 s + b_0)X_1 s$$
$$C(s) = (s^2 + 7s + 2)X_1 s$$

4. Take the inverse Laplace transform with zero initial conditions, then state the variables to conform to an output equation.

$$c(t) = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$$

State the variables,

$$x_1 = x_1$$
$$\dot{x}_1 = x_2$$
$$\ddot{x}_1 = x_3$$

Then,

$$y = x_3 + x_2 + x_1 = c(t)$$

5. Generate the output equation.

$$y = \begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

## State-Space Representation to Transfer Function

The following are the standard form of *State Equation* and *Output Equation* from the State-Space Representation knowing that ***A, B, C,***and ***D*** are matrices.

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$
$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$$

To convert the state-space representation into transfer function having the standard form of $G(s) = \frac{Y(s)}{U(s)}$, we need to find the Laplace transform of the state and output equations and solve for $X(s)$.

In solving for $X(s)$ we will obtain the final equation for the transfer function matrix that relates the input function $Y(s)$ to output function $U(s)$:

$$Y(s) = [C(sI - A)^{-}1B + D]U(s)$$

Where,

$$\boldsymbol{I} = \text{identity matrix of } s$$

**Example 3.2** Convert the state and output equations to a transfer function.

$$x = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$
$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} x$$

## Solution

1. *Substitute the values and matrices to the equation and perform the equation.*

$$\frac{Y(s)}{U(s))} = [C(sI - A)^{-}1B + D]$$
$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} \left[ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} \left[ \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix} \right]^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

2. *For a 2x2 matrix, solve the inverse of the matrix through co factor for the numerator and determinant for the denominator.*

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} \frac{\begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}}{s^2 + 4s + 6} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

3. *Solve the equation to arrive at the final transfer function.*

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} \frac{\begin{bmatrix} 2s \\ 8 \end{bmatrix}}{s^2 + 4s + 6}$$
$$\frac{Y(s)}{U(s)} = \frac{3s + 5}{s^2 + 4s + 6}$$

# 3.3 Laboratory Experiment

## Module Exercises

1. Find the state equations and output equations in vector matrix form of the following transfer functions below. Solve each analytically and cross check using MATLAB.

   a. $G(s) = \frac{10s+7}{s^3+5s^2+7s+2}$

   b. $G(s) = \frac{s+4}{s^2+3s+2}$

   c. $G(s) = \frac{8}{s^3+5s^2+11s+6}$

   d. $G(s) = \frac{s+2}{s^2+7s+12}$

   e. $G(s) = \frac{s^2+12s+8}{s^4+4s^3+39s^2+5s+108}$

2. Find the transfer function of the state space equations below. Solve it analytically and cross check using MATLAB.

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 3 & -5 & 5 \\ 0 & 1 & 2 \end{bmatrix} B = \begin{bmatrix} 14 \\ 0 \\ 0 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

3. Convert the state-space form below to a transfer function without using the direct MATLAB command "ss2tf".

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 4 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ -1 & -6 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 12 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

## Simulation Activity

## Generating the system response of a state space model

## Objective:

- To find the state space model of a translational mechanical system
- To observe the system response of a state space model

## Procedures

1. Find the state space representation of the translational mechanical system below analytically/-mathematically.

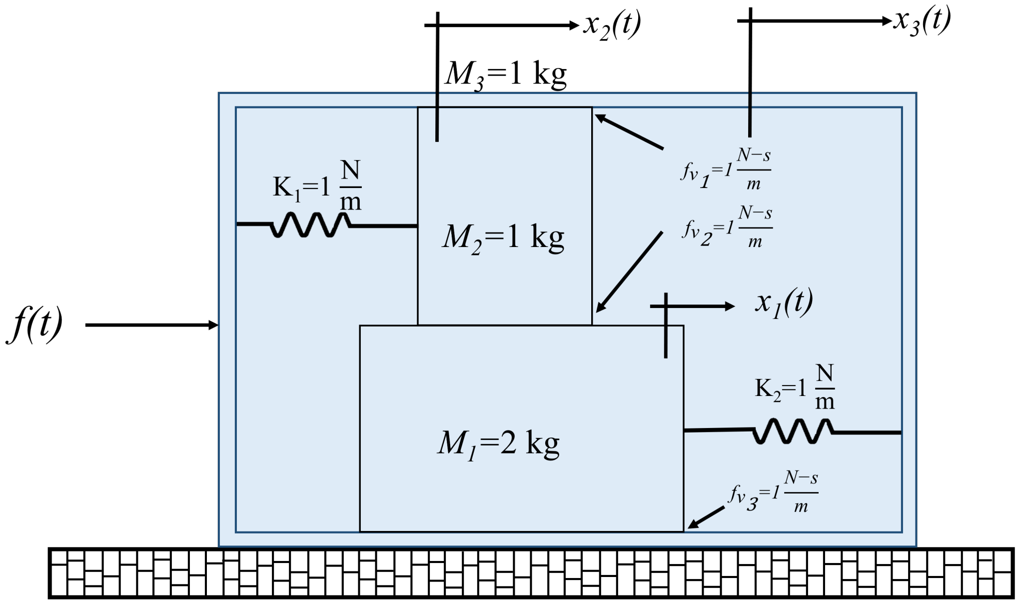



**Figure 3.4:** Translational Mechanical System

2. After obtaining the state space representation of the system, start the **MATLAB software**
3. Open **Simulink** and create a blank model.
4. Go to the **Library Browser** and click **Sources**.
5. Drag the Step block to the Simulink editor canvas.
6. Double click on the canvas and type State-space to insert a transfer function block.
7. Double click on the **State space block** to change the parameters.
8. Modify the parameters (A,B, C, D) based on the obtained state space representation of the system above. *Hint : Separate the rows with semicolons to create the matrix.*
9. Click **Apply** then **Ok**.
10. Double click on the canvas and type **Scope** to insert a **Scope** block.
11. Interconnect the blocks by dragging the cursor from the output of one block to the input of another block.
12. Click **Run** to simulate the model.
13. To view the output plot, double click the scope.

## 3.4 Questions to Ponder

1. What are the factors that influence in choosing state variables in a system?
2. Give an advantage of using the transfer function approach instead of the state space approach.

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Dorf, Richard C. and Robert H. Bishop (2017). Modern Control Systems. 13th ed. Hoboken, NJ: Pearson Education, Inc. ISBN: 9780134407623.

Nise, Norman (2015). Control Systems Engineering. 7th ed. Hoboken, NJ: John Wiley & Sons, Inc. ISBN: 9781118170519.

# Chapter 4
# Time Response

## 4.1 Objective

- To determine a control system's time response, using the poles and zeros of transfer functions.
- To describe first-order systems' time response quantitatively.
- To formulate the general response of second-order systems with the given pole location
- To get the natural frequency and damping ratio of a second-order system
- To find the settling time, peak time, percent overshoot, and rise time for an underdamped second-order system

## 4.2 Theory

### Forced response and Natural Response

The output response of a system is the summation of the forced response and natural response.

1. **Forced response *(particular solution)***

   This response is also referred to as steady-state response. It is simply the part of the total response due to the input. In addition, it has typically the same form as the input and its derivatives.

2. **Natural response *(homogeneous solution)***

   The natural response or homogeneous solution is the part of the total response function due to the system and the way the system acquires or dissipates energy.

### Poles, Zeros, and their relationship with a system's time response

1. **Poles of a transfer function**

   The poles of a transfer function are values of the Laplace transform variable, $s$, for which the transfer function becomes infinite, or any roots of the denominator that are common to the roots of the numerator.

   **Example:**

$$G(s) = \frac{s+4}{(s+2)(s+5)}$$

   **Solution:**

   Equate the denominator to zero.

$$\text{Poles: } s = -2; s = -5$$

2. **Zeros of a transfer function**

The zeros of a transfer function are values of the Laplace transform variable, $s$, for the transfer function becomes zero, or any roots of the numerator that are common to the roots of the denominator.

**Example:**

$$G(s) = \frac{s+4}{(s+2)(s+5)}$$

**Solution:**

Equate the denominator to zero.

$$\text{Poles: } s = -4;$$

**Example 4.1** Determine the poles, zeros, and system response of the system below if the input is a step function,$R(s) = \frac{1}{s}$.

$$G(s) = \frac{s+2}{s+5}$$

**Solution**

1. *First find the poles and zeros of the given transfer function.*
   *To get the pole, equate the denominator to zero.*

$$s + 5 = 0$$
$$s = -5 \leftarrow Pole$$

   *To get the zero, equate the numerator to zero.*

$$s + 2 = 0$$
$$s = -2 \leftarrow Zero$$

2. *To get the system response, multiply the transfer function by a step function,* $R(s) = \frac{1}{s}$.

$$C(s) = G(s)R(s)$$
$$C(s) = \left(\frac{s+2}{s+5}\right)\left(\frac{1}{s}\right)$$
$$C(s) = \frac{s+2}{s(s+5)}$$

3. *Next, find the partial fraction expansion of* $C(s) = \frac{s+2}{s(s+5)}$.

$$C(s) = \frac{s+2}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

*when* $s = 0$,

$$A = \frac{s+2}{s+5} = \frac{0+2}{0+5} = \frac{2}{5}$$

*when* $s = -5$,

$$B = \frac{s+2}{s} = \frac{-5+2}{-5} = \frac{2}{3}$$

*Therefore,*

$$C(s) = \frac{\frac{2}{5}}{s} + \frac{\frac{2}{3}}{s+5}$$

4. *Find the inverse laplace transform of* $C(s) = \frac{\frac{2}{5}}{s} + \frac{\frac{2}{3}}{s+5} = \frac{2}{3}$.

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

## Time Response of First-order Systems without Zeros

A first order system without zeros is illustrated by the transfer function shown below.

$$G(s) = \frac{a}{s+a}$$

When the input is a unit step ($R(s) = \frac{1}{s}$), the Laplace transform of the step response is :

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

By taking the inverse laplace transform, the step response of the system is:

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

### Transient response performance specifications of first-order systems

1. **Time constant**

   The first time response specification of a first-order system is time constant ($T_c$). This is the time for $e^{-at}$ to decay to 37 % of its original value. It can be computed using the quantity :

$$T_c = \frac{1}{a}$$

**Definition 4.1**

***Time constant*** *is the time it takes for the output to exhibit a 63.2 % change due to a step input. In other words, time constant is the speed at which a system responds to initial conditions.* ♣

2. **Rise time**

Another parameter is rise time $T_r$ which refers to the time required for the system step response to go from 10% to 90% (0.1 to 0.9) of its final value.

$$T_r = \frac{2.2}{a}$$

**Definition 4.2**

***Rise Time*** *is the time for a system to respond to a step input and attain a response equal to a percentage of the magnitude of the input.* ♣

3. **Settling time**

The amount of time for the step response to reach and stay within 2% of the steady-state value is called the settling time $T_s$ . Other percentages can also be used, for instance, 5%.

$$T_s = \frac{4}{a}$$

**Definition 4.3**

***Settling Time*** *is the time required for the system output to settle within a certain percentage of the input amplitude.* ♣

## Second-Order Systems

The responses of second-order systems have a wide range and must be described and analyzed. Unlike in first-order systems where varying the parameters only changes the speed of the response, variations in a second-order system parameters can affect the form of the response.

There are four kinds of second order systems responses:

1. **Overdamped responses**

Poles: Two real at $-\sigma_1, -\sigma_2$,

Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. **Underdamped responses**

Poles: Two complex at $-\sigma_d \pm j\omega_d$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t} cos(\omega_d t - \phi)$$

3. **Undamped responsese**

   Poles: Two imaginary at $\pm j\omega_1$

   Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\ cos(\omega_1 t - \phi)$$

4. **Critically damped responses**

   Poles: Two real at $-\sigma_1$ Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_2 t}$$

Fig. 4.1 illustrates the step responses of the four damping cases discussed above. This figure shows that the critically damped case is the division between the underdamped cases and the overdamped cases. Critically damped case is also the fastest response without overshoot.

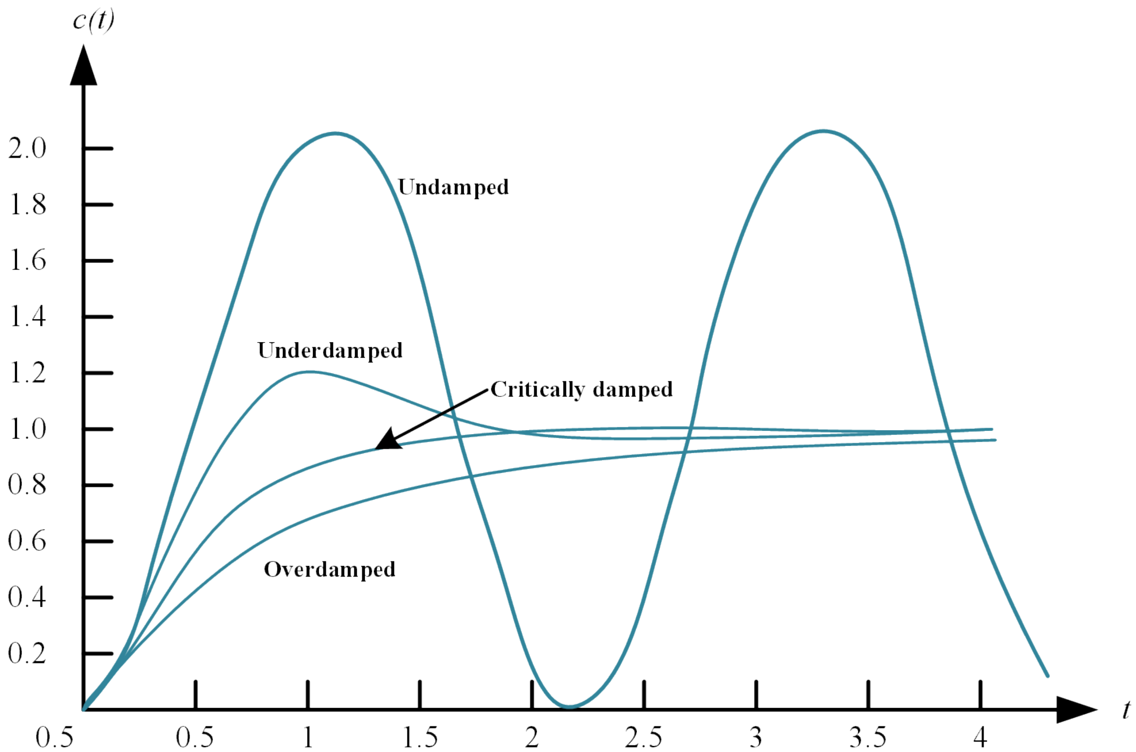



**Figure 4.1:** Damping cases of second-order system

Second-order systems are generally described by two physically meaningful specifications: natural frequency and damping ratio. Natural frequency and damping ratio are quantities used to define the characteristics of the transient response of second-order systems similarly as rise time and time constant characterize first-order system response.

**Definition 4.4**

***Natural frequency***, *$\omega_n$, is the frequency of oscillation of the system without damping.*

$$\omega_n = \sqrt{b}$$

♣

**Definition 4.5**

***Damping ratio***, *$\zeta$, the ratio of the exponential decay frequency to the natural frequency of the system. It generally separates the four types of responses*

$$\zeta = \frac{a}{2\omega_n}$$

♣

The general second-order transfer function in terms of $\omega_n$ and $\zeta$ is

$$G(s) = \frac{b}{s^2 + as + b} \rightarrow \boldsymbol{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \tag{4.1}$$

(4.2)

**Example 4.2** Determine the $\omega_n$ and $\zeta$ of the system below.

$$G(s) = \frac{400}{s^2 + 12s + 400}$$

**Solution**

1. *Natural frequency, $\omega_n$*

$$\omega_n = \sqrt{b}$$
$$\omega_n = \sqrt{400}$$
$$\omega_n = \sqrt{20}$$

2. *Damping ratio, $\zeta$*

$$\zeta = \frac{a}{2\omega_n}$$
$$\zeta = \frac{12}{2(2)}$$

## Underdamped Second-Order Systems Specifications

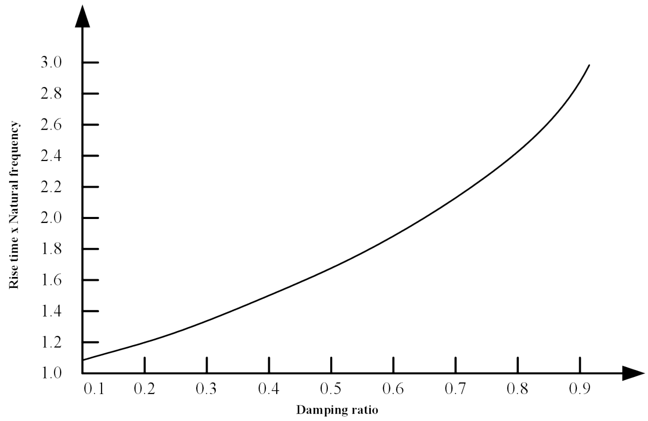
Aside from the two second-order system specifications, $(\omega_n)$ and $(\zeta)$, there are other parameters associated with underdamped responses.

These specifications are:

1. **Rise time**

   Again, rise time $T_r$ refers to the time required for the system step response to go from 10% to 90% (0.1 to 0.9) of its final value.

   Fig. 4.2presents the normalized rise time vs the damping ratio for an underdamped response. The plotted response was obtained through a computer since a precise analytical relationship between damping ratio and rise time cannot be determined.




| Damping ratio | Normalized rise time |
|---|---|
| 0.1 | 1.104 |
| 0.2 | 1.203 |
| 0.3 | 1.321 |
| 0.4 | 1.463 |
| 0.5 | 1.638 |
| 0.6 | 1.854 |
| 0.7 | 2.126 |
| 0.8 | 2.467 |
| 0.9 | 2.883 |

**Figure 4.2:** Rise versus damping ratio

2. **Peak time**

   The amount of time required for the underdamped response to reach the first, maximum, or peak is referred to as the peak time $T_p$.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

3. **Percent overshoot**

   Percent overshoot (%OS) refers to the amount the step response overshoots the steady-state, final value at the peak time. This is expressed as a percentage of the steady-state value.

$$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \cdot 100$$

4. **Settling time**

   The amount of time for the step response to reach and stay within 2% of the steady-state value is called the settling time $T_s$ . Other percentages can also be used, for instance, 5%.

$$T_s = \frac{4}{\zeta\omega_n}$$

*The **rise time, settling time,** and **peak time** give information regarding the speed of the transient response. This information aids a designer to find out whether the speed and the nature of the response do or do not reduce the system performance.*

**Example 4.3** For the given second-order system below, find the following transient response specifications:

$$G(s) = \frac{484}{s^2 + 32s + 484}$$

a. $\omega_n$
b. $\zeta$
c. $T_p$
d. %OS
e. $T_s$
e. $T_r$

### Solution

*a. Natural frequency $\omega_n$*

$$\omega_n = \sqrt{b}$$
$$\omega_n = \sqrt{484}$$
$$\omega_n = \sqrt{22}$$

*b. Damping ratio $\zeta$*

$$\zeta = \frac{a}{2\omega_n}$$
$$\zeta = \frac{32}{2(22)}$$
$$\zeta = 0.73$$

*c. Peak time $T_p$*

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$
$$T_p = \frac{\pi}{22\sqrt{1-0.73^2}}$$
$$T_p = 0.209\,\text{sec}$$

d. *Percent overshoot %OS*

$$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \cdot 100$$
$$\%OS = e^{-\left(\frac{0.73\pi}{\sqrt{1-0.73^2}}\right)} \cdot 100$$
$$\%OS = 3.489\%$$

e. *Settling time $T_s$*

$$T_s = \frac{4}{\zeta\omega_n}$$
$$T_s = \frac{4}{0.7322}$$
$$T_s = 0.249\,\text{sec}$$

e. *Rise time $T_r$ Referring from the table in Figure 4.2 , the normalized rise time is approximately 2.2 seconds. Dividing it by the natural frequency yields to 0.1*

$$T_r = \frac{2.2}{22}$$
$$T_r = 0.1\,\text{sec}$$

## 3 steps in Finding and Plotting Poles (p), Zeros (z), and Gain (k) using zpk and pzmap command in MATLAB Scripts

| | | |
|---|---|---|
| p | Pole | roots of the denominator polynomial |
| z | Zero | roots of the numerator polynomial |
| k | Gain | ratio of output to input; usually used to describe the amplification |
| num | Numerator of Polynomial | array of coefficient of NUMERATOR polynomial |
| den | Denominator of Polynomial | array of coefficient of DENOMINATOR polynomial |
| zpk | Zero-pole-gain | used to create zero-pole-gain models, or to convert dynamic system models to zero-pole-gain form. |
| tf2zp | Transfer Function to Zero-Pole-Gain Conversion | convert transfer function filter parameters to zero-pole-gain form |

1. Given the transfer function, list the polynomials in the numerator and denominator.

$$\frac{s^2 + 6s + 5}{s^3 + 16s^2 + 10s + 25}$$

**Numerator** : $s^2 + 6s + 5$
**Denominator**: $s^3 + 16s^2 + 10s + 25$

2. In MATLAB, define the coefficients of numerator and denominator in an array.
**Example:**

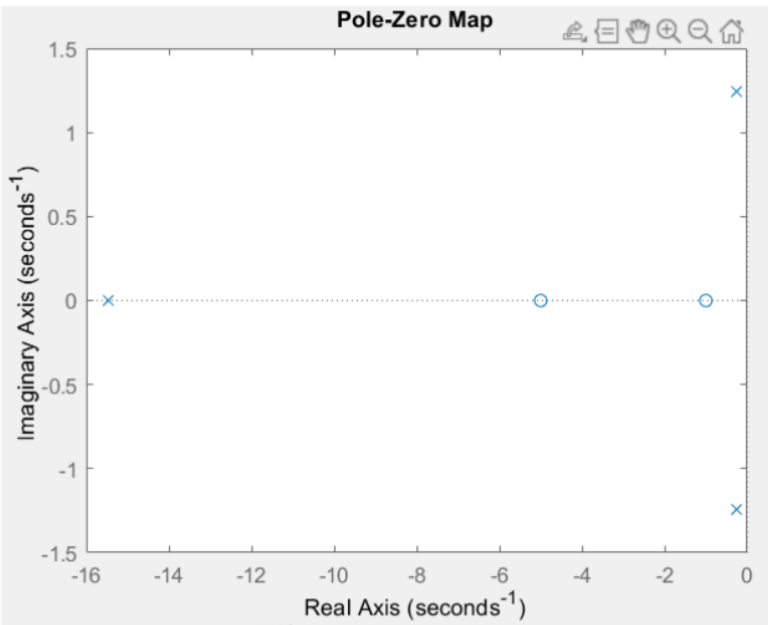
```
num=[1 6 5] % Coefficient of the numerator in an array
den=[1 16 10 25] % Coefficient of the denominator in an array
```

3. Enter the syntax to get the zeros(z), pole(p) and gain(k) of the transfer function.
**Example:**

```
[z,p,k] = tf2zp(num,den)} % Zero-pole-Gain syntax
```

4. To plot the pole and zeros in a pole-zero map, use the pzmap command. Click Run and the z, p, k values will be displayed in the command window and the pole-zero map will be shown.

```
num=[1 6 5]
den=[1 16 10 25]
[z,p,k]=tf2zp(num,den)
pzmap(p,z)
```



## Two easy steps in finding Natural Frequency and Damping Ratio of transfer function using damp(sys) command in MATLAB Scripts

`sys = tf(numerator,denominator)` creates a continuous-time transfer function model, setting the Numerator and Denominator properties. For instance, consider a continuous-time SISO dynamic system represented by the transfer function sys(s) = $\frac{N(s)}{D(s)}$, the input arguments numerator and denominator are the coefficients of $N(s)$ and $D(s)$, respectively.

1. Identify the transfer function using the transfer function model syntax.

$$\frac{36}{s^2 + 4.2s + 36}$$

```
sys = tf([36],[1,4.2,36])
```

2. Use the `damp(sys)` command and run to generate the pole, damping ratio, natural frequency and time constant of the system.

```
sys = tf([36],[1,4.2,36]
damp(sys)
```

**Result:**

```
          Pole              Damping      Frequency      Time constant
                                       (rad\seconds)     (seconds)

 -2.10e+00 + 5.26e+00i  3.50e0-01     6.00e+00         4.76e-01
 -2.10e+00 - 5.26e+00i  3.50e0-01     6.00e+00         4.76e-01
```

$$\omega_n = 6\ \frac{rad}{sec}$$
$$\zeta = 3.50$$

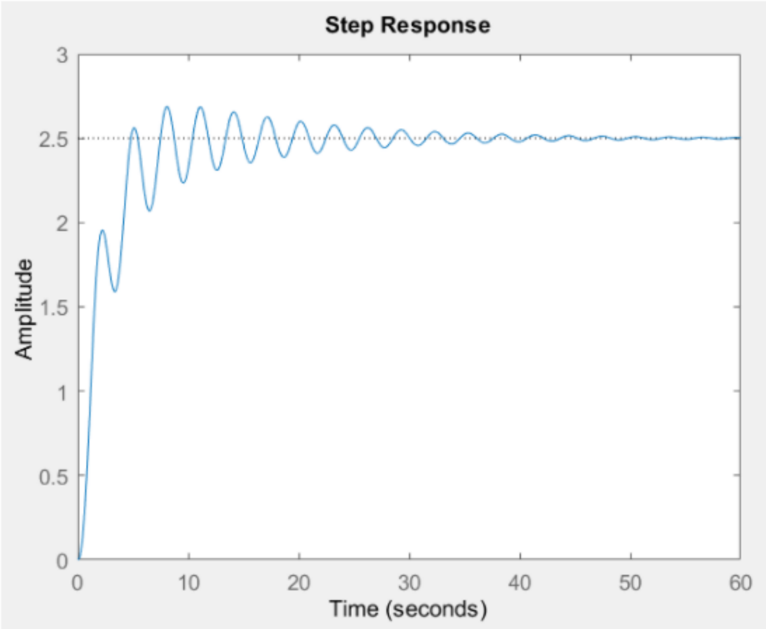
## Finding $T_p$ , %OS, $T_s$ , and $T_r$ from a Transfer Function using step(sys) and stepinfo(sys) command in MATLAB

**Given transfer function:**

$$G(s) = \frac{s^2 + 5s + 5}{s^4 + 1.65s^3 + 5s^2 + 6.5s + 2}$$

1. Create the transfer function and examine its step response using `step(sys)` command.

```
sys = tf([1 5 5],[1 1.65 5 6.5 2]);
step(sys)
```




2. The plot shows above that the response rises in a few seconds, and then rings down to a steady-state value of about 2.5. To compute the characteristics of this response, use the `stepinfo(sys)` command.

```
sys = tf([1 5 5],[1 1.65 5 6.5 2]);
stepinfo(sys)
```

```
    RiseTime: 3.8456
SettlingTime: 27.9762
 SettlingMin: 2.0689
 SettlingMin: 2.6873
   Overshoot: 7.4915
  Undershoot: 0
        Peak: 2.6873
    PeakTime: 8.0530
```

3. (Optional) You can also determine the characteristics of the response in the graph. Right click the graph, move your cursor to Characteristics and mark the Peak response, Settling Time, and Rise Time check to display the parameter points.

## 4.3 Laboratory Experiment

### Module Exercises

1. Find the state equations and output equations in vector matrix form of the following transfer functions below. Solve each analytically and cross check using MATLAB.

   a. $G(s) = \frac{400}{s^2+12s+400}$

   b. $G(s) = \frac{900}{s^2+90s+900}$

   c. $G(s) = \frac{225}{s^2+30s+225}$

   d. $G(s) = \frac{625}{s^2+625}$

2. With the help of poles inspection, identify the type of system response of the second order systems given in Item #1.
3. By inspection, formulate the general form of the step response of the given transfer functions in Item #1 and plot the step response using MATLAB.
4. Identify the natural frequency ($\omega_n$) and damping ratio ($\zeta$) of the given transfer functions in Item #1 using analytical computation and MATLAB.

   a. $T(s) = \frac{16}{s^2+3s+16}$

   b. $T(s) = \frac{0.04}{s^2+0.02s+0.04}$

   c. $T(s) = \frac{1.05 \cdot 10^7}{s^2+1.6\times10^3 s+1.05x\times10^7}$

### Simulation Activity

### Transient Response Analysis of an RLC Circuit

### Objective:

- To analyze the transient response of a series RLC Circuit

### Procedures

1. Download the **"RLC_Simscape.slx"** file from the e-Learning Google Drive.
2. Open the Simscape file in the MATLAB Software.

3. Make another 2 models below by duplicating the given circuit. Label the first, second and third circuit as **“Circuit A”**, **“Circuit B”**, and **“Circuit C”**, respectively.
4. Set the value of the three passive elements, resistor, inductor and capacitor with the values shown on table below:

| | Circuit A | Circuit B | Circuit C |
|---|---|---|---|
| **Resistor** | 50 W | 1000 W | 200 W |
| **Inductor** | 10mH | 10mH | 10mH |
| **Capacitor** | 1mF | 1mF | 1mF |

5. Connect the three Voltage Measurements in a scope with 3 inputs: Circuit A, Circuit B and Circuit C as illustrated below:

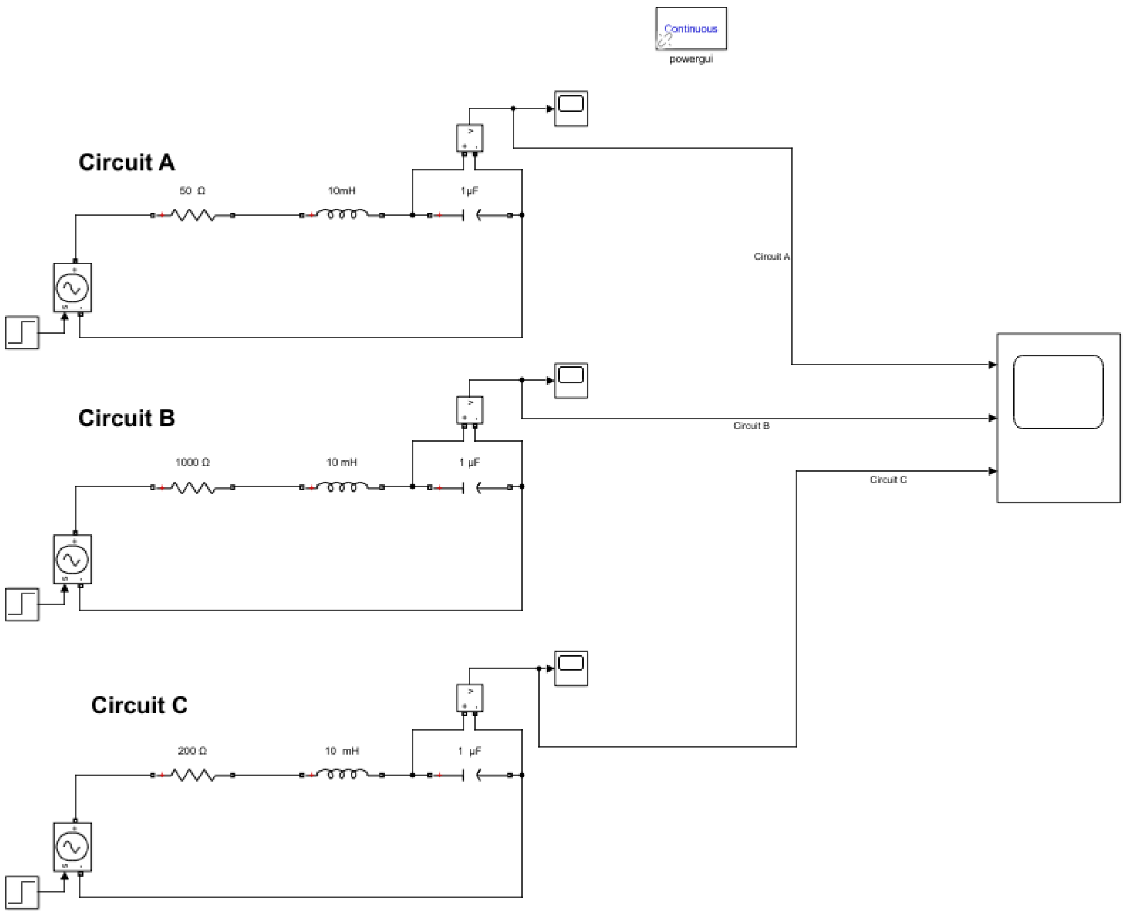



**Figure 4.3**

6. **Run** the simulation and double-click the **scope** to visualize the transient response of the three circuit. Save the system response.
7. Given the formula of damping factor of Series RLC Circuit, $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$, complete the table below:

| | **Damping factor,** $\zeta$ | **Transient response** |
|---|---|---|
| **Circuit A** | | |
| **Circuit B** | | |
| **Circuit C** | | |

## 4.4 Questions to Ponder

1. How can the values of the passive elements affect the transient response of the system?
2. What is the relationship between the damping factor and the transient response of an RLC circuit?

# Bibliography

Dorf, Richard C. and Robert H. Bishop (2017). Modern Control Systems. 13th ed. Hoboken, NJ: Pearson Education, Inc. ISBN: 9780134407623.

Nise, Norman (2015). Control Systems Engineering. 7th ed. Hoboken, NJ: John Wiley & Sons, Inc. ISBN: 9781118170519.

# Chapter 5
# Reduction of Multiple Subsystems

## 5.1 Objective

- To simplify a block diagram consisting of multiple subsystems to a single block that represents a transfer function
- To understand signal-flow graphs
- To obtain the transfer function of multiple subsystems by utilizing Mason's rule
- To show state equations in the form of signal-flow graphs.

## 5.2 Theory

### Block Diagrams

**Definition 5.1**

*Blocks diagrams graphically show the relationship of system variables. They are basically visual representations of interconnection of multiple subsystems forming control systems.* ♣

A block diagram of a linear, time invariant system mainly consists of four elements. These are:

- **Summing Junctions**

  The polarity (+ +, + -, ++-, etc.) of a summing junction is critical. If the polarity is positive, then the signal will be positive. When the polarity is negative, then the signal is negative.

  **Definition 5.2**

  *A summing junction, also referred as summing point, is a block diagram element that represents the algebraic summation of two (or more) signals. Therefore, whenever two or more signals will be added together a summing junction or summing point shall be used.* ♣

  Fig. 5.1 shows a summing joint with three input signals,$R_1s$, $R_2s$, $R_3s$ and a resulting signal. The polarity of $R_1s$ and $R_2s$ are both positive, while $R_3s$ is negative. Hence, the resulting signal of $C(s) = R_1(s) + R_2(s)$ - $R_3(s)$.

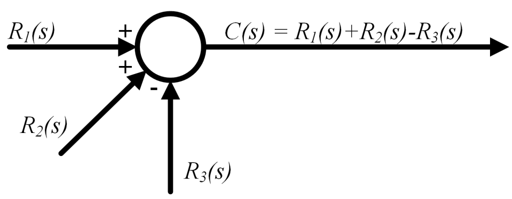



**Figure 5.1:** Polarity of a summing junction

- **Pick-off points**

  On the other hand, a pick off point is a block diagram element that represents the distribution of one signal to several branches or subsystems as shown in Fig. 5.2. The pick off point distributes the signal Rs into three more Rs branches.

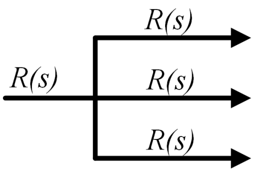



**Figure 5.2:** Pickoff point

- **Signal**

  In a block diagram, arrows, as shown in Fig. 5.3represent input and output signals as shown in the figure below. They indicate the signal flow direction throughout the system. These signals can be force, voltage velocity, etc.


**Figure 5.3:** Signals

- **System**

  A system is represented as a block with a single input, one output and labeled with a transfer function. The Fig. 5.4 presents a block with an input $R(s)$, output $C(s)$, and the transfer function $G(s)$.

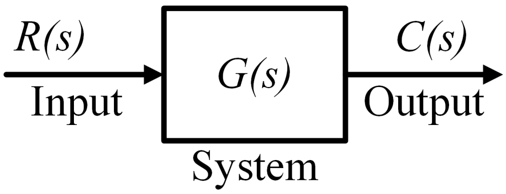



**Figure 5.4:** System

## Block Diagram Configurations

The basic elements, ***blocks, signals, summing junctions, and pick off points***, can be constructed into different topologies – cascade, parallel, and feedback form.

Physical systems include complex integration of these three configurations.Therefore, understanding these three topologies is essential in finding the equivalent transfer function of complex physical systems.

The common topologies will serve as the basis for reducing complicated interconnecting subsystems into a single block or transfer function.

1. **Cascade Form**

   Cascade form is a common arrangement in block diagrams wherein the subsystems are in series. Fig. 5.5 shows an example of cascaded subsystems. The intermediate variables, $X_2$ and $X_1$, are indicated at the output of every block/subsystem and are obtained by multiplying every input by each transfer function.

$$X_2(s) = G_1(s)R(s)$$
$$X_1(s) = G_2(s)G_1(s)R(s)$$
$$C(s) = G_3(s)G_2(s)G_1(s)R(s)$$

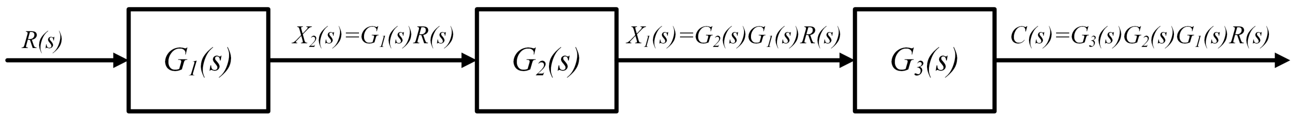



**Figure 5.5:** Subsystems in cascaded form

   Therefore the three blocks/subsystems can be reduced into a single block with a transfer function equivalent to the product of each individual block as shown in Fig. **??**

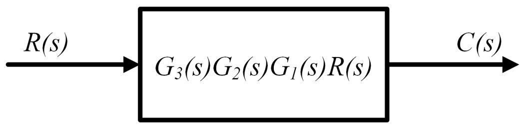



**Figure 5.6:** Equivalent transfer function

2. **Parallel Form**

   Fig. 5.7 illustrates another topology having two or more blocks/subsystems connected in parallel. In parallel form, the subsystems have a common input wherein again the output of each subsystem is derived from the product of the input and the transfer function.

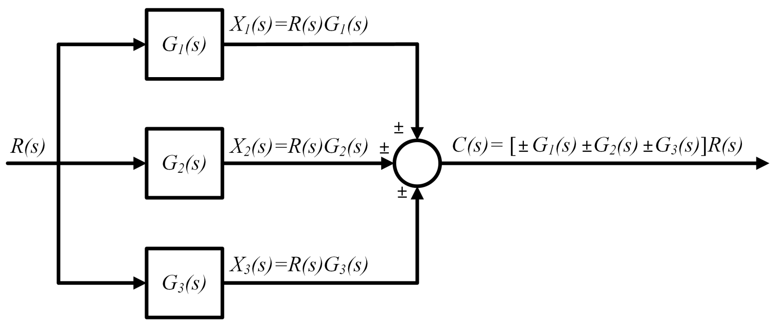



**Figure 5.7:** Subsystems in parallel form

The outputs,$X_1$(s),$X_2$(s), and $X_3(s)$ are then summed up to obtain the equivalent transfer function as shown in Fig. 5.7. In other words, the equivalent transfer function of parallel subsystems is the algebraic sum of the individual transfer functions.

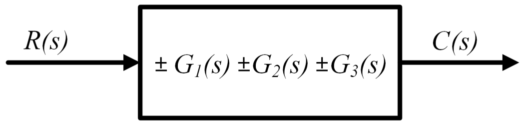



**Figure 5.8:** Equivalent transfer function

### 3. Feedback Form

Fig. 5.9 shows a typical and a simplified model of a closed loop or feedback system. As discussed in Module 1–Introduction to Control Systems, feedback systems utilize feedback signal and output measurement.

**Note** *The system has negative feedback if the sign at the summing junction is negative and positive feedback if the sign is positive.*

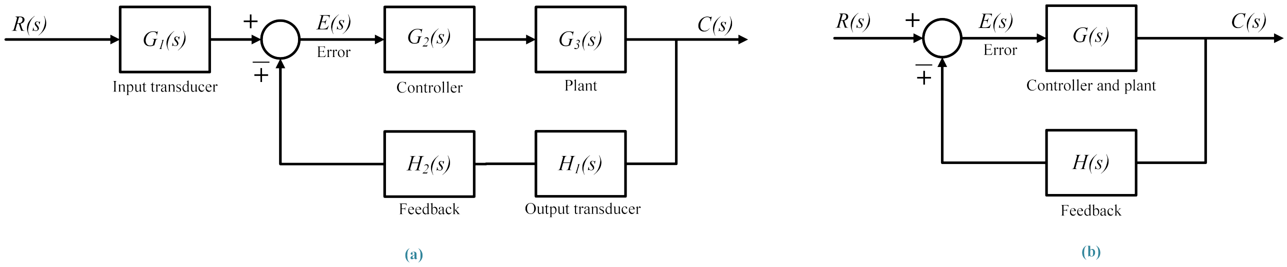



**Figure 5.9:** **a.**typical feedback system **b.**simplified feedback system

The simplified model of the feedback system is given by $E(s)=R(s)\mp C(s)H(s)$, by substituting $E(s)=\frac{C(s)}{G(s)}$ to this equation the equivalent transfer function can be obtained.

$$E(s) = \frac{C(s)}{G(s)}$$

$$C(s) = G(s)[R(s)C(s) \mp H(s)]$$

$$C(s)[1 \pm G(s)H(s) = G(s)R(s)$$

Therefore the equivalent transfer function is $\frac{C(s)}{R(s)}=\frac{G(s)}{1\pm G(s)H(s)}$. Its block diagram is shown below.

