

- The matrix that relates the end-effector velocities to joint velocities.
- Partial derivatives of the Forward Kinematics equation.  $H T M = 4 \times 4$

$\theta_n$  at  $d_n$

Constant Velocity - displacement over time (t).  $v = \frac{d}{t}$

Average Velocity - change in displacement over change in time (t).  $v = \frac{D_2 - D_1}{t_2 - t_1} \text{ or } v = \frac{\Delta D}{\Delta t}$

Instantaneous Velocity - The rate of change in displacement with respect to time (t).

$$\begin{array}{ll} \text{a.} & \text{b.} \\ J_1 = 0^\circ/\text{s} & J_1 = 90^\circ/\text{s} \\ J_2 = 90^\circ/\text{s} & J_2 = 90^\circ/\text{s} \end{array}$$

$$\text{Velocity} = \frac{dD}{dt}$$

angular velocity  $\rightarrow \dot{\theta}$  angular displacement  $\rightarrow \frac{d\theta}{dt} \rightarrow \ddot{\theta}$

linear velocity  $\rightarrow \dot{x}$  linear displacement  $\rightarrow \frac{dx}{dt} \rightarrow \dot{x}, \dot{y}, \dot{z} \text{ and } \dot{d}$

end-effector velocity vector

$$\begin{array}{l} \text{linear velocities of EE at } x, y, z \text{ direction} \\ \text{angular velocities of EE at } x, y, z \text{ axis} \end{array} \left\{ \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \right. = J \left[ \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \right] \left. \begin{array}{l} \text{Jacobian Matrix} \\ \text{Joint Velocity Vector} \end{array} \right]$$

$\dot{q}_n = \text{either } \dot{\theta}_n \text{ or } d_n$

The concept of the Jacobian matrix was first introduced by the mathematician Carl Gustav Jacob Jacobi in the 19th century.

Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

rows = no. of rows in EE velocity vector (always 6)  
columns = no. of rows in DH parametric table.

$$J = 6 \times 4$$

	n	$\theta$	$\alpha$	r	d
${}^0 H \rightarrow 1$		$0^\circ$	$270^\circ$	0	$a_1$
${}^1 H \rightarrow 2$		$270^\circ$	$270^\circ$	0	$a_2 + d_1$
${}^2 H \rightarrow 3$		$90^\circ$	$270^\circ$	0	$a_3 + d_2$
${}^3 H \rightarrow 4$		$0^\circ$	$0^\circ$	0	$a_4 + d_3$

$$J = 6 \times 3$$

	n	$\theta$	$\alpha$	r	d
${}^0 H \rightarrow 1$		$0^\circ + \theta_1$	$90^\circ$	0	$a_1$
${}^1 H \rightarrow 2$		$0^\circ + \theta_2$	$0^\circ$	$a_2$	0
${}^2 H \rightarrow 3$		$0^\circ + \theta_3$	$0^\circ$	$a_3$	0

$$J = 6 \times 6$$

	n	$\theta$	$\alpha$	r	d
1	$0^\circ$	$0^\circ$	0	$a_1 +$	
2	$90^\circ$	$0^\circ$	$a_2$	0	
3	$0^\circ$	$0^\circ$	$a_3 + a_2$	0	
4	$0^\circ$	$90^\circ$	0	$a_4$	
5	$90^\circ + \theta_4$	$90^\circ$	0	0	
6	$0^\circ$	$0^\circ$	0	$a_4 +$	

## Methods of obtaining Jacobian Matrix

1. Partial Derivative Method
2. Propagation Method
3. Etc.

### Forward Kinematics Equation

$$H_{0\_2} = \begin{bmatrix} x_2 \\ [(\cos(t_1)\cos(t_2) - \sin(t_1)\sin(t_2)), \\ (\cos(t_1)\sin(t_2) + \cos(t_2)\sin(t_1)), \\ 0, \\ 0, \\ 0, \\ 0] \end{bmatrix}$$

### Linear Algebra Method

	Prismatic	Revolute
Linear	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_n^0 - d_{i-1}^0)$
Rotational	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$R$  = Rotation Matrix (follow the superscripts and subscripts)

