Time domain — Frequency Domain

$\mathcal{L}\{f(t)\} \longrightarrow F(s)$

D.E. $\longrightarrow$ F(s) Algebra

2nd year college

8th grade: s^2+s+1, $(s-1)^2$ ← polynomials, quadratic equations, perfect square

Inverse Laplace

D.E. $\longleftarrow$ F(s)

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

Table 2.1 Laplace transform table

Item no.	f(t)	F(s)
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform of Differential Equations

$$\mathcal{L}[y(t)] = Y(s)$$

Frequency Domain for Transfer Function Analysis
↳ Initial conditions are zero, $sy(0), y'(0) = 0$

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$$

$$\frac{dy}{dt} \longrightarrow \mathcal{L}[\dot{y}(t)] = sY(s) - \cancel{y(0-)}^{0}$$

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$$

$$\frac{d^2y}{dt^2} \longrightarrow \mathcal{L}[\ddot{y}(t)] = s^2Y(s) - \cancel{sy(0-)}^{0} - \cancel{y'(0-)}^{0}$$

$$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$$

$$\frac{d^3y}{dt^3} \longrightarrow \mathcal{L}\{\dddot{y}(t)\} = s^3Y(s)$$

$$\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} = s^4Y(s) \qquad \mathcal{L}\left\{\frac{d^{46}y}{dt^{46}}\right\} = s^{46}Y(s)$$

$$\mathcal{L}\left[\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32x(t)\right] = \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 12\cdot\mathcal{L}\left\{\frac{dy}{dt}\right\} + 32\cdot\mathcal{L}\{y\} = 32\cdot\mathcal{L}\{x(t)\}$$

$$s^2Y(s) + 12sY(s) + 32Y(s) = 32X(s)$$

$$\boxed{Y(s)\left[s^2+12s+32\right] = 32X(s)}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{32\} = \frac{32}{s}$$

$$\mathcal{L}\left[5\frac{d^5y}{dt^5} - 6\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 13\dot{y} - y = \frac{d^2x}{dt^2} + \frac{dx}{dt} - 3\right]$$

$$5s^5Y(s) - 6s^3Y(s) + s^2Y(s) + 13sY(s) - Y(s) = s^2X(s) + sX(s) - \frac{3}{s}$$

$$\boxed{Y(s)\left[5s^5 - 6s^3 + s^2 + 13s - 1\right] = sX(s)\left[s+1\right] - \frac{3}{s}}$$

Laplace Transform of
Integral Equations

$$\mathcal{L}\left[\int f(t)\right] = \frac{F(s)}{s} = \mathcal{L}\left[f(t)\right]\cdot\frac{1}{s}$$

$$\mathcal{L}\left\{\int 6\frac{d^4y}{dt^4}\right\} = 6\cdot\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\}\cdot\frac{1}{s}$$

$$= 6\cdot s^{\cancel{4}\,3}Y(s)\cdot\frac{1}{\cancel{s}}$$

$$= \boxed{6s^3Y(s)}$$

$$\mathcal{L}\left\{\int 32\frac{d^3y}{dt^3} - \int 6\ddot{y} + \int 3y\right\} = 32\cdot\frac{s^{\cancel{3}\,2}Y(s)}{\cancel{s}} - 6\cdot\frac{s^{\cancel{2}}Y(s)}{\cancel{s}} + 3\cdot\frac{Y(s)}{s}$$

$$= 32s^2Y(s) - 6sY(s) + \frac{3}{s}Y(s)$$

$$= Y(s)\left[32s^2 - 6s + \frac{3}{s}\right]$$

$$= Y(s)\left[\frac{32s^3 - 6s^2 + 3}{s}\right]$$

$$\mathcal{L}\left\{\int\frac{d^5x}{dt^5} + 6\frac{d^3x}{d^3} - 5\int\frac{d^2x}{dt^2} + \frac{\dot{x}}{2} - x + \int 3 = y(t)\right\}$$

$$\frac{s^{\cancel{5}\,4}X(s)}{\cancel{s}} + 6\cdot s^3X(s) - 5\cdot\frac{s^{\cancel{2}}}{\cancel{s}}X(s) + \frac{1}{2}\cdot sX(s) - X(s) + \frac{3}{s^2} = Y(s)$$

$$\boxed{X(s)\left[s^4 + 6s^3 - 5s + \frac{s}{2} - 1\right] + \frac{3}{s^2} = Y(s)}$$