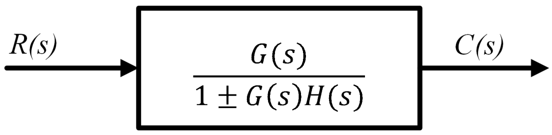



**Figure 5.10:** Equivalent transfer function

If G and H are constants, they are called **gains**.

## Basic Block Moves

In block diagrams, cascade, parallel, and feedback forms for subsystems may appear but are not always apparent.

In order to create these familiar configurations, basic block movements can be done, for instance, moving blocks left and right, as well as moving summing junctions ahead and behind blocks.

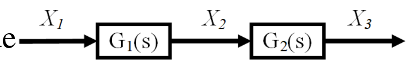
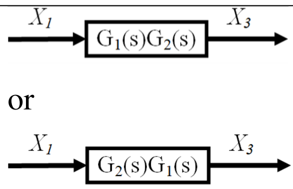
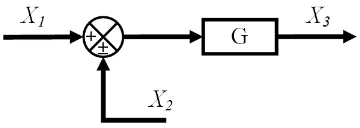
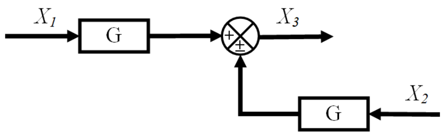
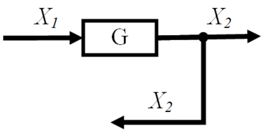
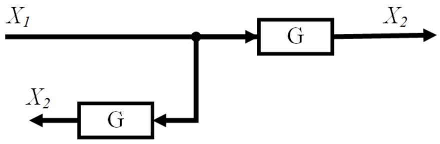
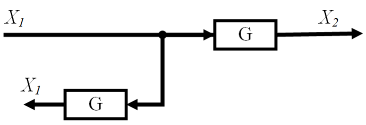
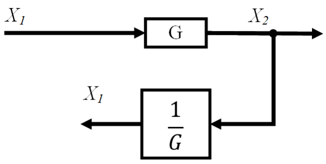
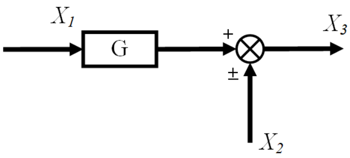
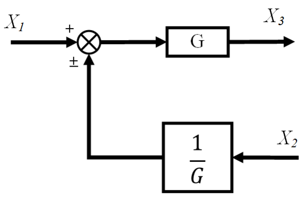
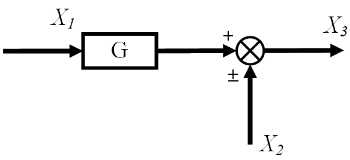
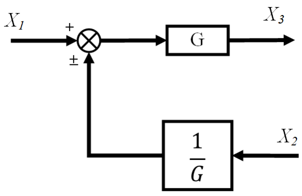
| Transformation | Original Diagram | Equivalent Diagram |
|---|---|---|
| 1. Combining blocks in cascade |  |  |
| 2. Moving a summing point behind a block |  |  |
| 3. Moving a pickoff point ahead of a block |  |  |
| 4. Moving a pickoff point behind a block |  |  |
| 5. Moving a summing point ahead of a block |  |  |
| 6. Eliminating a feedback loop |  |  |

**Table 5.1:** Summary of block movements

## Block Diagram Reduction

Again, block diagrams of certain physical systems are quite complex, thereby requiring simplification of block diagrams to evaluate the systems' performance which is performed by manipulating the blocks.To have a better understanding on block diagram reduction, let us have an example.

**Example 5.1** Reducing a Block Diagram with Familiar Topologies

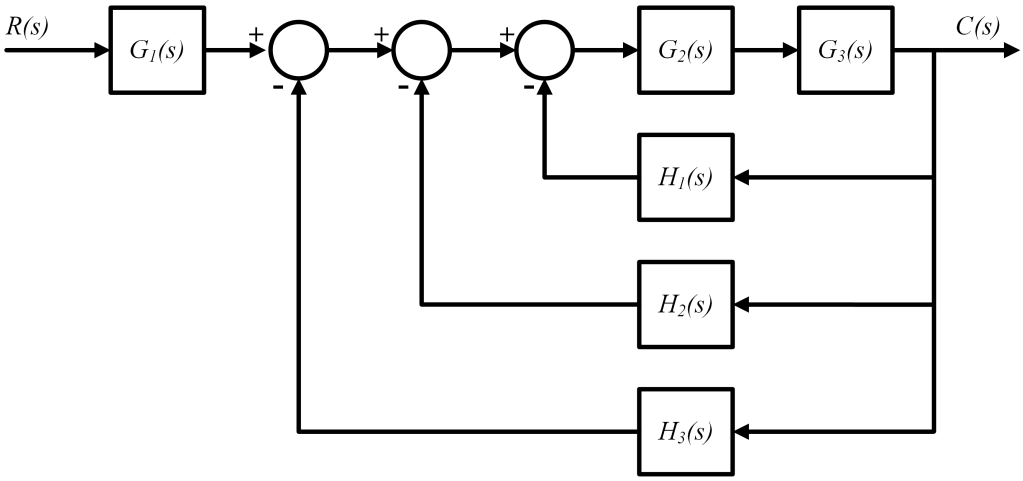



**Figure 5.11**

## Solution

1. *Collapse the three succeeding summing junctions into one summing junction*

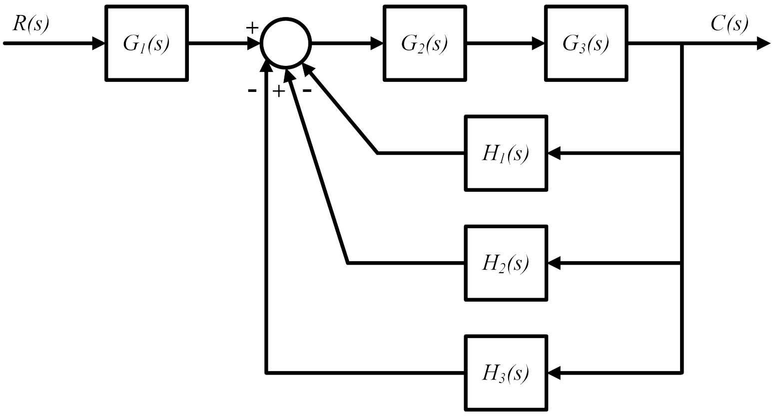



**Figure 5.12**

2. *Since $G_2(s)$ and $G_3(s)$ are cascaded, multiply the two blocks to obtain the equivalent transfer function.*
3. *Add $H_1(s)$, $H_2(s)$, and $H_3(s)$ since these feedback functions have a common input and are connected in parallel.*

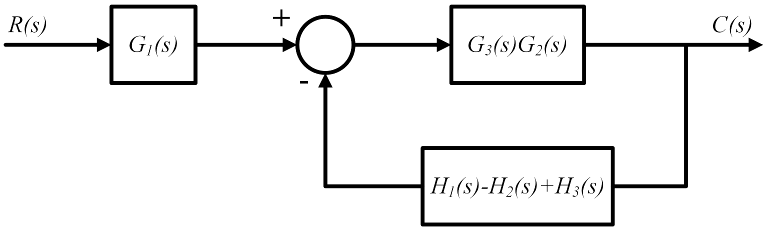



**Figure 5.13**

4. *Eliminate the feedback loop using the formula* $G(s) = \frac{(G(s)}{1 \pm G(s)H(s)}$, *then multiply by* $G_1(s)$ *since they became cascaded.*
*The figure below is the equivalent transfer function.*

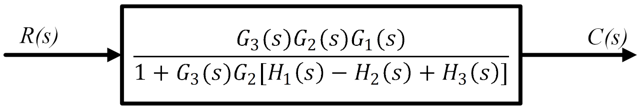



**Figure 5.14**

**Example 5.2** Simplifying a Block Diagram by Moving Blocks

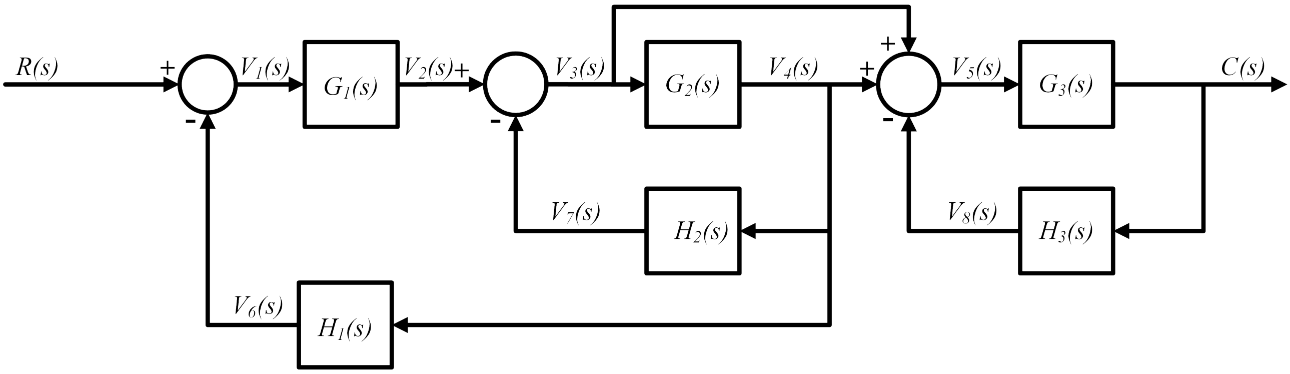



**Figure 5.15**

For easier visualization of block transformations, refer to Table **??**

**Solution**

1. *To create a parallel connection of subsystems, move* $G_2(s)$ *to the left past the pickoff point. (Transformation #4), then eliminate the feedback loop* $G_3(s)$ *and* $H_3(s)$ *(Transformation #6).*

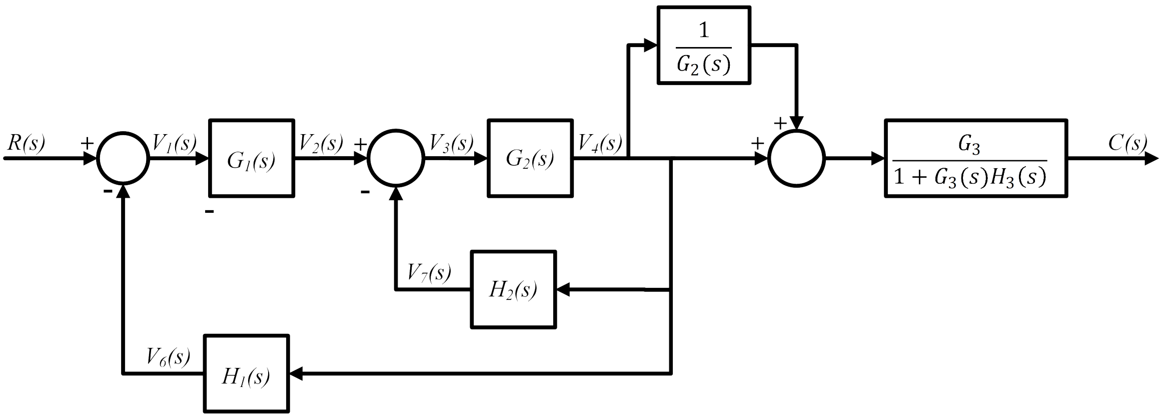



**Figure 5.16**

2. *Next, move the summing junction ahead block* $G_1(s)$ *to the left, and multiply the cascaded subsystems,* $G_1(s)$ *and* $G_2(s)$ *(Transformation #5). Then simplify the parallel pair of* $\frac{1}{G_2(s)}$ *and unity.*

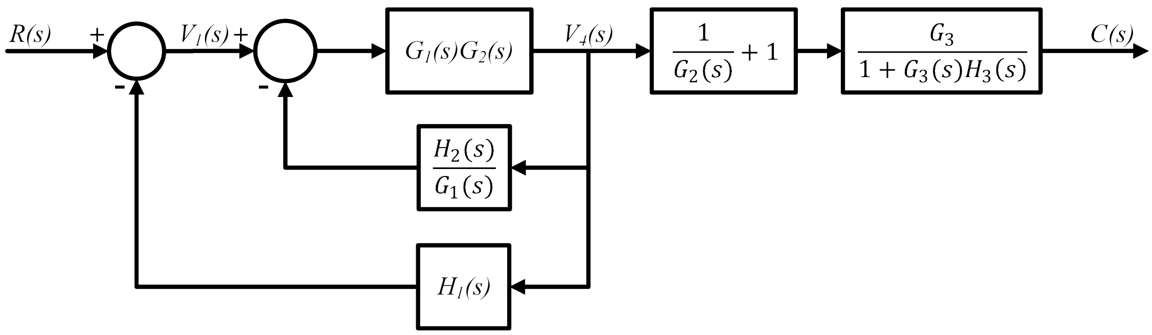



**Figure 5.17**

3. *Simplify the two summing junctions, add the two feedback elements in parallel form, then multiply the two cascaded blocks/subsystems (Transformation #1).*

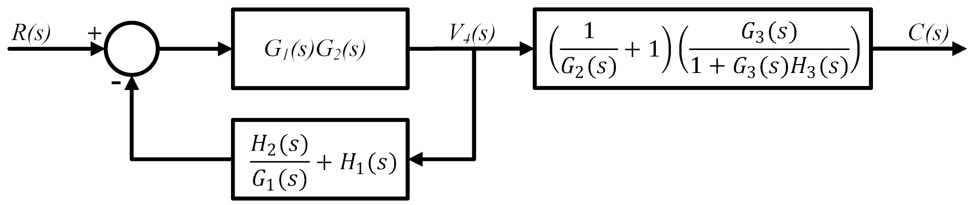
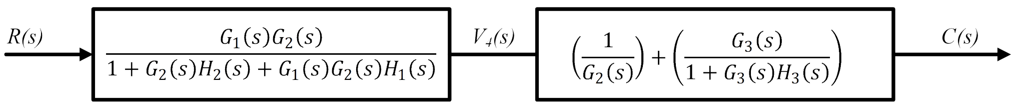



**Figure 5.18**

4. *Apply the feedback formula to simplify the feedback loop (Transformation #6).*




5. *Multiply the two blocks in series (transformation #1) to obtain the equivalent transfer function.*

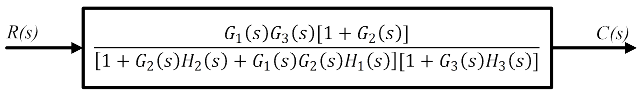



**Figure 5.19**

## Signal Flow Graphs

**Definition 5.3**

*A signal-flow graph is a graphical representation of a set of linear relations. They are primarily useful for feedback control systems because feedback theory is particularly focused on the flow and processing of signals in systems. [1, ch.2]* ♣

A signal-flow graph only consists of interconnected branches and nodes unlike block diagrams that consist of summing junctions, blocks, etc.

### Branches

Systems in signal-flow graphs are represented as branches or lines with arrows. The arrows then depict the signal flow direction within the system as shown in the Fig. 5.20. The transfer function is indicated adjacent to the line.


**Figure 5.20:** Branch

### Nodes

On the other hand, nodes represent signals. Fig. 5.21 shows that the name of the signal is always written adjacent to the node.


**Figure 5.21:** Node

Fig. 5.22 illustrates the interconnection of the signals and the systems where each signal is equal to the summation of all signals flowing into it.

For example,

$$C_1(s) = R_1(s)G_1(s)G_4(s) - R_2(s)G_2(s)G_4(s) + R_3(s)G_3(s)G_4(s)$$

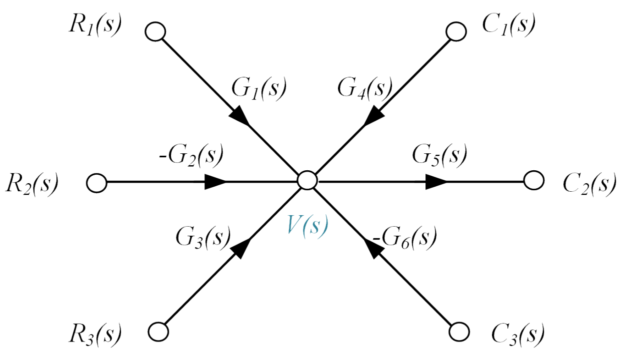



**Figure 5.22:** Signal-flow graph

## Mason's Rule

**Definition 5.4**

*Mason's rule is a technique for reducing a signal-flow graph into a single transfer function using a formula derived by S.J Mason which creates simultaneous equations that can be written from the graph. Through the use of loops in the system, we will find Mason's rule easier to use than block diagram reduction.* ♣

Consider the given signal-flow graph in understanding the following definitions.

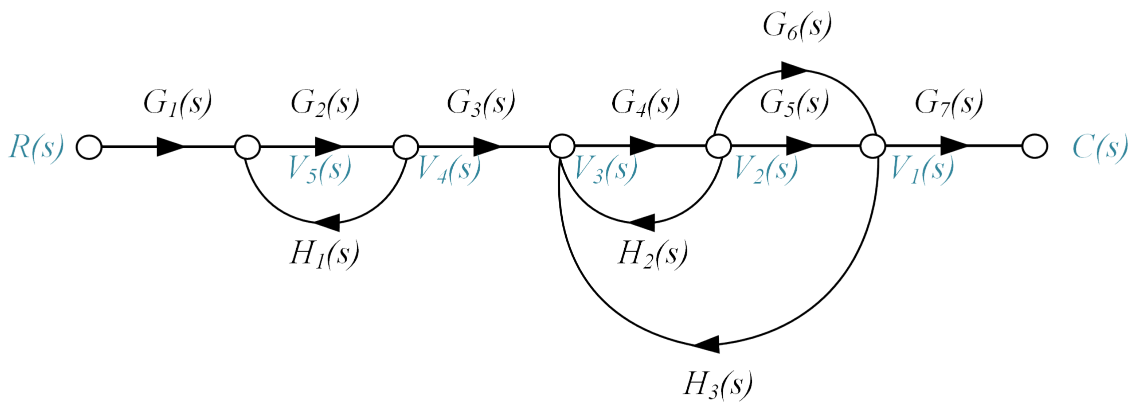



**Figure 5.23:** Signal-flow graph

1. **Loop Gain**

    It is the loop found by traversing a path starting from a node and ending up at the same node, without passing through any other node more than once that follows the direction of a signal flow.

    From the figure, loop gains are:

    a. $G_2(s)H_1(s)$
    b. $G_4(s)H_2(s)$
    c. $G_4(s)G_5(s)H_3(s)$
    d. $G_4(s)G_6(s)H_3(s)$

2. **Forward-path gain**

    It is a gain in a forward direction found by traversing a path from the input node to the output node of a signal-flow graph.

    From the figure, forward-path gains are:

    a. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$
    b. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

3. **Non Touching loops**

    These are the loops in a signal-flow graph that do not have any nodes in common. In the given graph, the loop $G_2(s)H_1(s)$ does not touch the loops $G_4(s)H_2(s)$, $G_4(s)G_5(s)H_3(s)$, and $G_4(s)G_6(s)H_3(s)$.

4. **Non Touching-Loop gain**

   It is the product of loop gains from the non-touching loops taken two, three, or more at a time. For example, the following are the non-touching loop gains taken two at a time:

   a. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
   b. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
   c. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

From the definitions, we can now apply the Mason's rule.

## MASON'S RULE

The equivalent transfer function $\frac{C(s)}{R(s)}$ to the Mason's rule is given by the equation:

$$G(s) = \frac{C(s)}{R(s)} = \frac{\Sigma_k T_k \Delta_k}{\Delta}$$

where:
$k$ = number of forward paths
$T_k$ = the $kth$ forward-path gain
$\Delta = 1 - \Sigma$ loop gains $+\Sigma$ non-touching loop gains taken two at a time $-\Sigma$ non-touching loop gains taken three at a time $+\cdots$ (alternating signs for the components)
$\Delta_k = \Delta - \Sigma$ loop gain terms in $\Delta$ that touch the $kth$ forward-path

Steps in finding the transfer function from a signal-flow group using Mason's rule:

1. Identify the forward-path gain(s).
2. Identify the loop gain(s).
3. Identify the non-touching loop gains taken two at a time, taken three at a time, and so on.
4. Form the $\Delta$ and $\Delta_k$.
5. Substitute the values to the formula.

### Example 5.3

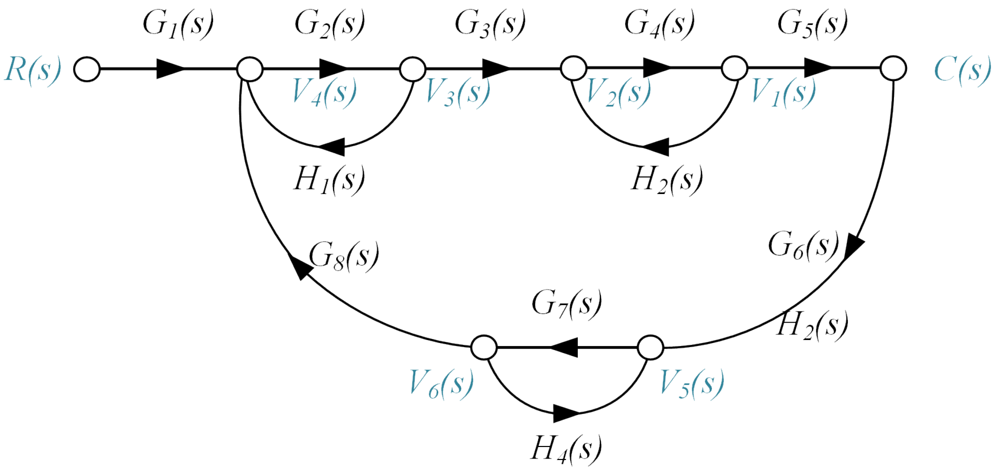



**Figure 5.24**

### Solution

1. *Identify forward path gain.*
   *In the example, we only have one forward-path gain,* $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$
2. *Identify loop gains.*
   *In the example, there are four namely:*
   a. $G_2(s)H_1(s)$
   b. $G_4(s)H_2(s)$
   c. $G_7(s)H_4(s)$
   d. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$
3. *Identify non-touching loops taken $nth$ at a time*
   *Taken two at a time:*
   *Loop 1 and 2 :* $G_2(s)H + 1(s)G_4(s)H_2(s)$
   *Loop 1 and 3 :* $G_2(s)H_1(s)G_7(s)H_4(s)$
   *Loop 2 and 3 :* $G_4(s)H_2(s)G_7(s)H_4(s)$

   *Taken three at a time:*
   *Loop 1, 2, and 3:* $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

4. *Form the $\Delta$ and $\Delta_k$*

$$\begin{aligned}\Delta = 1 &- [G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] \\ &+ [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) + G_4(s)H_2(s)G_7(s)H_4(s)] \\ &- [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]\end{aligned}$$

*$\Delta_k$ is obtain by eliminating from $\Delta$ the loop gains that touch the kth forward path:*

$$\Delta_1 = 1 - G_7(s)H_4(s) \tag{5.1}$$

$$\tag{5.2}$$

5. *Substitute the values to the formula.*

$$G(s) = \frac{T_1\Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)][1 - G_7(s)H_4(s)]}{\Delta}$$

*Note* *The equation is only consisted of one term since there is only one forward path*

## Signal Flow Graphs of State Equations

We have learned that Signal-Flow Graphs are the graphical representation of block diagrams that are used to determine the overall transfer function of a control system. In this section, we will draw and create signal-flow graphs from a given state equation.

Steps in creating Signal-Flow Graph from State Equations

1. Identify the nodes which are equivalent to the state variables of the state equation and their respective derivatives placed to their left. Also, identify a node as the input, $r$, and an output node, $y$.
2. Connect the nodes and their corresponding derivatives with then defining integration, $\frac{1}{s}$.
3. Using the equation of each state variable, feed to each node the incated signals. *(In feeding the signal to each node, it's important to note that the signal flow of each state equation must be towards the state variable itself.)*

**Example 5.4** Draw a signal flow graph for the following state and output equations.

$$\begin{aligned}
\dot{x}_1 &= 2x_1 - 5x_2 + 3x_3 + 2r \\
\dot{x}_2 &= -6x_1 - 2x_2 + 2x_3 + 5r \\
\dot{x}_3 &= x_1 - 3x_2 + -4x_3 + 7r \\
y &= -4x_1 + 6x_2 + 9x_3
\end{aligned}$$

**Solution**

1. *Identify the nodes of the signal-flow graphs.*
   *In the given state and output equations, the nodes of the signal flow graph are the three state*

*variables, $x_1$, $x_2$, and $x_3$, and their respective derivatives placed on their left. Also, identify the input and output nodes of the graph.*

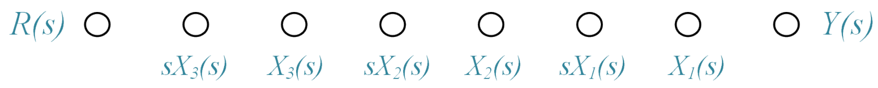



**Figure 5.25**

2. *Connect the nodes and their corresponding derivatives with then defining integration, $\frac{1}{s}$.*

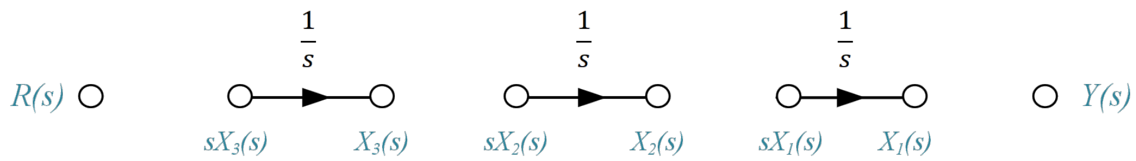



**Figure 5.26**

3. *Using the equation of each state variable, feed to each node the incated signals.*
   *In creating the branches, state variables are the dependent variables that need to relate to its given equation.*
   *For $\frac{dx_1}{dt}$, $x_1$ receives $2x_1 - 5x_2 + 3x_3 + 2r$*

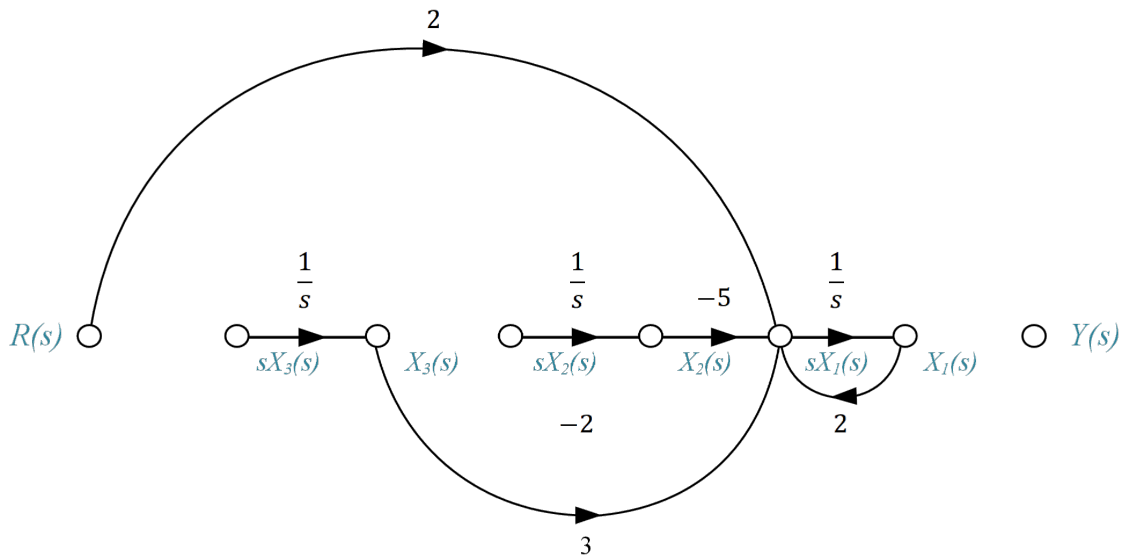



**Figure 5.27**

*Similarly, $x_2$ receives $-6x_1 - 2x_2 + 2x_3 + 5r$*

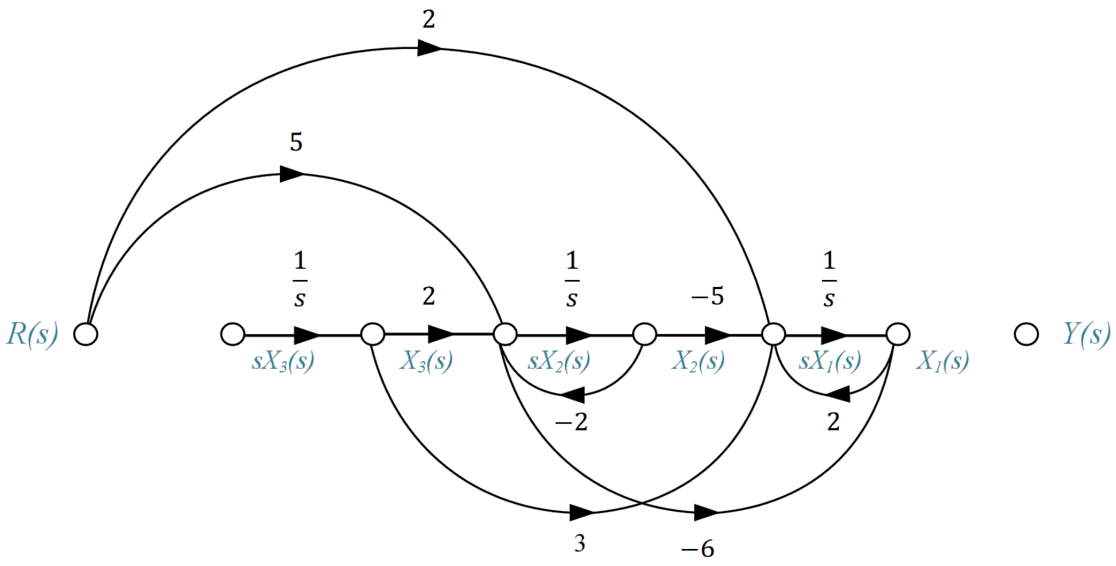



**Figure 5.28**

*$x_3$ receives* $x_1 - 3x_2 - 4x_3 + 7r$

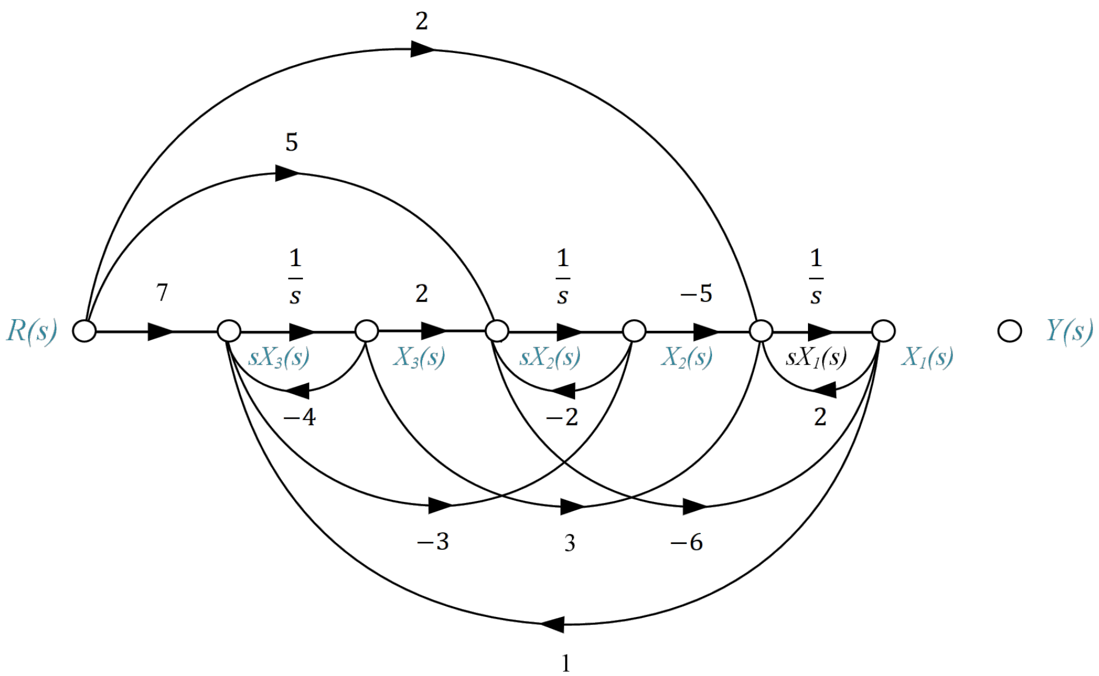



**Figure 5.29**

*Lastly, the output, $y$ receives* $-4x_1 + 6x_2 + 9x_3$

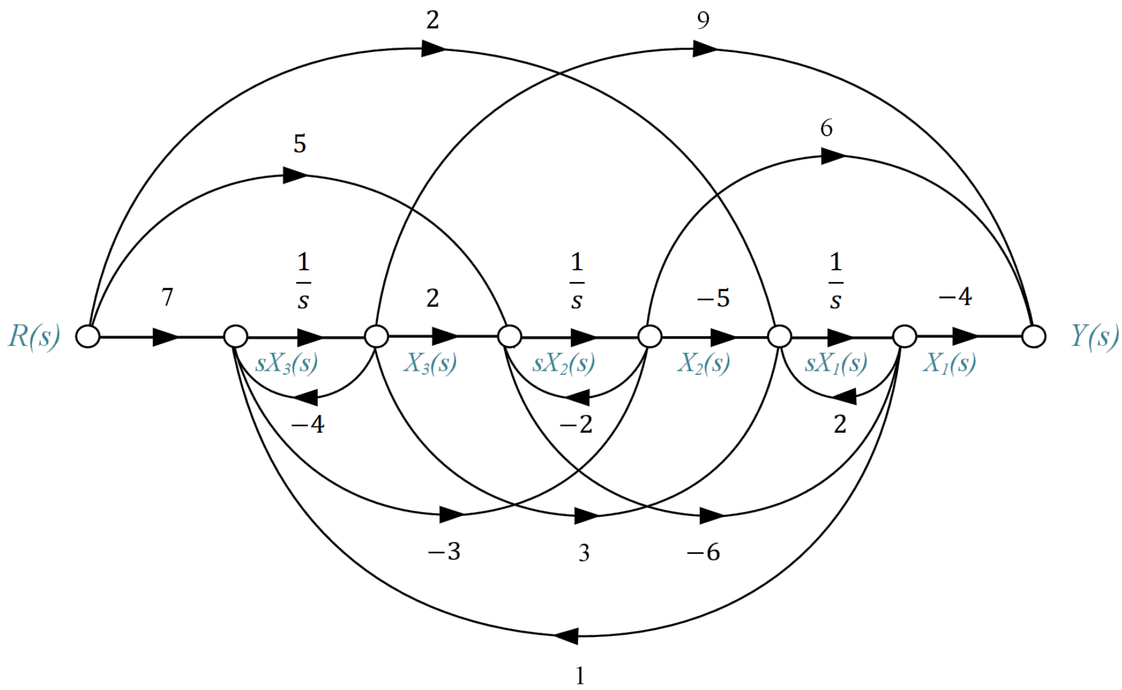



**Figure 5.30**

## 5.3 Laboratory Experiment

### Module Exercises

1. Reduce the block diagram below to find the transfer function $\frac{Y(s)}{X(s)}$.

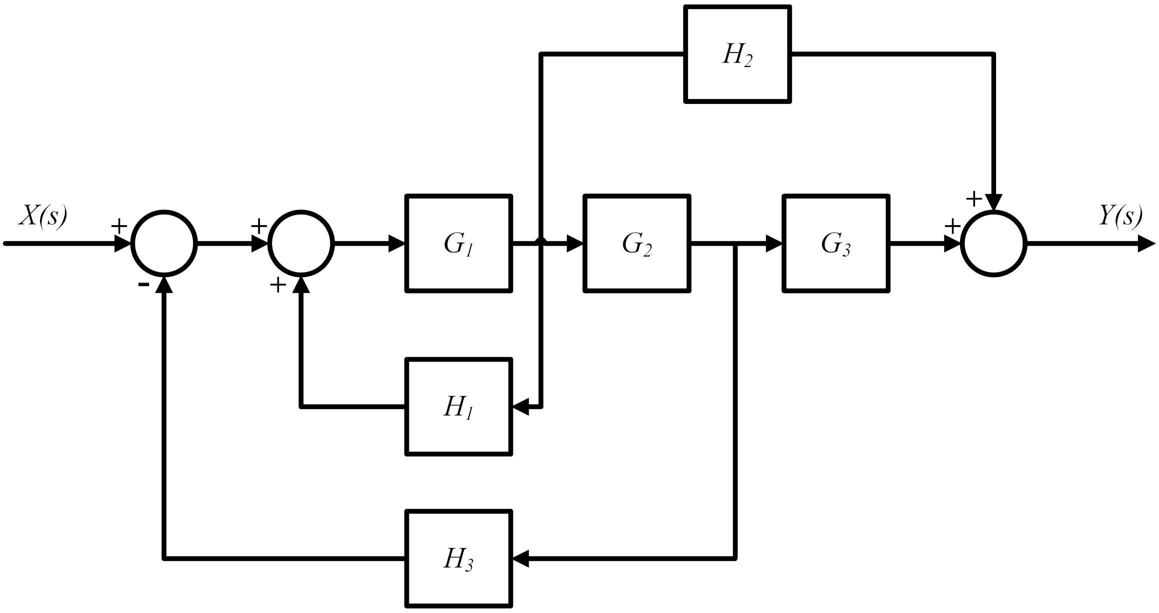



Figure 5.31

2. Use Mason's gain formula to determine the transfer function of the signal-flow graphs below.

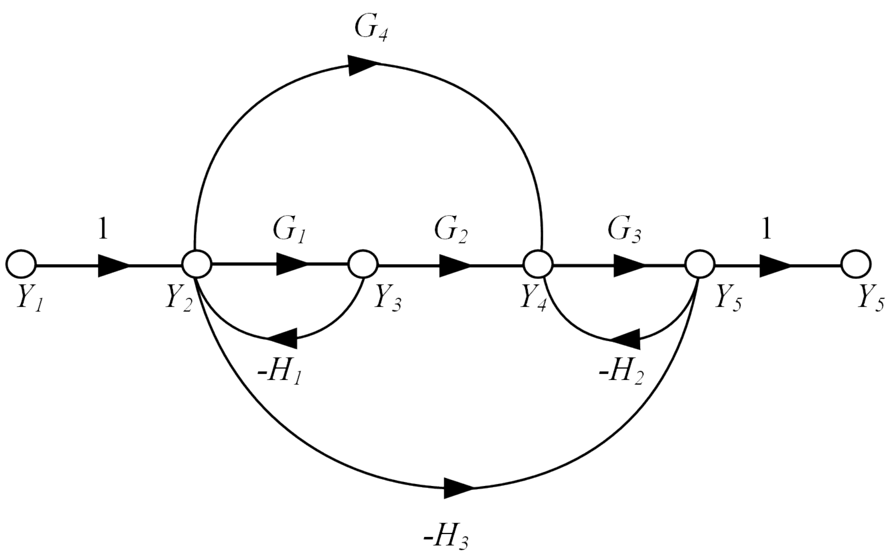



Figure 5.32

3. Reduce the block diagram and find the transfer function of the system below.

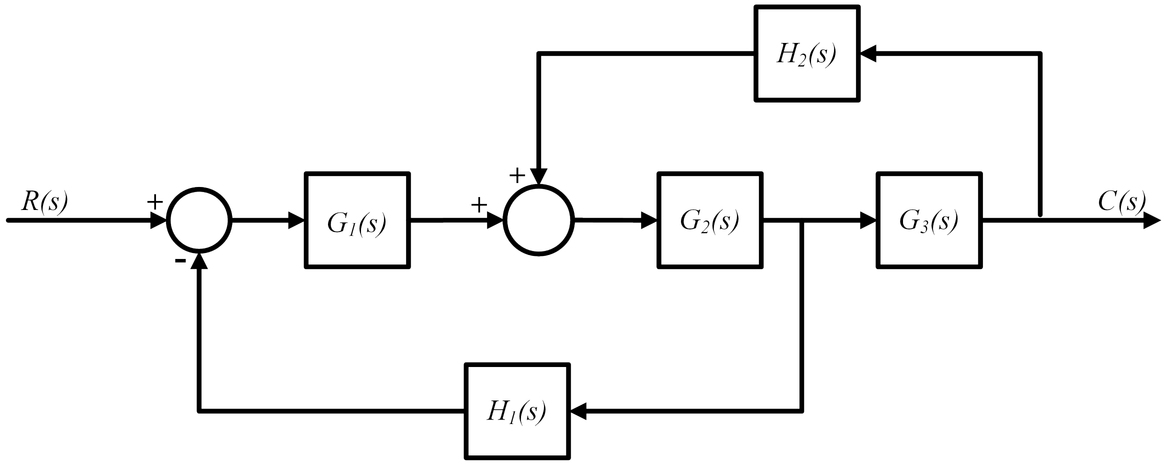



**Figure 5.33**

4. Reduce the block diagram and find the transfer function of the system below. Solve it analytically and crosscheck using MATLAB.