$d$  = Position Vectors (follow the superscripts and subscripts)

$i$  = which column/joint you are Solving at J

$$H_1^0 \rightarrow R_{1-1}^0$$

$$H_2^0 \rightarrow R_{2-1}^0$$

$$H_3^0 \rightarrow R_{3-1}^0$$

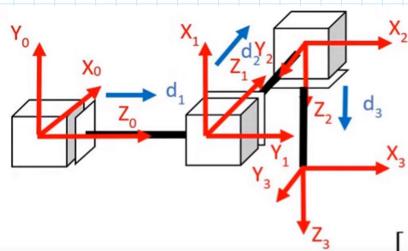
$n$  = no. of columns/joints of J

$$J^{6 \times 3} \rightarrow n=3$$

$$J^{6 \times 4} \rightarrow n=4$$

$$J^{6 \times 6} \rightarrow n=6$$

### Example No. 1

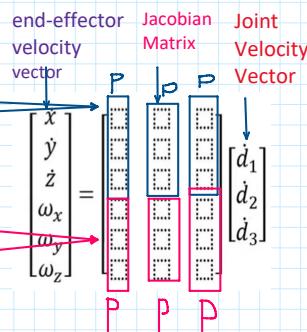


$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \text{Matrix} & \text{Matrix} & \text{Matrix} \\ \text{Matrix} & \text{Matrix} & \text{Matrix} \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

PT : 3 -rows

### Linear Algebra Method

	Prismatic	Revolute
Linear	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_n^0 - d_{i-1}^0)$
Rotational	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



end-effector velocity vector

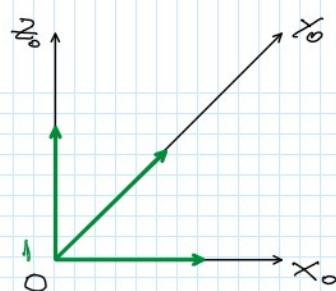
Jacobian Matrix

Joint Velocity Vector

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} R_0^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_2^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$



$$R_0^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$3 \times 3$        $3 \times 1$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & R_1^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_2^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$R_1^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$3 \times 3$        $3 \times 1$

$$R_2^o = R_x^o \times R_z^o$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 1 & R_2^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$R_2^o = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_z^o = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_x^o = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

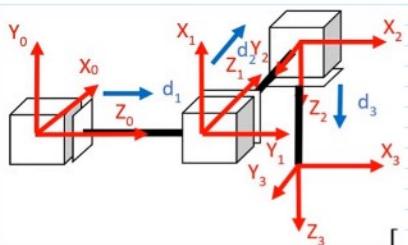
$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$\begin{aligned}\dot{x} &= 0 + \dot{d}_2 + 0 \\ \dot{y} &= 0 + 0 - \dot{d}_3 \\ \dot{z} &= \dot{d}_1 + 0 + 0 \\ \omega_x &= 0 \\ \omega_y &= 0 \\ \omega_z &= 0\end{aligned}$$

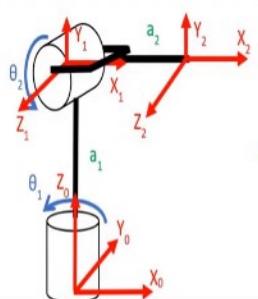
$$\begin{aligned}\dot{x} &= \dot{d}_2 \\ \dot{y} &= -\dot{d}_3 \\ \dot{z} &= \dot{d}_1 \\ \omega_x &= 0 \\ \omega_y &= 0 \\ \omega_z &= 0\end{aligned}$$

$$\begin{aligned}\dot{x}_3^0 &= \dot{d}_2 \\ \dot{y}_3^0 &= -\dot{d}_3 \\ \dot{z}_3^0 &= \dot{d}_1 \\ \omega_x &= 0 \\ \omega_y &= 0 \\ \omega_z &= 0\end{aligned}$$



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \end{bmatrix} \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}$$

