A Transmission Line Taper of Improved Design*

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Summary—**The theory of the design of optimal cascaded transformer arrangements can be extended to the design of continuous transmission-line tapers. Convenient relationships have been obtained from which the characteristic impedance contour for an optimal transmission-line taper can be found.**

The performance of the Dolph-Tchebycheff transmission-line taper treated here is optimum in the sense that it has minimum reflection coefficient magnitude in the pass band for a specified length of taper, and, likewise, for a specified maximum magnitude reflection coefficient in the pass band, the Dolph-Tchebycheff taper has minimum length.

A sample design has been carried out for the purposes of illustration, and its performance has been compared with that of other tapers. In addition, a table of values of a transcendental function used in the design of these tapers is given.

Introduction

THE ANALYSIS of nonuniform transmission lines has been a subject of interest for a considerable period of time. One of the uses for such nonuniform lines is in the matching of unequal resistances over a broadband of frequencies. It has recently been shown that the theory of Fourier transforms is applicable to the design of transmission-line tapers.[1] It is the purpose of this paper to present a transmission-line taper design of improved characteristics. The performance of this taper is optimum in the sense that for a given taper length the input reflection coefficient has minimum magnitude throughout the pass band, and for a specified tolerance of the reflection coefficient magnitude the taper has minimum length.

For any transmission line system the applicable equations are

$$\frac{dV}{dx} = -ZI$$
$$\frac{dI}{dx} = -YV, \tag{1}$$

where

V = the voltage across the transmission line,
I = the current in the transmission line,
Z = the series impedance per unit length of line,
and
Y = the shunt admittance per unit length of line.

Fig. 1 illustrates the configuration to which the above equations are to be applied.

For nonuniform lines, the quantities Z and Y are known nonconstant functions of position along the line, and the properties of the system are determined through a solution of (1) along with the pertinent boundary conditions. Through use of the waveguide formalism[2] (1) is applicable to uniconductor waveguide as well as to transmission line. Strictly speaking, of course, (1) is not precisely applicable to any system since it accounts for the propagation of a single mode only. It furnishes an excellent description, however, as long as all modes but dominant mode are well below cutoff.

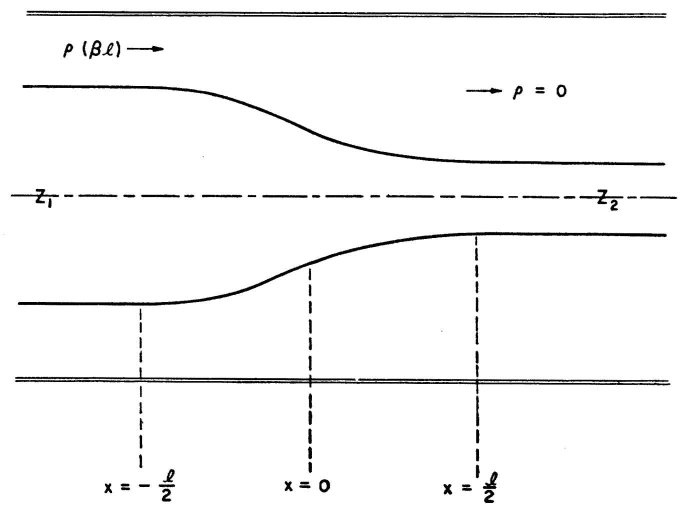

Fig. 1—Tapered transmission-line matching section.

Eq. (1) can be recast in a more directly useful form through the introduction of the quantities

$\gamma = \sqrt{ZY}$ = the propagation constant of the line,

$Z_0 = \sqrt{Z/Y}$ = the characteristic impedance of the line,

and

$$\rho = \frac{V/I - Z_0}{V/I + Z_0} = \text{the reflection coefficient at any point along the line.} \tag{2}$$

These lead to first order nonlinear differential equation[3]

$$\frac{d\rho}{dx} - 2\gamma\rho + \frac{1}{2}(1-\rho^2)\frac{d(\ln Z_0)}{dx} = 0. \tag{3}$$