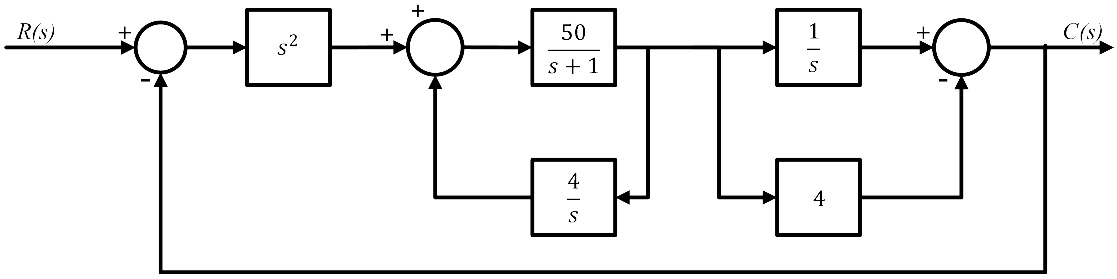



**Figure 5.34**

5. Draw a signal-flow graph for the following state and output equations.

$$\dot{x} = \begin{bmatrix} -\frac{6}{9} & -\frac{2}{9} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} x + \begin{bmatrix} \frac{1}{9} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \end{bmatrix} x$$

## Simulation Activity

## Reducing a Block Diagram with Multiple Subsystems

## Objective:

- To find reduce a block diagram with multiple subsystems into a single transfer function.
- To find reduce a block diagram using MATLAB commands.

a. Reduce the block diagram below manually to find the transfer function

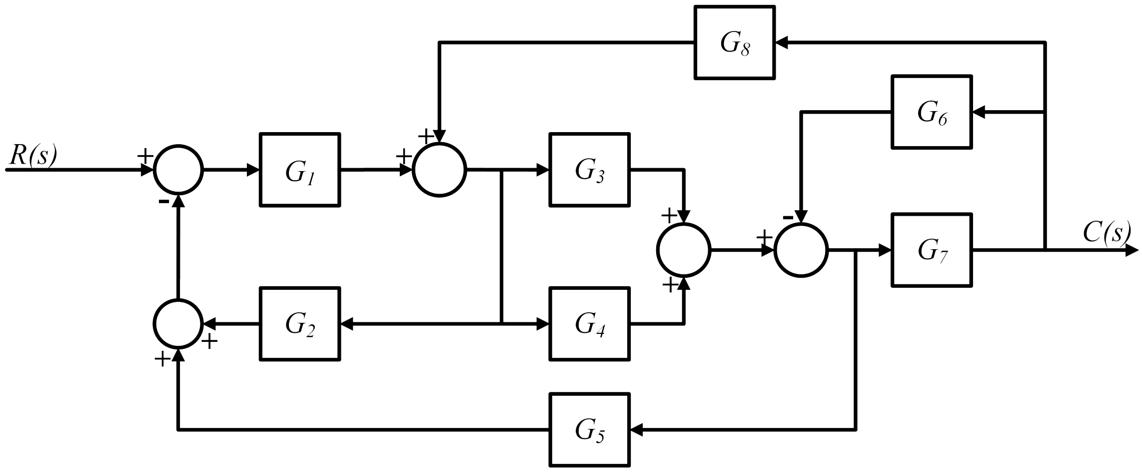



Figure 5.35

b. Using `append` and `connect` MATLAB commands to reduce the block diagram above.

| | |
|---|---|
| `append` | *appends the inputs and outputs of the models* $sys_1, \ldots, sys_N$ |
| `connect` | *connects the block diagram elements* $sys_1, \ldots, sys_N$ |

In reference to the block diagram above, use the following transfer functions:

$$G_1 = \frac{1}{s+3}$$
$$G_2 = \frac{1}{s^2+3s+2}$$
$$G_3 = \frac{1}{s+1}$$
$$G_4 = \frac{1}{s}$$
$$G_5 = \frac{8}{s+4}$$
$$G_6 = \frac{1}{s^2+6s+9}$$
$$G_7 = \frac{5}{s+3}$$
$$G_8 = \frac{1}{s+5}$$

## Procedures

1. Launch the **MATLAB** software.
2. Define the transfer functions $G_1, G_2, G_3, G_4, G_5, G_6, G_7, and G_8$ using `sys = tf(num,den)` Example:

```
G1=tf([0 1], [1 7]);
G2=tf([ 0 0 1],[1 2 3]);
```

3. Append all the transfer functions from $G_1$ to $G_8$. Assign it to variable T1.

```
sys =append (sys1,sys2,...,sysN)
```

4. Describe the inputs for the transfer function $G_1, G_2, G_3, G_4, G_5, G_6, G_7,$ *and* $G_8$.
   *(the number of columns is equal to the number of greatest summing point inputs plus 1)*

```
Q= = [ 1 -2 -5 9; 2 1 8 0;.....]
```

   *The first column denotes the block. For block 2 $G2(s)$ the inputs are from block 1 ($G1(s)$)and 8 ($G8(s)$).*
5. Specify the input and output of the block diagram.

```
inputs= =(block where the input signal is applied );
outputs= =(block where the output signal is);
```

6. Connect the blocks appended in T1 as specified by Q taking into consideration that the input and output are specified as input and output respectively. Assign it to variable Ts.
7. Obtain the transfer function T.

## 5.4 Questions to Ponder

1. What is the purpose of reducing multiple subsystems in a control system?
2. What are the advantages of the various techniques of reducing multiple subsystems?

# Bibliography


Dorf, Richard C. and Robert H. Bishop (2017). Modern Control Systems. 13th ed. Hoboken, NJ: Pearson Education, Inc. ISBN: 9780134407623.

Farag, Samir A. (Jan. 2021). Block diagram reduction using MATLAB part 1. Online. URL: https://www.youtube.com/watch?v=C7BDm5Yq-9k.

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# Chapter 6
# Stability

## 6.1 Objective

- To create and interpret a basic Routh table in order to determine a system's stability.
- To create and analyze a Routh table in which the first element of a row or the entire row is zero.

## 6.2 Theory

### Stable, Marginally Stable, and Unstable Linear Systems

The total response of a system is given by,

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

#### *Stability for LTI systems*

- Natural response as $t \to \infty$
  - Stable: $\to 0$
  - Unstable: Grows without bound
  - Marginally Stable: Neither decays nor grows but remains constant
- Total Response (BIBO- Bounded Input, Bounded Output)
  - Stable: Every bounded input yields a bounded output
  - Unstable: Any bounded inputs yields an unbounded output
  - Marginally stable: Some bounded inputs yield unstable outputs
- Stability $\Rightarrow$ only the forced response remains

#### *Stability for LTI systems in terms of pole locations*

- Closed-loop Transfer Function poles
  - Stable: Only in Left-Hand Plane (LHP)
  - Unstable: At least 1 in Right-hand plane (RHP) and/or multiplicity greater than 1 on the imaginary axis.
  - Marginally stable: Only imaginary axis poles of multiplicity 1 and poles in the LHP

## Routh-Hurwitz Criterion - History Interlude

1. Edward John Routh (1831-1907)
    - English Mathematician
    - 1876- Proposed what became the Routh-Hurwitz stability criterion
2. Adolf Hurwitz (1859-1919)
    - German Mathematician
    - 1895- Determined the Routh-Hurwitz stability criterion

## Routh-Hurwitz Stability Criterion

- Stability information without the need to solve for the Closed-loop system poles
- How many closed-loop system poles are in the LHP, RHP, and on the imaginary axis.

Two steps required in this method:

1. Generate a data table called a Routh table
2. Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $j\omega$-axis.

## Generating a basic Routh Table

Procedure;

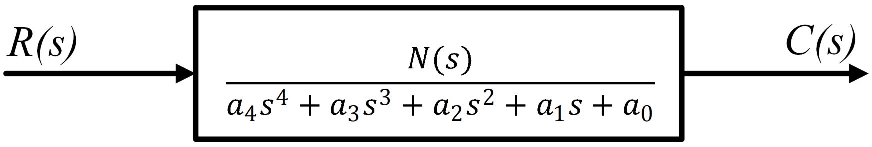


**Figure 6.1:** Equivalent CL transfer function

1. Label rows with powers of $s$ from the highest power of the denominator of the closed-loop transfer function down to $s^0$.
2. In the $1^{st}$ row, horizontally list every other coefficient starting with the coefficient starting with the coefficient of the highest power of $s$.
3. In the $2^{nd}$ row, horizontally list every other coefficient of the next highest power of $s$.

| | | | |
|---|---|---|---|
| $s^4$ | $a_4$ | $a_2$ | $a_0$ |
| $s^3$ | $a_3$ | $a_1$ | 0 |
| $s^2$ | | | |
| $s^1$ | | | |
| $s^0$ | | | |

**Table 6.1:** Routh table

4. Remaining row entries are filled with the negative determinant of entries in the previous 2 rows divided by entry in the 1st column directly above the calculated row. The left-hand column of the determinant is always the 1st column of the previous 2 rows, and the right-hand column is the elements of the column above and to the right.

| | | | |
|---|---|---|---|
| $s^4$ | $a_4$ | $a_2$ | $a_0$ |
| $s^3$ | $a_3$ | $a_1$ | $0$ |
| $s^2$ | $\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$ | $\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$ | $\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$ |
| $s^1$ | $\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$ | $\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$ | $\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$ |
| $s^0$ | $\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$ | $\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$ | $\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$ |

**Table 6.2:** Routh table

Interpreting a basic Routh table

- The number of roots of the polynomial that are in the RHP is equal to the number of signs changes in the $1^{st}$ column of a Routh table
- A system is stable if there are no sign changes in the first column of the Routh table

## Routh-Hurwitz Criterion: Special Cases

Two special cases can occur:

1. Zero only in the $1^{st}$ column
   - If the 1st element of a row is a zero, division by zero would be required to form the next row
2. Entire row of zeros
   - Result of there being a purely even polynomial that is a factor of the original polynomial

### *Zero only in the $1^{st}$ column*

1. Epsilon procedure
   - To avoid this phenomenon, an epsilon, $\varepsilon$, is assigned to replace zero in the 1st column.
   - The value $\varepsilon$ is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the $1^{st}$ column can be determined.

2. Reciprocal roots procedure
   - A polynomial that has the reciprocal roots of the original polynomial has its roots distributed the same–RHP, LHP, or imaginary axis–because taking the reciprocal of the root value does not move it to another region.
   - The polynomial that has the reciprocal roots of the original may not have a zero in the $1^{st}$ column.
   - Replacing $s$ with $\frac{1}{d}$ results in the original polynomial with its coefficient written in reverse order.

**Example 6.1** Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the feedback control system shown below:

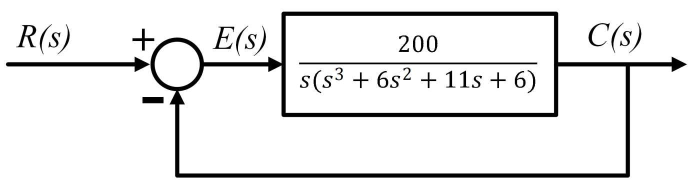


**Figure 6.2**

## Solution

1. *Find the closed-loop transfer function.*

$$G(s) = \frac{C(s)}{R(s)} = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

2. *Determine the characteristic equation.*

$$C.E. = s^4 + 6s^3 + 11s^2 + 6s + 200$$

3. *Generate the Routh table.*

| | | | | |
|---|---|---|---|---|
| $s^4$ | 1 | + | 11 | 200 |
| $s^3$ | 1 ~~6~~ | + | 1 ~~6~~ | 0 |
| $s^2$ | 1 ~~10~~ | + | 20 ~~200~~ | 0 |
| $s^1$ | -19 | + | 0 | 0 |
| $s^0$ | 20 | - | 0 | 0 |

**Table 6.3**

4. *Interpret the Routh Table.*

   *At the $s^1$ row there is a negative coefficient; thus, there are **two sign changes**. The system is **UNSTABLE**, since it has **two right-half-plane poles** and **two left-half-plane poles**. The system cannot have $j\omega$ poles since a row of zeros did not appear in the Routh table.*

5. *Verify the results of the stability of the system using simulation.*

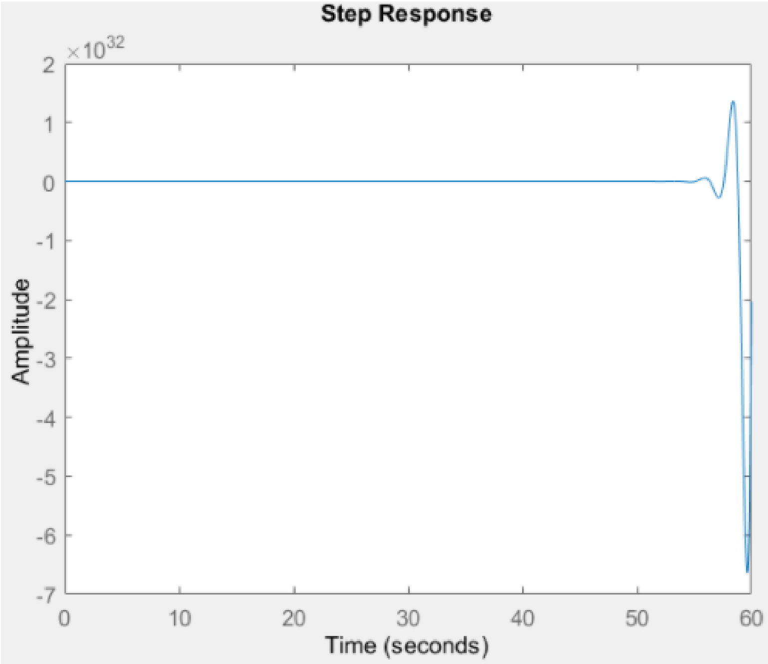


**Figure 6.3:** Step response

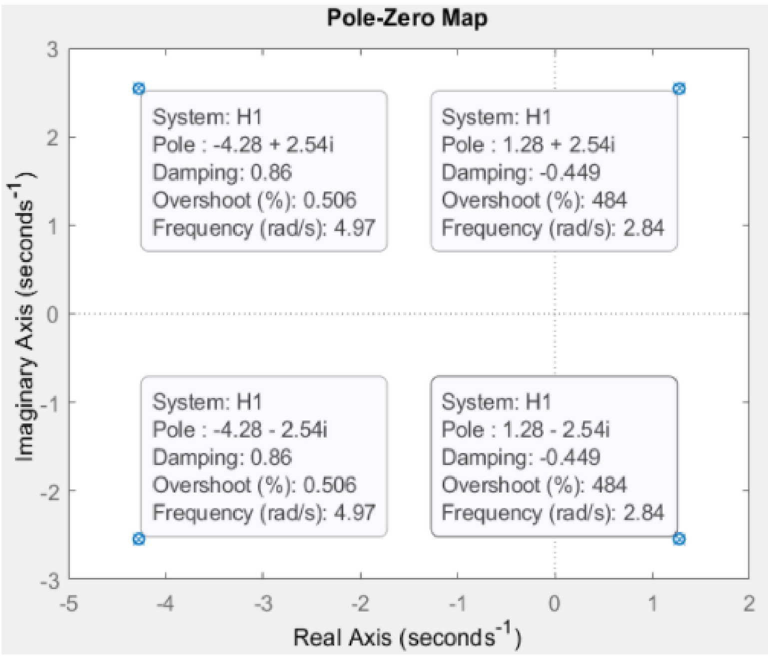


**Figure 6.4:** Pole-Zero map

## *Entire row of Zeros*

- Purely even polynomials: Only have roots that are symmetrical and real
  - Root positions to generate even polynomials (symmetrical about the origin)

1. Symmetrical and real
2. Symmetrical and imaginary
3. Quadrantal

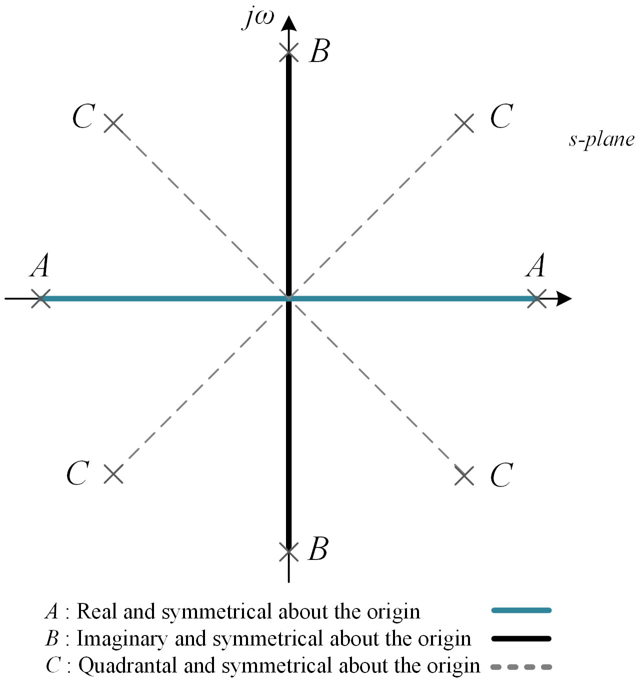


**Figure 6.5**

- Even polynomial appears in the row directly above the row of zeros.
- Every entry in the table from the even polynomial's row to the end of the chart applies only to the even polynomial.
- Number of sign changes from the even polynomial to the end of the table equals the number of RHP roots of the even polynomial.
- Even polynomial must have the same number of LHP roots as it does RHP roots.
- Remaining poles must be on the imaginary axis.
- The number of sign changes, from the beginning of the table down to the even polynomial, equals the number of RHP roots
- Remaining roots are LHP roots.
- The other polynomial can contain no roots on the imaginary axis.

**Example 6.2** Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the closed-loop Transfer Function, $T(s)$ shown below:

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

**Solution**

1. *Determine the characteristic equation.*

$$C.E. = s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20$$

2. *Generate the Routh Table*

| | | | | | | |
|---|---|---|---|---|---|---|
| $s^8$ | 1 ~~6~~ | + | 12 | 39 | 48 | 20 |
| $s^7$ | 1 | | 22 | 59 | 38 | 0 |
| $s^6$ | -1 ~~10~~ | | -2 ~~20~~ | 1 ~~10~~ | 2 ~~20~~ | 0 |
| $s^5$ | 1 ~~20~~ | | 3 ~~60~~ | 2 ~~40~~ | 0 | 0 |
| $s^4$ | 1 | | 3 | 2 | 0 | 0 |
| $s^3$ | 0 | | 0 | 0 | 0 | 0 |
| $s^2$ | | | | | | |
| $s^1$ | | | | | | |
| $s^0$ | | | | | | |

**Table 6.4**

3. *At the $s^3$ row we obtain a row of zeros. Moving back one row to $s^4$, extract the even polynomial, $P(s)$, as*

$$P(s) = s^4 + 3s^2 + 2$$

4. *Take the derivative of the even polynomial with respect to s to obtain the coefficients that replace the row of zeros in the $s^3$ row.*

$$\frac{dP(s)}{ds} = 4s^3 + 6s + 0$$

5. *Replace the row of zeros with the coefficients of the derivative 4, 6, and 0 , simplify the row for convenience and continue the table to the $s^0$ row, using the standard procedure.*

| | | | | | | |
|---|---|---|---|---|---|---|
| $s^8$ | 1 ~~6~~ | + | 12 | 39 | 48 | 20 |
| $s^7$ | 1 | + | 22 | 59 | 38 | 0 |
| $s^6$ | -1 ~~10~~ | - | -2 ~~20~~ | 1 ~~10~~ | 2 ~~20~~ | 0 |
| $s^5$ | 1 ~~20~~ | + | 3 ~~60~~ | 2 ~~40~~ | 0 | 0 |
| $s^4$ | 1 | + | 3 | 2 | 0 | 0 |
| $s^3$ | 2 ~~40~~ | + | 3 ~~60~~ | 0 | 0 | 0 |
| $s^2$ | 3 $\cancel{\frac{3}{2}}$ | + | 4 ~~2~~ | 0 | 0 | 0 |
| $s^1$ | $\frac{1}{3}$ | + | 0 | 0 | 0 | 0 |
| $s^0$ | 4 | + | 0 | 0 | 0 | 0 |

**Table 6.5**

6. *Interpret the Routh Table*

*It can be noted that $s^4$ down to $s^0$ is a test of even polynomials and is a factor of the transfer function. Since there is no sign changes from $s^4$ down to $s^0$, then the* ***even polynomial does not have right-half plane poles****. Since there are no right-half-plane poles,* ***no left-half-plane poles are present*** *on even polynomials because of the requirement for symmetry. Since the even polynomial is of fourth order, the* ***four remaining poles must be on the $j\omega$-axis****.*

*On the other hand, the remaining roots of the total polynomial are evaluated from the $s^8$ down to $s^4$ row. We notice two sign changes: one from the $s^7$ row to the $s^6$ row and the other from the $s^6$ row to the $s^5$ row. Thus, the other polynomial must have* ***two roots in the right half-plane and 2 roots on the left half plane for symmetry****.*

| Polynomial | | | |
|---|---|---|---|
| Location | Even (Fourth Order) | Other (Fourth Order) | Total (Eight-Order) |
| RHP | 0 | 2 | **2** |
| LHP | 0 | 2 | **2** |
| $j\omega$-axis | 4 | 0 | **4** |

**Table 6.6:** Summary table of pole locations

*Thus, the system has* ***two poles in the right half-plane, two poles in the left half-plane****, and* ***four poles on the $j\omega$-axis****; it is* ***UNSTABLE*** *because of the existence of right-half plane poles.*

7. *Verify the results of the stability of the system using simulation.*

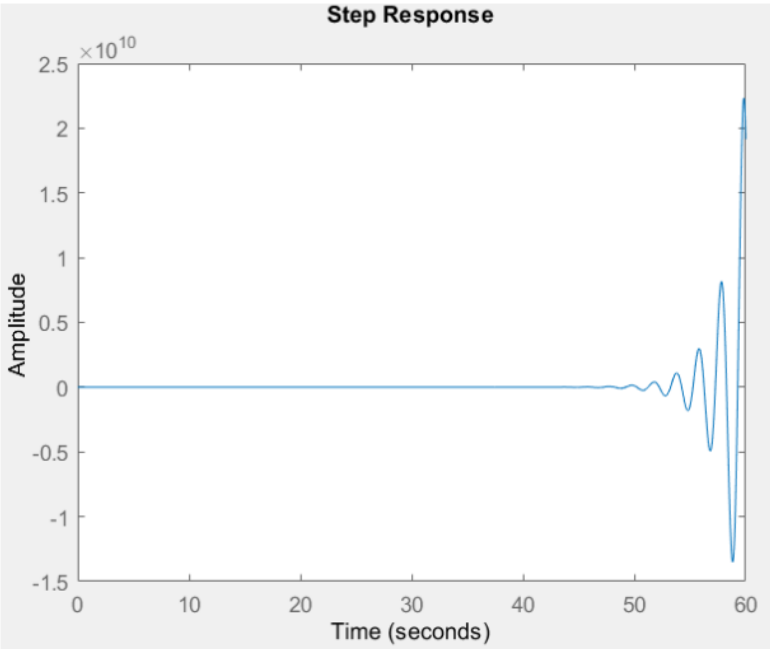


**Figure 6.6:** Step response

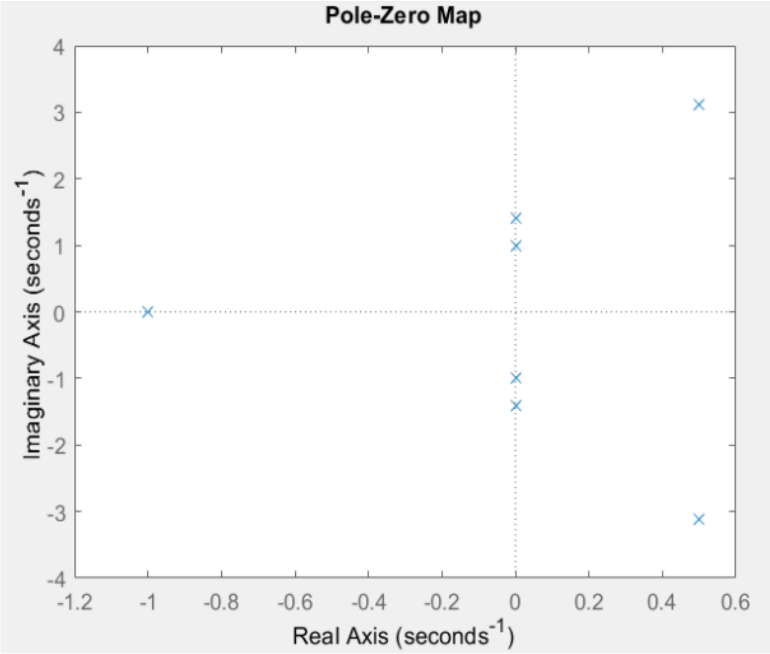


**Figure 6.7:** Pole-Zero map

## Stability in State Space

Eigenvalues, $\lambda$, of system matrix, A

- System poles
- Values that permit a nontrivial solution (other than 0) for eigenvectors, $x$, in the equation

$$\begin{aligned} Ax &= \lambda x \\ x &= (\lambda I - A)^{-1} 0 \\ &= \frac{adj(\lambda I - A)}{det(\lambda I - A)} 0 \end{aligned}$$

- All solutions will be the null vector except for the occurrence of zero in the denominator
- This is the only condition where elements of $x$ will be $\frac{0}{0}$ or indeterminate, it is the only case where a nonzero solution is possible.
- Solutions of the $det(\lambda I - A)$ will be the characteristic equation which is a polynomial.

## 6.3 Laboratory Experiment

### Module Exercises

1. Given the transfer function below, tell how many roots of the following polynomial are in the right half-plane, in the left half-plane, and on the $j\omega$-axis, and interpret the result whether it is stable, marginally stable or unstable.

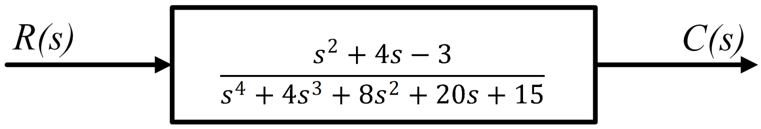


**Figure 6.8**

2. Designers have developed small, fast, vertical- takeoff fighter aircraft that are invisible to radar (stealth aircraft). This aircraft concept uses quickly turning jet nozzles to steer the airplane. The control system for the heading or direction control is shown below. Determine the maximum gain of the system for stable operation and compare two gains, $K_1$ and $K_2$ that would result in an unstable and stable operation using MATLAB.

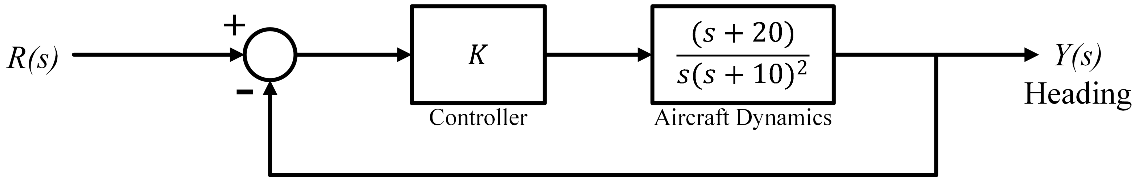


**Figure 6.9**

3. For the following system represented in state space, find out how many poles are in the left half plane, in the right half-plane, and on the $j\omega$-axis. Verify your answer by Plotting the Pole-Zero Map and Step Response of the system.

$$\dot{x} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x$$

### Simulation Activity

### Stability Analysis using Routh Hurwitz Methods

### Objective:

- To determine the stability of a system using the Routh Hurwitz method in MATLAB.

## Procedures

1. Download the **"RouthTable_Script.m"** file from the e-learning platform Google drive.
2. Open the file in the **MATLAB** software.
3. Load the script and **run** the program in the editor window.

```
%Generating Basic Routh Table using MATLAB Scripts

%  Instructions
%  ------------
%  In this program, you must enter your coefficients of the characteristics
%  equation, and the Routh-Hurwitz table will be displayed.

%% Initialization
clear ; close all; clc

% Taking coefficients vector and organizing the first two rows
coeffVector = input('Enter the coefficients of the Characteristic Equation= ');
ceoffLength = length(coeffVector);
RouthTableColumn = round(ceoffLength/2);

%  Initialize Routh-Hurwitz table with empty zero array
RouthTable = zeros(ceoffLength,RouthTableColumn);

%  Compute first row of the table
RouthTable(1,:) = coeffVector(1,1:2:ceoffLength);

%  Check if length of coefficients vector is even or odd
if (rem(ceoffLength,2) ~= 0)
    % if odd, second row of table will be
    RouthTable(2,1:RouthTableColumn - 1) = coeffVector(1,2:2:ceoffLength);
else
    % if even, second row of table will be
    RouthTable(2,:) = coeffVector(1,2:2:ceoffLength);
end

%% Calculate Routh-Hurwitz table's rows

%  Set epss as a small value
epss = 0.01;
```

```
% Calculate other elements of the table
for i = 3:ceoffLength

    % special case: row of all zeros
    if RouthTable(i-1,:) == 0
        order = (ceoffLength - i);
        cnt1 = 0;
        cnt2 = 1;
        for j = 1:RouthTableColumn - 1
            RouthTable(i-1,j) = (order - cnt1) * RouthTable(i-2,cnt2);
            cnt2 = cnt2 + 1;
            cnt1 = cnt1 + 2;
        end
    end

    for j = 1:RouthTableColumn - 1
        % first element of upper row
        firstElemUpperRow = RouthTable(i-1,1);

        % compute each element of the table
        RouthTable(i,j) = ((RouthTable(i-1,1) * RouthTable(i-2,j+1)) - ....
            (RouthTable(i-2,1) * RouthTable(i-1,j+1))) / firstElemUpperRow;
    end


    % special case: zero in the first column
    if RouthTable(i,1) == 0
        RouthTable(i,1) = epss;
    end
end

%% Compute number of right hand side poles(unstable poles)
% Initialize unstable poles with zero
unstablePoles = 0;

% Check change in signs
for i = 1:ceoffLength - 1
    if sign(RouthTable(i,1)) * sign(RouthTable(i+1,1)) == -1
        unstablePoles = unstablePoles + 1;
    end
```

```
end

%   Print calculated data on screen
fprintf('\n Routh-Hurwitz Table:\n')
RouthTable

%   Print the stability result on screen
if unstablePoles == 0
    fprintf('~~~~~> The system is STABLE! <~~~~~\n')
else
    fprintf('~~~~~> The system is UNSTABLE! <~~~~~\n')
end

fprintf('\n Number of Right hand side poles =%2.0f\n',unstablePoles)

reply = input('Do you want roots of system be shown? Y/N ', 's');
if reply == 'Y' || reply == 'y'
    sysRoots = roots(coeffVector);
    fprintf('\n Given polynomial coefficients roots :\n')
    sysRoots
end
```

4. Given the function $T(s) = \dfrac{s^2+4s+5}{s^4+s^3+3s^2+s+6}$, enter the coefficients of the characteristic equation in the command window. Be sure it is enclosed in open and closed brackets.
5. Press **Enter** and the **Routh Table** and its interpretation are generated.
6. Record the number of Right hand side poles and the system roots.

## 6.4 Questions to Ponder

1. What would happen to a physical system that becomes unstable?
2. How important is stability in a specific real life application?

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# Chapter 7
# Steady-State Errors

## 7.1 Objective

- To find the steady-state error for a unity feedback system
- To specify a system's steady-state error performance and design the gain of a closed-loop system to meet a steady-state error specification
- To find the steady-state error for disturbance inputs
- To find the steady-state error for non-unity feedback systems

## 7.2 Theory

### Steady State Error

The previous modules discussed the first two design specifications, mainly the *transient response* and *system stability*. In this module we will tackle the third design objective which is the **steady state error**. Errors in any control systems can be caused by a variety of factors, for instance, changes in the reference input of the system. A change can generate unpreventable steady state errors and errors in transient periods. Another cause of steady state error is control system flaws such as aging, static friction, backlash in gears, etc.

In this module we will not focus on system imperfections that cause steady state error, but instead on the type of system and test inputs (step, ramp, parabola) that cause these errors. By definition, **steady state error** is the difference between the input and the output of a system after the natural response has decayed to zero. It occurs when the output of the control system deviates from the desired response as shown in the Fig.7.1. As we can observe, there is a discrepancy between the desired output and the actual output. In other words, steady state error measures the accuracy of how the system follows the input command.

### Steady State Errors for a Unity Feedback System

For unity feedback systems, steady state error can be calculated in terms of:

1. **Closed-loop transfer function** Fig. 7.2 shows the general closed loop block diagram with the transfer function $T(s)$, and error $E(s)$.

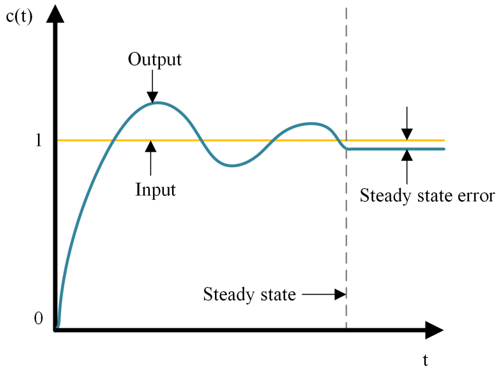



**Figure 7.1:** Steady state error

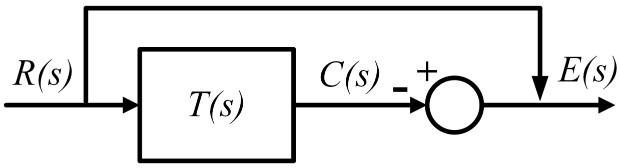



**Figure 7.2:** General block diagram of a closed loop control system error

Considering the figure above, the error $E(s)$ can be obtained by,

$$C(s) = R(s)T(s)$$
$$E(s) = R(s) - C(s)$$
$$E(s) = R(s)[1 - T(s)]$$

To find the final value of the error, the final value theorem is applied,

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$
$$e(\infty) = \lim_{s \to 0} sR(s)[1 - T(s)]$$

**Example 7.1** Solve the steady state error for the system below if R(s) is a step input.

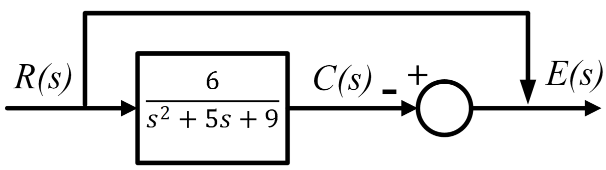



**Figure 7.3**

### Solution

*(a). Solve for $E(s)$.*

$$E(s) = R(s)[1 - T(s)]$$
$$E(s) = \frac{1}{s}[1 - \frac{6}{s^2 + 5s + 9}]$$
$$E(s) = [1 - \frac{s^2 + 5s + 9 - 6}{s^2 + 5s + 9}]$$
$$E(s) = \frac{s^2 + 5s + 3}{s(s^2 + 5s + 9)}$$

*(b). Solve for the final value of the steady state error*

$$e(\infty) = \lim_{s \to 0} sR(s)[1 - T(s)]$$
$$e_(\infty) = \frac{s(s^2 + 5s + 3)}{s(s^2 + 5s + 9)}$$
$$= \frac{0^2 + 5(0) + 3}{0^2 + 5(0) + 9}$$
$$e_(\infty) = \frac{1}{3}$$

### 2. Open-loop transfer function

Fig.7.4 is a feedback control system with an error, $E(s)$, feedback, $H(s)$, and system, $G(s)$. Although the transfer function $T(s)$ can be obtained and then proceed in finding the steady state error, calculating the steady state error in terms of $G(s)$ provides greater insight for analysis and design.

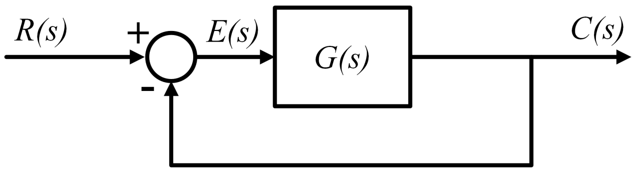



**Figure 7.4:** Block diagram of a unity feedback system with error

Considering the Figure above, the error $E(s)$ can be obtained by,

$$
\begin{aligned}
C(s) &= E(s)G(s) \\
E(s) &= R(s) - C(s) \\
E(s) &= R(s) - E(s)G(s) \\
E(s)[1 + G(s)] &= R(s) \\
E(s) &= \frac{R(s)}{1 + G(s)}
\end{aligned}
$$

Then applying the final value theorem to obtain the final steady state error value,

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

With this equation, conclusions can now be drawn about the relationships between the steady state error and the open loop system.

## *Test Signals*

Steady state errors of control systems depend upon the given type of input. The table below shows the three test inputs used to define the specifications for the steady state error characteristics of a control system. These test signals are unit step, ram, and parabolic inputs.

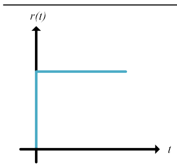
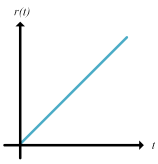
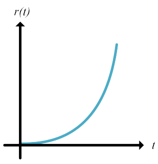
| Waveform | Name | Physical Interpretation | Time function | Laplace transform |
|---|---|---|---|---|
|  | Step | Constant position | $1$ | $\frac{1}{s}$ |
|  | Ramp | Constant velocity | $t$ | $\frac{1}{s^2}$ |
|  | Parabola | Constant acceleration | $\frac{1}{2}t^2$ | $\frac{1}{s^3}$ |

**Table 7.1:** Summary of block movements

1. **Step,** $u(t)$

For step input, $R(s)$ is equivalent to $\frac{1}{s}$. Using the equation $e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$ , the steady state error for a step input is ,

$$e(\infty) = e_{step}(\infty) = \lim_{s\to 0} \frac{s(\frac{1}{s})}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}$$

Wherein $1 + \lim_{s\to 0} G(s)$ is the dc gain of the forward transfer function. To have a zero steady state error, $G(s)$ must be in the form of $G(s) = \frac{(s+z_1)(s+z_2)...}{s^2(s+p_1)(s+p_2)}$. There must be at least one pure integration ($n \geq 1$).

2. **Ramp,** $tu(t)$ For ramp input, $R(s)$ is equivalent to $\frac{1}{s^2}$. Using the equation $e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$, the steady state error for a step input is ,

$$e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(\frac{1}{s^2})}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}$$

To have a zero steady state error, there must be at least two pure integration ($n \geq 2$).

3. **Parabola,** $t^2u(t)$ For parabolic input, R(s) is equivalent to $\frac{1}{s^3}$. Using the equation $e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$ , the steady state error for a parabolic input is ,

$$e(\infty) = e_{parabola}(\infty) = \lim_{s\to 0} \frac{s(\frac{1}{s^3})}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)}$$

To have a zero steady state error, there must be at least two pure integration ($n \geq 3$).

