- The matrix that relates the end-effector velocities to joint velocities.
- Partial derivatives of the Forward Kinematics equation. HTM = 4x4

θ_n at d_n

Constant Velocity - displacement over time (t). $V = \frac{D}{t}$

Average Velocity - change in displacement over change in time (t). $V = \frac{D_2 - D_1}{t_2 - t_1}$ or $V = \frac{\Delta D}{\Delta t}$

Instantaneous Velocity - The rate of change in displacement with respect to time (t). $V = \frac{dD}{dt}$ or $V = \dot{D}$

a.
$J_1 = 0°/s$
$J_2 = 90°/s$

b
$J_1 = 90°/s$
$J_2 = 90°/s$

Velocity $= \frac{dD}{dt}$

angular velocity $\longrightarrow \theta \longrightarrow \frac{d\theta}{dt} \longrightarrow \dot{\theta}$
angular displacement

linear velocity $\longrightarrow x \longrightarrow \frac{dx}{dt} \longrightarrow \dot{x}, \dot{y}, \dot{z}$ and $\dot{d}$
linear displacement

end-effector velocity vector

Jacobian Matrix

Joint Velocity Vector

linear velocities of EE at x, y, z direction

angular velocities of EE at x, y, z axis

$$\begin{array}{c} d\left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \\ \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \end{array} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$\dot{q}_n =$ either $\dot{\theta}_n$ or $\dot{d}_n$

The concept of the Jacobian matrix was first introduced by the mathematician Carl Gustav Jacob Jacobi in the 19th century.

Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \end{bmatrix}$$

rows = no. of rows in EE velocity vector (always 6)

columns = no. of rows in DH parametric table.

J = 6x4

	n	θ	α	r	d
${}^0_1H \rightarrow$	1	0°	270°	0	a_1
${}^1_2H \rightarrow$	2	270°	270°	0	$a_2 + d_1$
${}^2_3H \rightarrow$	3	90°	270°	0	$a_3 + d_2$
${}^3_4H \rightarrow$	4	0°	0°	0	$a_4 + d_3$

J = 6x3

	n	θ	α	r	d
${}^0_1H \rightarrow$	1	$0° + \theta_1$	90°	0	a_1
${}^1_2H \rightarrow$	2	$0° + \theta_2$	0°	a_2	0
${}^2_3H \rightarrow$	3	$0° + \theta_3$	0°	a_3	0

J = 6x6

n	θ	α	r	d
1	0°	0°	0	$a_1 + d_1$
2	$0° + \theta_2$	0°	a_2	0
3	$0° + \theta_3$	0°	a_4	a_3
4	$0° + \theta_4$	90°	0	$-a_5$
5	$90° + \theta_5$	90°	0	0
6	$0° + \theta_6$	0°	0	$a_6 + a_7$

Methods of obtaining Jacobian Matrix
1. Partial Derivative Method
2. Propagation Method
3. Etc.

Forward Kinematics Equation

X_2 Y_2 Z_2

```
H0_2 =
[cos(t1)*cos(t2) - sin(t1)*sin(t2), - cos(t1)*sin(t2) - cos(t2)*sin(t1), 0, a2*cos(t1) + a4*cos(t1)*cos(t2) - a4*sin(t1)*sin(t2)]
[cos(t1)*sin(t2) + cos(t2)*sin(t1),   cos(t1)*cos(t2) - sin(t1)*sin(t2), 0, a2*sin(t1) + a4*cos(t1)*sin(t2) + a4*cos(t2)*sin(t1)]
[                                0,                                   0, 1,                                                a1 + a3]
[                                0,                                   0, 0,                                                      1]
```

Linear Algebra Method

	Prismatic	Revolute
Linear	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times(d^0_n-d^0_{i-1})$
Rotational	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\\\omega_x\\\omega_y\\\omega_z\end{bmatrix}=\begin{array}{c}\begin{array}{ccc}P & R & R\end{array}\\\begin{bmatrix}\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\end{bmatrix}\end{array}\begin{bmatrix}\dot{\theta}_1\\\dot{\theta}_2\\\dot{d}_3\end{bmatrix}$$

R = Rotation Matrix (follow the superscripts and subscripts)

d = Position Vectors (follow the superscripts and subscripts)

i = which column/joint you are Solving at J

$H^0_1 \rightarrow R^0_{1-1}$

$H^1_2 \rightarrow R^0_{2-1}$

$H^2_3 \rightarrow R^0_{3-1}$

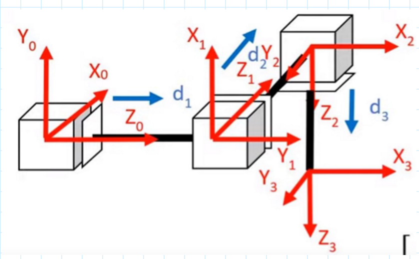
n = no. of columns/joints of J

$J^{6\times3} \rightarrow n=3$

$J^{6\times4} \rightarrow n=4$

$J^{6\times6} \rightarrow n=6$

Example No. 1



$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\\\omega_x\\\omega_y\\\omega_z\end{bmatrix}=\begin{bmatrix}\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\end{bmatrix}\begin{bmatrix}\dot{d}_1\\\dot{d}_2\\\dot{d}_3\end{bmatrix}$$