This equation has the advantage that it is in terms of the quantity of direct interest in impedance matching problems. Likewise, a very natural approximation for impedance matching purposes can be made directly in this equation. If it is assumed that $\rho^2 \ll 1$, (3) becomes

$$\frac{d\rho}{dx} - 2\gamma\rho + F(x) = 0,$$

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[1] F. Bolinder, "Fourier transforms in the theory of inhomogeneous transmission lines," Proc. IRE, vol. 38, p. 1354; November, 1950.
[2] N. Marcuvitz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 1, p. 7; 1951.
[3] L. R. Walker and N. Wax, "Nonuniform transmission lines and reflection coefficients," *Jour. Appl. Phys.*, vol. 17, pp. 1043–1045; December, 1946.

where

$$F(x) = \frac{1}{2} \frac{d(\ln Z_0)}{dx}, \tag{4}$$

which is a first-order linear differential equation in ρ.

Solution of the Differential Equation

A solution of the differential equation (4) is sought which satisfies the boundary condition $\rho = 0$ at $x = l/2$. An integrating factor for this equation is

$$G(x) = \exp\left[-2\int_a^x \gamma d\xi\right], \tag{5}$$

where the lower limit of the integral is arbitrary. Applying this to (4) it is found that the solution satisfying the boundary condition is given by[4]

$$\rho(x) = \int_x^{l/2} F(z) \exp\left[-2\int_x^z \gamma(\xi)d\xi\right] dz, \tag{6}$$

and, hence, the input reflection coefficient is

$$\rho = \int_{-l/2}^{l/2} F(z) \exp\left[-2\int_{-l/2}^z \gamma(\xi)d\xi\right] dz. \tag{7}$$

The solution given above is subject only to the restriction that the reflection coefficient is relatively small. It is equally applicable to lossless transmission line, lossy transmission line, and waveguide tapers. The physical interpretation of the solution as given is evident. The incremental reflection at each cross section is given by $F(z)$, and the exponential term expresses the total delay and attenuation of this reflected component at the input of the tapered section relative to incident input wave.

Optimal Design of Transmission-Line Tapers

An important special case of the general situation considered above is the lossless transmission-line taper as illustrated in Fig. 1. In this case, the characteristic impedance is a real number and is independent of frequency. The wave propagated in the line is essentially TEM in character, and the propagation constant is purely imaginary and proportional to the frequency.

Under these conditions the interior integration of (7) can be carried out, and the input reflection coefficient becomes

$$\rho \exp(j\beta l) = \int_{-l/2}^{l/2} F(z) \exp(-j2\beta z)dz. \tag{8}$$

This relationship can be inverted through the theory of Fourier transforms to obtain

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} [\rho \exp(j\beta l)] \exp(j2\beta x)d\beta. \tag{9}$$

The analogy between the present problem and the synthesis of radiation patterns from line sources is evident. In each case, the quantities of interest are related through the Fourier transform.

In the use of (9) the reflection coefficient is specified so that its value at negative frequencies is equal to the complex conjugate of its value at the corresponding positive frequencies. This is necessary in order that the specified reflection coefficient shall correspond to a physically realizable structure.[5] This requirement then insures that the transform will be completely real as it must be in order that $F(x)$ have significance in terms of a transmission-line taper.

Collin has recently shown that optimum performance is obtained from a cascaded transformer structure when the power loss ratio is expressed in terms of the Tchebycheff polynomial of degree equal to the number of sections.[6] This is equivalent to having the input reflection coefficient proportional to the Tchebycheff polynomial of the same degree, when its square is small relative to one [an assumption already made in the derivation of transform pair (8) and (9)].

By allowing the number of sections to increase indefinitely for a fixed over-all length, the results of Collin can be extended to the case of a continuous transmission-line taper. In the case of a cascaded transformer arrangement, a secondary maximum in the reflection coefficient magnitude occurs at the first and at all succeeding frequencies where the individual section lengths become equal to a multiple of a half-wavelength. As the number of sections is allowed to increase without limit for a fixed over-all length, the frequency at which this first secondary maximum occurs also increases without limit so that the pass band consists of all frequencies beyond that for which the reflection coefficient first comes within the specified tolerance.