**Example 7.2** For the given system below, find the steady state error for each test input.

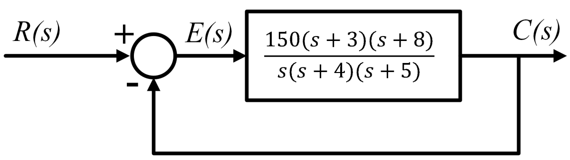



**Figure 7.5**

a. $7u(t)$
b. $7tu(t)$
c. $7t2u(t)$

**Solution** *Assume that the system is stable.*

a. $7u(t)$

$$e_{step}(\infty) = \frac{1}{1 + \lim_{s\to 0} G(s)} = \frac{7}{1 + \frac{110(s+3)(s+8)}{s(s+4)(s+5)}} = \frac{7}{1 + \frac{110(0+3)(0+8)}{0(0+4)(0+5)}} = \frac{7}{\infty} = 0$$

b. $7tu(t)$

$$e_{ramp}(\infty) = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{7}{\frac{s(110)(s+3)(s+8}{s(s+4)(s+5)}} = \frac{7}{\frac{110(0+3)(0+8)}{(0+4)(0+5)}} = \frac{7}{132} = 0.05$$

c. $7t^2u(t)$

$$e_{parabola}(\infty) = \frac{1}{\lim_{s\to 0} s^2G(s)} = \frac{14}{\frac{s^2(110)(s+3)(s+8)}{s(s+4)(s+5)}} = \frac{14}{\frac{0(110)(0+3)(0+8)}{(0+4)(0+5)}} = \frac{14}{0} = \infty$$

## Static Error Constants and Type of Systems

In Module 4 - Time Response, transient response is characterized by damping ratio, natural frequency, and specifications such as settling time, peak, percent overshoot, etc. For this module, we will specify the steady state error performance specifications of control systems. These specifications are referred to as static error constants. These constants are the position **constant, $K_p$, velocity constant, $K_v$ , and acceleration constant, $K_a$**.

From the previous discussion above, we derived the following steady state error equations for every test input. In these equations, the three terms in the denominator taken to the limit that determine the steady state error are the **static error constants**.

1. **Step, $u(t)$**

$$e_{step}(\infty) = \frac{1}{1 + \lim_{s\to 0} G(s)}$$

The position constant, $K_p$, is defined by $K_p = \lim_{s\to 0} G(s)$ .

Therefore, the steady state error formula in terms of position constant is $e_{step}(\infty) = \dfrac{1}{1 + K_p}$

2. **Ramp, $tu(t)$**

$$e_{ramp}(\infty) = \frac{1}{\lim_{s\to 0} sG(s)}$$

The velocity constant, $K_v$, is defined by $K_v = \lim_{s\to 0} sG(s)$ .

Therefore, the steady state error formula in terms of velocity constant is $e_{ramp}(\infty) = \dfrac{1}{K_v}$

3. **Parabola, $t^2u(t)$**

$$e_{parabola}(\infty) = \frac{1}{\lim_{s\to 0} s^2G(s)}$$

The acceleration constant, $K_a$, is defined by $K_a = \lim_{s\to 0} s^2G(s)$ .

Therefore, the steady state error formula in terms of acceleration constant is $e_{parabola}(\infty) = \dfrac{1}{K_a}$

Based on these quantities, it can be concluded that **the higher the static error constants are, the smaller the steady-state error**, since they are in the denominator of the error equation.

**Note** *The static error constants discussed above are critical for the steady state error analysis only if the input signal is a step, ramp, or parabolic respectively*

**Example 7.3** For the system below, find the static error constants ($K_p, K_v, K_a$) and expected steady state error for every input signal (step, ramp,parabola). Assume that the system is stable.

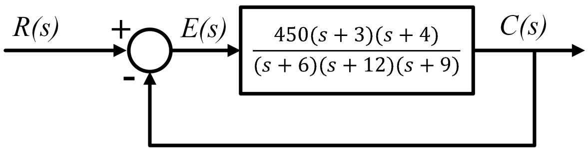



**Figure 7.6**

**Solution**

*Static error constants*

a. $K_p$

$$K_p = \lim_{s\to 0} G(s) = \frac{450(0+3)(0+4)}{(0+6)(0+12)(0+9)} = 8.33$$

b. $K_v$

$$K_v = \lim_{s\to 0} sG(s) = \frac{0(450)(0+3)(0+4)}{(0+6)(0+12)(0+9)} = 0$$

c. $K_a$

$$K_a = \lim_{s\to 0} s^2G(s) = \frac{0^2(450)(0+3)(0+4)}{(0+6)(0+12)(0+9)} = 0$$

*Steady State Error*

a. *Step*

$$e_{step}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+8.33} = 0.107$$

b. *Ramp*

$$e_{ramp}(\infty) = \frac{1}{K_v} = \frac{1}{0} = \infty$$

c. *Parabola*

$$e_{parabola}(\infty) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Fig. 7.7 is a representation of a feedback system for defining the system type.

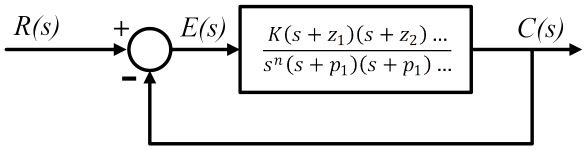



**Figure 7.7:** Feedback system for determining system type

The system type is the value of $n$ in the denominator. This refers to the number of pure integrations in the forward path of a unity feedback system. For instance, **if n = 0 or n =1, the corresponding system is Type 0 or Type 1**.

The figure below summarizes the concepts of static error constants and steady state errors for type 0, type 1, and type 2 systems due to different input signals.

| | | Type 0 | | Type 1 | | Type 2 | |
|---|---|---|---|---|---|---|---|
| **Input** | **Steady-state error formula** | **Static error constant** | **Error** | **Static error constant** | **Error** | **Static error constant** | **Error** |
| Step, $u(t)$ | $\frac{1}{1+K_p}$ | $K_p = \text{Constant}$ | $\frac{1}{1+K_p}$ | $K_p = \infty$ | 0 | $K_p = \infty$ | 0 |
| Ramp, $tu(t)$ | $\frac{1}{K_v}$ | $K_v = 0$ | $\infty$ | $K_v = \text{Constant}$ | $\frac{1}{K_v}$ | $K_v = \infty$ | 0 |
| Parabola, $\frac{1}{2}t^2u(t)$ | $\frac{1}{K_a}$ | $K_a = 0$ | $\infty$ | $K_a = 0$ | $\infty$ | $K_a = \text{Constant}$ | $\frac{1}{K_a}$ |

**Figure 7.8:** Summary of the relationships between signal input, system type, static error constants, and steady-state errors

By these relationships, a specific steady state error specification can provide a lot of information. Several conclusions can be drawn from a given static error constant.

For instance, a control system has a velocity constant of $K_v = 150$. We can conclude that:

a. The system is stable.
b. It is a Type 1 system, since Type 1 systems are the only system type where velocity constants are finite. Refer to the table above.
c. Since $K_v$ is finite and the error for a ramp input is inversely proportional to $K_v$, therefore the test signal is a ramp input.
d. The steady-state error between the input ramp and the output ramp is $1 = K_v$ per unit of input slope.

Let us now use these static error constants in designing the gain of a closed-loop system in order to meet a steady state error specification.

**Example 7.4** A feedback control system is given below. From the figure, determine the value of K (gain) to achieve a 10% in the steady state.

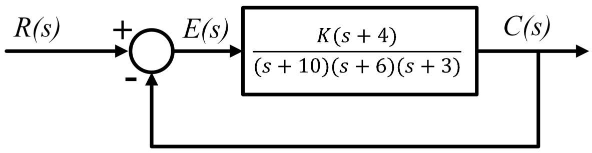



**Figure 7.9**

**Solution** *Since the system* $\dfrac{C(s)}{R(s)} = \dfrac{K(s+4)}{s(s+10)(s+6)(s+3)}$ *is Type 1, the test signal is a ramp input and yields a finite error ($K_v$, refer to the table above) . Therefore, the equation is*

$$e_{ramp}(\infty) = \frac{1}{K_v} = 0.1(10\%)$$

*Solve for $K_v$,*

$$K_v(0.1) = 1$$
$$K_v = 10.1 = 10$$

*Since $K_v = \lim_{s\to 0} sG(s)$,*

$$10 = \frac{K(0+4)}{(0+10)(0+6)(0+3)}$$

*Therefore the value of K is 450.*

## Steady State Error for Disturbances

During operations, physical systems are prone to disturbances. For instance, outside temperature fluctuations that affect a heating system, gravity forces, or wind gusts that affect the position of an antenna. They are undesirable signals that affect the input or output of a process. Closed-loop or feedback control systems are capable of compensating these influences that enter the system. The system can be designed to follow the input with minimal or zero error.

Fig. 7.10 illustrates a feedback control system that shows a disturbance between the controller and the plant. From this, the equation for steady state error with disturbance can be derived.

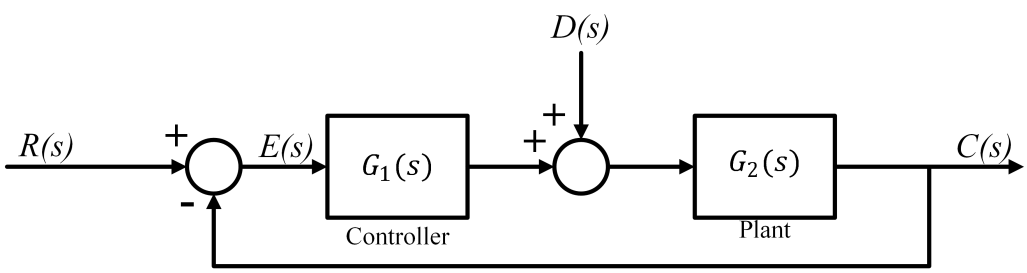



**Figure 7.10:** Feedback control system with disturbance

The total steady steady state error is the sum of the error due to the input $R(s)$ and error due to the disturbance $D(s)$.

$$e(\infty) = e_R(\infty) + e_D(\infty)$$
$$e_R(\infty) = \lim_{s\to 0} \frac{s}{1+G_1(s)G_2(s)} R(s)$$
$$e_D(\infty) = \lim_{s\to 0} \frac{sG_2(s)}{1+G_1(s)G_2(s)} D(s)$$

Assuming that the disturbance is a step signal, the steady state error due to the disturbance is

$$e_D(\infty) = -\frac{1}{\lim_{s\to 0} \frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)}$$

This means that the effect of a disturbance can be reduced by adjusting system gains, either increasing the gain of $G_1(s)$ or decreasing the gain of $G_2(s)$.

**Example 7.5** Determine the steady state error due to a step disturbance.

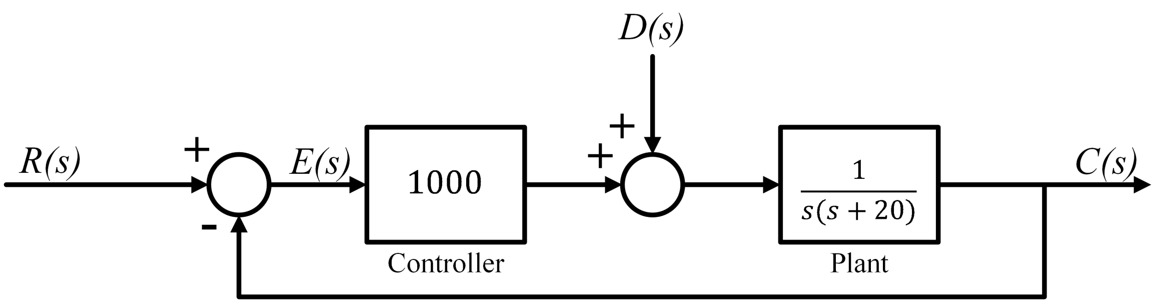



**Figure 7.11**

**Solution**

1. *Substitute the values into the equation.*

$$e_D(\infty) = -\frac{1}{\lim_{s\to 0} \dfrac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)}$$
$$e_D(\infty) = -\frac{1}{\dfrac{1}{s(s+20)} + 1000}$$

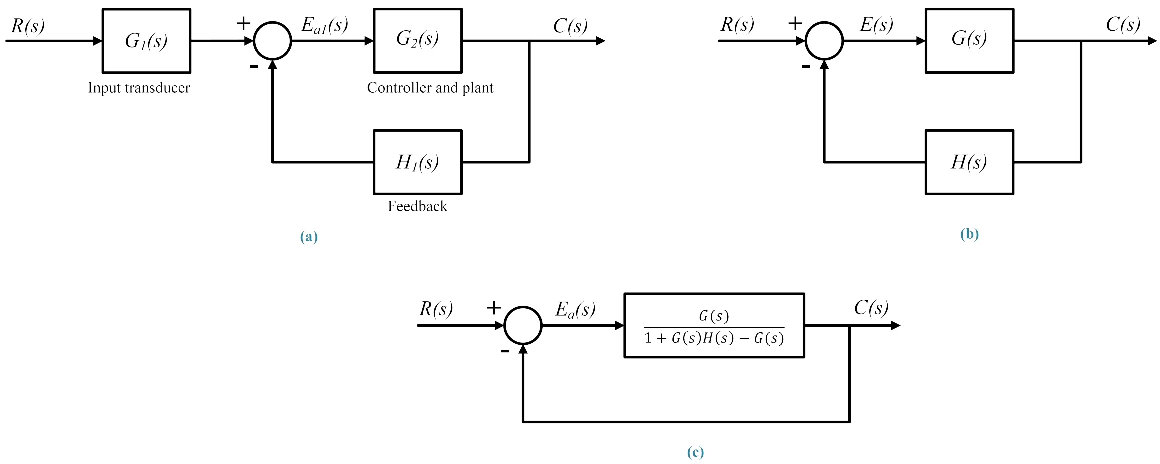



**Figure 7.12:** **a.** general feedback system **b.** equivalent nonunity feedback system **c.**equivalent unity feedback system

2. *Solve.*

$$e_D(\infty) = -\frac{1}{0(0+20)+1000}$$

$$e_D(\infty) = -\frac{1}{1000}$$

## Steady-State Error for Non Unity Feedback Systems

In a practical sense, most control systems do not have unity feedback. This is due to the system's physical model or the compensation utilized to improve system performance.

Fig. 7.12 illustrates the block diagram of a general feedback control system and its equivalent nonunity and unity feedback system. Forming an equivalent unity feedback system from a general nonunity feedback system.

As seen in the nonunity feedback system above, the difference between the input and output signal is not considered the error. This is referred to as actuating signal $E_a(s)$. The actuating signal can be obtained by,

$$e_{al} = \lim_{s\to 0} \frac{sR(s)G_1(s)}{1+G_2(s)H_1(s)}$$

**To solve for the steady state error, the nonunity feedback system must be converted into a unity feedback system.** The unity feedback is obtained by adding and subtracting unity feedback paths which yields to the equivalent transfer function in Fig. 7.12

For non unity feedback system with a disturbance, the steady state error is,

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\left\{\left[1-\frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}\right]R(s)-\left[\frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\right]\right\}$$

On the other hand, the steady state error due to a step disturbance is,

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\left\{\left[1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{\lim_{s\to 0}[1 + G_1(s)G_2(s)H(s)]}\right] - \left[\frac{\lim_{s\to 0} G_2(s)}{\lim_{s\to 0} 1 + G_1(s)G_2(s)H(s)}\right]\right\}$$

**Example 7.6** From the given nonunity feedback system, solve for the steady state error and steady state actuating signal for a unit step input.

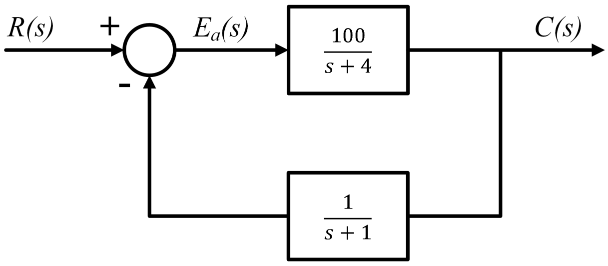



**Figure 7.13**

## Solution

1. *Convert the nonunity feedback system to unity feedback system using the equation:*

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

$$G_e(s) = \frac{\frac{100}{s+5}}{1 + (\frac{100}{s+5})(\frac{1}{s+1}) - (\frac{100}{s+5})}$$

$$G_e(s) = \frac{100(s+1)}{s^2 - 94s + 5}$$

2. *There are no pure integrations in $G_e(s)$, therefore the system is Type 0 and the static error constant is $K_p$.*

$$K_p = \frac{100(0+1)}{0^2 - 94(0) + 5}$$

$$K_p = 20$$

3. *Solve the steady state error for a step input.*

$$e_{step}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+20} = \frac{1}{21}$$

$$e_{step}(\infty) = 4.545 \times 10^{-2} or\ 0.04545$$

4. *Solve for the steady state actuating signal for a step input using the formula.*

$$e_{al}(\infty) = \lim_{s\to 0} \frac{sR(s)G_1(s)}{1 + G_2(s)H_1(s)}$$

$$e_{al}(\infty) = \frac{s\frac{1}{s}(1)}{1 + (\frac{100}{s+5})(\frac{1}{s+1})} = \frac{1}{1 + (\frac{100}{0+5})(\frac{1}{0+1})} = \frac{1}{21}$$

$$e_{al}(\infty) = 4.545 \times 10^{-2} or\ 0.04545$$

# 7.3 Laboratory Experiment

## Module Exercises

1. For the given system below. Determine the following:

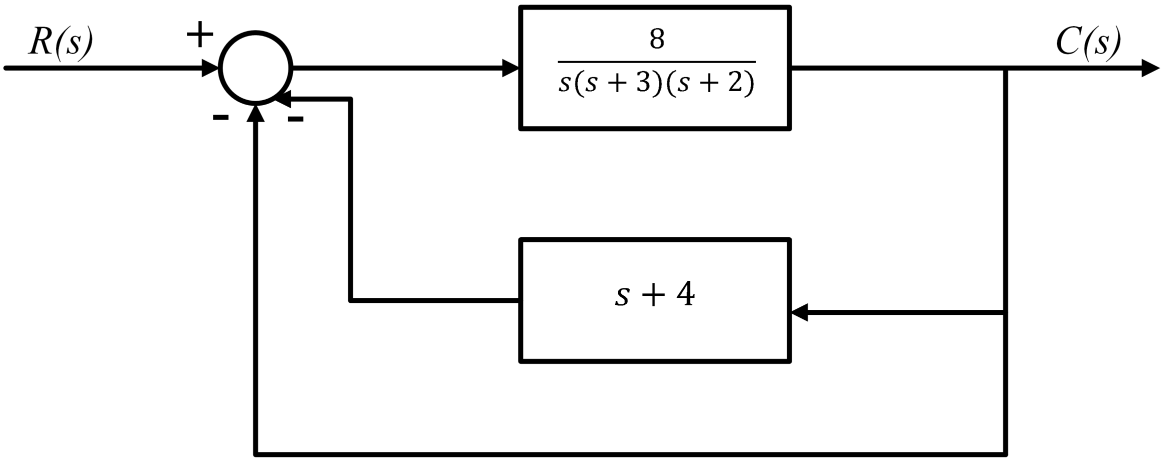



**Figure 7.14**

   a. Static error constants
      - $K_p$
      - $K_v$
      - $K_a$

   b. Steady state error for an input of:
      - $45u(t)$
      - $45tu(t)$
      - $45t^2u(t)$

   c. System type

2. Determine the following for the given system below.

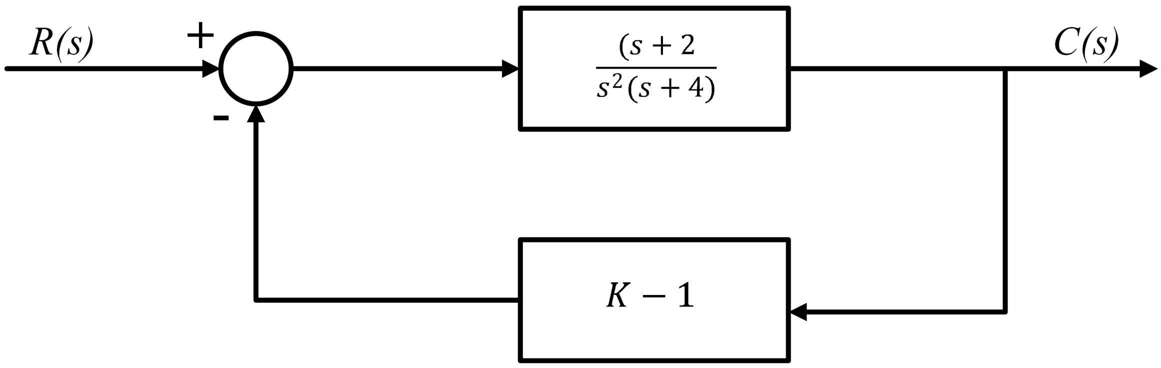



**Figure 7.15**

   a. System type

b. The gain value (K) so that there is 0.1% error in the steady state.

## Simulation Activity

## Steady State Error due to a Step input and Disturbance

## Objective:

- To observe the relationship of step input, gains, disturbance and steady state error

## Procedures

**Part I**

1. Consider the figure below that shows a feedback system with a disturbance. Find the the steady state error due to a step input $R(s) = \frac{10}{s}$ with a disturbance $D(s) = 0$ manually.

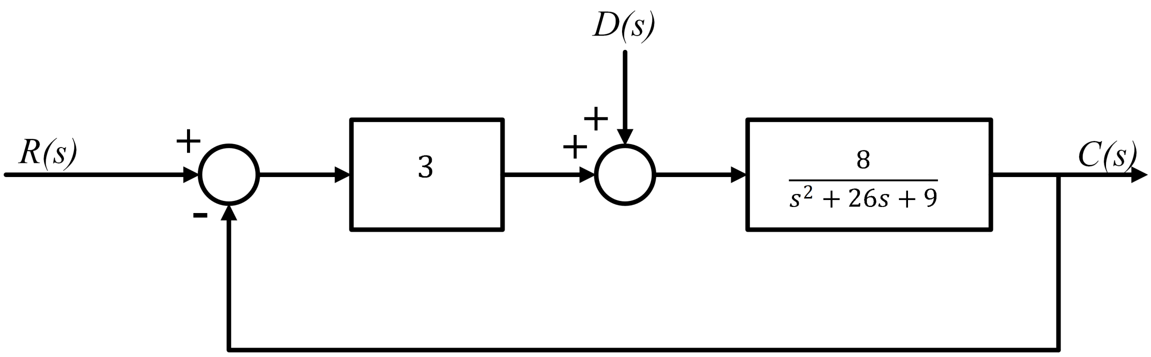



Figure 7.16

2. Open a blank Simulink model in MATLAB Software.
3. Add two **Step** blocks namely $R(s)$ and $D(s)$, two **Sum** blocks, a **Gain** block, a **Transfer Fcn** block, a **Mux** and a **Scope** block.
4. Connect the component blocks based on the configuration shown on Figure 7.17 below:

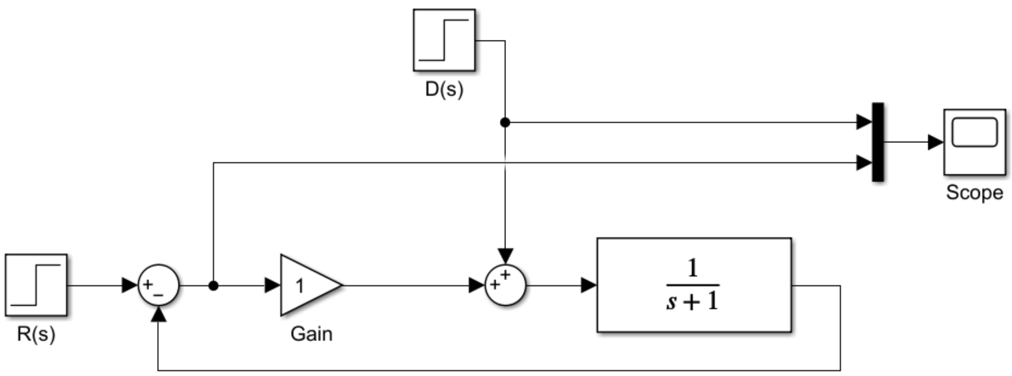



Figure 7.17

5. Modify the parameters of each block:
   a. Step block, $R(s)$: Final value = 10 ;Step time = 0
   b. Gain block = 3
   c. Transfer function block = $\dfrac{8}{s^2+26s+9}$
6. Click **Run** and double click the scope to observe the plot. Verify the result of procedure 1.
7. Modify the gain values and observe the resulting plots.
   - K= 5
   - K= 10
   - K= 20

**Part II**

1. Find the steady state error due to a step disturbance $D(s) = -\dfrac{2}{s}$ with $R(s) = 0$ manually.
2. Modify the parameters of each block:
   a. Step block $R(s) = 0$
   b. Step block $D(s) = -\dfrac{2}{s}$
   b. Gain block= 3
3. Make sure that Step $D(s)$ and the signal are connected before the gain to the MUX block.
4. Click **Run** and **double click** the scope to observe the plot. Verify the result of procedure 1.
5. Find the the steady state error due to a step input $R(s) = \dfrac{10}{s}$ with a disturbance $D(s) = -\dfrac{2}{s}$ manually.
6. Modify the number of inputs of the **MUX** block
   a. Number of inputs = 3
7. Connect the $D(s)$, $R(s)$, and transfer function block to the MUX block.
8. Modify the block parameters:
   a. Step block R(s) = $R(s) = \dfrac{10}{s}$
   b. Step block D(s) = $D(s) = -\dfrac{2}{s}$
9. Click **Run** and double click the scope to observe the plot.

## 7.4 Questions to Ponder

1. How will you compare the calculated and simulated results?
2. Based on the simulation, how does the gain value affect the steady state error? Explain their relationship.
3. From your observation, does a disturbance affect the system output? Does it affect the system's steady state error?

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# Chapter 8
# Root Locus Techniques

## 8.1 Objective

- To define a root locus
- To state the properties of a root locus
- To find the coordinates of points on the root locus and their associated gains
- To use the root locus to design a parameter value to meet a transient response specification for systems of order 2 and higher

## 8.2 Theory

### History Interlude

The location of the closed-loop roots of the characteristic equation in the s-plane is directly related to the relative stability and transient performance of a closed-loop control system. In order to obtain suitable root locations, one or more system parameters must frequently be adjusted. As a result, it is worthwhile to determine how the roots of a given system's characteristic equation migrate about the s-plane as the parameters are varied; in other words, it is useful to determine the locus of roots in the s-plane as a parameter is varied. Walter Richard Evans (1920 – 1999), an American control theorist, introduced the root locus method in 1948 and it has been developed and widely used in control engineering practice. He was awarded with Richard E. Bellman Control Heritage Award for his very significant contribution to the field of automatic control systems analysis and synthesis by inventing the root-locus technique.

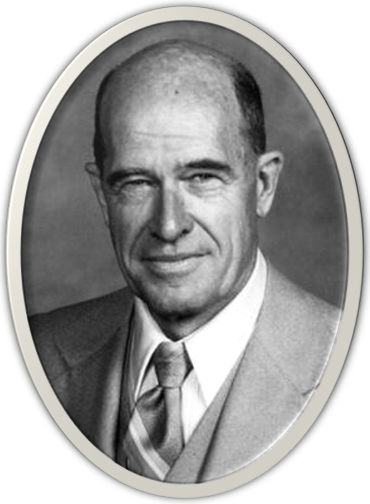


**Figure 8.1:** Walter Richard Evans (1920-1999)

### Introduction

### *What is Root Locus?*

The root locus technique is a graphical method for drawing the location of roots in the s-plane as a parameter is changed. The root locus method gives the engineer a measure of the sensitivity of the system's roots to a change in the parameter under consideration. The root locus technique can be very useful when combined with the Routh–Hurwitz criterion.

**Definition 8.1**

***Root Locus** is the location of closed-loop poles as a system parameter like gain, K, is varied.* ♣

General idea of Root Locus (RL)

- It uses the poles and zeros of the Open-loop transfer function (product of the forward path transfer function and feedback path transfer function) to analyze and design the poles of a closed-loop transfer function as a system (plant or controller) parameter, $K$, that shows up as a gain in the open-loop transfer function is varied.
- It is a graphical representation of:
  - Stability (closed-loop poles)
    - Range of stability, instability, & marginal stability
  - Transient response
    - Rise time ($T_r$), Settling time ($T_s$) and Percent Overshoot ($\%OS$)

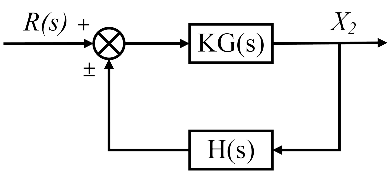


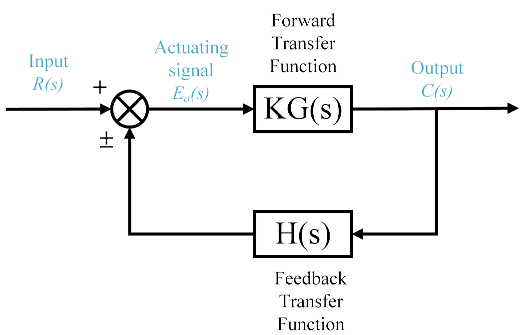

**Figure 8.2:** Feedback Control System

Two concepts that are essential in root locus:

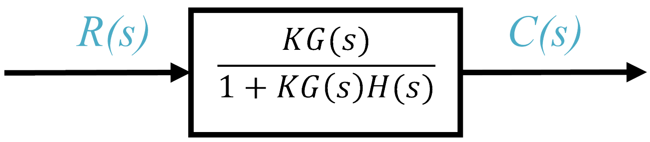
1. The control system problem
2. Complex numbers and their representation as vectors

## *The Control System Problem*

A typical closed-loop feedback control system is shown in Figure 8.3. The open-loop transfer function was defined $KG(s)H(s)$. Ordinarily, we can determine the poles of $KG(s)H(s)$, since these poles arise from simple cascaded first- or second-order subsystems. Further, variations in $K$ do not affect the location of any pole of this function. On the other hand, we cannot determine the poles of $T(s) = KG(s)/[1 + KG(s)H(s)]$ unless we factor the denominator. Also, the poles of $T(s)$ change with K.




**(a)** Negative Feedback Closed-loop system




**(b)** Equivalent Transfer Function

**Figure 8.3:** Closed-loop Feedback Control System

Since the Forward Transfer Function is ,

$$G(s) = \frac{N_G(s)}{D_G(s)} \tag{8.1}$$

and Feedback Transfer Function is

$$H(s) = \frac{N_H(s)}{D_H(s)} \tag{8.2}$$

Then, the Feedback Closed-loop Transfer Function is

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)} \tag{8.3}$$

where $N$ and $D$ are factored polynomials and signify numerator and denominator terms, respectively.

From these equations, we observe the following:

- The zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$
- The poles of $T(s)$ are not immediately known and in fact can change with $K$.
- The poles of $T(s)$ are not immediately known without factoring the denominator, and they are a function of $K$.
- Since the system's transient response and stability are dependent upon the poles of $T(s)$, we have no knowledge of the system's performance unless we factor the denominator for specific values of $K$.

## *Vector Representation of Complex Numbers*

Any complex number, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector, as shown in Fig. 8.4 The complex number also can be described in polar form with magnitude $M$ and angle $\theta$, as $M\angle\theta$. If the complex number is substituted into a complex function, $F(s)$, another complex number will result. For example, it is shown in Fig. 8.4 that if $F(s) = s + a$, then substituting the complex number $s = \sigma + j\omega$ yields $F(s) = (\sigma + a) + j\omega$, another complex number. We conclude that $s + a$ is a complex number and can be represented by a vector drawn from the zero of the function to the point $s$.

Now let us apply the concepts to a complicated function. Assume a function

$$F(s) = \frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = \frac{\prod numerator's\ complex\ factors}{\prod denominator's\ complex\ factors} \tag{8.4}$$

where the symbol $\prod$ means "product," $m$ = number of zeros; and $n$ number of poles. Since each complex factor can be thought of as a vector, the magnitude, $M$, of $F(s)$ at any point, $s$, is

$$M = \frac{\prod zero\ lengths}{\prod pole\ lengths} = \frac{\prod_{i=1}^{m}|(s + z_i)|}{\prod_{j=1}^{n}|(s + p_j)|} \tag{8.5}$$

where a zero length, $|(s + z_i)|$, is the magnitude of the vector drawn from the zero of $F(s)$ at $-z_i$ to the point $s$, and a pole length, $|(s + p_j)|$, is the magnitude of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point $s$.

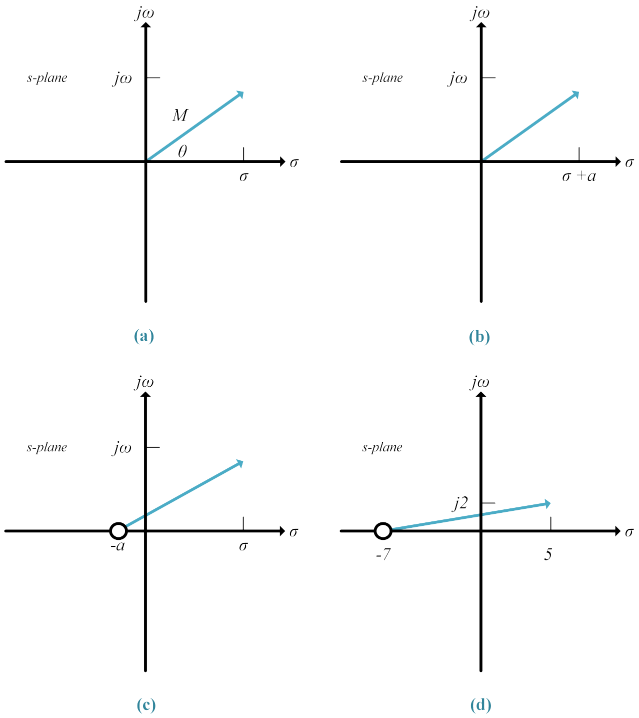



**Figure 8.4:** Vector representation of complex numbers a. $s = \sigma + j\omega$, b. $(s + a)$; c. alternate representation of $(s + a)$, d. $(s + 7)|_{s\to +j2}$

On the other hand, the angle, $\theta$, of $F(s)$ at any point, $s$, is

$$\theta = \sum zero\ angles - \sum pole\ angles = \sum_{i=1}^{m} \angle(s + z_i) - \sum_{j=1}^{n} \angle(s + p_j) \tag{8.6}$$

where a zero angle is the angle, measured from the positive extension of the real axis, of a vector drawn from the zero of $F(s)$ at $-z_i$ to the point $s$, and a pole angle is the angle, measured from the positive extension of the real axis, of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point $s$.

**Example 8.1** Given the function $G(s) = \dfrac{s+1}{s(s+2)}$, find F(s) at the point $s = -3 + j4$.

**Solution**

1. *Find the poles and zeros of the given transfer function*

$$Zeros : s = -1 \tag{8.7}$$

$$Poles : s = 0; s = -2 \tag{8.8}$$

2. *Identify the vectors (in polar form) by substituting $s = -3 + j4$*

- *L1 (zero vector)*

$$(s + 1) = (-3 + j4 + 1) = -2 + j4 \rightarrow 2\sqrt{5} < 116.57^\circ$$

- *L2 (Pole$(s = 0)$ vector)*

$$s = -3 + j4 \rightarrow 5 < 126.87^\circ$$

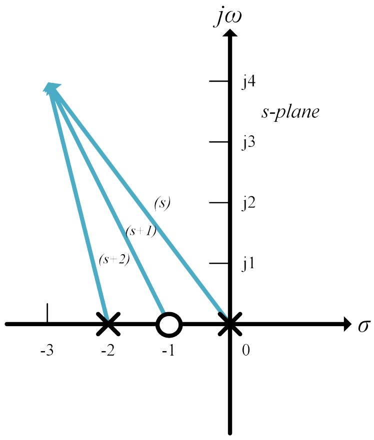



**Figure 8.5:** Vector representation

- *L3 (Pole $(s = -2)$ vector)*

$$(s+2) = (-3 + j4 + 2) = -1 + j4 \rightarrow \sqrt{17} < 104.04°$$

3. *Find the magnitude and angle*

$$M = \frac{\prod zerolengths}{\prod poleslengths} = \frac{2\sqrt{5}}{(5)(\sqrt{17})} = 0.217$$

$$\theta = \sigma zeroangles - \sigma poleangles = 116.57° - (126.87° + 104.04°) = -114.34°$$

$$M\angle\theta = 0.217\angle - 114.34°$$

## Defining Root Locus

The root locus technique can be used to analyze and design the effect of loop gain upon the system's transient response and stability.

A security camera system similar to that shown in Fig. 8.6 can automatically follow a subject. The tracking system monitors pixel changes and positions the camera to center the changes.

Assume the block diagram representation of a tracking system as shown in Fig. 8.6, where the closed-loop poles of the system change location as the gain, $K$, is varied. Table 8.1 , which was formed by applying the quadratic formula to the denominator of the transfer function in Fig. 8.6, shows the variation of pole location for different values of gain, $K$. The data of Table 8.1 is graphically displayed in Fig. 8.7, which shows each pole and its gain.

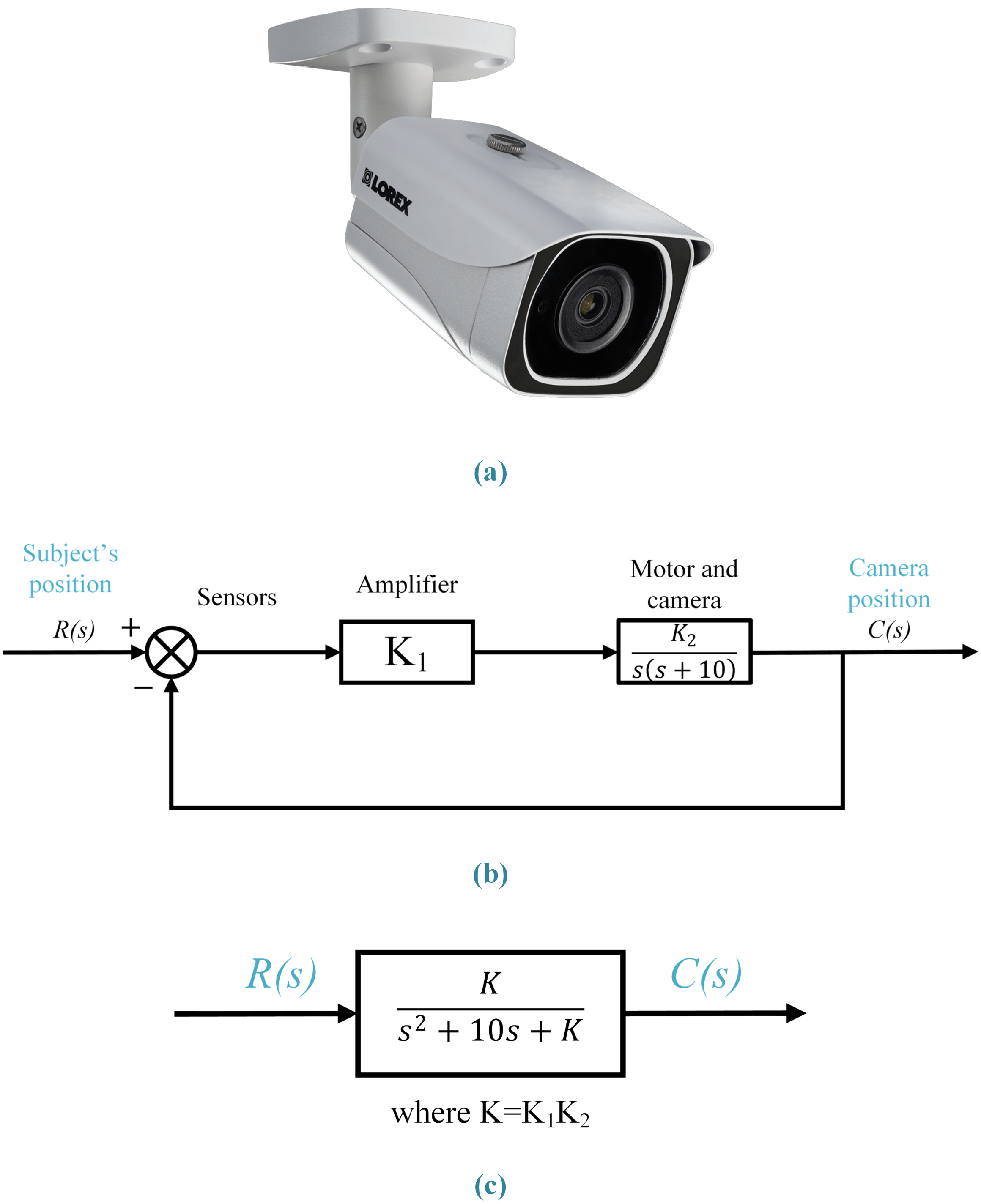



**Figure 8.6:** **a.** Security cameras with auto tracking can be used to follow moving objects automatically **b.** Negative Feedback Closed-loop system **c.** Equivalent Transfer Function

| K | Pole 1 | Pole 2 |
|---|---|---|
| 0 | -10 | 0 |
| 5 | -9.47 | -0.53 |
| 10 | -8.87 | -1.13 |
| 15 | -8.16 | -1.84 |
| 20 | -7.24 | -2.76 |
| 25 | -5 | -5 |
| 30 | $-5 + j2.24$ | $-5 - j2.24$ |
| 35 | $-5 + j3.16$ | $-5 - j3.16$ |
| 40 | $-5 + j3.87$ | $-5 - j3.87$ |
| 45 | $-5 + j4.47$ | $-5 - j4.47$ |
| 50 | $-5 + j5$ | $-5 - j5$ |

**Table 8.1:** Pole location as function of gain for the system

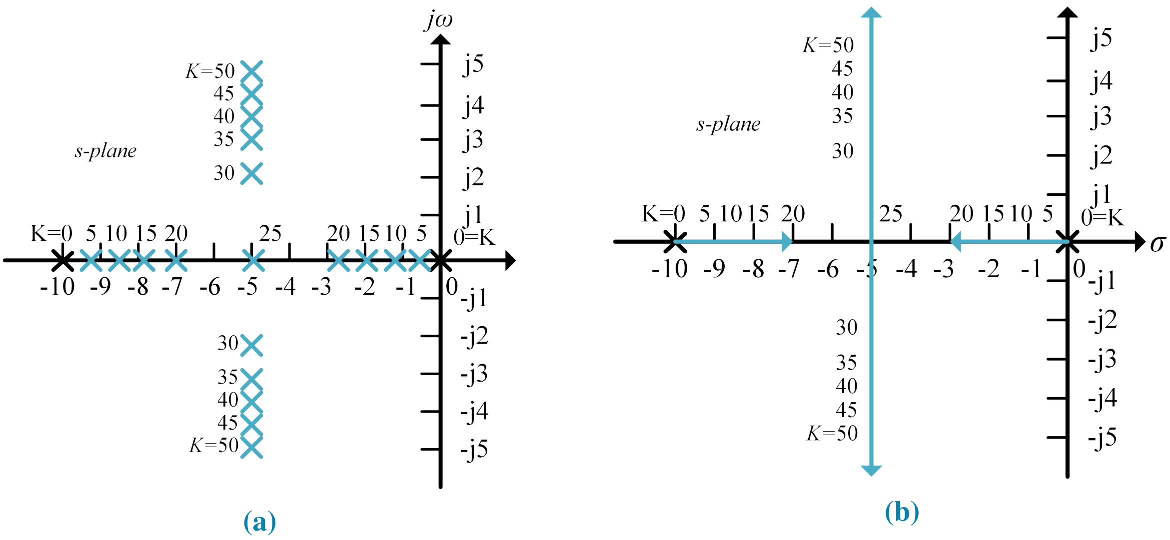



**Figure 8.7:** a. Pole Plot ; b. Root Locus

By inspecting the root locus, we could interpret that:

1. $\%OS$ is directly proportional to gain, $K$;
2. $T_s$ remains constant as the gain, $K$ increases (under all conditions of underdamped responses); and
3. $T_p$ is inversely proportional to gain, $K$.

