COMPUTING AND ESTIMATING THE RATE OF CONVERGENCE

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Abstract. Introduces the definition of rate of convergence for sequences and applies this to fixed-point root-finding iterative methods. Concludes with the development of a formula to estimate the rate of convergence for these methods when the actual root is not known.

1. Rate of Convergence

Definition 1. *If a sequence $x_1, x_2, \ldots, x_n$ converges to a value r and if there exist real numbers $\lambda > 0$ and $\alpha \geq 1$ such that*

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^\alpha} = \lambda \tag{1}$$

*then we say that α is the **rate of convergence** of the sequence.*

When $\alpha = 1$ we say the sequence converges *linearly* and when $\alpha = 2$ we say the sequence converges *quadratically.* If $1 < \alpha < 2$ then the sequence exhibits *superlinear* convergence.

2. Fixed-Point Iterations

Many root-finding methods are *fixed-point* iterations. These iterations have this name because the desired root r is a **fixed-point** of a function $g(x)$, i.e., $g(r) \to r$. To be useful for finding roots, a fixed-point iteration should have the property that, for x in some neighborhood of r, $g(x)$ is closer to r than x is. This leads to the iteration

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \ldots$$

Newton's method is an example of a fixed-point iteration since

$$x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)} \tag{2}$$

and clearly $g(r) = r$ since $f(r) = 0$.

Theorem 1. *Let r be a fixed-point of the iteration $x_{n+1} = g(x_n)$ and suppose that $g'(r) \neq 0$. Then the iteration will have a **linear** rate of convergence.*

Proof. Using Taylor's Theorem for an expansion about fixed-point r we find

$$g(x) = g(r) + g'(r)(x - r) + \frac{g''(\xi)}{2}(x - r)^2 \tag{3}$$

Date: February 2007. Revised February 2015, February 2019.

where ξ is some value between x and r. Evaluating at x_n and noting that $x_{n+1} = g(x_n)$ and $g(r) = r$ we obtain

$$x_{n+1} = r + g'(r)(x_n - r) + \frac{g''(\xi)}{2}(x_n - r)^2.$$

Subtracting r from both sides and dividing by $x_n - r$ gives

$$\frac{x_{n+1} - r}{x_n - r} = g'(r) + \frac{g''(\xi)}{2}(x_n - r)$$

which, as $n \to \infty$, yields

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = |g'(r)|. \tag{4}$$

Comparing this with Equation (1) we see that $\alpha = 1$ and $\lambda = |g'(r)|$, indicating that the method converges linearly. □

Next, consider the case when $g'(r) = 0$. This is important because it explains why Newton's method converges so quickly (when it converges at all).

Theorem 2. *Let r be a fixed-point of the iteration $x_{n+1} = g(x_n)$ and suppose that $g'(r) = 0$ but $g''(r) \neq 0$. Then the iteration will have a **quadratic** rate of convergence.*

Proof. Using Taylor's Theorem once again, but including one more term, we have

$$g(x) = g(r) + g'(r)(x - r) + \frac{g''(r)}{2}(x - r)^2 + \frac{g'''(\xi)}{6}(x - r)^3.$$

As before, we substitute x_n for x and use the facts that $x_{n+1} = g(x_n)$, $g(r) = r$, and $g'(r) = 0$ to obtain

$$x_{n+1} = r + \frac{g''(r)}{2}(x_n - r)^2 + \frac{g'''(\xi)}{6}(x_n - r)^3.$$

Subtracting r from both sides and dividing by $(x_n - r)^2$ gives

$$\frac{x_{n+1} - r}{(x_n - r)^2} = \frac{g''(r)}{2} + \frac{g'''(\xi)}{6}(x_n - r)$$

which, as $n \to \infty$, gives

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} = \frac{|g''(r)|}{2}. \tag{5}$$

Observe that $\alpha = 2$, which shows the iteration will converge quadratically. □

In most instances this situation applies to Newton's method. Computing $g'(x)$ from (2) we have

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

When this is evaluated at r, we find that $g'(r) = 0$ because $f(r) = 0$, provided $f'(r) \neq 0$, and so we expect Newton's method will converge quadratically. It is possible to show that

$$\lim_{x\to r} g'(x) = \frac{1}{2}$$

when $f'(r) = 0$, so in this case Newton's method exhibits only linear convergence.