For maximum bandwidth with a fixed maximum magnitude of reflection coefficient then, input reflection coefficient for a continuous taper takes form

$$\rho \exp(j\beta l) = \rho_0 \frac{\cos\left[\sqrt{(\beta l)^2 - A^2}\right]}{\cosh(A)}, \tag{10}$$

which is the limiting form of the Tchebycheff polynomial as its degree increases without limit.[7] The specification of the parameter A determines the maximum magnitude of reflection coefficient in the pass band which consists of all frequencies such that $\beta l \geq A$. The reflection coefficient magnitude takes on its maximum value $|\rho_0|$ at zero frequency, and it oscillates in the pass band with constant amplitude equal to $\rho_0/\cosh(A)$. A plot of the function given by (10) is shown in Fig. 2 for a number of different values of A.

The inversion of the above specified reflection coefficient through (9) yields[7]

[4] L. R. Ford, "Differential Equations," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 2, pp. 36–39; 1933.

[5] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., Inc., New York, N. Y., ch. 7, p. 106; 1945.

[6] R. E. Collin, "Theory and design of wide-band multisection quarter-wave transformers," Proc. IRE, vol. 43, pp. 179–185; February, 1955.

[7] T. T. Taylor, "Dolph arrays of many elements," Tech. Memo. No. 320, Hughes Aircraft Co., Culver City, Calif., August 18, 1953.

$$F(x) = \frac{\rho_0}{\cosh(A)} \left\{ \frac{A^2}{l} \frac{I_1[A\sqrt{1-(2x/l)^2}]}{A\sqrt{1-(2x/l)^2}} + \frac{1}{2}\left[\delta\left(x-\frac{l}{2}\right) + \delta\left(x+\frac{l}{2}\right)\right]\right\}, \quad |x| \leqq l/2,$$
$$= 0, \qquad |x| > l/2, \tag{11}$$

where I_1 is the first kind of modified Bessel function of the first order, and δ is the unit impulse function.

The variation of characteristic impedance along the taper can be found by direct integration of $F(x)$, and it is given by

$$\ln(Z_0) = \frac{1}{2}\ln(Z_1Z_2) + \frac{\rho_0}{\cosh(A)}\left\{A^2\phi(2x/l, A) + U\left(x-\frac{l}{2}\right) + U\left(x+\frac{l}{2}\right)\right\}, \quad |x| \leqq l/2,$$
$$= \ln(Z_2), \qquad x > l/2,$$
$$= \ln(Z_1), \qquad x < -l/2. \tag{12}$$

U is the unit step function defined by,

$$U(z) = 0, \qquad z < 0,$$
$$U(z) = 1, \qquad z \geqq 0, \tag{13}$$

and ϕ is defined by

$$\phi(z, A) = -\phi(-z, A) = \int_0^z \frac{I_1(A\sqrt{1-y^2})}{A\sqrt{1-y^2}}\,dy, \qquad |z| \leqq 1. \tag{14}$$

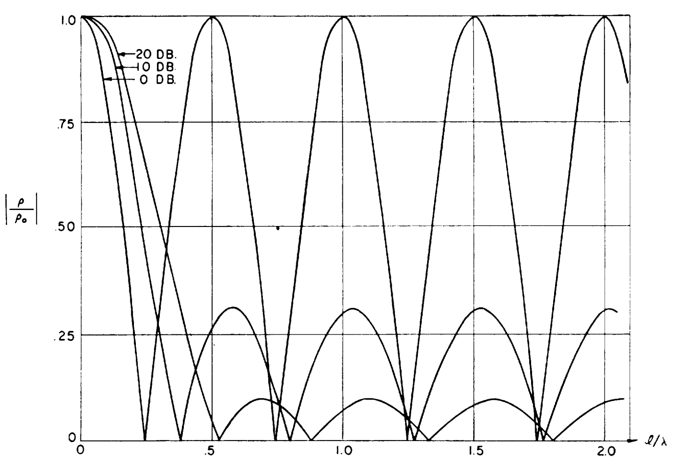

Fig. 2—Response of Dolph-Tchebycheff transmission-line tapers.