## Properties of Root Locus

In this section, we will examine the properties of the root locus which should enable us to make a rapid sketch of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function.

The properties of the root locus can be derived from the general control system of Figure 8.3. The closed-loop transfer function for the system is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \tag{8.9}$$

1. The angle of the complex number is an odd multiple of 180°

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad ; k = 0, \pm1, \pm2, \pm3, \ldots \tag{8.10}$$

2. The system gain, $K$, satisfies:
   (a). Magnitude criterion

$$|KG(s)H(s)| = 1 \tag{8.11}$$

   (b). Angle Criterion

$$\angle KG(s)H(s) = (2k+1)180^\circ \tag{8.12}$$

   Thus,

$$K = \frac{1}{|G(s)|\,|H(s)|} = \frac{1}{M} \tag{8.13}$$

**Example 8.2** Given a unity feedback system that has the forward transfer function below, do the following:

$$G(s) = \frac{K(s+2)}{(s^2+4s+13)}$$

a. Calculate the angle of $G(s)$ at the point $(-3 + j0)$ by finding the algebraic sum of angles of the vectors drawn from the zeros and poles of $G(s)$ to the given point.

   **Solution** *When $s = -3 + j0$ is substituted to the forward transfer function $G(s)$, it will have a magnitude of $M = 0.1$, and an angle of $\angle G(s) = 180^\circ$.*

b. Determine if the point specified in a is on the root locus

   **Solution** *Since $\angle G(s)$ satisfies $(2k+1)180^\circ$, the point is on the root locus.*

c. If the point specified in a is on the root locus, find the gain, $K$, using the lengths of the vectors.

   **Solution**

$$K = \frac{1}{|G(s)|\,|H(s)|} = \frac{1}{M} = \frac{1}{0.1} = 10$$

## Sketching Root Locus

The following five rules allow us to sketch the root locus using minimal calculations. The rules yield a sketch that gives intuitive insight into the behavior of a control system.

1. **Number of branches.** The number of branches of the root locus equals the number of closed-loop poles.
2. **Symmetry.** The root locus is symmetrical about the real axis.
3. **Real-axis segments.** On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.
4. **Starting and ending points.** The root locus begins at the finite and infinite poles of $G(s)H(s)$

and ends at the finite and infinite zeros of $G(s)H(s)$.

5. **Behavior at infinity.** The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, $\sigma_a$ and angle, $\theta_a$ as follows:

$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{\#finite\ poles - \#finite\ zeros} \tag{8.14}$$

$$\theta_a = \frac{(2k+1)\pi}{\#finite\ poles - \#finite\ zeros} \tag{8.15}$$

where $k = 0, \pm 1, \pm 2, \pm 3$ and the angle is given in radians with respect to the positive extension of the real axis.

**Example 8.3** Sketch the root locus for the system shown on Figure 8.8 below:

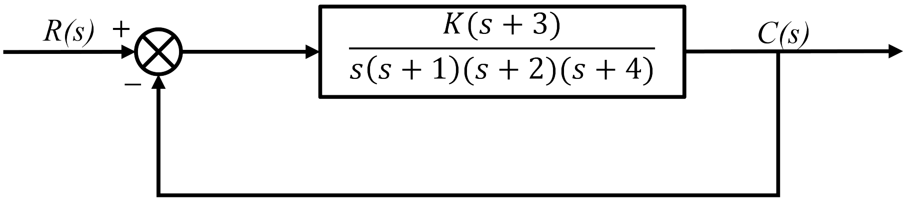



**Figure 8.8:** Given system for Example 8.2

**Solution**

1. *Calculate the asymptotes in terms of $\sigma_a$ and $\theta_a$*

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

$$\begin{aligned}
\theta_a &= \frac{(2k+1)\pi}{\#finite\ poles - \#finite\ zeros} \\
&= \pi/3 && for\ k = 0 \\
&= \pi && for\ k = 1 \\
&= 5\pi/3 && for\ k = 2
\end{aligned}$$

2. *Incorporate the rules for sketching.*
*As per Rule 4, the locus is identified as beginning at the open-loop poles and ending at the open-loop zeros. If there are more open-loop poles than open-loop zeros, then there must be zeros at infinity. Rule 5, then, tells us that the three zeros at infinity are at the ends of the asymptotes. The calculated asymptotes explain how we arrive at these zeros at infinity. The real-axis segments should also lie to the left of an odd number of poles and/or zeros.*

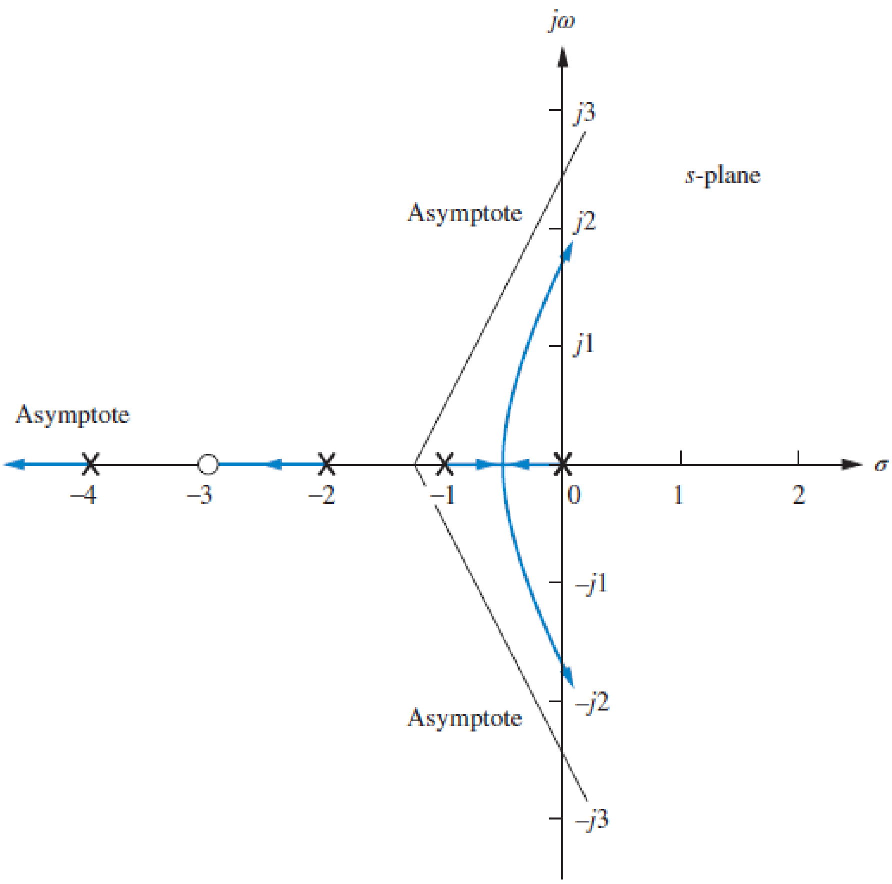



**Figure 8.9:** Root locus and asymptotes for the given system

## Refining the Sketch

In this section, we will discuss how to refine our root locus sketch by calculating real-axis breakaway and break-in points, $j\omega$-axis crossings, angles of departure from complex poles, and angles of arrival to complex zeros. We will conclude by showing how to find accurately any point on the root locus and calculate the gain.

1. **Real-axis breakaway & break-in points:** At the breakaway or break-in point, the branches of the root locus form an angle of $180^\circ/n$ with the real axis, where $n$ is the number of closed-loop poles arriving at or departing from the single breakaway or break-in point on the real axis.

   **Definition 8.2**
   ***Breakaway Point*** *is the point where the locus leaves the real axis,* $-\sigma_1$.

   **Definition 8.3**
   ***Break-in point*** *is the point where the locus returns to the real axis,* $\sigma_2$

   (a). **Differential Calculus procedure:** Maximize & minimize the gain, $K$, using differential calculus: The root locus breaks away from the real-axis at a point where the gain is **maximum** and breaks into the real-axis at a point where the gain is **minimum**. For all points on the Root Locus,

$$K = -\frac{1}{G(s)H(s)}$$

For points along the real-axis segment of the root locus where breakaway and break-in points could exist, $s = \sigma$. Differentiating with respect to $\sigma$ and setting the derivative equal to zero, results in points of maximum and minimum gain and hence the breakaway and break-in points.

(b). **Transition Procedure:** This eliminates the need to differentiate. Breakaway and break-in points satisfy the relationship

$$\sum_{i=1}^{m} \frac{1}{\sigma + z_i} = \sum_{j=1}^{n} \frac{1}{\sigma + p_j}$$

where $z_i$ and $p_j$ are the negative of the zero and pole values, respectively, of $G(s)H(s)$.

2. **The $j\omega$ crossings:** To find the $j\omega$ crossings, Routh-Hurwitz criterion can be used. Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

**Definition 8.4**

*The $j\omega$ **crossing** is a point on the Root Locus that separates the stable operation of the system from the unstable operation.* ♣

Procedures for finding $j\omega$ crossings:

(a). Using the Routh-Hurwitz criterion, forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

(b). At the $j\omega$ crossing, the sum of angles from the finite open loop poles and zeros must add to $(2k + 1)180°$. Search the $j\omega$-axis for a point that meets this angle condition.

3. **Angles of departure & arrival:** The value of $\omega$ at the axis crossing yields the frequency of oscillation, while the gain, $K$, at the $j\omega$-axis crossing yields the maximum or minimum positive gain for system stability.

(a). Assume a point $\epsilon$ close to the complex pole or zero. Add all angles drawn from all Open loop poles and zeros to this point. The sum equals $(2k + 1)180°$. The only unknown angle is that drawn from the $\epsilon$ close pole or zero, since the vectors drawn from all other poles and zeros can be considered drawn to the complex pole or zero that is $\epsilon$ close to the point. Solving for the unknown angle yields the angle of departure or arrival.

4. **Plotting & calibrating the Root Locus:** All points on the root locus satisfy the angle criterion, which can be used to solve for the gain, $K$, at any point on the root locus. Procedure:

(a). Search a given line for a point yielding

$$\sum zero\ angles - \sum pole\ angles = (2k + 1)180°$$

or

$$\angle G(s)H(s) = (2k + 1)180°$$

The gain at that point on the root locus satisfies

$$K = \frac{1}{|G(s)H(s)|} = \frac{\prod finite\ pole\ lengths}{\prod finite\ zero\ lengths}$$

## Control system architecture

**Control system architecture** defines how your controllers interact with the system under control. The architecture comprises the tunable control elements of your system, additional filter and sensor components, the system under control, and the interconnections among all these elements. For example, a common control system architecture is the **single-loop feedback configuration** shown in Figure 8.10 below:

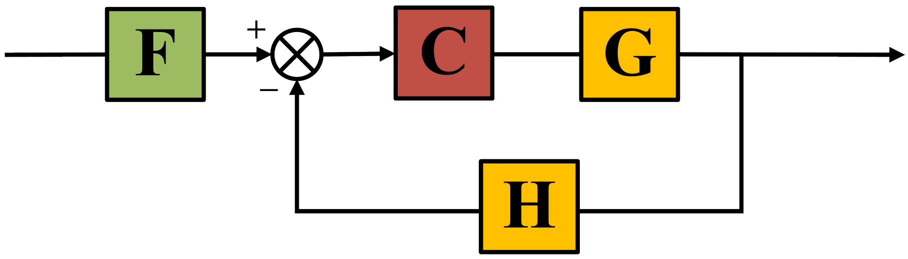



**Figure 8.10:** Single-loop feedback configuration

where the $G$ block is the transfer function, and the $H$ block represents the measurements, which will also be 1 for the proportional feedback system we are considering. The $C$ block represents the compensator, which will be the gain $K$ for our proportional feedback system. Because control systems are so conveniently expressed in this block diagram form, these elements are referred to as fixed blocks and tunable blocks.

## Sketching a Root Locus and Finding Critical Points using Control Systems Designer App

**Example 8.4** Sketch the root locus for the system shown on Figure 8.11 and find the following:

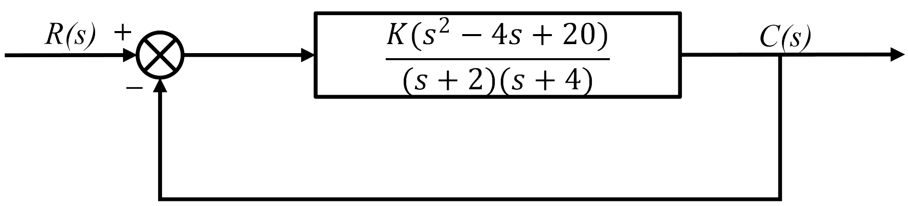



**Figure 8.11:** Given system for Example 8.3

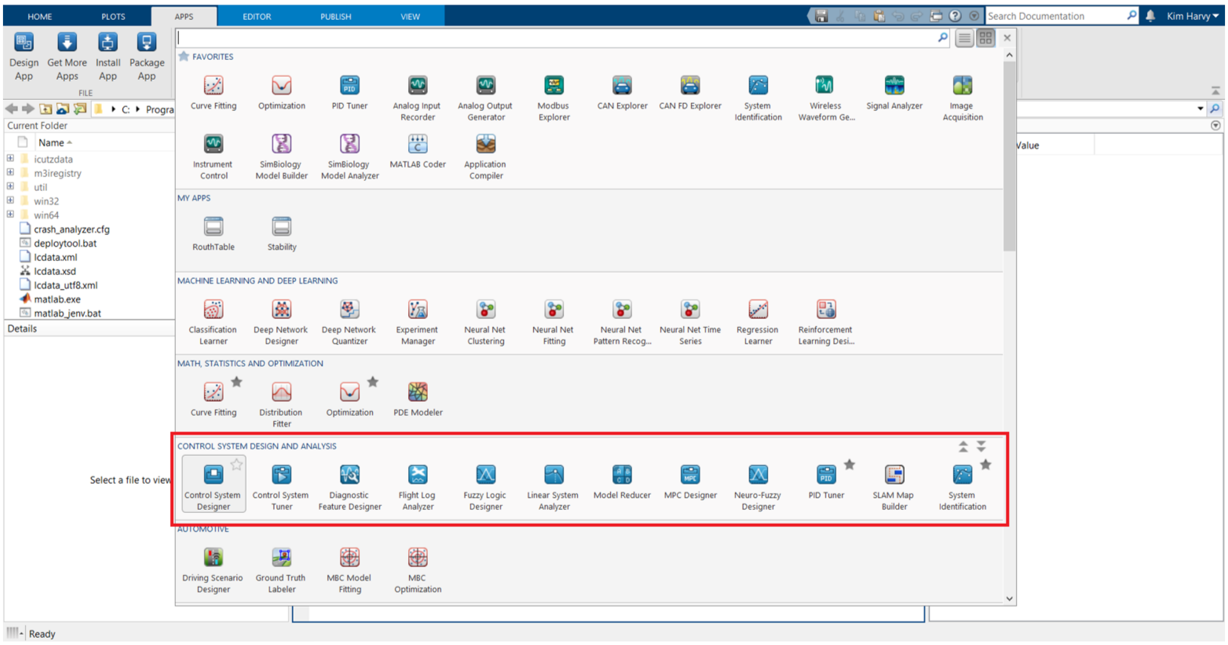
a. The exact point and gain where the locus crosses the 0.45 damping ratio line
b. The exact point and gain where the locus crosses the $j\omega - axis$
c. The breakaway point on the real axis

d. The range of $K$ within which the system is stable

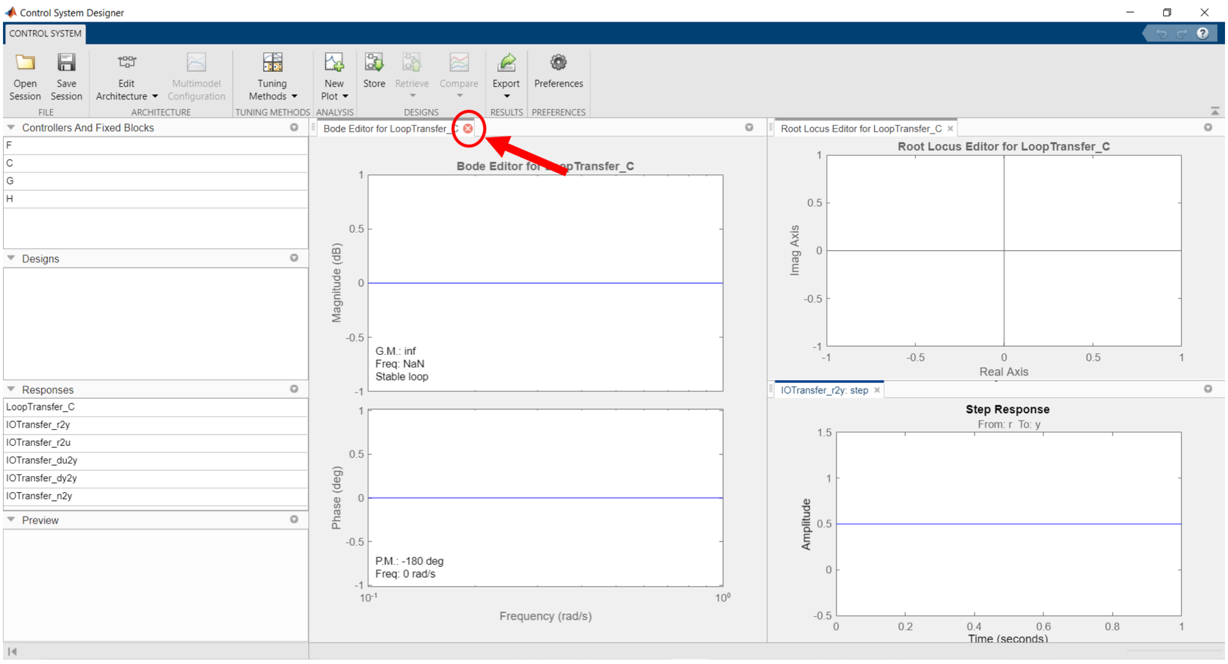
## Solution

A. *Generating the Root Locus*

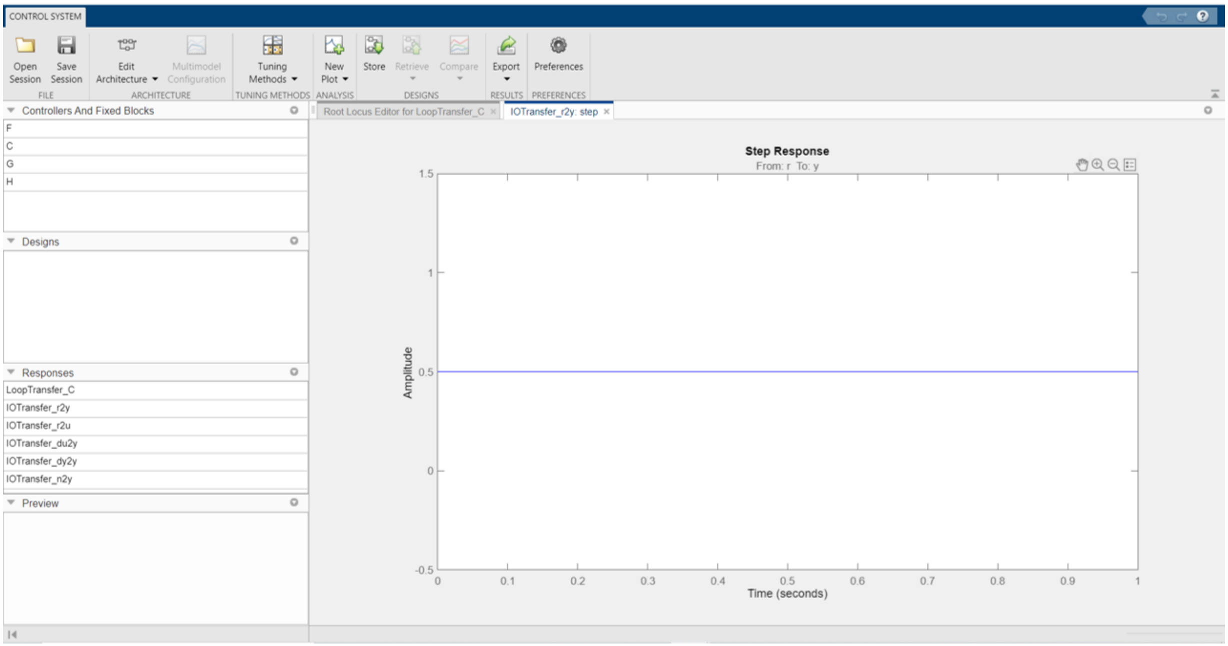
1. *Open MATLAB Software*
2. *Under the* ***MATLAB Toolstrip****, go to the* ***Apps tab****, under* ***Control System Design and Analysis****, click the* ***Control Systems Designer app*** *icon.*




3. *Close the* ***Bode Editor for Loop Transfer*** *since we are going to focus on Root Locus Plot and Step Response of the system.*



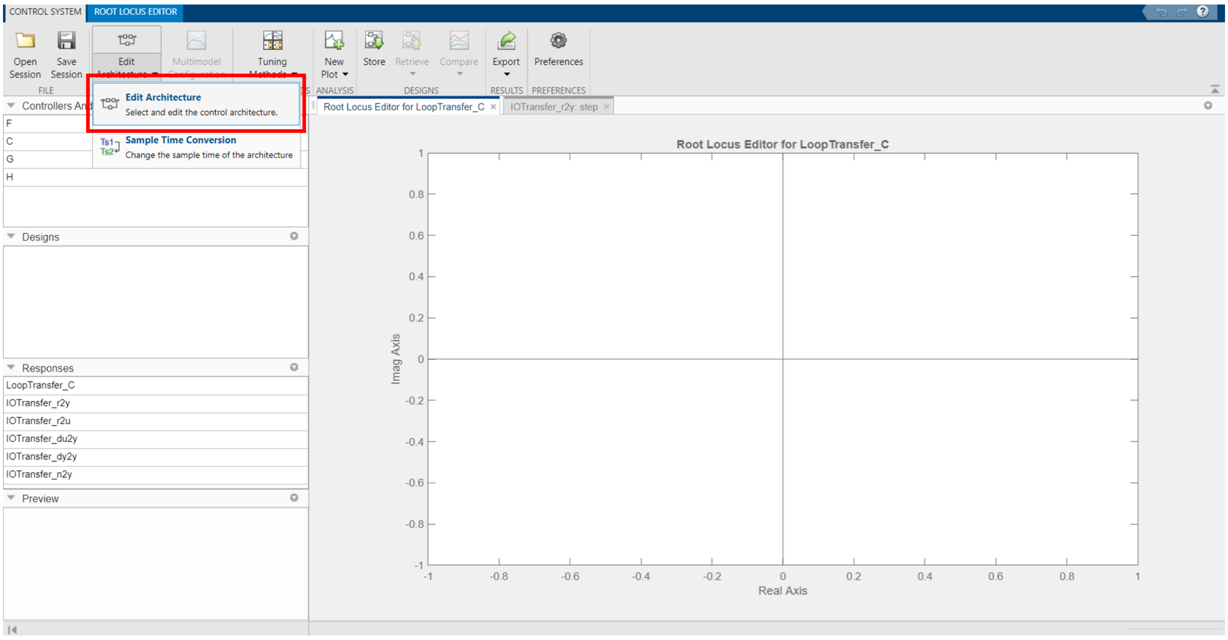
4. *Drag the **IOTransfer_r2y:step** into the root locus to separate it as a tab.*



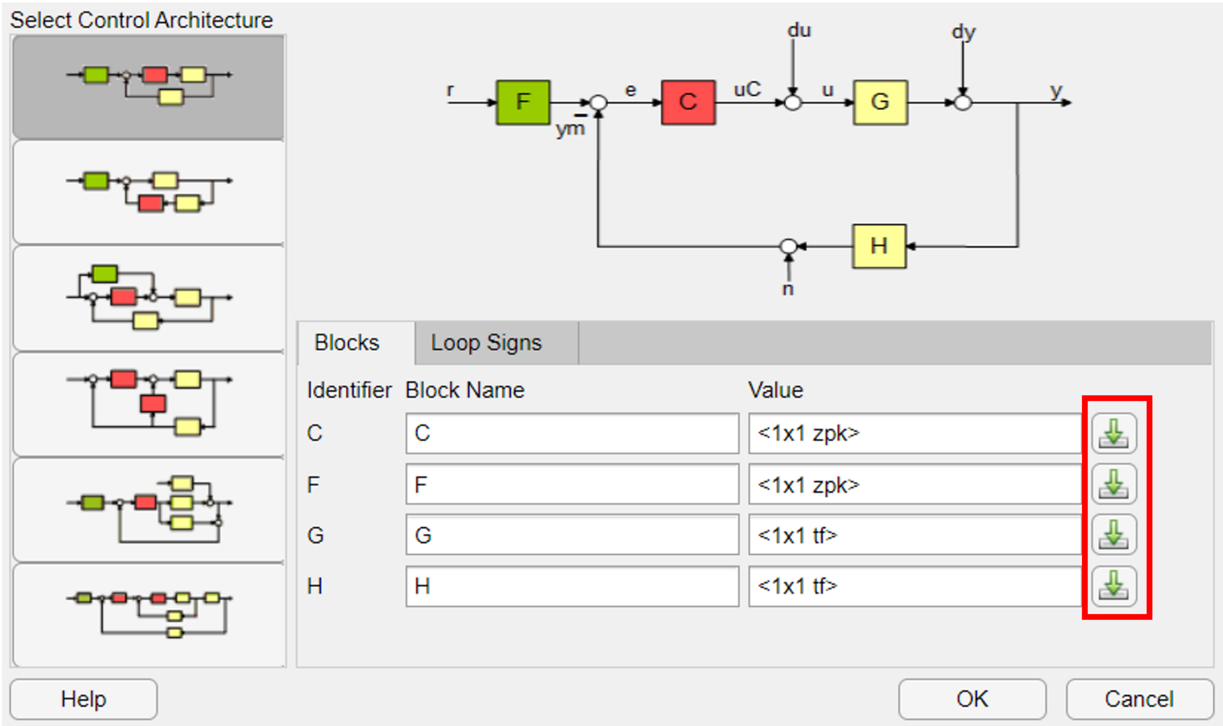
5. *Using a single-loop feedback configuration shown on Figure 8.11, identify the fixed components of the system namely $F, G, andH$ from the given transfer function.*
   *F= 1*
   *H= 1 (since it is unity feedback)*
   $G = \dfrac{(s^2 - 4s + 20)}{(s+2)(s+4)}$
6. *Define and enter the variables in **MATLAB Workspace**.*
   *In the command window type,*

```
F=tf(1); %Prefilter
H=tf(1); %Sensor Dynamics (Feedback)
num=[1,-4,20]; %Numerator of the Plant Model
den=[1,6,8]; %Denominator of the Plant Model
G=tf(num,den); %Plant Model
```

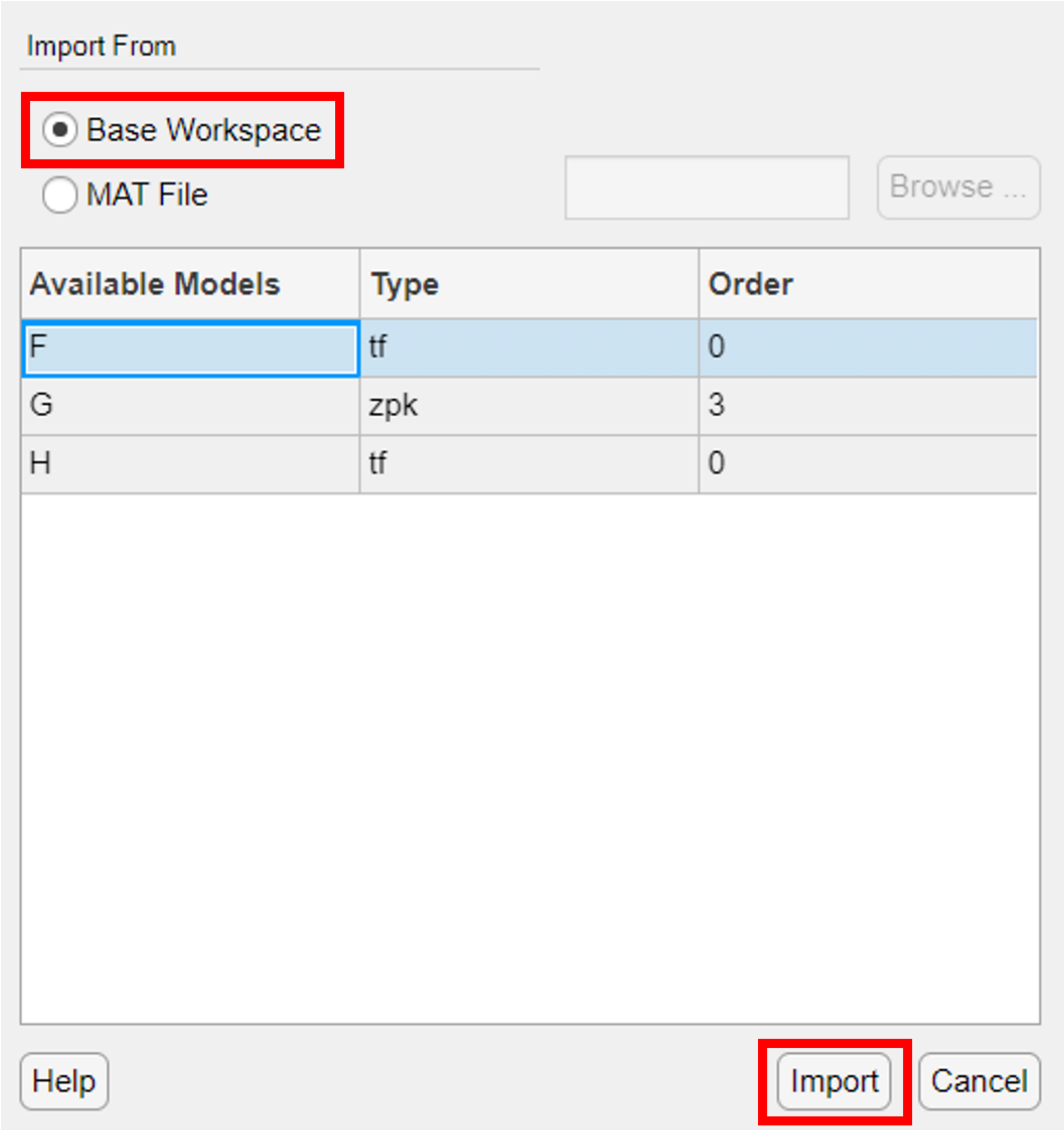
7. *On the Control System Designer App, go to the **Control System Tab** and click **Edit Architecture**.*



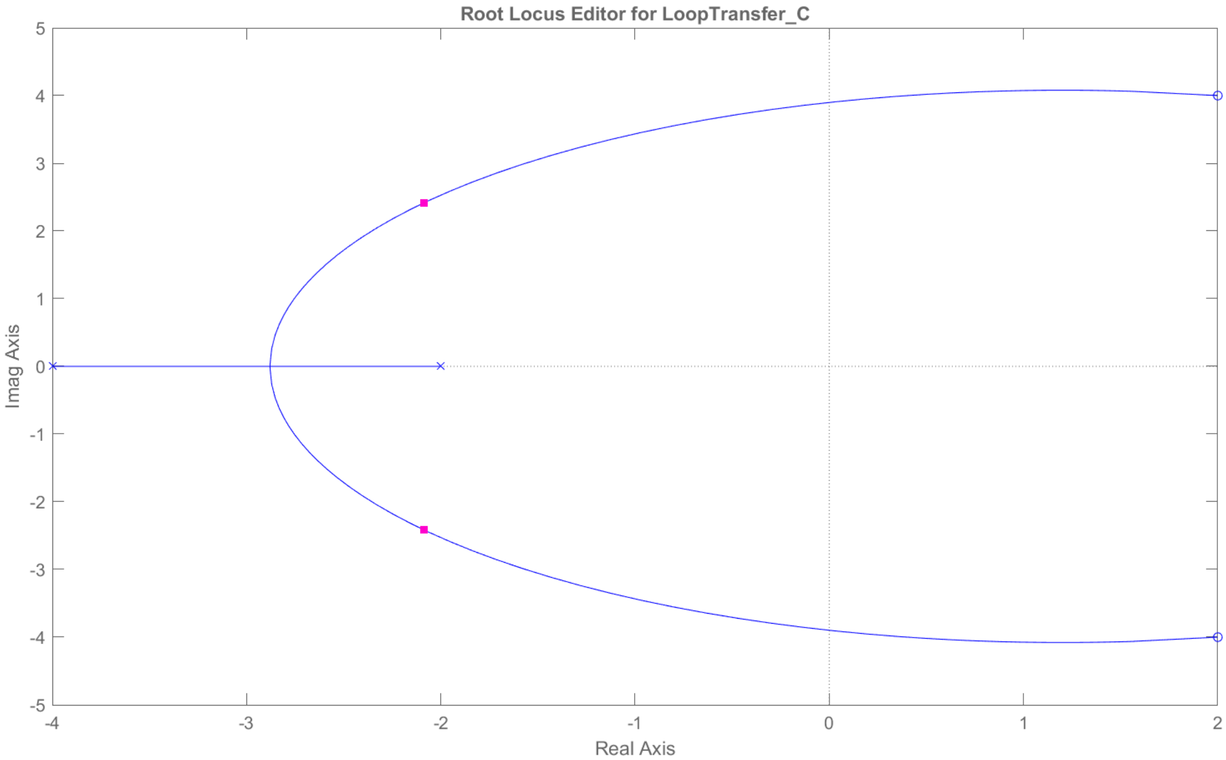
8. *Using the default single-loop feedback configuration, click the **import button** on each variable F, G and H to import data that are defined in the MATLAB Workspace.*



9. *Select **Base Workspace** under the **Import From** options and click the **respective variable** under the **Available Models** column. Click **Import**.*



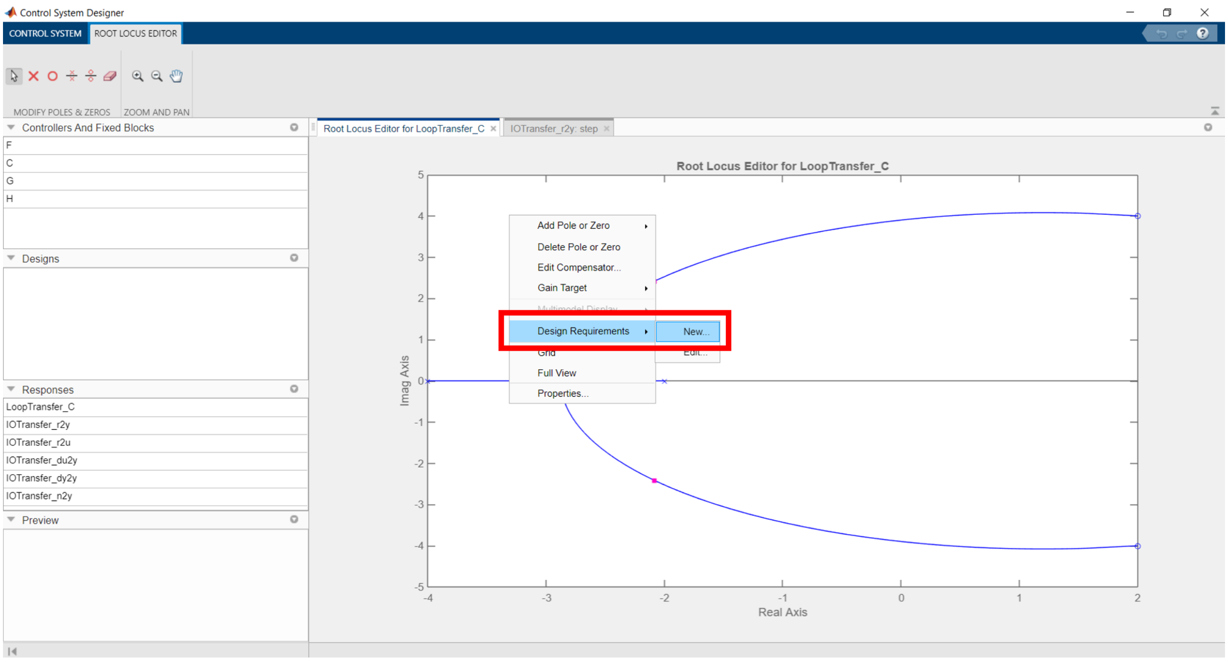
10. *Repeat steps for the remaining variables. Once done, click **Ok** on the **Edit Architecture-Configuration** Window to generate the Root Locus of the system.*



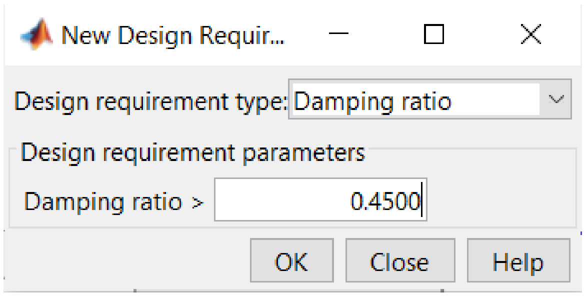

B. *Finding the coordinates of points on the root locus and their associated gains*

(a). *The exact point and gain where the locus crosses the 0.45 damping ratio line*

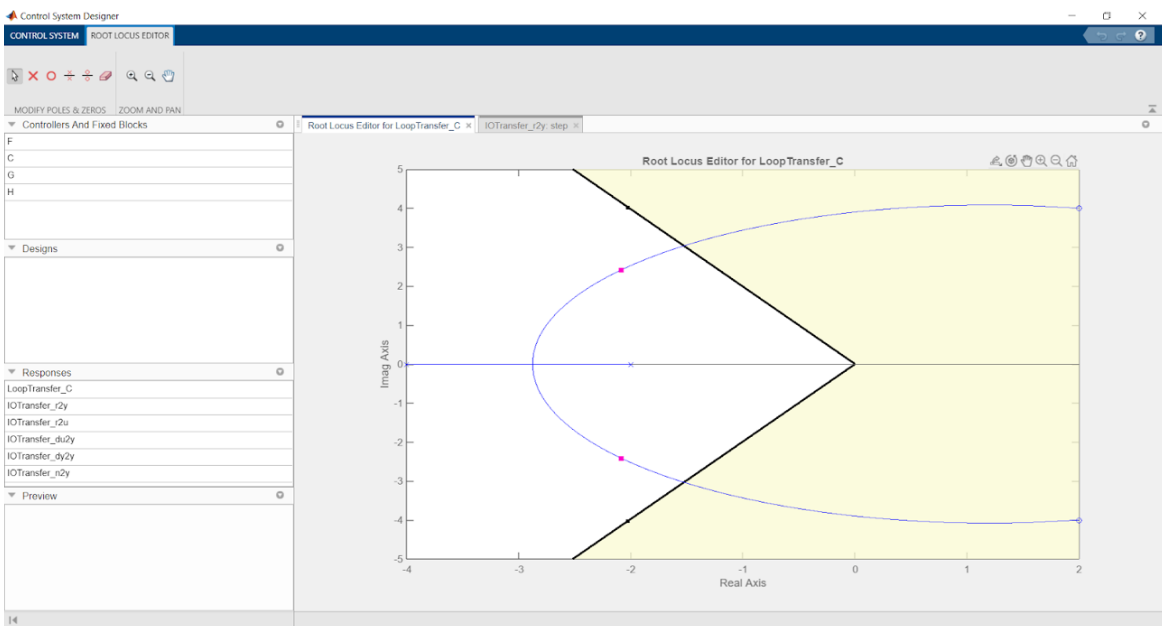
1. *Right click on the Root Locus Editor and click **Design Requirements** → **New**.*