3. Estimating the Rate of Convergence

It is convenient to define the error after n steps of an iterative root-finding algorithm as $e_n = x_n - r$. As $n \to \infty$ we see from Equation (1) that

$$|e_{n+1}| \approx \lambda |e_n|^\alpha \quad \text{and} \quad |e_n| \approx \lambda |e_{n-1}|^\alpha$$

and so

$$\frac{|e_{n+1}|}{|e_n|} \approx \frac{\lambda |e_n|^\alpha}{\lambda |e_{n-1}|^\alpha} \approx \left| \frac{e_n}{e_{n-1}} \right|^\alpha .$$

Solving for α yields

$$\alpha \approx \frac{\log |e_{n+1}/e_n|}{\log |e_n/e_{n-1}|} = \frac{\log |(x_{n+1} - r)/(x_n - r)|}{\log |(x_n - r)/(x_{n-1} - r)|}. \tag{6}$$

To make use of this formula we need to know the ratios of consecutive errors. While we cannot compute these ratios exactly (since we do not know the exact value of the root r), we can approximate them with ratios of the differences of consecutive estimates of the root. To see this, first substitute x_n and x_{n-1} into Equation (3) to obtain the two expansions

$$x_{n+1} = r + g'(r)(x_n - r) + \frac{g''(\xi_1)}{2}(x_n - r)^2 \tag{7}$$

$$x_n = r + g'(r)(x_{n-1} - r) + \frac{g''(\xi_2)}{2}(x_{n-1} - r)^2. \tag{8}$$

Subtracting (8) from (7), dividing by $(x_n - x_{n-1})$, and taking the limit as $n \to \infty$ we have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{x_{n+1} - x_n}{x_n - x_{n-1}} \right| &= \lim_{n\to\infty} \left| g'(r) + \frac{g''(\xi_1)}{2} \frac{(x_n - r)^2}{x_n - x_{n-1}} - \frac{g''(\xi_2)}{2} \frac{(x_{n-1} - r)^2}{x_n - x_{n-1}} \right| \\ &= |g'(r)| \end{aligned}$$

since both $(x_n - r)^2$ and $(x_{n-1} - r)^2$ go to zero more quickly than $x_n - x_{n-1}$ as the sequence $\{x_n\}$ converges to r. Comparing this result with Equation (4) we conclude, for suitably large values of n, that

$$\frac{e_{n+1}}{e_n} = \frac{x_{n+1} - r}{x_n - r} \approx \frac{x_{n+1} - x_n}{x_n - x_{n-1}}$$

which allows us to approximate α with

$$\alpha \approx \frac{\log |(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\log |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}. \tag{9}$$

Even though this only gives an estimate of α, we note that in practice it agrees well with the theoretical convergence rates of bisection and Newton's method and gives us a good measure of the efficiency of various forms of fixed-point algorithm.

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6.4.4. Other Related Methods

The secant method does not achieve quadratic convergence. Indeed, it has been shown under very weak restrictions that no iterative method using only one function evaluation per step can have $p = 2$. However, **Steffensen's method**,

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \quad g(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}, \tag{6.4.5}$$

which requires two function evaluations but no derivatives, will be shown to be of second order. This method, which is closely related to the secant method, is of particular interest for solving systems of nonlinear equations in several variables (see Sec. 6.9).

If we put $\beta_n = f(x_n)$ and expand $g(x_n)$ in a Taylor series about x_n, we get

$$g(x_n) = \frac{f(x_n + \beta_n) - f(x_n)}{\beta_n} = f'(x_n)\left(1 - \frac{1}{2}h_n f''(x_n) + O(\beta_n^2)\right),$$

where $h_n = -f(x_n)/f'(x_n)$ is the Newton correction. Thus

$$x_{n+1} = x_n + h_n(1 + \tfrac{1}{2}h_n f''(x_n) + O(\beta_n^2)),$$

and using the error equation Eq. (6.3.2), for Newton's method, which can be written

$$h_n = -\epsilon_n + \frac{1}{2}\epsilon_n^2 \frac{f''(\xi)}{f'(x_n)}, \quad \epsilon_n = x_n - \alpha,$$

we get

$$\frac{\epsilon_{n+1}}{\epsilon_n^2} \longrightarrow \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}(1 + f'(\alpha)).$$

This proves that Steffensen's method is of second order.