Eq. (12) furnishes the information required for the design of a Dolph-Tchebycheff tapered transition. The quantity ρ_0 is determined by the two impedances Z_2 and Z_1 which are to be matched, and A is selected on the basis of the allowed maximum reflection coefficient magnitude in the pass band. One of the interesting aspects of this design is that the taper has a discontinuous change of characteristic impedance at each end as well as a continuous change along the length of the taper. It is interesting to note that when the tolerated reflection coefficient approaches the initial reflection coefficient ρ_0, the parameter A approaches zero, and the bandwidth comprises all frequencies from zero to infinity. In this case, the Dolph-Tchebycheff taper design degenerates into the usual quarter-wavelength transformer design with a discontinuous change of characteristic impedance at each end and a constant characteristic impedance at intermediate points.

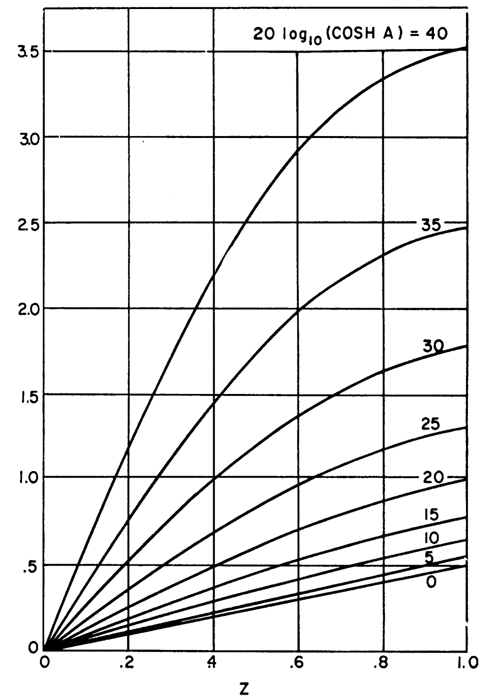

Fig. 3—Plot of the function $\phi(z, A)$.

The function $\phi(z, A)$ is not expressible in closed form except for special values of the parameters. Therefore, this function has been computed through standard integration formulas on an IBM CPC digital computer for a suitable range of values of the parameters. Tabulated values of $\phi(z, A)$ are given to six decimal places in Table I, p. 34, and the function is shown in Fig. 3. The special closed-form relationships

$$\phi(0, A) = 0,$$
$$\phi(z, 0) = z/2,$$

and

$$\phi(1, A) = \frac{\cosh(A) - 1}{A^2}, \tag{15}$$

are obtained for the end points of the parameter ranges.

One more comment should be made in regard to the application of the preceding design procedure. If one uses the natural value

$$\rho_0 = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \tag{16}$$

in entering the equation (12), it will be found that the designed taper does not quite fit the final impedances

TABLE I

VALUES OF THE FUNCTION $\phi(z, A)$ FOR $z=0(0.05)1.00$ AND $20 \log_{10} (\cosh A)=0(5)40$

	$\phi(z, A)$								
	$20 \log_{10} (\cosh A)$								
z	0	5	10	15	20	25	30	35	40
0.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.05	0.025000	0.029593	0.036848	0.048140	0.065590	0.092539	0.134313	0.199460	0.301772
0.10	0.050000	0.059161	0.073629	0.096137	0.130902	0.184564	0.267698	0.397268	0.600625
0.15	0.075000	0.088681	0.110276	0.143848	0.195661	0.275567	0.399242	0.591802	0.893707
0.20	0.100000	0.118128	0.146721	0.191132	0.259597	0.365055	0.528062	0.781511	1.178306
0.25	0.125000	0.147479	0.182899	0.237850	0.322448	0.452552	0.653321	0.964936	1.451913
0.30	0.150000	0.176708	0.218746	0.283869	0.383962	0.537610	0.774237	1.140744	1.712272
0.35	0.175000	0.205794	0.254197	0.329059	0.443899	0.619809	0.890101	1.307751	1.957437
0.40	0.200000	0.234711	0.289191	0.373296	0.502035	0.698767	1.000282	1.464942	2.185803
0.45	0.225000	0.263438	0.323667	0.416460	0.558163	0.774142	1.104238	1.611487	2.396134
0.50	0.250000	0.291950	0.357568	0.458441	0.612094	0.845635	1.201523	1.746753	2.587582
0.55	0.275000	0.320226	0.390837	0.499134	0.663658	0.912994	1.291792	1.870306	2.759684
0.60	0.300000	0.348244	0.423420	0.538444	0.712709	0.976019	1.374800	1.981918	2.912359
0.65	0.325000	0.375982	0.455266	0.576284	0.759120	1.034555	1.450409	2.081555	3.045886
0.70	0.350000	0.403418	0.486328	0.612574	0.802790	1.088504	1.518582	2.169376	3.160875
0.75	0.375000	0.430533	0.516559	0.647248	0.843641	1.137814	1.579377	2.245710	3.258228
0.80	0.400000	0.457305	0.545918	0.680245	0.881619	1.182484	1.632947	2.311049	3.339098
0.85	0.425000	0.483716	0.574365	0.711518	0.916692	1.222564	1.679531	2.366019	3.404835
0.90	0.450000	0.509746	0.601865	0.741027	0.948855	1.258145	1.719443	2.411361	3.456938
0.95	0.475000	0.535377	0.628386	0.768745	0.978123	1.289363	1.753065	2.447905	3.497000
1.00	0.500000	0.560591	0.653899	0.794653	1.004533	1.316391	1.780835	2.476547	3.526658