2. *Select Damping Ratio on the* ***Design requirement type*** *and set the damping ratio of 0.45 as specification. Click* ***OK****.*



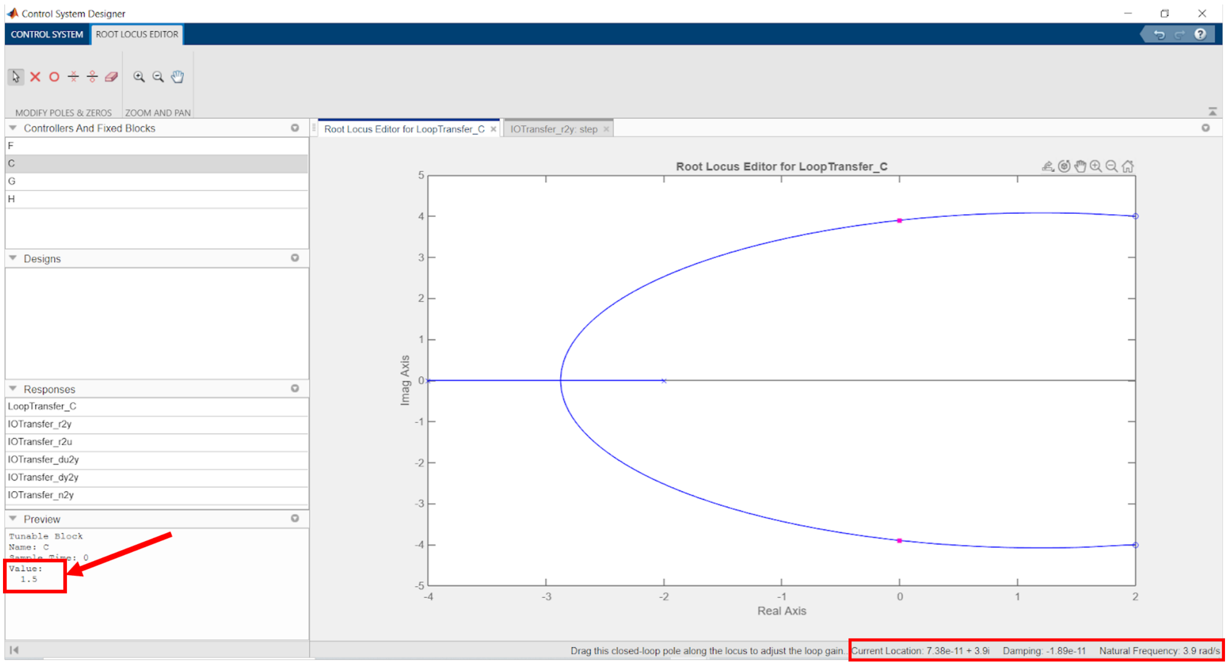

3. *Each requirement defines an exclusion region, indicated by a yellow shaded area. To satisfy a requirement, a response plot must remain on or outside of the associated exclusion region. To find the exact point and gain where the locus crosses the 0.45 damping ratio line, drag the* ***pink dots*** *(closed-loop pole locations) to the* ***damping ratio line****.*




4. *On the bottom-right and bottom left part of the Control system Designer window, the exact location at the damping ratio line and its equivalent value of $K$ are indicated respectively.*

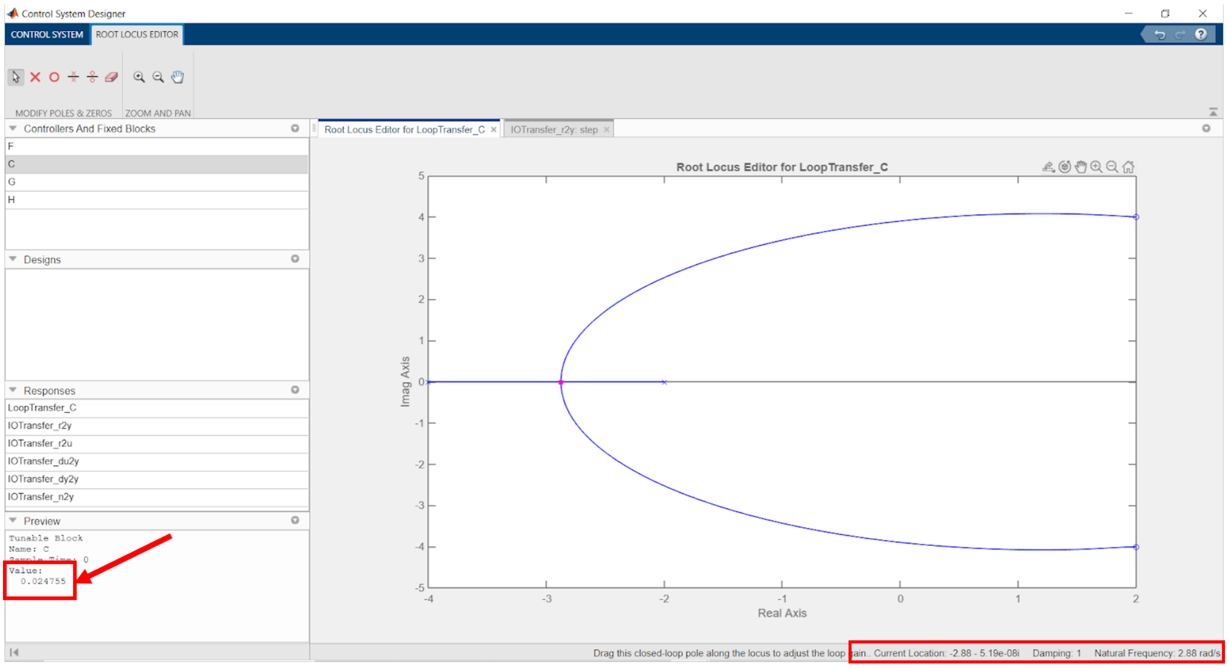
   ***In polar form,*** $-1.53 \pm j3.03$ ***is equal to*** $3.39\angle 116.79°$ ***and*** $K = 0.417$

(b). *To find the exact point where the locus crosses the $j\omega$-axis, drag the pink dots (closed-loop pole locations) to the $j\omega$-axis, where $\theta = 90°$.Thus, we find that the root locus crosses the $j\omega$-axis at $\pm j3.9$ with a gain of $K = 1.5$.*




*(c). The breakaway point on the real axis*

1. *To find the breakaway point, use the Control System Designer app to search the real axis between 2 and 4 for the point that yields maximum gain. Simply drag the pink dots (closed-loop pole locations) to the real axis between -2 and -4 that will yield into maximum gain. In this case, a maximum gain of* $K = 0.0248$ *is found at the point* $-2.88$*. Therefore, the breakaway point is between the open-loop poles on the real axis at -2.88.*




*(d). The range of K within which the system is stable*

*From the answer to b, the system is stable for K between 0 and 1.5 or* $0 \leq K < 1.5$.

## Transient Response Design via Gain Adjustment

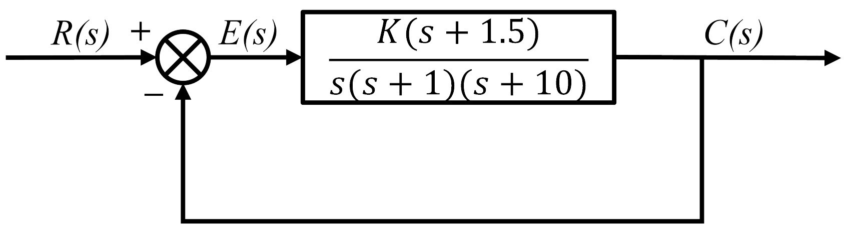
### *Conditions justifying a second-order approximation*

1. Higher-order poles are much farther into the left half of the s-plane than the dominant second-order pair of poles. The response that results from a higher-order pole does not appreciably change the transient response expected from the dominant second-order poles.
2. Closed-loop zeros near the closed-loop second-order pole pair are nearly canceled by the close proximity of higher-order closed-loop poles.
3. Closed-loop zeros not canceled by the close proximity of higher-order closed-loop poles are far removed from the closed-loop second-order pole pair.

### *Summarizing the design procedure for higher-order systems*

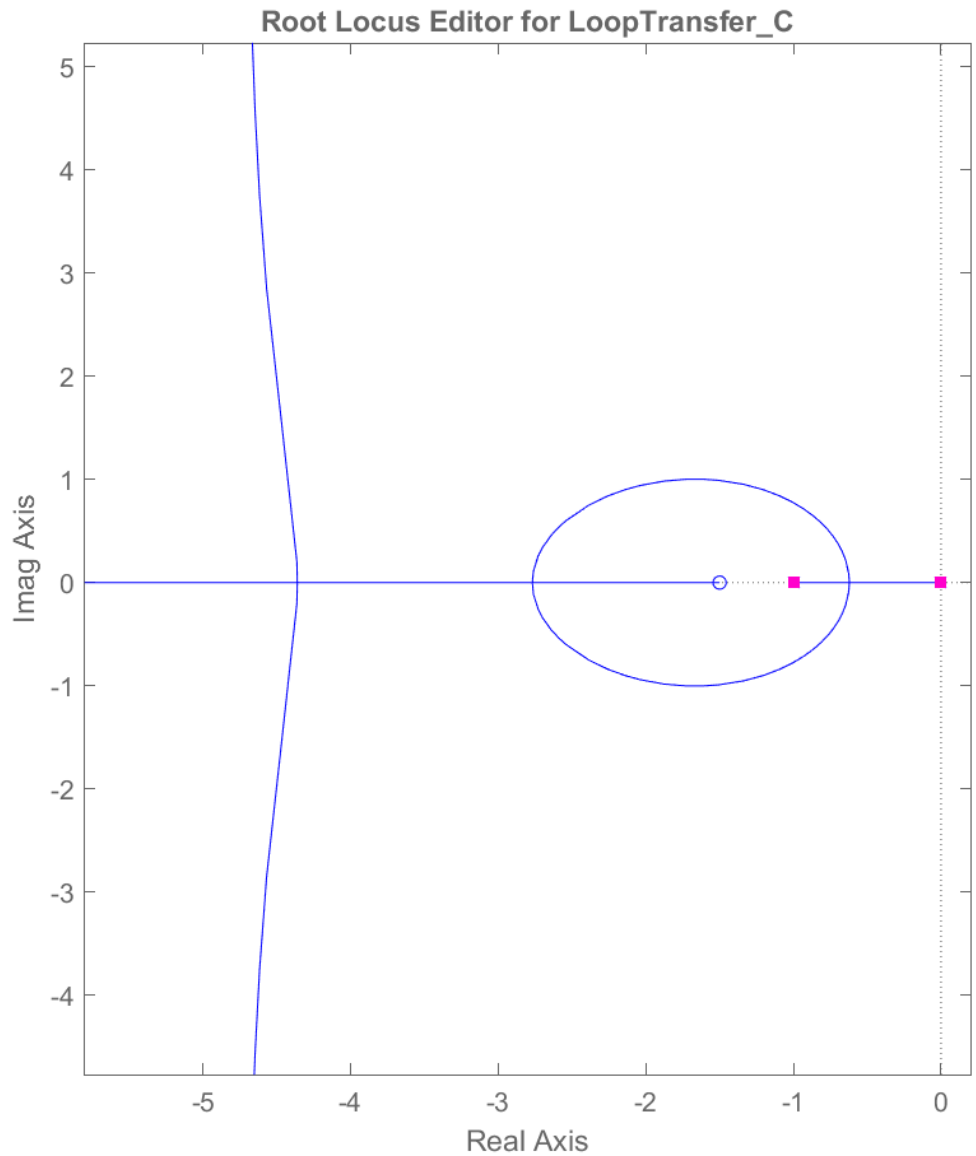
1. Sketch the root locus for the given system.
2. Assume the system is a second-order system without any zeros and then find the gain to meet the transient response specification.
3. Justify your second-order assumption by finding the location of all higher-order poles and evaluating the fact that they are much farther from the $j\omega$-axis than the dominant second-order pair.
4. If the assumptions cannot be justified, your solution will have to be simulated in order to be sure it meets the transient response specification.

**Example 8.5** Given the system shown below, design the value of gain, $K$, to yield 1.52% overshoot. Also estimate the settling time, peak time, and steady-state error.




**Solution**

1. *Sketch the root locus in MATLAB using the Control System Designer.*

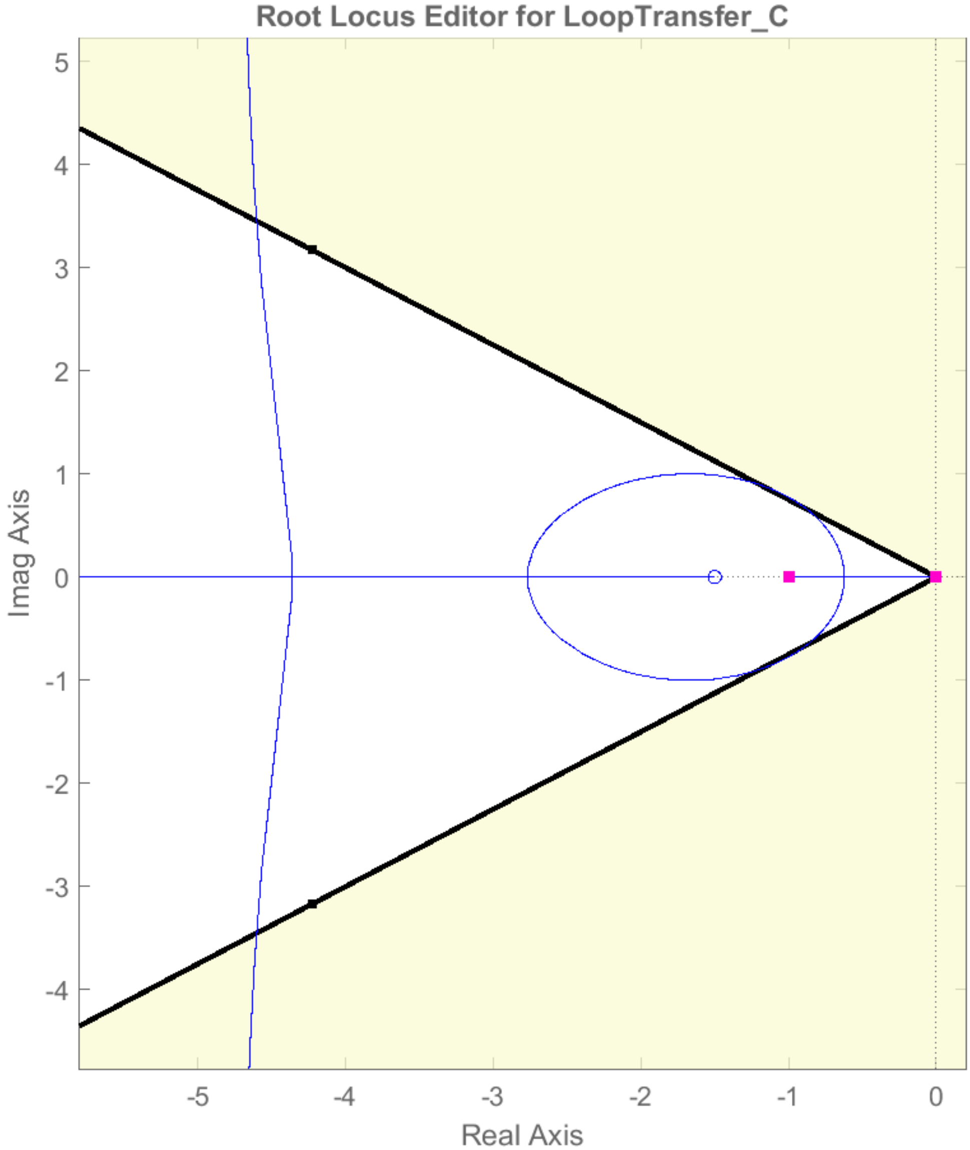



2. *By dragging the pink dots (closed-loop pole locations), determine the Break-away points and break-in points and their respective values of gain.*

| Break-away points (Maximum Gain) | | Break-in points (Minimum Gain) | |
|---|---|---|---|
| **Location** | **Gain** | **Location** | **Gain** |
| -0.62 | 2.511 | -2.8 | 27.91 |
| -4.4 | 28.89 | | |

**Table 8.2:** Summary of breakaway and break-in points

3. *Assume that the system can be approximated by a second order, under-damped system without any zeros. Set the design requirement of* 1.52% *overshoot.*




4. *Search for the Closed-Loop (CL) poles that touch the 0.8 damping ratio line or the 1.52 percent overshoot line and their respective gains by dragging the pink dots (closed-loop pole locations).*

| Case | CL Poles | CL Zeros | Gain |
|---|---|---|---|
| 1 | -0.87\pm j0.66 | -1.5+j0 | 7.36 |
| 2 | -1.19\pm j0.96 | -1.5+j0 | 12.79 |
| 3 | -4.6\pm j3.45 | -1.5+j0 | 39.64 |

5. *For each point, estimate the settling time, peak time using*

$$T_s = \frac{4}{\zeta \omega_n}$$

*where $\zeta \omega_n$ is the real part of the closed loop pole, and*

$$T_p = \frac{\Pi}{\omega_n \sqrt{1 - \zeta^2}}$$

*where $\omega_n\sqrt{1-\zeta^2}$ is the imaginary part of the closed-loop pole*

| Case | CL Poles | CL Zeros | Gain | Settling Time | Peak Time |
|---|---|---|---|---|---|
| 1 | -0.87\pm j0.66 | -1.5+j0 | 7.36 | 4.60 | 4.76 |
| 2 | -1.19\pm j0.96 | -1.5+j0 | 12.79 | 3.36 | 3.49 |
| 3 | -4.6\pm j3.45 | -1.5+j0 | 39.64 | 0.87 | 0.91 |

6. *To test our assumption of a second-order system, we must calculate the location of the third pole. Using the root locus program, search along the negative extension of the real axis between the zero at -1.5 and the pole at -10 for points that match the value of gain found at the second-order dominant poles.*

| Case | CL Poles | CL Zeros | Gain | Settling Time | Peak Time | Third-CL poles |
|---|---|---|---|---|---|---|
| 1 | -0.87\pm j0.66 | -1.5+j0 | 7.36 | 4.60 | 4.76 | -9.25 |
| 2 | -1.19\pm j0.96 | -1.5+j0 | 12.79 | 3.36 | 3.49 | -8.61 |
| 3 | -4.6\pm j3.45 | -1.5+j0 | 39.64 | 0.87 | 0.91 | -1.80 |

7. *Examine the steady-state error produced in each case given by $K_v$ and is calculated as* $K_v = \lim_{s\to 0} sG(s) = \frac{K(1.5)}{(1)(10)}$.

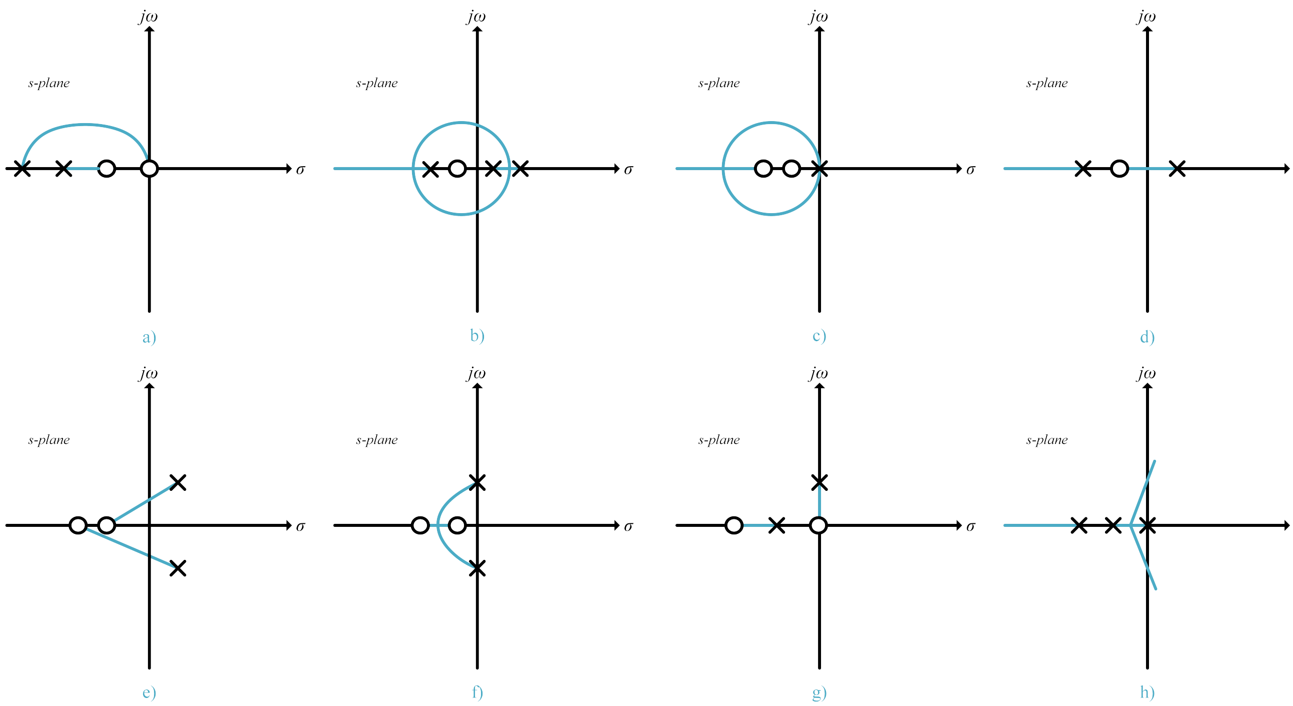
| Case | CL Poles | CL Zeros | Gain | Settling Time | Peak Time | Third-CL poles | $K_v$ |
|---|---|---|---|---|---|---|---|
| 1 | -0.87\pm j0.66 | -1.5+j0 | 7.36 | 4.60 | 4.76 | -9.25 | 1.1 |
| 2 | -1.19\pm j0.96 | -1.5+j0 | 12.79 | 3.36 | 3.49 | -8.61 | 1.9 |
| 3 | -4.6\pm j3.45 | -1.5+j0 | 39.64 | 0.87 | 0.91 | -1.80 | 5.9 |

8. *To determine the validity of the second-order approximation, compare the third-closed loop poles to the closed-loop zero.*
*Since Cases 1 and 2 yield third closed-loop poles that are relatively far from the closed-loop zero, there is no pole-zero cancellation, and a second-order system approximation is **not valid**. On the other hand, In Case 3, the third closed-loop pole and the closed-loop zero are relatively close to each other, and a second-order system approximation can be considered **valid**.*

# 8.3 Laboratory Experiment

## Module Exercises

1. Determine whether or not the sketch can be a root locus for each of the root loci shown below. Explain why the sketch cannot be a root locus. Give all of your reasons.




2. Given a unity feedback system that has the forward-path transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

   (a). Sketch the root locus.
   (b). Using a second-order approximation, design the value of $K$ to yield 10% overshoot for a unit-step input.
   (c). Estimate the settling time, peak time, rise time, and steady-state error for the value of $K$ designed in (b).
   (d). Determine the validity of your second-order approximation.

## Simulation Activity

## Plotting and analyzing Root Locus

## Objective:

- To analyze the stability of the system by using Root locus in MATLAB.
- To determine the coordinates of points on the root locus and the gains corresponding with them.

## Procedures

1. Open **MATLAB** software.
2. Save the script with a filename **'Lab_RootLocus'**.
3. Given the LTI transfer-function, $G(s) = \frac{K(s-2)(s-4)}{(s^2+6s+25)}$ Enter the program in the editor window.

```
%Laboratory8- Root Locus Techniques

clf % Clear graph.
numgh=[1 -6 8]; % Define numerator of G(s)H(s).
dengh=[1 6 25]; % Define denominator of G(s)H(s).
' G(s)H(s)' % Display label.
GH=tf(numgh,dengh) % Create G(s)H(s) and display.
rlocus(GH) % Draw root locus.
z=0:0.05:0.5; % Define damping ratio values:0.2 to 0.5 in steps of 0.05.
wn=0; % Define natural frequency values
sgrid(z,wn) % Damping ratio and natural frequency grid lines
title ('Root Locus') % Define title for root locus.
```

4. Execute and **Run** the Program.
5. By using the Root Locus Plot, fill up the table below:

| | **Location** | **Gain** |
|---|---|---|
| $j\omega$-axis crossing | | |
| Break-in point | | |
| $\zeta = 0.05$ | | |
| $\zeta = 0.10$ | | |
| $\zeta = 0.15$ | | |
| $\zeta = 0.20$ | | |
| $\zeta = 0.25$ | | |
| $\zeta = 0.30$ | | |
| $\zeta = 0.35$ | | |
| $\zeta = 0.40$ | | |
| $\zeta = 0.45$ | | |
| $\zeta = 0.50$ | | |

6. What is the range of gain, $K$, for which the system is stable?

## 8.4 Questions to Ponder

1. What can you observe about the value of gan as the daming factor increases?
2. Why root locus is important in determining the dynamic response of the system?

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# Chapter 9
# Frequency Response Techniques

## 9.1 Objective

- To define the frequency response of a system
- To define, plot, and find the stability and gain and phase margins of Bode Diagram
- To sketch and plot Nyquist criterion
- To find the stability and gain and phase margins using Nyquist diagrams

## 9.2 Theory

### Definition and Concept of Frequency Response

This chapter will focus on the frequency response approaches to the analysis and design of a control system. Previously, root locus techniques were discussed which is a complement of the frequency response, but the advantage of the latter approach is that we can utilize the data obtained from measurements on the physical system without deriving its mathematical model.

In order for us to further understand the frequency response, let us discuss its concept.

**Frequency response** is a steady state output of a system from a sinusoidal input. Meaning to say, a sinusoidal input will generate the same type of output as a response but may differ from the amplitude and phase angle. These are a function of a frequency. Basically, in frequency response methods, we will vary the frequency of the input signal over a certain range and thus study the resulting response.

Mentioning that we have a sinusoidal input, sinusoid can be represented as complex numbers as ***phasors***. It consists of a ***magnitude*** $M_i$ which is an amplitude of a sinusoid from 0 to peak value, and a ***phase angle*** $\phi_i$ which is the phase shift of the sinusoid. It is then represented by $M_i\angle\phi_i(\omega)$. Knowing that it is a sinusoidal transfer function - that is the transfer function $s$ is replaced by $j\omega$, where $\omega$ is the frequency. Since the system can cause both the amplitude and phase angle of the input to be changed, it will result to an amplified output as $M_0\angle\phi_0(\omega)$, where it can be represented as the product of the input and the system function, $M(\omega)\angle\phi(\omega)$.

From the equation, we can see that the system function is given by,

$$M_0(\omega)\angle\phi_0(\omega) = M_i(\omega)M(\omega)\angle[\phi_i(\omega) + \phi(\omega)]$$

$$M(\omega) = \frac{M_0(\omega)}{M_i(\omega)}$$

and,

$$\phi(\omega) = \phi_0(\omega) - \phi_i(\omega)$$

Thus, this is called the **frequency response** $(M(\omega)\angle\phi(\omega))$, where $M(\omega)$ is the magnitude frequency response or the ratio of the output sinusoid's magnitude to the input sinusoid's magnitude, and $\phi(\omega)$ as the *phase frequency response* or the difference between the phase angle between the output and input sinusoids.

For a system with a given transfer function $G(s)$, we will substitute $s$ into $G(j\omega)$ since the steady-state output of a transfer function system can be obtained directly from the sinusoidal transfer function, and it can be represented by the magnitude and phase angle with frequency as a parameter.

Suppose that the system function $G(j\omega)$ is put to the transfer function and hit a magnitude and phase shift, it yields to

$$G(j\omega) = M(\omega)\angle\phi(\omega) \tag{9.1}$$

Input phasor forms,

- Polar, $M_i\angle\phi_i$
  - $M_i = \sqrt{A^2 + B^2}$
  - $\phi_i = -arctan\frac{B}{A}$
- Rectangular, $A - jB$
- Euler's, $M_i e^{j\phi_i}$

Frequency response of system,

$$G(j\omega) = G(s)|_{s\to j\omega} \tag{9.2}$$

**Example 9.1**

Find the analytical expression for the magnitude frequency response and the phase frequency response for the systems below:

a. $G(s) = \dfrac{1}{s+2}$

b. $G(s) = \dfrac{1}{(s+2)(s+4)}$

**Solution**

*a.* $G(s) = \dfrac{1}{s+2}$

1. *Substitute $s = j\omega$ in the transfer function.*

$$G(s) = \frac{1}{(s+2)}$$
$$G(s) = \frac{1}{j\omega + 2}$$

2. *Find the magnitude and phase angle frequency response.*

$$|G(j\omega)| = M(\omega) = \frac{1}{\sqrt{\omega^2 + 4}}$$
$$\phi(\omega) = -tan^{-1}(\frac{\omega}{2})$$

*b.* $G(s) = \frac{1}{(s+2)(s+4)}$

1. *Substitute $s = j\omega$ in the transfer function.*

$$G(j\omega) = \frac{1}{(j\omega + 2)(j\omega + 4)}$$

2. *Find the magnitude and phase angle frequency response*

$$|G(j\omega)| = M(\omega) = \frac{1}{\sqrt{(8-\omega^2)^2 + (6\omega)^2}}$$
$$\phi(\omega) = -tan^{-1}(\frac{6\omega}{8-\omega^2})$$

## Bode Diagrams or Logarithmic Plots

### *History Interlude*

- **Hendrik Wade Bode (1930s)** - invented the Bode plots, gain margin, & phase margin
- **1944** – WWII anti-aircraft (including V-1 flying bombs) systems
- **1957** – Served on NACA (now NASA) with Wernher von Braun (inventor of V-1 flying bombs & V-2 rockets)

### *Introduction*

Bode Plot is one of the frequency response techniques where two graphs are included - plot of the logarithm of the magnitude of the sinusoidal transfer function, and the other is plot of phase angle. It tends to separate the magnitude and phase plots across the entire spectrum for visualization that are plotted against the frequency on logarithmic scale.

- Magnitude - decibels (dB) vs log , where $G(j\omega) = dB = 20log|M|$
- Phase angle - phase angle (in degrees) vs $log\omega$

**Definition 9.1**

***Break frequency*** *is the point where low-frequency and high-frequency asymptotes breaks*

**Definition 9.2**

***Asymptotes*** *refers to the straight line approximations.*

**Definition 9.3**

***Low-frequency asymptote*** *refers to the low frequency approximation*

**Definition 9.4**

***High-frequency asymptote*** *refers to the high frequency approximation*

## *Simple Bode Plots*

- Bode Plot Approximation for $G(s) = s + a$

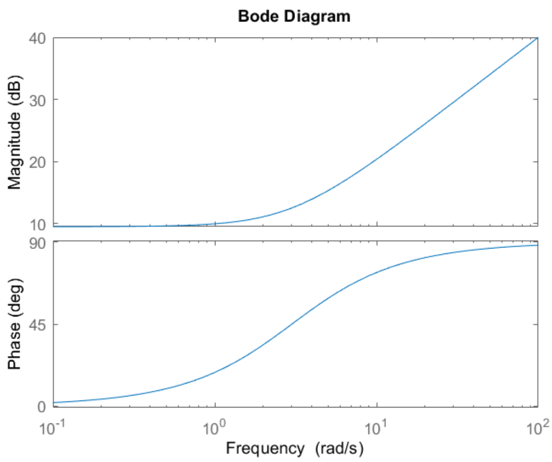


**Figure 9.1:** Magnitude and phase plot of $G(s) = s + a$

- Bode Plot Approximation for $G(s) = \dfrac{1}{(s+a)}$

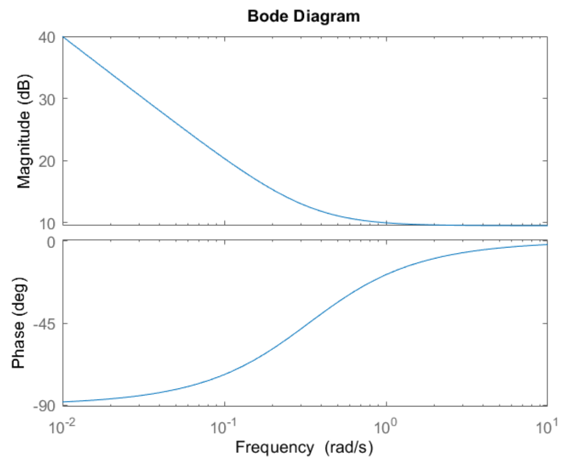


**Figure 9.2:** Magnitude and phase plot of $G(s) = \dfrac{1}{(s+a)}$

- Bode Plot Approximation for $G(s) = s$

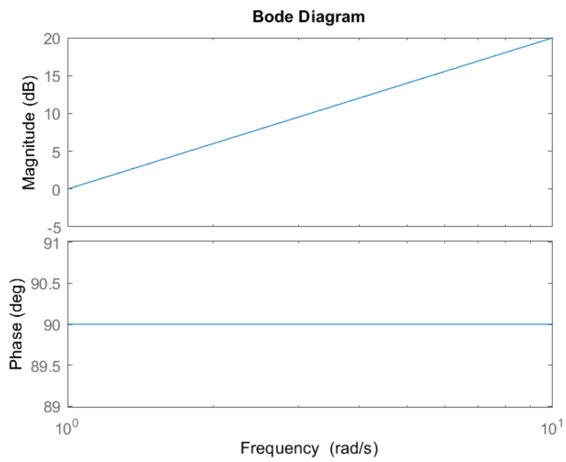


**Figure 9.3:** Magnitude and phase plot of $G(s) = s$

- Bode Plot Approximation for $G(s) = \dfrac{1}{s}$

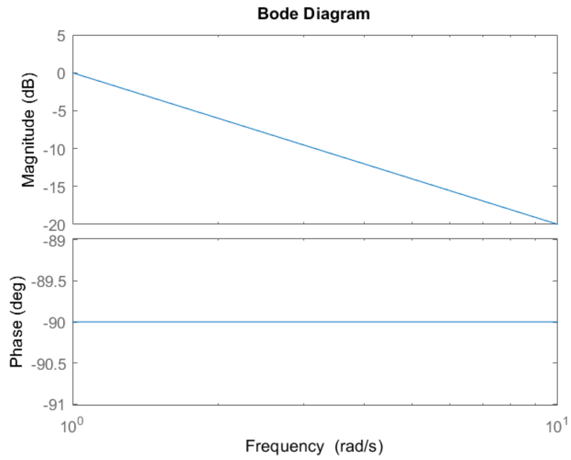


**Figure 9.4:** Magnitude and phase plot of $G(s) = \dfrac{1}{s}$

- Bode Plot of $G(s) = \dfrac{s+a}{a}$

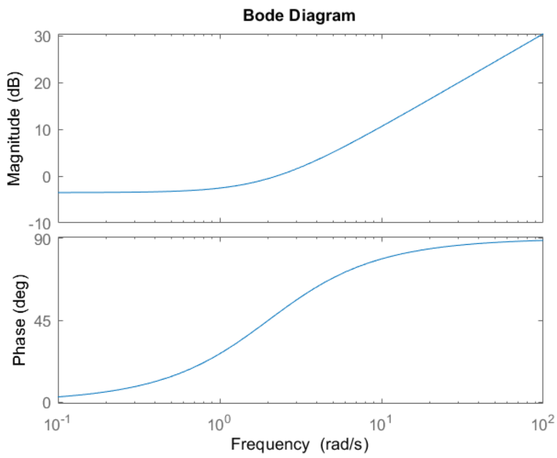


**Figure 9.5:** Magnitude and phase plot of $G(s) = \dfrac{s+a}{a}$

- Bode Plot of $G(s) = \dfrac{a}{s+a}$

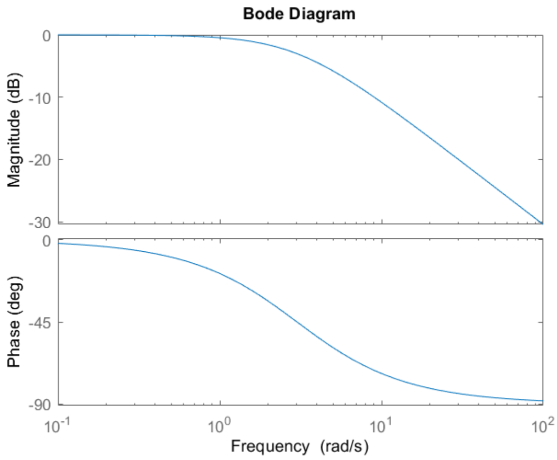


**Figure 9.6:** Magnitude and phase plot of $G(s) = \dfrac{a}{s+a}$

## *Bode Plot by Hand*

Steps in plotting Magnitude and Phase angle. Consider the following transfer function:

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

1. Check the transfer function whether it is in standard form $\dfrac{1}{1+sT}$.

$$(1+0.4s)(1+0.1s) \textbf{\textit{ is in standard form.}}$$

2. Substitute $s = j\omega$

$$G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+0.1j\omega)}$$

3. Plot the Magnitude by finding the corner frequency.
   *Remember that the relationship of T in the standard form* $\dfrac{1}{1+sT}$ *to the frequency is*

$$\omega = \frac{1}{T}$$

for $(1+0.4j\omega)$,

$$\omega_{c1} = (\frac{1}{T}) = \frac{1}{0.4} = 2.5\frac{rad}{sec}$$

for $(1+0.1j\omega)$,

$$\omega_{c2} = (\frac{1}{T}) = \frac{1}{0.1} = 10\frac{rad}{sec}$$

**Definition 9.5**

***Corner frequency or the break frequency*** *in Bode plot is where the two asymptotes cut or meet each other.* ♣

4. Made a table containing the terms of the function, corner frequency, slope, and the change in slope.

| Term | Corner Frequency | Slope | Change in Slope |
|---|---|---|---|
| $\frac{10}{j\omega}$ | - | $-20$ dB/decade | - |
| $\frac{1}{1+0.4j\omega}$ | $2.5\frac{rad}{sec}$ | $-20$ dB/decade | $-40$ dB/decade |
| $\frac{1}{1+0.1j\omega}$ | $10\frac{rad}{sec}$ | $-20$ dB/decade | $-60$ dB/decade |

**Table 9.1**

***Key points***

- When $s$ is in the denominator, the slope of the plot decreases by -20 dB/decade.
- When $s$ is in the numerator, the slope of the plot increases by +20 dB/decade.
- The degree of $s$ determines the slope of the plot (i.e. $s^2 = 40dB/decade$, $s^3 = 60dB/decade$, $\frac{1}{s^2} = -40dB/decade$, $\frac{1}{s^3} = -60dB/decade$).