PT : 3 -rows

Linear Algebra Method

	Prismatic	Revolute
Linear	$\boxed{R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}}$	$\boxed{R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times(d^0_n-d^0_{i-1})}$
Rotational	$\boxed{\begin{bmatrix}0\\0\\0\end{bmatrix}}$	$\boxed{R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}}$

end-effector velocity vector

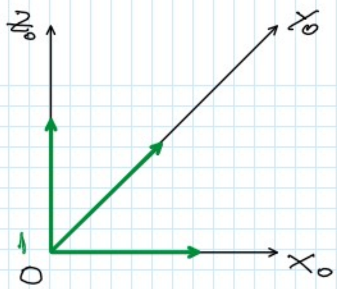
Jacobian Matrix

Joint Velocity Vector

$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\\\omega_x\\\omega_y\\\omega_z\end{bmatrix}=\begin{array}{c}\begin{array}{ccc}P & P & P\end{array}\\\begin{bmatrix}\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\end{bmatrix}\\\begin{array}{ccc}P & P & P\end{array}\end{array}\begin{bmatrix}\dot{d}_1\\\dot{d}_2\\\dot{d}_3\end{bmatrix}$$

end-effector velocity vector
Jacobian Matrix
Joint Velocity Vector

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\\ \square & \square & \square\end{bmatrix}\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix} \Rightarrow \begin{bmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} R^0_{0\,(1-1)}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{1\,(2-1)}\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_{2\,(3-1)}\begin{bmatrix}0\\0\\1\end{bmatrix}\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix}$$



$$R^0_{0\,(1-1)}\begin{bmatrix}0\\0\\1\end{bmatrix} = \overset{R^0_0}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}} \times \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{matrix}0\\0\\1\end{matrix}$$

$$\underset{3\times3}{} \qquad \underset{3\times1}{} \qquad \cdot\, 3\times1$$

$$\begin{bmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} 0 & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_2\begin{bmatrix}0\\0\\1\end{bmatrix}\\ 0 & & \\ 1 & & \\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix}$$

$$R^0_{1\,(2-1)}\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \overset{R^0_1}{\begin{bmatrix}0&0&\boxed{1}\\1&0&\boxed{0}\\0&1&\boxed{0}\end{bmatrix}} \times \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{matrix}1\\0\\0\end{matrix}$$

$$3\times3 \qquad 3\times1$$

$$\begin{bmatrix}\dot{X}\\ \dot{Y}\\ \dot{Z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} 0 & 1 & R^0_2\begin{bmatrix}0\\0\\1\end{bmatrix}\\ 0 & 0 & \\ 1 & 0 & \\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix}$$

$$R^0_2 = R^0_1 \times R^1_2$$

$$\overset{R^0_1}{\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}} \times \overset{R^1_2}{\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}} = \begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix}$$

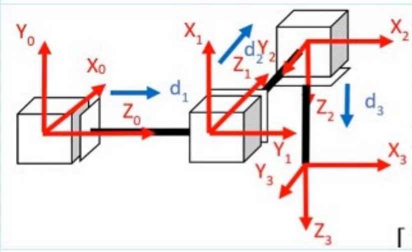
$$\overset{R^0_2}{\begin{bmatrix}0&-1&0\\0&0&-1\\1&0&0\end{bmatrix}} \times \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{matrix}0\\-1\\0\end{matrix}$$

$$\begin{bmatrix}\dot{X}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \overbrace{\begin{bmatrix}0&1&0\\0&0&-1\\1&0&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}}^{6\times3}\overset{3\times1}{\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix}}$$

$$\begin{aligned}\dot{X} &= 0 + \dot{d}_2 + 0\\ \dot{y} &= 0 + 0 - \dot{d}_3\\ \dot{z} &= \dot{d}_1 + 0 + 0\\ \omega_x &= 0\\ \omega_y &= 0\\ \omega_z &= 0\end{aligned}$$

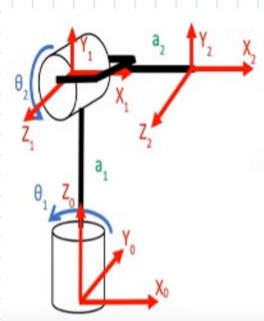
$$\boxed{\begin{aligned}\dot{X} &= \dot{d}_2\\ \dot{y} &= -\dot{d}_3\\ \dot{z} &= \dot{d}_1\end{aligned}}$$
$$\begin{aligned}\omega_x &= 0\\ \omega_y &= 0\\ \omega_z &= 0\end{aligned}$$

$$\boxed{\begin{aligned}\dot{X}^0_3 &= \dot{d}_2\\ \dot{y}^0_3 &= -\dot{d}_3\\ \dot{z}^0_3 &= \dot{d}_1\end{aligned}}$$
$$\begin{aligned}\omega_x &= 0\\ \omega_y &= 0\\ \omega_z &= 0\end{aligned}$$