Z_2 and Z_1 at the end points. This fact is an evidence of the approximation $\rho^2 \ll 1$ which was made at the outset in differential equation (4). Discrepancy becomes larger as value of magnitude of ρ_0 increases. This design inconvenience can be eliminated, however, by taking

$$\rho_0 = \tfrac{1}{2} \ln (Z_2/Z_1), \tag{17}$$

as the initial value of the reflection coefficient instead of the true value of (16). The two expressions are identical to a second order of approximation for small differences between Z_2 and Z_1, and the use of the second expression will yield a taper design which exactly fits its end-point impedances for all values of Z_2 and Z_1. The effect of the approximation $\rho^2 \ll 1$ will then be evidenced by a slight deviation from the performance given by (10) in the low-frequency range outside the pass band.

COAXIAL TRANSMISSION-LINE TAPER FROM 50 TO 75 OHMS

As an application of the preceding results, the design of an optimal 50–75 ohm coaxial transmission line taper will be indicated in detail. The taper is to be designed so that the input reflection coefficient magnitude does not exceed about one per cent in the pass band.

The initial value of the reflection coefficient in this case is equal to 0.2. The value of ρ_0 for use in the design of the taper is found from (17) to be

$$\rho_0 = \tfrac{1}{2} \ln(1.5) = 0.20274. \tag{18}$$

As observed previously, this value does not differ markedly from the zero frequency reflection coefficient.

It will be required that the maximum reflection coefficient magnitude in the pass band shall not exceed one-twentieth of ρ_0. Thus, from (10)

$$\cosh (A) = 20,$$

so that

$$A = 3.6887. \tag{19}$$

The characteristic impedance contour can now be obtained directly from (12). The resulting Z_0 curve is illustrated in Fig. 4, and the corresponding coaxial line-conductor contour is shown in Fig. 5.

Characteristic impedance has a discontinuous jump from 50 to 50.52 ohms at left-hand end and a corresponding jump from 74.24 to 75 ohms at right-hand end. Characteristic impedance at center of taper is equal to 61.24 ohms, geometric mean between 50 and 75 ohms.

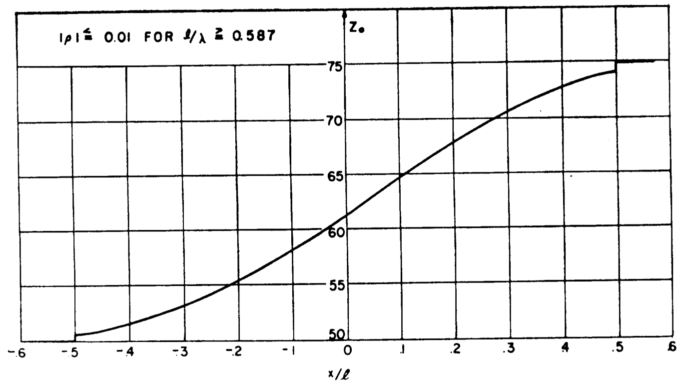

Fig. 4—50–75 ohm Dolph-Tchebycheff tapered transition.