5. Assign a frequency ($\omega_l$) lower than the $\omega_{c1}$ and frequency ($\omega_h$) higher than the $\omega_{c2}$. Solve for the total magnitude of the transfer function.

$$\omega_l = 0.1\frac{rad}{sec}$$
$$\omega_h = 100\frac{rad}{sec}$$

for $\frac{10}{j\omega}$, let $\omega = (\omega_l = 0.1)$

$$M_1 = 20log|\frac{10}{j\omega}| = 20log|\frac{10}{0.1}| = 40dB$$

$\omega = (\omega_{c1} = 2.5)$

$$M_2 = 20log|\frac{10}{j\omega}| = 20log|\frac{10}{2.5}| = 12dB$$

$\omega = (\omega_{c2} = 10)$

$$M_3 = [slope\ from\ \omega_{c1}\ to\ \omega_{c2} \times log\frac{\omega_{c2}}{\omega_{c1}}] + M_2$$

$$M_3 = [-40 \times log\frac{10}{2.5}] + 12$$

$$M_3 = -12dB$$

$\omega = (\omega_h = 100)$

$$M_4 = [slope\ from\ \omega_h\ to\ \omega_{c2} \times log\frac{\omega_h}{\omega_{c2}}] + M_3$$

$$M_4 = [-60 \times log\frac{100}{10}] - 12$$

$$M_4 = -72dB$$

6. Plot the magnitude in the log scale.

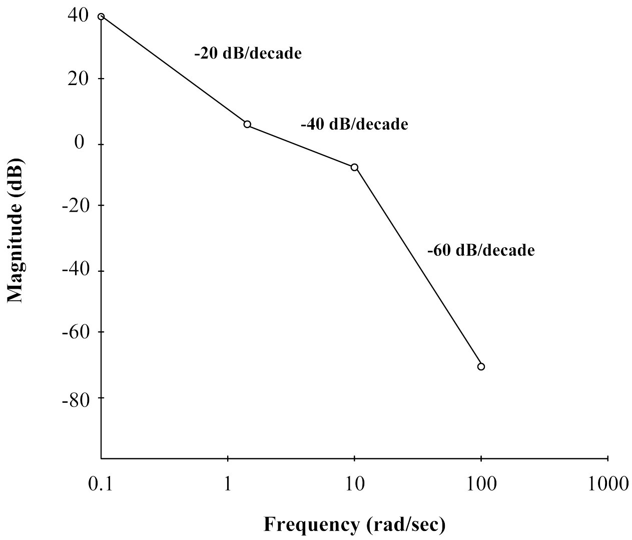


**Figure 9.7:** Magnitude plot

For Phase plot,

1. Find the value of $\phi$ from the transfer function $G(j\omega)$
   For each term,

| | |
|---|---|
| $\frac{10}{j\omega}$ | $-90°$ |
| $(1 + 0.4j\omega)$ | $-tan^{-1}(0.4\omega)$ |
| $(1 + 0.1j\omega)$ | $-tan^{-1}(0.1\omega)$ |

**Table 9.2**

Hence,

$$\phi = -90 - tan^{-1}(0.4\omega) - tan^{-1}(0.1\omega)$$

***Key points***

- When $s$ is in the numerator, the angle is 90° ($s^2 = +180°$, and so on).
- When $s$ is in the denominator, the angle is $-90°$ ($\frac{1}{s^2} = -180°$, and so on).

2. Substitute the values of frequency ($\omega$) to find the phase angles

| $\omega$ | $\phi$ |
|---|---|
| $\omega_l = 0.1$ | $-93°$ |
| $\omega_{c1} = 2.5$ | $-149°$ |
| $\omega_{c2} = 10$ | $-210°$ |
| $\omega_h = 100$ | $-263°$ |

**Table 9.3**

3. Plot the phase angles in the log scale.

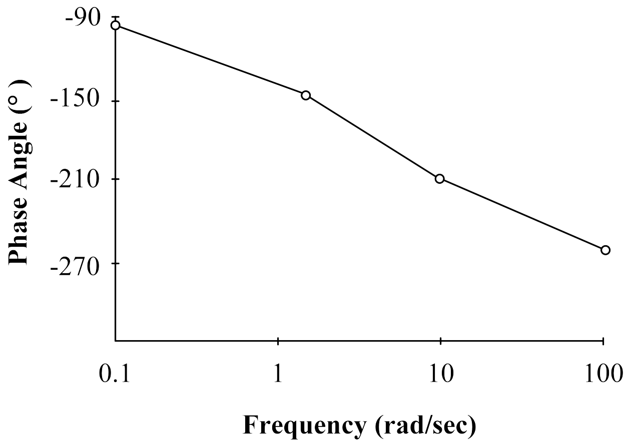


**Figure 9.8:** Phase plot

## *Gain and Phase Margin of the Bode Plot*

Considering the previous plots namely the ***magnitude*** and ***phase plots***, we can identify the ***gain*** and ***phase margins*** as well as the ***gain*** and ***phase crossover frequency***.

**Definition 9.6**

***Gain Margin*** *($G_M$) is the change in open-loop gain, expressed in decibels(dB), required at 180 degrees of phase shift to make the closed-loop system unstable.* ♣

**Definition 9.7**

***Phase Margin*** $(G_M)$ *is the change in open-loop gain, expressed in decibels(dB), required at 180 degrees of phase shift to make the closed-loop system unstable.* ♣

Steps in identifying Gain and Phase Margins

Considering the previous Bode Plot

1. Draw a horizontal line along the 0 dB in magnitude plot.
2. Identify the gain crossover frequency ($\omega_{gc}$) by plotting the point where the magnitude plot and horizontal line intersects with each other at 0 dB.
3. Draw another horizontal line along the $-180°$ in the phase plot.
4. Identify the phase crossover frequency ($\omega_{pc}$) by plotting the point where the phase plot and horizontal line intersects with each other at 180°.

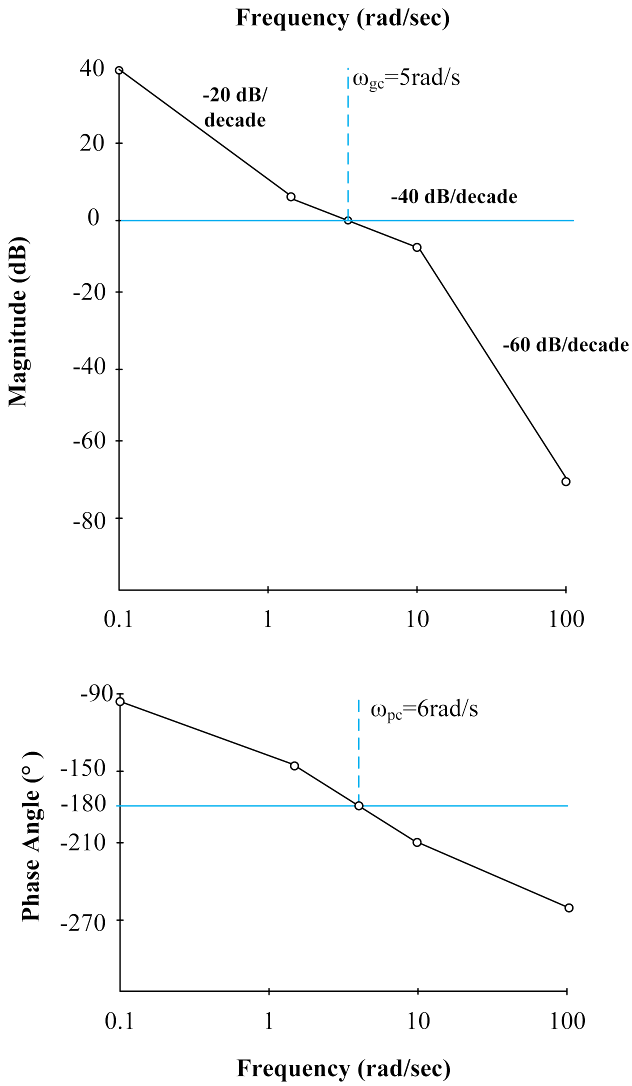


**Figure 9.9:** Gain and phase crossover frequency

5. To identify the gain margin from the phase crossover frequency, extend the line until it intersects the magnitude plot.

**Note** *Gain margin is equal to the negative of the point where $\omega_{pc}$ and magnitude plot intersects.*

On the other hand, the phase margin is obtained by extending the gain crossover frequency until it reaches the phase plot.

**Note** *Phase margin is equal to 180 degrees plus the point where the $\omega_{gc}$ and phase plot intersects.*

$$G_M = -(-1.9dB) \Rightarrow 1.9dB$$
$$P_M = 180° + \phi_{gc} = 180° + (-175°) = 5°$$

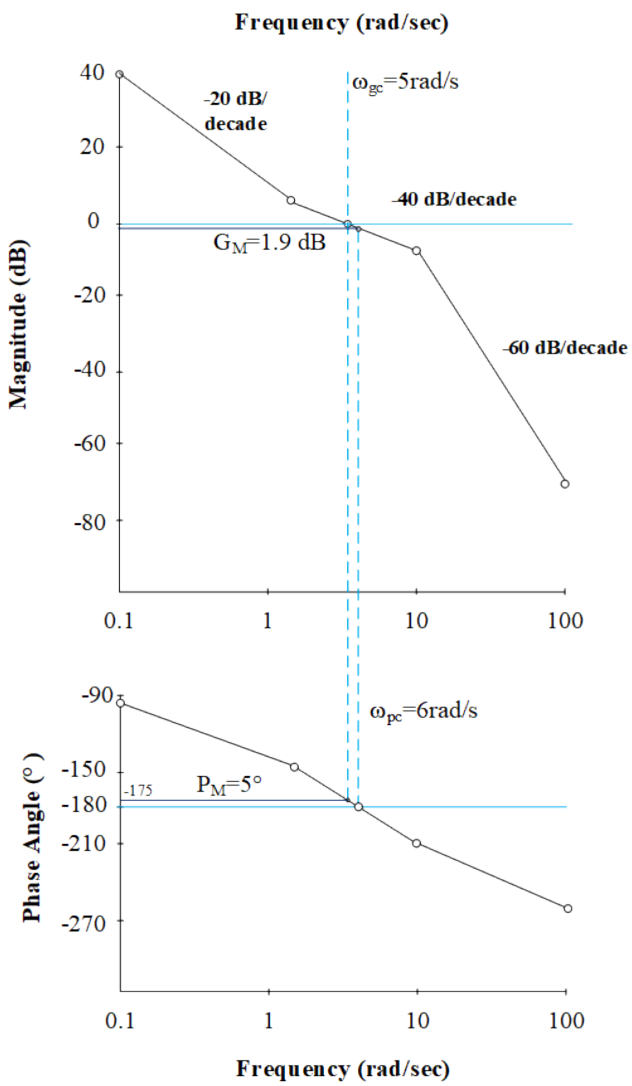


**Figure 9.10:** Gain and phase margin

### *Stability of the System*

We will be able to determine the stability of the system depending on the values of the margins and crossover frequencies.

| **STABLE** | **UNSTABLE** | **MARGINALLY STABLE** |
|---|---|---|
| $G_M \wedge P_M = (+)$ | $G_M \vee P_M = (-)$ | $G_M \wedge P_M = 0$ |
| $\omega_{gc} < \omega_{pc}$ | $\omega_{gc} > \omega_{pc}$ | $\omega_{gc} = \omega_{pc}$ |

**Table 9.4**

From the example,

$$G_M = 1.9dB$$
$$P_M = 5°$$

The system is **stable** since both are positive.

$$\omega_{gc} = 5\ rad/s$$
$$\omega_{pc} = 6\ rad/s$$

Since $\omega_{gc} = 5rad/s$ is less than $\omega_{pc} = 6rad/s$, the system is **stable**.

### *Bode Plot using MATLAB*

Since we are now familiar with the Bode plot by hand, in this section, we are now plotting the Bode plot of a transfer function using MATLAB software. This method is much easier compared to the previous method since the response is instant.

Using the same transfer function, $G(s) = \dfrac{10}{s(1+0.4s)(1+0.1s)}$, we will use the MATLAB script to see if the two methods will yield the same Bode plot.

1. Open the MATLAB software and create a new script.
2. Define the transfer function using the syntax `s=tf('s')`.

```
s=tf('s');
g=10/[s*(1+0.4*s)*(1+0.1*s)];
```

3. To see the Bode plot, use the command `bode(g)`.

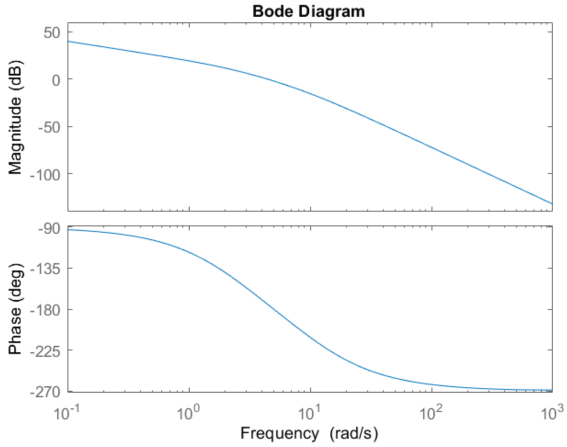


**Figure 9.11:** Bode plot of $G(s) = \dfrac{10}{s(1+0.4s)(1+0.1s)}$

## *Gain and Phase Margin using MATLAB*

To determine the margins of the bode plot:

1. Open the bode plot of the transfer function.

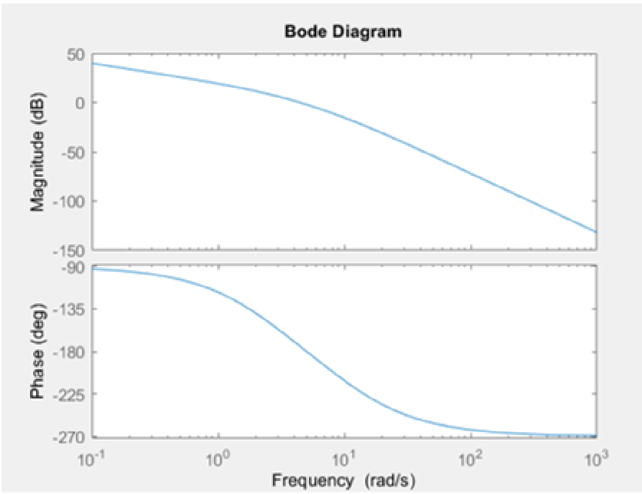


**Figure 9.12:** Bode plot

2. Right click the plot and select **“Characteristics » All Stability Margins”**.

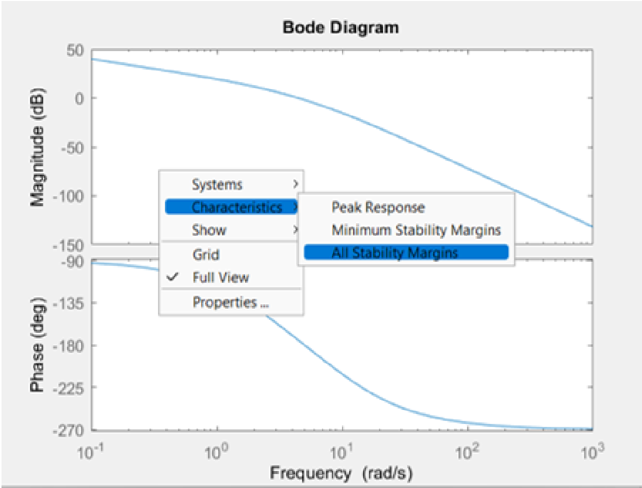


Figure 9.13

3. The result will show the points of the gain margin and phase margin as well as the gain and phase crossover frequency and the stability of the system.

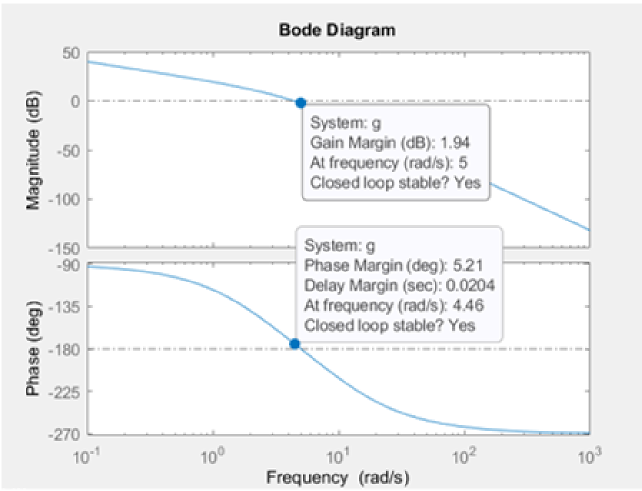


**Figure 9.14:** Gain margin and phase margin with stability

Another method is through MATLAB script. Using the same syntax to get the bode plot, enter the command `margin(g)` to show the Gain and Phase margins of the transfer function.

```
s=tf('s');
g=10/[s*(1+0.4*s)*(1+0.1*s)];
bode(g)
margin(g)
```

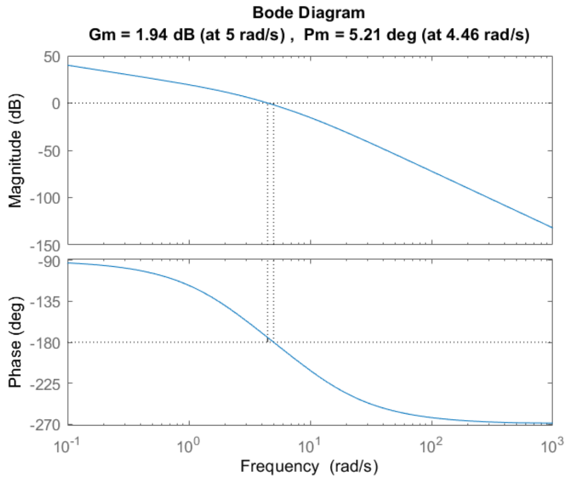


**Figure 9.15:** Bode plot with gain and phase margins

**Definition 9.8**

***Resonance*** *occurs when a system has a large gain in a very small frequency range that means there are certain frequencies that are amplified disproportionately to those around them and recognized by having a sharp peak in the Bode gain plot.* ♣

**Definition 9.9**

***Resonant Peak*** *is the maximum value of the magnitude of the frequency response expressed in decibels* ♣

**Definition 9.10**

***Resonant Frequency*** *is the frequency at which the frequency response has reached the maximum value of magnitude* ♣

**Definition 9.11**

***Bandwidth*** *is the frequency at which the magnitude response curve is 3 dB down from its value at zero frequency.* ♣

## Nyquist Criterion

### *History Interlude*

The Nyquist Criterion is a useful technique most especially in control systems engineering since it can determine the absolute stability of a closed-loop system graphically through the frequency response of an open-loop system even without determining the closed-loop poles. This technique was derived and developed by Harry Theodor Nyquist (1889 -1976) who was an American Engineer where his discovery of criterion (1932) had a great impact and importance during World War II as it helped to control artillery employing electromechanical feedback systems.

## Introduction

Nyquist Criterion uses the open-loop frequency response and open-loop pole location in order to determine the stability of a closed-loop system.

## Derivation of Nyquist Criterion

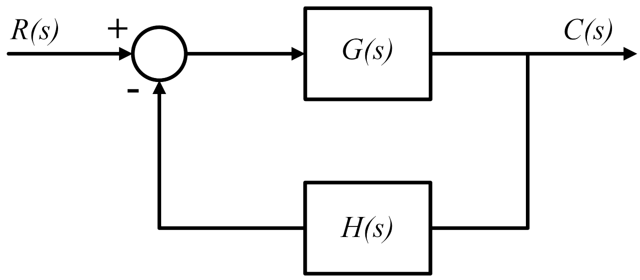


**Figure 9.16:** Closed-loop control system

Considering Fig. 9.16, let us define the given system with open-loop basic notation,

$$G(s) = \frac{N_G}{D_G} \tag{9.3}$$

$$H(s) = \frac{N_H}{D_H} \tag{9.4}$$

we find

$$G(s)H(s) = \frac{N_G N_H}{D_G D_H} \tag{9.5}$$

In Closed-loop

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H} \tag{9.6}$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G D_H}{D_G D_H + N_G N_H} \tag{9.7}$$

***Four Important Concepts in Deriving Nyquist Criterion***

1. The relationship between the poles of $(1 + G(s)H(s))$ and the poles of $(G(s)H(s))$;
2. The relationship between the zeros of $(1 + G(s)H(s))$ and the poles of the closed-loop transfer function, $(T(s))$;
3. The concept of *mapping* points; and
4. The concept of mapping *contours*.

From the equations, we can see that

1. The poles of $(1 + G(s)H(s))$ are the same as the poles of $(G(s)H(s))$, open-loop systems.
2. The zeros of $(1 + G(s)H(s))$ are the same as the poles of $(T(s))$, the closed-loop system.

In addition, the Nyquist stability criterion states that if there are open-loop poles P (poles) and closed-loop poles Z (zeros) are enclosed by the s plane closed path in Fig. 9.17, then the corresponding $G(s)H(s)$ plane should encircle the origin. The number of N encirclements is given by

$$N = Z - P$$

where $N$ equals the number of clockwise rotations of the resulting contour about the -1 point.

- N is positive for clockwise encirclement around (-1,0)
- N is negative for counterclockwise encirclement around (-1,0)
- If any poles are found on the right-half plane, the system will get unstable

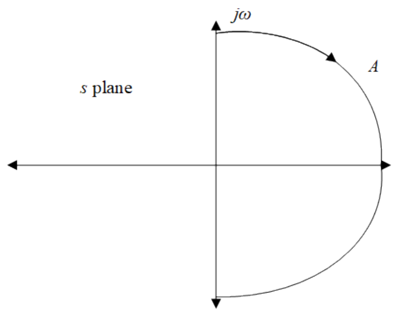


**Figure 9.17:** Nyquist contour enclosing right half s-plane

**Definition 9.12**

***Contour*** *is a collection of points that can be mapped through the given function resulting into another contour in the s-plane* ♣

## *Basic Contour Mapping*

- If G(s)H(s) has only zeros or only poles that are *not encircled* by the closed path in s plane, then the resulting contour maps in a *clockwise* direction
- If G(s)H(s) has *only zeros* that are *encircled* by the closed path in s plane, then the resulting contour maps in a *clockwise* direction
- If G(s)H(s) has *only poles* that are *encircled* by the closed path in s plane, then the resulting contour maps in a *counterclockwise* direction
- If G(s)H(s) has *only zeros* or *only poles* that *encircled* by the closed path in s plane, then the resulting contour map does *encircle the origin*
- If G(s)H(s) has *#poles = #zeros* that are *encircled* by the closed path in s plane, the resulting contour map *does not encircle the origin*

Mapping $G(s)H(s)$ to the s plane as $s \to j\omega$ instead of $1 + G(s)H(s)$ since all the poles and zeros are already known, the resulting contour is the same as a mapping through 1 + G(s)H(s), except that it is translated one unit to the left. Hence, we count rotations about -1, thus Nyquist stability criterion is as follows:

*If a contour, A, encircles the entire right half-plane is mapped through $G(s)H(s)$, then the number of closed-loop poles, Z, in the right half-plane equals the number of open-loop poles, P, that are in the right half-plane plus the number of clockwise revolutions, N, around -1 of the mapping; $Z = P + N$. The mapping is called the* ***Nyquist Diagram or Nyquist plot of*** *$G(s)H(s)$.*

This method is considered as a frequency response technique as the mapping of the points on the s plane through the function $G(s)H(s)$ is the same as letting $s \rightarrow j\omega$ to form the frequency response function $G(j\omega)H(j\omega)$. So, we are finding the frequency response of $G(s)H(s)$ over the part of contour A in the figure on the positive $j\omega$-axis. Part of the Nyquist diagram is the polar plot of the frequency response of $G(s)H(s)$.

***Polar Plot*** is a plot that can be drawn between the magnitude and phase angle of $G(j\omega)H(j\omega)$ by varying $\omega$ from 0 to $\infty$. In a graph, the positive angles are represented in counterclockwise direction, while the negative values are in clockwise direction. For example, the angle 90° in counterclockwise direction is equal to the angle $-270°$ in clockwise direction.

**Rules for drawing Polar Plots**

Consider the open loop transfer function for a closed loop control system

$$G(s)H(s) = \frac{5}{s(s+1)(s+2)}$$

- Let $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+2)}$$

- Write the expression for magnitude and phase of the transfer function.
  Magnitude is

$$M = \frac{5}{\omega(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})}$$

  Phase angle is

$$\phi = -90° - tan^{-1}\omega - tan^{-1}\frac{\omega}{2}$$

- Find the magnitude and phase angle of the transfer function by substituting $\omega = 0$ and $\omega = \infty$ to the expression.

| Frequency (rad/sec) | Magnitude | Phase angle ° |
|---|---|---|
| 0 | $\infty$ | -90 or 270 |
| $\infty$ | 0 | -90 or 270 |

Table 9.5

The polar starts at $(\infty, -90°)$ and ends at $(0, -270°)$.

- Determine whether the plot intersects the real axis by setting the imaginary components equal to 0 and find the value of $\omega$.

$$
\begin{aligned}
G(j\omega)H(j\omega) &= \frac{5}{(j\omega^3 + 3(j\omega)^2 + 2j\omega)} \\
&= \frac{5}{-3\omega^2 + j\omega(2+\omega^2)} \times \frac{-3\omega^2 - j\omega(2+\omega^2)}{-3\omega^2 - j\omega(2+\omega^2)} \\
G(j\omega)H(j\omega) &= \frac{5[-3\omega^2 - j\omega(2+\omega^2)]}{9\omega^4 + \omega^2(2+\omega^2)^2}
\end{aligned}
$$

By setting $\omega = 0$ we will obtain a value of $\omega = \sqrt{2}$ and this polar plot will intersect the negative real axis. The phase angle corresponding to the negative real axis is $-180°$.

Substituting $\omega = \sqrt{2}$ to the magnitude of open loop transfer function

$$
\begin{aligned}
M &= \frac{5}{\sqrt{2}(\sqrt{(\sqrt{2})^2 + 1})(\sqrt{(\sqrt{2})^2 + 4})} \\
M &= 0.83
\end{aligned}
$$

Hence, the polar plot intersects the negative real axis when $\omega = \sqrt{2}$ and the polar coordinate is $(0.83, -180°)$

## *Sketching Nyquist Plot by Hand*

Consider the transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2}$$

1. Change $s = j\omega$

$$(G(j\omega) = \frac{1}{(j\omega)^2 + 3j\omega + 2}$$

2. Plot 4 points of frequency to the transfer function.
   a. $\omega = 0$

$$G(0) = \frac{1}{(0)^2 + 3(0) + 2} = \frac{1}{2}$$

   b. $\omega = \infty$

$$G(0) = \frac{1}{(\infty)^2 + 3(\infty) + 2} = 0$$

c. Imaginary intercept

$$G(j\omega) = \frac{1}{(j\omega)^2 + 3(j\omega) + 2}$$
$$= \frac{1}{(2-\omega^2) + 3j\omega} \times \frac{(2-\omega^2) - 3j\omega}{(2-\omega^2) - 3j\omega}$$
$$G(j\omega) = \frac{(2-\omega^2) - 3j\omega}{(2-\omega^2)^2 + 9\omega^2}$$

The imaginary intercept can be obtained by letting $\omega = 0$ in the real component,

$$real \Rightarrow \frac{2-\omega^2}{(2-\omega^2)^2 + 9\omega^2} = 0\omega = \sqrt{2}$$

Substitute the value to the imaginary component to get the imaginary intercept,

$$imaginary = \frac{-3\omega}{(2-\omega^2)^2 + 9\omega^2}$$
$$= \frac{-3(\sqrt{2})}{(2-\sqrt{2}^2)^2 + 9\sqrt{2}^2}$$
$$= -\frac{-3\sqrt{2}}{18}$$
$$imaginary = -0.236$$

d. Real Intercept The real intercept can be obtained by letting $\omega = 0$ in the imaginary component

$$imaginary = \frac{-3\omega}{(2-\omega^2)^2 + 9\omega^2}$$
$$= \frac{-3(0)}{(2-(0)^2)^2 + 9(0)^2}$$
$$imaginary = 0$$

Substituting $\omega = 0$ to real intercept yields $G(j\omega = 0)$ which is already obtained.

3. Plot the points on the s-plane and draw its reflection about the real axis to create the Nyquist Diagram.

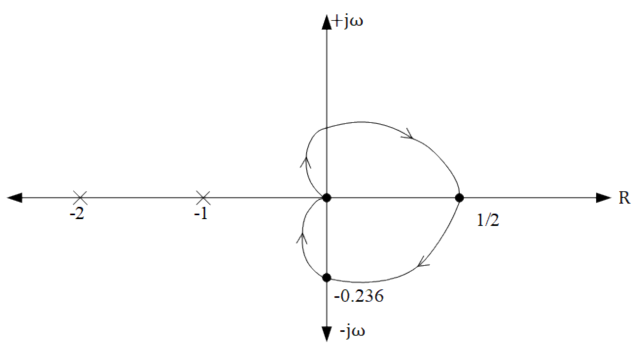


**Figure 9.18:** Nyquist Diagram of G(s)

Since there are no closed-loop poles located inside the contour, and there are no encirclements around the point (-1,0), the system is **stable**.

Analytically, we can solve the number of closed-loop poles to determine if the system is ***stable***.

Based on the Nyquist Diagram, we have open-loop poles outside the contour,

$$P = 0$$

In addition, we have no encirclements around -1,

$$N = 0$$

Hence, the number of closed-loop poles (Z) is zero that makes the system stable,

$$Z = P + N$$
$$Z = 0$$

## *Nyquist Diagram in MATLAB*

Just like in the Bode plot, it is much easier to determine the exact and accurate Nyquist Diagram when using the MATLAB software.

In this case, define the same transfer function in MATLAB script.

```
sys = tf([1], [1 3 2])\\
```

Using the command `nyquist(sys)` and running the simulation, MATLAB will show the Nyquist Diagram of the transfer function.

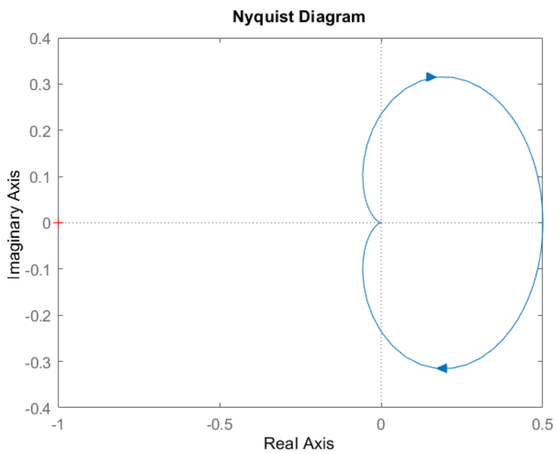


**Figure 9.19:** Nyquist Diagram of G(s) in MATLAB

We can see that the Nyquist plot started in the point 0.5 and continues to loop in a clockwise direction since there is no encirclement of the closed-loop poles and -1, the system is ***stable.***

## *Stability via the Nyquist Diagram*

In this section, we will discuss the stability of a system via the Nyquist diagram if a closed-loop system has a variable gain in the loop. For this, what range of gain would make a system stable? The general approach is to set the loop gain equal to unity and draw the Nyquist diagram. Since gain is simply a multiplying factor, the effect of the gain is to multiply the resultant by a constant anywhere along the Nyquist diagram.

For example,

Consider the transfer function below and determine for what range of K is the $G_{CL}$ stable.

$$G(s) = \frac{1}{(s+10)(s+2)^2}$$

let $s = j\omega$

$$|G(j\omega)| = \frac{1}{(j\omega+10)(j\omega+2)^2}$$

for $\omega = 0$

$$|G(j\omega)| = \frac{1}{(0+10)(0+2)^2}$$

for $\omega = \infty$

$$\begin{aligned}|G(j\omega)| &= \frac{1}{(\infty+10)(\infty+2)^2}\\ &= 0\\ \measuredangle G_{OL} &= \frac{1}{-j\infty} = -90^\circ\\ \measuredangle G_{OL} &= 0^\circ - (90^\circ) = 90^\circ\end{aligned}$$

*For Stability:* where is $G_j\omega$ real only?

$$\begin{aligned}G(j\omega) &= \frac{1}{(j\omega+10)(-\omega^2+4j\omega+4)}\\ &= \frac{1}{-j\omega^3-4\omega^2+4j\omega-10\omega^2+40j\omega+40}\\ G(j\omega) &= \frac{1}{40-14\omega^2+j(44\omega-\omega^3)}\end{aligned}$$

setting the imaginary component equal to 0, the frequency $\omega$ becomes $\sqrt{44}$

by substitution

$$\begin{aligned}|G(j\sqrt{44})| &= \frac{1}{40-14(\sqrt{44})^2} = -\frac{1}{576}\\ \measuredangle G(j\sqrt{44}) &= 180^\circ\end{aligned}$$

*For Marginal Stability*: Where is $G(j\omega)$ imaginary only?

setting the real component equal to 0,

$$40 - 14\omega^2 = 0$$
$$\omega^2 = \frac{40}{14}$$
$$\omega = \sqrt{\frac{20}{7}}$$

by substitution,

$$|G(j\sqrt{44})| = \frac{1}{j[44(\sqrt{\frac{20}{7}} - (\sqrt{\frac{20}{7}})^3]} = 0.0144$$
$$\measuredangle G(j\sqrt{\frac{20}{7}}) = -90°$$

## *Sketching Nyquist Diagram*

Plotting the poles and zeros of the transfer function

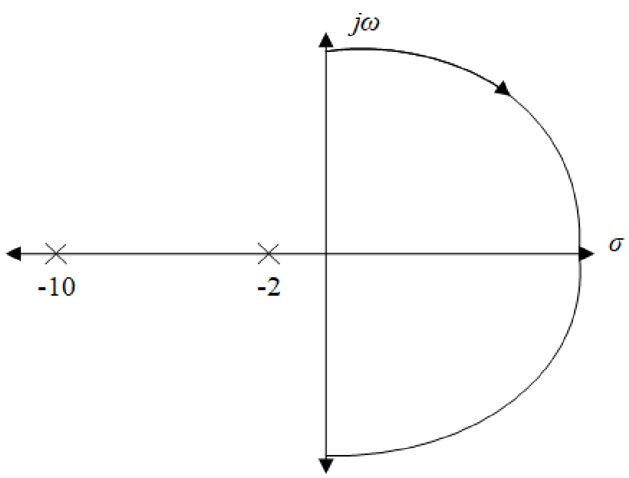


**Figure 9.20:** Contour of $G(s)$

$$P = 0$$
$$N = 0$$
$$Z = P + N = 0$$

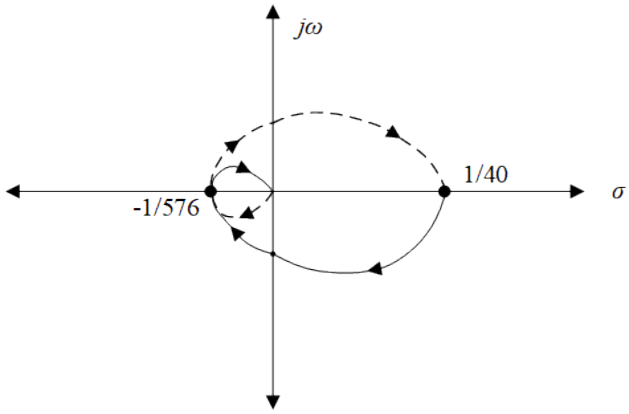


**Figure 9.21:** Nyquist Diagram of $G(s)$

If K=-1, we have to ensure that we do not encircle the point -1 and we did not since we only encircle the point $-\frac{1}{576}$. So for stability,

$$-\frac{1}{576} > -\frac{1}{K} \Rightarrow \frac{1}{576} < \frac{1}{K} \Rightarrow K < 576$$
$$\frac{1}{40} > -\frac{1}{K} \Rightarrow -\frac{1}{40} < \frac{1}{K} \Rightarrow K > -40$$

Hence, the range of gain to make the system stable is,

$$-40 < K < 576$$

### *Gain and Phase Margin via the Nyquist Diagram*

Two quantitative measures to determine how stable a system is using the Nyquist diagram are the gain and phase margins. Systems with greater gain and phase margins can withstand greater changes in system parameters before becoming unstable. In the same sense, these margins can be qualitatively related to the root locus, in that systems whose poles are farther from the imaginary axis have a greater degree of stability.

Assume a system is stable without encirclements of point -1. Using Fig. 9.22, the gain difference between the Nyquist diagram's crossing of the real axis at $-\frac{1}{a}$ and the -1 critical point determines the proximity of the system to instability. Hence, if the gain of the system were multiplied by $a$ units, the Nyquist diagram would intersect the critical point. We then say that the gain margin is $a$ units, or, expressed in dB, $G_M = 20loga$. Notice that the gain margin is the reciprocal of the real-axis crossing expressed in dB.

In terms of the phase margin, at point $Q$', where the gain is unity, $a$ represents the system's proximity to instability. At unity gain, if a phase shift of $\alpha$ degrees occurs, the system becomes unstable. Thus, the amount of phase margin is $\alpha$.

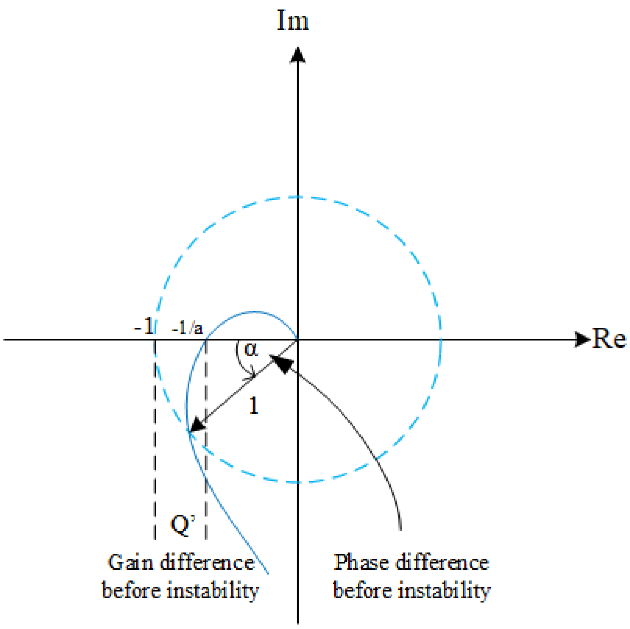


**Figure 9.22:** Nyquist diagram showing gain and phase margins

$$Gainmargin = G_M = 20 loga$$
$$Phasemargin = \phi_M = \alpha$$

The stability of the control system based on the relation between the gain and phase margins is the following:

- If the gain margin $G_M$ is greater than one $G_M > 1$ and the phase margin $P_M$ is positive (+), then the control system is stable.
- If the gain margin $G_M$ is equal to one $G_M = 1$ and the phase margin $P_M$ is zero degrees, then the control system is marginally stable.
- If the gain margin $G_M$ is less than one $G_M < 1$ and/or the phase margin $P_M$ is negative (-), then the control system is stable.

**Example 9.2** Find the gain and phase margins for the system $G(s) = \dfrac{K}{(s^2+2s+2)(s+2)}$ if $K = 6$.

**Solution** *Find the frequency where the Nyquist diagram crosses the negative real axis.*

$$s \longrightarrow j\omega$$

$$\begin{aligned}
G(j\omega)H(j\omega) &= \frac{6}{[(j\omega)^2 + 2(j\omega) + 2][(j\omega) + 2]} \\
&= \frac{6}{(j\omega)^3 + 4(j\omega)^2 + 6(j\omega) + 4} \\
&= \frac{6}{-j\omega^3 - 4\omega^2 + 6(j\omega) + 4} \\
&= \frac{6}{4(1-\omega^2) + j\omega(6-\omega^2)} \times \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{4(1-\omega^2) - j\omega(6-\omega^2)} \\
G(j\omega)H(j\omega) &= \frac{6[4(1-\omega^2) - j\omega(6-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}
\end{aligned}$$

*Setting the imaginary part equal to zero, we find* $\omega = \sqrt{6} rad/sec$.
*By substitution to real part*

$$\begin{aligned}
real &= \frac{6[4(1-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} \\
&= \frac{6[4(1-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} \\
&= \frac{6[4(1-(\sqrt{6})^2)]}{16[1-(\sqrt{6})^2)]^2 + (\sqrt{6})^2[6-(\sqrt{6})^2]^2} \\
real &= -\frac{3}{10}
\end{aligned}$$

*Thus, the gain can be increased by*

$$\begin{aligned}
\frac{1}{K} &= -0.3 \\
&= \frac{1}{0.3} \\
K &= 3.33
\end{aligned}$$

*before the real part becomes -1. Hence,*

$$\begin{aligned}
G_M &= 20 log 3.33 \\
G_M &= 10.45 dB
\end{aligned}$$

*To determine the phase margin, find the frequency for which the magnitude is unity.*

$$G(j\omega)H(j\omega) = \frac{6[4(1-\omega^2) - j\omega(6-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

*In this equation, it has a unity gain at a frequency of 1.253 rad/s. At this frequency, the phase angle is* $-112.3°$.

$$\begin{aligned} P_M &= 180° + \phi \\ &= 180° + -122.3° \\ P_M &= 67.7° \end{aligned}$$

*Therefore the system is stable since the gain margin is greater than 1 and the phase margin is positive.*

## *Gain and Phase Margins through MATLAB*

Considering the same open loop transfer function of a control system where $K = 6$.

$$\begin{aligned} G(s) &= \frac{K}{(s^2+2s+2)(s+2)} \\ &= \frac{6}{(s^2+2s+2)(s+2)} \end{aligned}$$

1. Define the transfer function in the MATLAB with the zeros, poles, and value of K.
2. Use the command `nyquist(G)` to plot the nyquist diagram.

```
G=zpk([],[-1+i, -1-i], 6);
nyquist(G)
```

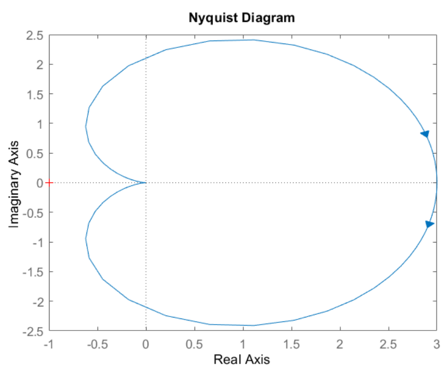


**Figure 9.23:** Nyquist diagram of $G(s) = \dfrac{6}{(s^2+2s+2)(s+2)}$

3. After the Nyquist diagram appears, to determine the margin points and read the gain and phase margins of the system,
   a. Right-click in the graph area then select the **"Characteristics > All Stability Margins"**.