$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\\\square&\square&\square\end{bmatrix}\begin{bmatrix}\dot{d}_1\\ \dot{d}_2\\ \dot{d}_3\end{bmatrix}$$

Example No. 2



$M = 2\text{-DOF}$

$M = 12-10$

$M = 6(2) - [(6\cdot 1)+(6-1)]$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{array}{c}\begin{array}{cc}R & R\end{array}\\ \begin{bmatrix}\square&\square\\\square&\square\\\square&\square\\\square&\square\\\square&\square\\\square&\square\end{bmatrix}\\ \begin{array}{cc}R & R\end{array}\end{array}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

	Prismatic	Revolute
Linear	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}\times(d^0_n - d^0_{i-1})$
Rotational	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix}\square&\square\\\square&\square\\\square&\square\\\square&\square\\\square&\square\\\square&\square\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{X}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} R^0_{\underset{1-1}{0}}\begin{bmatrix}0\\0\\1\end{bmatrix}\times(d^0_2 - d^0_0) & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix}\times(d^0_2 - d^0_1)\\ R^0_0\begin{bmatrix}0\\0\\1\end{bmatrix} & R^0_1\begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\underset{3\times3}{\overset{R_0^0}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}} \times \underset{3\times1}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\overset{d_2^0}{\begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 + a_1 \end{bmatrix}} - \overset{d_0^0}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 + a_1 \end{bmatrix}$$

$$\begin{vmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{vmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 + a_1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_1^0) \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$H_2^0 = H_1^0 * H_2^1$$

$$R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_2^0 - d_1^0)$$

$$\overset{R_1^0}{\begin{bmatrix} c\theta_1 & 0 & s\theta_1 \\ s\theta_1 & 0 & -c\theta_1 \\ 0 & 1 & 0 \end{bmatrix}} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s\theta_1 \\ -c\theta_1 \\ 0 \end{bmatrix}$$

$$\overset{3\times3 \quad d_2^0}{\begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 + a_1 \end{bmatrix}} - \overset{3\times1 \quad d_1^0}{\begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix}} = \begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 \end{bmatrix}$$

$$\begin{vmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{vmatrix} = \begin{bmatrix} \overset{3\times1}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \times \overset{3\times1}{\begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 + a_1 \end{bmatrix}} & \overset{3\times1}{\begin{bmatrix} s\theta_1 \\ -c\theta_1 \\ 0 \end{bmatrix}} \times \overset{3\times1}{\begin{bmatrix} a_2 c\theta_1 c\theta_2 \\ a_2 s\theta_1 c\theta_2 \\ a_2 s\theta_2 \end{bmatrix}} \\ \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{bmatrix} s\theta_1 \\ -c\theta_1 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Cross Product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Cross Product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{array}{cc} 3\times 1 & 3\times 1 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times & \begin{bmatrix} a_2 C\theta_1 C\theta_2 \\ a_2 S\theta_1 C\theta_2 \\ a_2 S\theta_2 + a_1 \end{bmatrix} \end{array} = \begin{matrix} -a_2 S\theta_1 C\theta_2 \\ a_2 C\theta_1 C\theta_2 \\ 0 \end{matrix}$$

Cross Product

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{array}{cc} 3\times 1 & 3\times 1 \\ \begin{bmatrix} S\theta_1 \\ -C\theta_1 \\ 0 \end{bmatrix} \times & \begin{bmatrix} a_2 C\theta_1 C\theta_2 \\ a_2 S\theta_1 C\theta_2 \\ a_2 S\theta_2 \end{bmatrix} \end{array} = \begin{matrix} -a_2 C\theta_1 S\theta_2 \\ -a_2 S\theta_1 S\theta_2 \\ a_2 S\theta_1^2 C\theta_2 + a_2 C\theta_1^2 C\theta_2 = a_2 C\theta_2 \overbrace{(S\theta_1^2 + C\theta_1^2)}^{1} \end{matrix}$$

$$\begin{array}{ccc} 6\times 1 & 6\times 2 & 2\times 1 \\ \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = & \begin{bmatrix} -a_2 S\theta_1 C\theta_2 & -a_2 C\theta_1 S\theta_2 \\ a_2 C\theta_1 C\theta_2 & -a_2 S\theta_1 S\theta_2 \\ 0 & a_2 C\theta_2 \\ 0 & S\theta_1 \\ 0 & -C\theta_1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \end{array}$$

$$\dot{x} = -a_2 S\theta_1 C\theta_2 \dot{\theta}_1 - a_2 C\theta_1 S\theta_2 \dot{\theta}_2$$

$$\dot{y} = a_2 C\theta_1 C\theta_2 \dot{\theta}_1 - a_2 S\theta_1 S\theta_2 \dot{\theta}_2$$

$$\dot{z} = a_2 C\theta_2 \dot{\theta}_2$$

$$\omega_x = S\theta_1 \dot{\theta}_2$$

$$\omega_y = -C\theta_1 \dot{\theta}_2$$

$$\omega_z = \dot{\theta}_1$$