Example No. 2



$$\begin{aligned}M &= 2 - \text{DOF} \\ M &= 12 - 10 \\ M &= 6(z) - [(6 \cdot j) + (6 - l)]\end{aligned}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} R & R \\ R & R \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

	Prismatic	Revolute
Linear	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_n^0 - d_{i-1}^0)$
Rotational	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_0^0) & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_1^0) \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$R^o$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$d_2^o$$

$$\begin{bmatrix} a_2 c \theta_1 c \theta_2 \\ a_2 s \theta_1 c \theta_2 \\ a_2 s \theta_2 + a_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_2 c \theta_1 c \theta_2 \\ a_2 s \theta_1 c \theta_2 \\ a_2 s \theta_2 + a_1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c \theta_1 c \theta_2 \\ a_2 s \theta_1 c \theta_2 \\ a_2 s \theta_2 + a_1 \end{bmatrix} R_i^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^o - d_1^o)$$

$$R_i^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \cdot \\ \ddot{\theta}_1 \\ \cdot \\ \ddot{\theta}_2 \end{bmatrix}$$

$$H_2^o = H_1^o * H_2^o$$

$$R_i^o \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^o - d_1^o)$$

$$R_i^o$$

$$\begin{bmatrix} \cos_i & 0 & \sin_i \\ \sin_i & 0 & -\cos_i \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin_i \\ -\cos_i \\ 0 \end{bmatrix}$$

$$d_2^o$$

$$d_1^o$$

$$\begin{bmatrix} a_2 c \theta_1 c \theta_2 \\ a_2 s \theta_1 c \theta_2 \\ a_2 s \theta_2 + a_1 \end{bmatrix}_{3 \times 1} - \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} a_2 c \theta_1 c \theta_2 \\ a_2 s \theta_1 c \theta_2 \\ a_2 s \theta_2 + a_1 \end{bmatrix}_{3 \times 1}$$

$$\begin{bmatrix} x \\ y \\ z \\ w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c \theta_1 c \theta_2 \\ a_2 s \theta_1 c \theta_2 \\ a_2 s \theta_2 + a_1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} \sin_i \\ -\cos_i \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c \theta_1 c \theta_2 \\ a_2 s \theta_1 c \theta_2 \\ a_2 s \theta_2 \\ \sin_i \\ -\cos_i \\ 0 \end{bmatrix}$$

Cross Product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Cross Product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$3 \times 1$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 + a_1 \end{bmatrix} = \begin{array}{l} -a_2 s\theta_1 c\theta_2 \\ a_2 c\theta_1 c\theta_2 \\ 0 \end{array}$$

Cross Product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$3 \times 1$

$$\begin{bmatrix} s\theta_1 \\ -c\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 \end{bmatrix} = \begin{array}{l} -a_2 c\theta_1 s\theta_2 \\ -a_2 s\theta_1 s\theta_2 \\ a_2 s\theta_1^2 c\theta_2 + a_2 c\theta_1^2 c\theta_2 = a_2 c\theta_2 (s\theta_1^2 + c\theta_1^2) \end{array}$$

$6 \times 1$

$6 \times 2$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} -a_2 s\theta_1 c\theta_2 & -a_2 c\theta_1 s\theta_2 \\ a_2 c\theta_1 c\theta_2 & -a_2 s\theta_1 s\theta_2 \\ 0 & a_2 c\theta_2 \\ 0 & s\theta_1 \\ 0 & -c\theta_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$2 \times 1$

$$\begin{aligned} \dot{x} &= -a_2 s\theta_1 c\theta_2 \dot{\theta}_1 - a_2 c\theta_1 s\theta_2 \dot{\theta}_2 \\ \dot{y} &= a_2 c\theta_1 c\theta_2 \dot{\theta}_1 - a_2 s\theta_1 s\theta_2 \dot{\theta}_2 \\ \dot{z} &= a_2 c\theta_2 \dot{\theta}_2 \\ w_x &= s\theta_1 \dot{\theta}_2 \\ w_y &= -c\theta_1 \dot{\theta}_2 \\ w_z &= \dot{\theta}_1 \end{aligned}$$