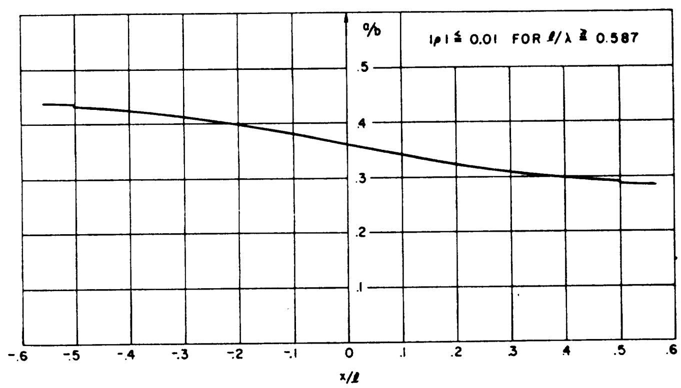

Fig. 5—Inner conductor contour for 50–75 ohm Dolph-Tchebycheff coaxial-line transition.

The performance of this taper is plotted in Fig. 6. The pass band consists of all frequencies greater than that for which $\beta l = 0.587$. For comparison, the performance of an exponential taper[8] and a hyperbolic taper[9] has been indicated on the same curve.

[8] C. R. Burrows, "The exponential transmission line," *Bell Sys. Tech. Jour.*, vol. 17, pp. 555–573; October, 1938.

[9] H. J. Scott, "The hyperbolic transmission line as a matching section," Proc. IRE, vol. 41, pp. 1654–1657; November, 1953.

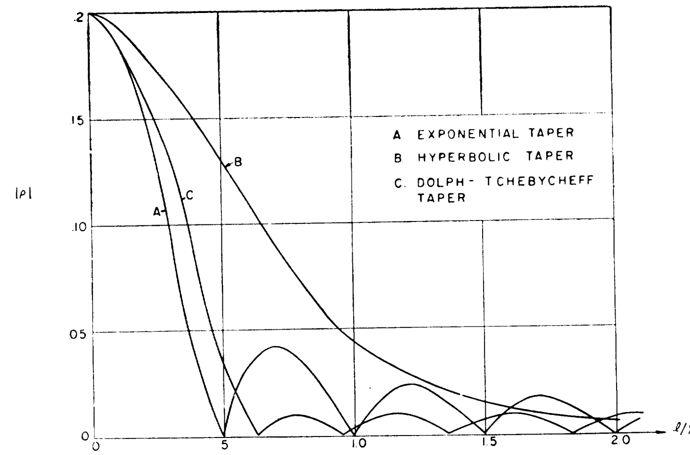

Fig. 6—Performance of 50–75 ohm Dolph-Tchebycheff tapered transition.

Conclusion

The theory of the design of optimal cascaded transformer arrangements can be extended to the design of continuous transmission-line tapers. Convenient relationships have been obtained from which the characteristic impedance contour for an optimal transmission-line taper can be found. Alternatively, this impedance contour can be thought of as the envelope of the pointwise specified characteristic impedance of a discrete cascaded transformer arrangement.

The performance of the Dolph-Tchebycheff transmission-line taper treated here is optimum in the sense that it has minimum reflection coefficient magnitude in the pass band for a specified length of taper, and, likewise, for specified maximum magnitude reflection coefficient in the pass band the Dolph-Tchebycheff taper has minimum length.

A sample design has been carried out for the purposes of illustration, and its performance has been compared with that of other tapers. In addition, a table of values of a transcendental function used in the design of these tapers is given in Table I.

A Precision Resonance Method for Measuring Dielectric Properties of Low-Loss Solid Materials in the Microwave Region*

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Summary—**A precision resonance method for measuring the dielectric properties of low loss solid materials has been developed in our laboratory. The dielectric sample to be measured is shaped into a cylindrical disk and inserted into a cylindrical cavity resonator oscillating in the T_{10p} mode. ϵ can be measured from the difference between the axial lengths of the cavity tuned to the same frequency with and without the sample, and tan δ can be found from the difference between the Q's of the cavity with and without the sample. By making use of a special marker of a resonance point on an oscilloscope, the measurements accuracy can be improved to yield only 1 per cent error in ϵ and 3 per cent error in tan δ for various low-loss samples. Such materials as polystyrol, polyethylene, teflon, and glass for high-frequency use were tested at 4,000 mc, 9,000 mc and 24,000 mc.**

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