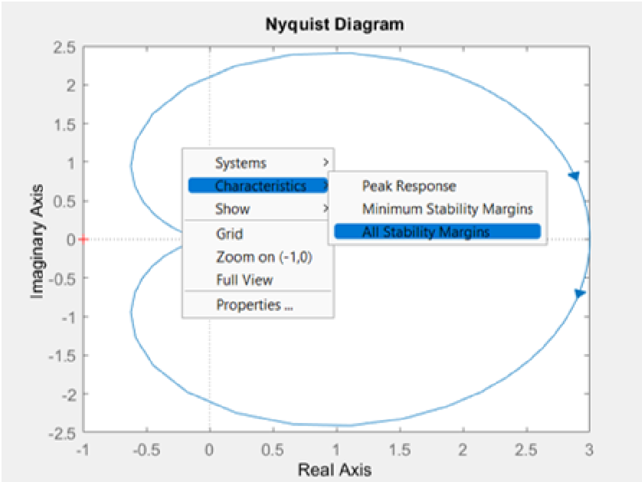


**Figure 9.24**

   b. Rest on the margin points to read the gain and phase margins.

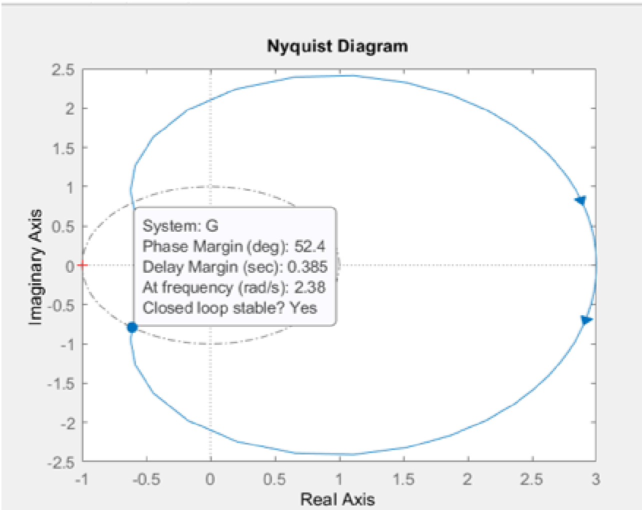


**Figure 9.25:** Margins of $G(s) = \dfrac{6}{(s^2+2s+2)(s+2)}$

# 9.3 Laboratory Experiment

## Module Exercises

1. Find the analytical expression for the magnitude and phase response for each $G(s)$ below.

   a. $G(s) = \frac{100}{s(s+10)(s+20)}$

   b. $G(s) = \frac{1}{s(s+2)(s+1)}$

   c. $G(s) = \frac{(s+10)}{(s+1)(s+40)}$

2. For each function in Problem #1, sketch the Bode plot of the following transfer function by hand and cross check using MATLAB.
3. Find the gain and phase margins of the transfer function given in Problem #1.
4. Sketch the Nyquist diagram of a control systems transfer function below and cross check using MATLAB

a.

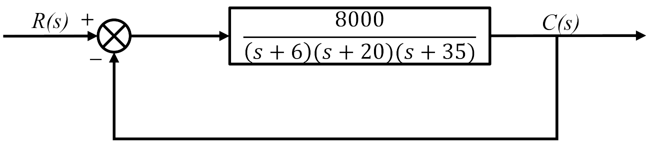


**Figure 9.26**

5. Find the gain and phase margins of the transfer function given in Problem #4.

## Simulation Activity

## Analysis of Bode Plot and Nyquist Diagram

## Objective:

- To analyze the effect of gain values to the stability of the system
- To be familiarized with the frequency response through Bode plot and Nyquist diagram

## Procedures

1. Download the **"FrequencyDomain_Livescript.mlx"** file from the e-Learning Google Drive.
2. Open the livescript file in **MATLAB** software.
3. For the **Gain and Phase Margin section**, change the value of gain *(K)* and complete the data inside the table.

| **Gain (*K*)** | **Gain Margin** | **Phase Margin** | **Gain Crossover Frequency** | **Phase Crossover Frequency** | **Stability based on Step response** |
|---|---|---|---|---|---|
| *1* | *13.3 dB* | *101°* | *1.85 rad/s* | *5.48 rad/s* | *Stable* |
| 1.5 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |

4. For the **Bandwidth frequency section**, change the value of the frequency on the slider and observe the system's output response and record the date on the table below.

| | $\omega$ | **Remarks on the response** |
|---|---|---|
| $\omega < \omega_{bw}$ | | |
| | | |
| | | |
| $\omega > \omega_{bw}$ | | |
| | | |
| | | |

5. On the **stability section**, adjust the loop gain K and observe the diagram based on the encirclement on point -1.

| **Loop Gain (*K*)** | **Stability of the System** |
|---|---|
| 0.5 | |
| 0.7 | |
| 0.9 | |
| 1.2 | |
| 1.4 | |
| 1.6 | |
| 1.8 | |
| 2.0 | |

6. For the gain margin section, adjust the gain $a$ and observe the nyquist diagram as well as the closed-loop step-response of the system.
7. For the last section, Phase Margin, adjust the phase and find the phase margin of the system where it becomes unstable when -1 is being encircled.

## 9.4 Questions to Ponder

1. What is the effect of increasing the gain value to the stability of the system?
2. What have you observed on the system's response by varying the input frequency based on the bandwidth?
3. In what value of gain does the system become stable or unstable? What is the behavior of the system as the gain increases or decreases?
4. What happens to the system as we decrease the value of a from the gain margin in Nyquist diagram? Explain briefly.

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# Chapter 10
# Design via State Space

## 10.1 Objective

- To use pole placement in designing a state-feedback controller for systems represented and not represented in phase-variable form in order to meet transient response specifications.
- To know if a system is controllable
- To use pole placement in designing a state-feedback observer for systems represented and not represented in observer canonical form
- To determine if a system is observable
- To design steady-state error characteristics for systems represented in state space

## 10.2 Theory

The drawback of frequency response and root locus techniques is that after designing the second-order poles, it is not clear how a higher order pair of poles affect the second-order approximation. State space methods do not allow the specification of closed-loop zero locations, which frequency domain methods do allow through replacement of the lead compensator zero. The drawback of state-space design is that it may be more sensitive to parameter changes.

### Controller Design

An nth-order feedback control system has an nth-order closed loop characteristic equation of the form

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0.$$

#### *Topology for Pole Placement*

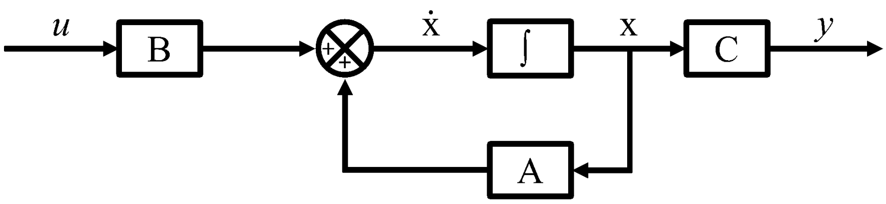



**Figure 10.1:** State-space representation of a plant

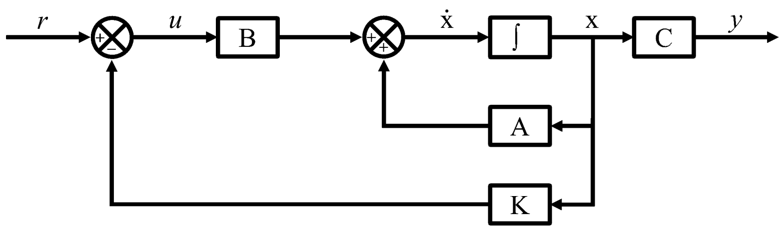



**Figure 10.2:** Plant with state-variable feedback

**Roots for an open–loop system:** $|(sI - A)| = 0$

If each state variable is fed back through a gain, there would be $n$ gains that can be adjusted to yield the required closed–loop pole values.

The state equations for the closed-loop system of Figure 10.1 can be written by inspection as:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Referring to Fig. 10.2, $u$ can be written as:

$$u = (r - Kx)$$

$$\dot{x} = Ax + B(r - Kx)$$

$$\dot{x} = (A - BK)x + Br$$

where $r$ is the vector of the desired state variables and $K$ is the state feedback gain matrix.

**Roots for an open–loop system:** $|(sI - A + BK)| = 0$

## *Pole Placement for Plants in Phase Variable Form*

To apply pole-placement methodology to plants represented in phase-variable form, we take the following steps:

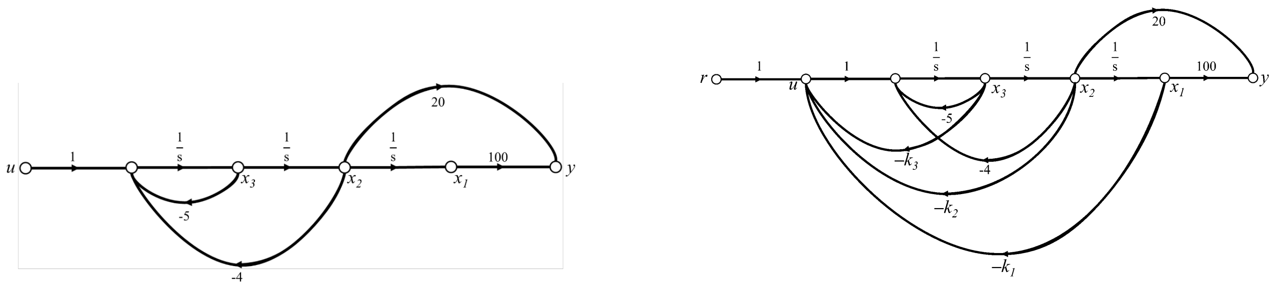
1. Represent the plant in phase-variable form.
2. Feedback each phase variable to the input of the plant through a gain, $k_i$.
3. Find the characteristic equation for the closed-loop system
4. Decide upon all closed-loop pole locations and determine an equivalent characteristic equation
5. Equate like coefficients of the characteristic and solve for $k_i$.

**Example 10.1**

Design the phase-variable feedback gains to yield 9.5% overshoot and a settling time of 0.74 second of the given plant below.

$$G(s) = \frac{20(s+5)}{s(s+1)(s+4)}$$

**Solution**




**(a)** Phase-variable representation **(b)** Plant with state-variable feedback

**Figure 10.3**

1. *Based on Fig. 10.3a, the phase-variable form is:*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -4 & -5 \end{bmatrix} ; B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} ; C = \begin{bmatrix} 100 & 20 & 0 \end{bmatrix}$$

2. *Closed-loop system's state equations of Fig. 10.3b*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} 100 & 20 & 0 \end{bmatrix} x$$

3. *Closed-loop system's characteristic equation:* $det(sI - (A - BK))$

$$(sI - (A - BK)) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix}$$

$$(sI - (A - BK)) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_1 & 4+k_2 & s+5+k_3 \end{bmatrix}$$

$$det(sI - (A - BK)) = s^3 + (5+k_3)s^2 + (4+k_2)s + k_1$$

4. *using* $T_s = 0.74$ *and 9.5% overshoot:*

$$T_s = \frac{4}{\sigma_d}$$
$$\sigma_d = \frac{4}{T_s}$$
$$\sigma_d = \frac{4}{0.74}$$
$$\sigma_d = 5.405$$

*9.5% OS yields to* $\zeta = 0.6$ *and* $\theta = 53.16°$

$$\omega_d = \sigma_d \tan(53.16°)$$
$$\omega_d = 7.214$$

*Desired poles:* $5.405 \pm j7.214$

*Desired characteristic equation*

$$= (s + 5.405 + j7.214)(s + 5.405 - j7.214)(s + 5)$$
$$= s^3 + 15.81s^2 + 135.31s + 406.3$$

*Closed-loop system's characteristic equation:*

$$(sI - (A - BK)) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4 + k_2) & -(5 + k_3) \end{bmatrix}$$
$$= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_1 & 4 + k_2 & s + 5 + k_3 \end{bmatrix}$$
$$\det(sI - (A - BK)) = s^3 + (5 + k_3)s^2 + (4 + k_2)s + k_1$$

5. *Equating the coefficients*

$$(5 + k_3)s^2 = 15.81s^2$$
$$\boxed{k_3 = 10.81}$$
$$(4 + k_2)s = 135.31s$$
$$\boxed{k_2 = 131.31}$$
$$\boxed{k_1 = 406.3}$$

*Rewriting the closed-loop system's state equations:*

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 406.3 & 131.31 & 10.81 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = \begin{bmatrix} 100 & 20 & 0 \end{bmatrix} x$$

*The transfer function will be:*

$$\boxed{T(s) = \frac{20(s+5)}{(s^3+10.81s^2+131.31s+406.3)}}$$

## Controllability

### *Controllability by Inspection*

A system with distinct eigenvalues and a diagonal system matrix is controllable if the input coupling matrix $B$ does not have any rows that are zero.

**Example 10.2**

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

**Answer:**
From the given system, $x_3$ is not controlled by $u$, therefore it is **uncontrollable**.

### *The Controllability Matrix*

**Definition 10.1**
*The controllability matrix is used as a tool for transforming a system to phase-variable form for the design of state-variable feedback.* ♣

If an input can be found to take every state variable from a desired initial state to a desired final state, the system is **controllable**; otherwise, the system is **uncontrollable**. A system is controllable if the controllability matrix, $C_M$ is of full rank. A matrix that is full rank has a nonzero determinant.

$$C_M = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

**Example 10.3**
Determine if the given system is controllable.

$$\dot{x} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 3 & -4 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

**Solution**

$$C_M = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$C_M = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & -9 \\ 1 & -1 & 16 \end{bmatrix}$$

$$\det C_M = -80$$

*This system is controllable, since the determinant is non-zero.*

## Alternative Approaches to Controller Design

This section will show and discuss on how to design state-variable feedback for systems not represented in phase-variable form.

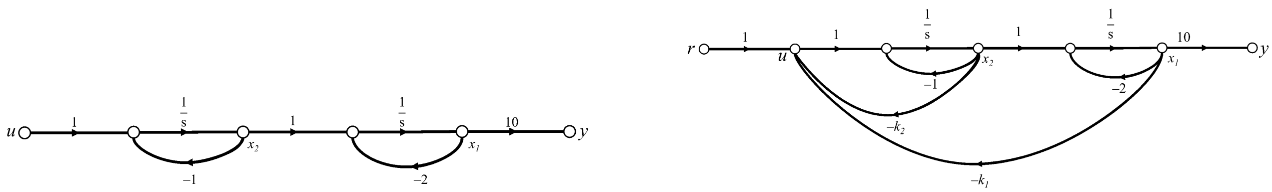
### *Controller Design by Matching Coefficients*

It uses the same method we used for systems represented in phase variables. However, it leads to difficult calculations of the control gains, especially for higher-order systems not represented with phase variables.

**Example 10.4**

Given a plant, $\dfrac{Y(s)}{U(s)} = 10[s+1s+2]$, design state feedback for the plant represented in cascade form to yield a 15% overshoot with a settling time of 0.5 second.

**Solution**




**(a)** Signal-flow graph in cascade form

**(b)** System with state feedback

**Figure 10.4**

*State equations:*

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -k_1 & -(k_2+1) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 10 & 0 \end{bmatrix} x$$

*Characteristic equation:* $s^2 + (k_2+3)\,s + (2k_2 + k_1 + 2) = 0$

*Using the transient response given in the problem, the desired characteristic equation is:*

$$s^2 + 16s + 239.5 = 0$$

*Equating the coefficients:*

$$(k_2+3)\,s = 16s$$

$$\boxed{k_2 = 13}$$

$$2k_2 + k_1 + 2 = 239.5$$

$$\boxed{k_1 = 211.5}$$

*Closed-loop system's state equations:*

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 211.5 & 13 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 10 & 0 \end{bmatrix} x$$

*The transfer function will be:* $\boxed{T(s) = \frac{10}{(s^2+13s+211.5)}}$

## *Controller Design by Transformation*

This method consists of transforming the system to phase variables, designing the feedback gains, and transforming the designed system back to its original state-variable representation. It requires that we first develop the transformation between a system and its representation in phase-variable form.

### Steps in using the Transformation Method:

1. Transform the system to phase–variable representation
   Plant not represented in phase-variable form
   $$\dot{z} = Az + Bu$$
   $$y = Cz$$
   Controllability Matrix: $C_{Mz} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$
   Assume that the plant can be transformed into the phase-variable $(x)$ representation with the transformation: $z = Px$
   Transformed Plant:
   $$C_{Mx} = P^{-1}C_{Mz}$$
   $$P = C_{Mz}{C_{Mx}}^{-1}$$
2. Design the state-variable control feedback gain
   Including both feedback and input $u = -K_x x + r$
   Transform plant with state-variable control feedback:
   $$\dot{x} = \left(P^{-1}AP - P^{-1}BK_x\right)x + P^{-1}Br$$
   $$y = CPx$$
   The zeros of this closed-loop system are determined from the polynomial formed from the elements of $CP$
3. Transform the system in phase-variable representation back to the original representation.

Plant not in phase-variable representation with state-variable control:

$$\dot{z} = \left(A - BK_x P^{-1}\right) z + Br$$
$$y = Cz$$

The state variable feedback gain: $K_z = K_x P^{-1}$

Zeros of the closed-loop transfer function are the same as the zeros of the uncompensated plant.

### Example 10.5

Design a state-variable feedback controller to yield a 20.8% overshoot and a settling time of 4 seconds for a plant, $G(s) = \dfrac{(s+4)}{(s+1)(s+2)(s+5)}$.

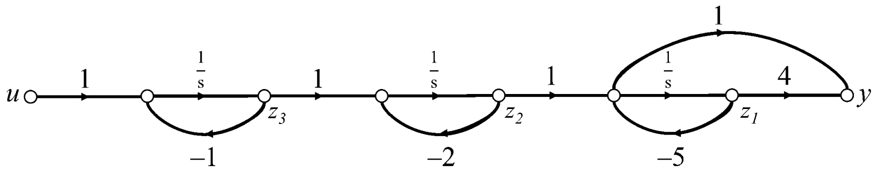



**Figure 10.5:** Signal-flow graph for plant

**Solution** *State Equation:*

$$\dot{z} = A_z z + B_z u$$
$$\dot{z} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = C_z z$$
$$y = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} z$$

*Controllability Matrix:*

$$C_{Mz} = \begin{bmatrix} B_z & A_Z B_Z & A_z^{\ 2} B_z \end{bmatrix}$$
$$C_{Mz} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$
$$\det C_{Mz} = -1$$

*The system is controllable*

*Characteristic equation:* $\det(sI - A_z) = s^3 + 8s^2 + 17s + 10 = 0$

*Phase-variable representation:*

$$\dot{x} = A_x x + B_x u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} x$$

*Controllability Matrix for the phase-variable system:*

$$C_{Mx} = \begin{bmatrix} B_x & A_x B_x & {A_x}^2 B_x \end{bmatrix}$$

$$C_{Mx} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -8 \\ 1 & -8 & -47 \end{bmatrix}$$

*Transformation Matrix:*

$$P = C_{Mz} C_{Mx}^{-1}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 7 & 1 \end{bmatrix}$$

*For a 20.8% overshoot and a settling time of 4 seconds, a factor of the characteristic equation of the designed closed-loop system is* $s^2 + 2s + 5$. *Since the closed-loop zero will be at s=-4, we choose the third closed-loop pole to cancel the closed-loop zero. Total characteristic equation:*

$$D(s) = (s + 4)\left(s^2 + 2s + 5\right)$$

$$D(s) = s^3 + 6s^2 + 13s + 20 = 0$$

*The state equation for the phase-variable form with state-variable feedback are:*

$$\dot{x} = (A_x - B_x K_x)\, x$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\left(10 + k_{1_x}\right) & -\left(17 + k_{2_x}\right) & -\left(8 + k_{3_x}\right) \end{bmatrix}$$

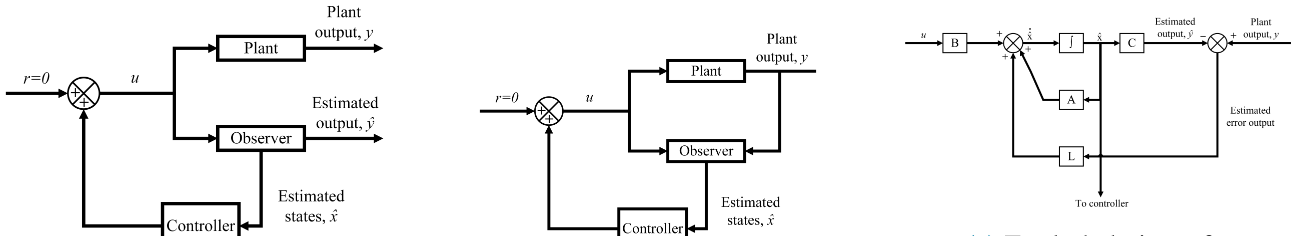
$$y = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} x$$

## Observer

Controller design relies upon access to the state variables for feedback through adjustable gains. In other applications, some of the state variables may not be available at all, or it is too costly to measure them or send them to the controller. If the state variables are not available because of system configuration or cost, it is possible to estimate the states. An observer or estimator is used to calculate state variables that are not accessible from the plant.

**State-space Equations**

| Plant | Observer |
|---|---|
| $\dot{x} = Ax + Bu$ | $\dot{\hat{x}} = A\hat{x} + Bu$ |
| $y = Cx$ | $\hat{y} = C\hat{x}$ |




**(a)** Open-loop observer **(b)** Closed-loop observer **(c)** Exploded view of a closed-loop observer

**Figure 10.6:** State-feedback design using an observer to estimate unavailable state variables

The dynamics of the difference between the actual and estimated states is unforced, and if the plant is stable, this difference, due to differences in initial state vectors, approaches zero. The speed of convergence between the actual state and the estimated state is the same as the transient response of the plant. Since the convergence is too slow, we seek a way to speed up the observer and make its response time much faster than that of the controlled closed-loop system, so that, the controller will receive the estimated states instantaneously.

The use of feedback in Fig. 10.6b, is to increase the speed of convergence between the actual and estimated estates. The error between the outputs of the plant and the observer is fed back to the derivatives of the observer's states. The system corrects to drive the error to 0.

When the controller is implemented, the phase-variable or controller canonical form yielded an easy solution for the controller gains. In designing an observer, it is the observer canonical form that yields the easy solution for the observer gains.

Similar to the design of the controller vector, $K$, the design of the observer consists of evaluating the constant vector, $L$, so that the transient response of the observer is faster than the response of the controlled loop in order to yield a rapidly updated estimate of the state vector.

### Procedure in Designing an Observer

1. Find the state equations for the error between the actual state vector and the estimated state vector, $x - \hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L\,(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$

2. Find the characteristic equation for the error system and evaluate the required L to meet a rapid transient response for the observer.

To obtain the state equation for the error between the estimated state vector and the actual state vector, we substitute the output equation to the state equation:

$$\left(\dot{x} - \dot{\hat{x}}\right) = (A - LC)\,(\dot{x} - \hat{x})$$

$$(y - \hat{y}) = C\,(x - \hat{x})$$

Letting $e_x = (x - \hat{x})$:

$$\dot{e}_x = (A - LC)\,e_x$$

$$y - \hat{y} = Ce_x$$

$e_x = (A - LC)\,e_x$ is unforced. If the eigenvalues are all negative, the estimated state vector error, $e_x$, will decay to zero.

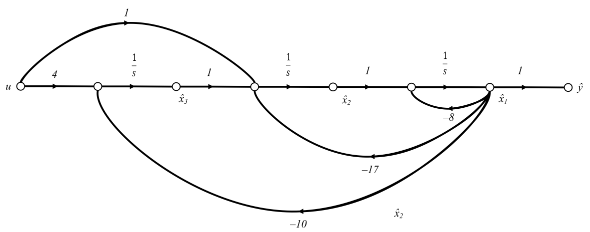
The characteristic equation is: $\det\left[\lambda I - (A - LC)\right] = 0$

3. Evaluate required observer feedback gain, L, to meet rapid transient response for observer.
4. Select eigenvalues of observer to yield stability & desired transient response that is faster than controlled closed-loop response.

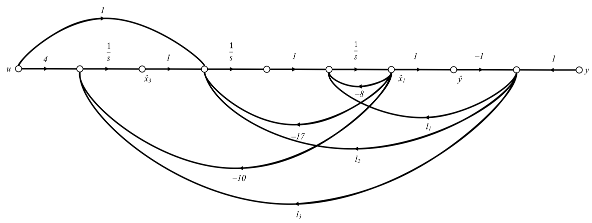
### Example 10.6

Design an observer for the plant $G(s) = \dfrac{(s+4)}{s^3 + 8s^2 + 17s + 10}$ which is represented in observer canonical form. The observer will respond 10 times faster than the closed-loop design.

### Solution



**(a)** Signal-flow graph of a system using observer canonical form variables



**(b)** Additional feedback to create observer

**Figure 10.7**

1. *Fig. 10.7a shows the estimated plant in observer canonical form.*
2. *Fig. 10.7b represents the difference between the plant's actual output, y, and the observer's estimated output, $\hat{y}$, and add the feedback paths from this difference to the derivative of each state variable.*
3. *Find the characteristic polynomial.*

*State equations for the estimated plant shown in Fig. 10.7a:*

$$\dot{\hat{x}} = Ax + Bu = \begin{bmatrix} -8 & -1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} u$$

$$\hat{y} = C\hat{x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \hat{x}$$

*Observer error:*

$$\dot{e}_x = (A - LC)\, e_x = \begin{bmatrix} -(8+l_1) & -1 & 0 \\ -(17+l_2) & 0 & 1 \\ -(10+l_3) & 0 & 0 \end{bmatrix} e_x$$

*Characteristic equation:* $s^3 + (8+l_1)\, s^2 + (17+l_2)\, s + (10+l_3)$

4. *Evaluate the desired polynomial, set the equal coefficients, and solve for the gains,* $l_i$*. The closed-loop controlled system has dominant second-order poles at* $-1 \pm j2$*. To make our observer 10 times faster, we design the observer poles to be at* $-10 \pm j20$*. We select the third pole to be 10 times the real part of the dominant second-order poles, or -100. Hence, the desired characteristic polynomial is:* $(s+100)\left(s^2+20s+500\right) = s^3 + 120s^2 + 2500s + 50000$

$$(8 + l_1) = 120$$

$$\boxed{l_1 = 112}$$

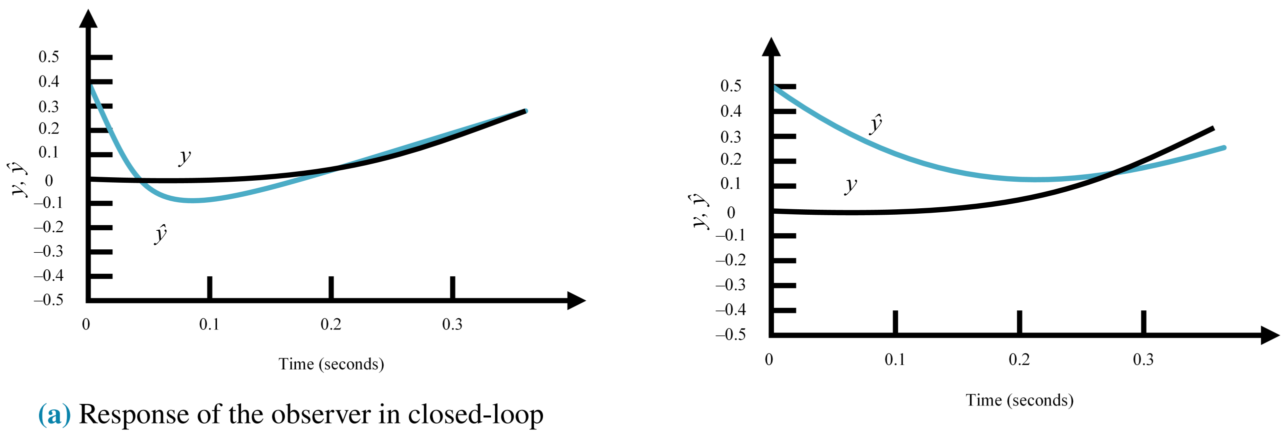
$$17 + l_2 = 2500$$

$$\boxed{l_2 = 2483}$$

$$10 + l_3 = 50000$$

$$\boxed{l_3 = 49990}$$

*A simulation of the observer with an input of* $r(t) = 100t$ *is shown in Fig.10.8a.*
*The initial conditions of the plant were all zero, and the initial condition of* $\hat{x}_1$ *was 0.5. Since the dominant pole of the observer is* $-10 \pm j20$*, the expected settling time should be about 0.4 second.*




**(a)** Response of the observer in closed-loop simulation

**(b)** Plant with state-variable feedback

**Figure 10.8**

## Observability

The ability to observe a state variable from the output is best seen from the diagonalized system. If the initial-state vector, $x(t_0)$, can be found from $u(t)$ and $y(t)$ measured over a finite interval of time from $t_0$, the system is said to be ***observable***, otherwise the system is said to be ***unobservable***.

**Definition 10.2**

***Observability*** *is the ability to deduce the state variables from a knowledge of the input, $u(t)$ and the output, $y(t)$.* ♣

Pole placement for an observer is a viable design technique only for systems that are observable.

## Observability by Inspection

We can also explore observability from the output equation of a diagonalized system. When the system matrix, A, is in diagonal or parallel form, it is apparent whether or not the system is B. A system with distinct (no repeat) eigenvalues and a diagonal system matrix, A, is observable if the output coupling matrix, C, does not have any columns that are zero.

## The Observability Matrix

In order to determine observability for systems under any representation or choice of state variables, a matrix can be derived that must have a particular property if all state variables are to be observed at the output.

State and output equation of nth-order plant:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

is completely observable if the matrix $(O_M)$, is of rank $n$.

$$O_M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

**Example 10.7** Determine if the system in the figure below is observable.

**Solution**

$$\dot{x} = Ax + Bu$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{21}{4} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = Cx$$

$$y = \begin{bmatrix} 5 & 4 \end{bmatrix}$$

$$O_M = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$O_M = \begin{bmatrix} 5 & 4 \\ -20 & -16 \end{bmatrix}$$

*Thus, the observability matrix does not have full rank, and the system is not observable.*

*Again, you might conclude by inspection that the system is observable because all states feed the output. Remember that observability by inspection is valid only for a diagonalized representation of a system with distinct eigenvalues.*

## Alternative Approaches to Observer Design

In this section, we use a similar idea for the design of observers not represented in observer canonical form.

### *Observer Design by Matching Coefficients*

Matching the coefficients of $det((sI-(A-LC)))$ with the coefficients of the desired characteristic polynomial. Again, this method can yield difficult calculations for higher-order systems.

**Example 10.8** Design an observer feedback gain for the system in phase-variable representation with a transient response described by $\zeta = 0.7$ and $\omega_n = 100$.

$$G(s) = \frac{407(s+0.916)}{(s+1.27)(s+2.69)}$$

**Solution**

$$\dot{x} = Ax + Bu$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3.416 & -3.96 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = Cx$$

$$y = \begin{bmatrix} 372.81 & 407 \end{bmatrix} x$$

$$O_M = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$O_M = \begin{bmatrix} 372.81 & 407 \\ -1390.31 & -1238.91 \end{bmatrix}$$

$$\det O_M = 103978.13$$

*The system is observable.*

$$A - LC = \begin{bmatrix} 0 & 1 \\ -3.416 & -3.96 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \times \begin{bmatrix} 372.81 & 407 \end{bmatrix}$$

$$A - LC = \begin{bmatrix} 372.81l_1 & 1 - 407l_2 \\ -(3.416 + 372.81l_1) & -(3.96 + 407l_2) \end{bmatrix}$$

$$\det\left((sI - (A - LC))\right) = s^2 + (3.96 + 372.81l_1 + 407l_2)\, s + (3.416 + 84.39l_1 + 372.81l_2) = 0$$

*Using* $\zeta = 0.7$ *and* $\omega_n = 100$

*The desired characteristic equation is:* $s^2 + 140s + 10000 = 0$

$$3.96 + 372.81l_1 + 407l_2 = 140$$

$$372.81l_1 = 136.04 - 407l_2$$

$$l_1 = \frac{136.04 - 407l_2}{372.81}$$

$$3.416 + 84.39\left(\frac{136.04 - 407l_2}{372.81}\right) + 372.81l_2 = 10000$$

$$\boxed{l_2 = 35.51}$$

*Substituting* $l_2 = 35.51$

$$3.96 + 372.81l_1 + 407l_2 = 140$$

$$\boxed{l_1 = -38.40}$$

## *Observer Design by Transformation*

**Example 10.9** Design an observer for the plant $G(s) = \dfrac{1}{(s+1)(s+2)(s+5)}$ represented in cascade form. The closed-loop performance of the observer is governed by the characteristic polynomial: $s^3 + 120s^2 + 2500s + 50000$.

**Solution** *Plant representation in its original cascade form*

$$\dot{z} = Az + Bu$$

$$\dot{z} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = Cz$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} z$$

*Observability Matrix:*

$$O_M = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$O_{Mz} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 25 & -7 & 1 \end{bmatrix}$$

$$\det(O_{Mz}) = 1; The\ system\ is\ observable$$

*Characteristic equation:* $\det(sI - A) = s^3 + 8s^2 + 17s + 10 = 0.$

$$A_x = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix}; C_x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

*Observability Matrix for Canonical form:*

$$O_{Mx} = \begin{bmatrix} C_x \\ C_x A_x \\ C_x {A_X}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 47 & -8 & 1 \end{bmatrix}$$

*Design for the observer canonical form::*

$$A_x - L_x C_x = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$A_x - L_x C_x = \begin{bmatrix} -(8 + l_1) & 1 & 0 \\ -(17 + l_2) & 0 & 1 \\ -(10 + l_3) & 0 & 0 \end{bmatrix}$$

*Characteristic polynomial:* $\det[sI - (A_x - L_x C_x)] = s^3 + (8 + l_1)s^2 + (17 + l_2)s + (10 + l_3)$

*Equating this polynomial to the desired closed-loop observer characteristic equation:* $s^3 + 120s^2 + 2500s + 50000$. *We find:* $L_x = \begin{bmatrix} 112 \\ 2483 \\ 49990 \end{bmatrix}$.

*Transforming back to the original representation:*

$$P = {O_{Mz}}^{-1} O_{Mx}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

*Transforming $L_x$ to the original representation:*

$$L_z = P L_x = \begin{bmatrix} 112 \\ 2147 \\ 47619 \end{bmatrix}$$

## Steady-State Error Design via Integral Control

In this section, we will discuss how to design systems represented in state space for steady-state error. A feedback path from the output is added to form the error, $e$, which is fed forward to the controlled plant via an integrator.

Additional state variable: $\dot{x}_N = r - Cx$

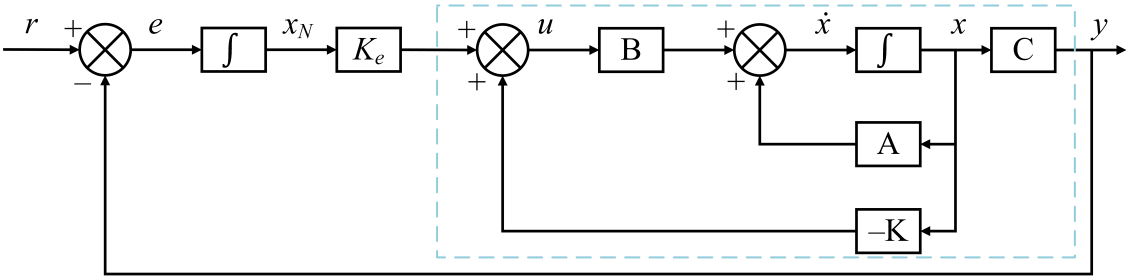



**Figure 10.9:** Integral control for steady-state error design

State equations of Fig. 10.9 can be represented as:

$$\dot{x} = Ax + Bu$$

$$\dot{x}_N = r - Cx$$

$$y = Cx$$

Augmented vectors and matrices:

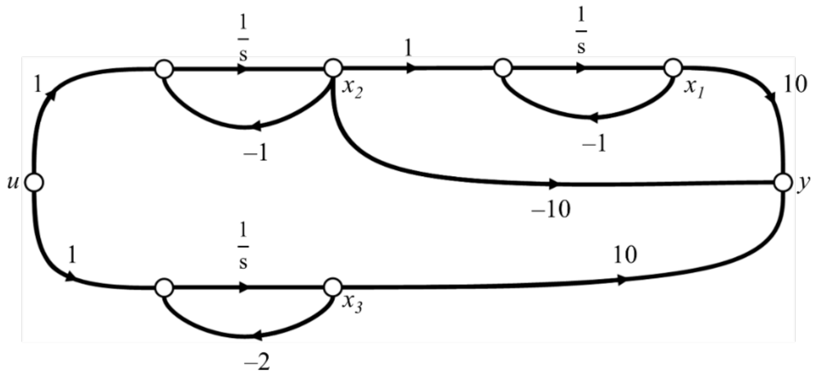
$$\begin{bmatrix} \dot{x} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} (A - BK) & BK_e \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix}$$

# 10.3 Laboratory Experiment

## Module Exercises

1. Given the system below, represented by a signal-flow diagram, determine its controllability.




2. Determine if the following systems are controllable and observable:

a. $\dot{x} = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

b. $\dot{x} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$

## Simulation Activity

## State Feedback with Integral Action

## Objective:

- To analyze and understand the effect of using an integral action in MATLAB.

## Procedures

### Part I.

1. Launch **MATLAB** software
2. Save the script with a filename **'Lab10_Feedback'**
3. Enter the syntax in MATLAB Command Window

```
A=[2 3; -1 4]
B=[0;1]
```

```
C=[1 0]
D=0

%desired poles in closed loop
pdes=[-1+i*2;-1-i*2]

%to find the state feedback gain
F = acker(A,B,pdes)
%To find the feedforward gain G
G=-inv(C*inv(A-B*F)*B)
```

4. Launch **Simulink**
5. Double click the canvas and type state-space to add a State-Space block.
6. Modify the parameters of the state-space by double clicking the state-space block.
   Parameters:
   A: A
   B: B
   C: eye(2)
   D: zeros(2,1)
7. Add a sum block (first sum block) by double clicking the canvas. Connect the output of the sum block to the input of the state-space block.
8. Add another sum block ($2^{nd}$ sum block). Modify the signs to ($| + -$). Connect the output of it to the (+) of the first sum block
9. Then add a gain block by double clicking the canvas. The value is G. Connect the output of this gain to the (+) sign of the second sum block.
10. Add a step input block by double clicking the canvas and typing step. Connect it to the gain block $G$.
11. Add another gain block having a value of C. Modify the multiplication of this gain block to Matrix (K*u). Connect the output of state-space block to the input of this gain block.
12. Add a mux.
13. Connect the output of the C gain to the first input of mux.
14. Connect also the step block to the second input of the mux.
15. Double click the canvas and type scope. Connect the output of the mux to the scope.
16. Add another gain having a value of F. The multiplication of this gain is Matrix (K*u). Connect the output of the state-space to the input of this block. While, the output of this block is connected to the negative sign of the $2^{nd}$ sum block.
17. Add again a step block and modify its step time to 4 and final value is 0.1.
18. **Run** the Simulink model and view the output by clicking the Scope.

### Part II

1. Return to the command window of MATLAB.
2. Enter the syntax:

```
%System definition
A=[2 3; -1 4]
B=[0;1]
C=[1 0]
D=0

%Augmented system
Aint = [A zeros(2,1);
        -C,0]
Bint= [B; 0]

% poles for the control
%three poles
pint=[-1+i*2;-1-i*2;-1]

Kint = acker(Aint, Bint, pint)

%state feedback
K=Kint(1:2)
H = Kint(3)
```

3. Copy and paste the Simulink model.
4. On the copied model, delete the connection between the step block and $G$ gain.
5. Duplicate the sum block (|+-) and add an integrator block.
6. Connect the step input to the positive sign of the new added sum block and connect the output of the sum block to integrator.
7. Change the value of G gain block to H.
8. Change the sign of the sum block after H gain to |- -.
9. Connect again the step block to the mux.
10. Connect the output of C gain to the negatve sign of the new added sum b
11. Change the value of F gain block to K.
12. **Run** the Simulink model and compare the output

## 10.4 Questions to Ponder

1. How does integral action affect the steady-state error of the system?
2. What is the purpose of integrator to a system?

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