

Odvodi

$f(x)$	$f'(x)$
x^n	nx^{n-1}
a^x	$a^x \ln(a)$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{sh} x$	$\operatorname{ch} x$
$\operatorname{ch} x$	$-\operatorname{sh} x$
$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{cth} x$	$-\frac{1}{\operatorname{sh}^2 x}$
$\operatorname{arsh} x$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arth} x$	$\frac{1}{1-x^2}$

Integrali

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} (n \neq -1)$
$\frac{1}{x}$	$\ln x $
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\frac{1}{\cos^2 x}$	$\tan x$
$\frac{1}{\sin^2 x}$	$-\cot x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\frac{1}{\sqrt{1+x^2}}$	$\operatorname{arsh} x = \ln \left x + \sqrt{x^2+1} \right $
$\frac{1}{1+x^2}$	$\arctan x$

Per Partes

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

$$\int u dv = uv - \int v du$$

Racionalne funkcije

$$\int \frac{p(x)}{q(x)}dx,$$

$$p(x), q(x) \text{ sta polinoma}$$

1. Če je $st(p(x)) \leq st(q(x))$ polinoma delimo
2. $q(x)$ razdelimo na linearne in kvadratne faktorje
3. Izraz pod integralom razcepimo na parcialne ulomke
$$\frac{p(x)}{q(x)} = \left[\frac{A_1}{x-a_1} + \dots + \frac{A_{n_1}}{(x-a_1)^{n_1}} \right] + \dots + \left[\frac{Z_1}{x-a_k} + \dots + \frac{Z_{n_k}}{(x-a_k)^{n_k}} \right] + \left[\frac{\alpha_1 x + \beta_1}{x^2 + b_1 x + c_1} + \dots + \frac{\alpha_{m_1} x + \beta_{m_1}}{(x^2 + b_1 x + c_1)^{m_1}} \right] + \dots + \left[\frac{\varphi_1 x + \omega_1}{x^2 + b_l x + c_l} + \dots + \frac{\varphi_{m_l} x + \omega_{m_l}}{(x^2 + b_l x + c_l)^{m_l}} \right]$$
Dolzina krivulj
4. Integriramo vsakega zase

$$k \geq 2 \qquad st(p(x)) \leq 2k - 1$$

$$st(q(x)) \leq 2k - 3 \qquad (ax^2 + bx + c) \qquad \text{nerazcepen v } \mathbb{R}$$

$$I = \int \frac{p(x)}{(ax^2 + bx + c)^k} = \int \frac{Ax + B}{ax^2 + bx + c} + \frac{q(x)}{(ax^2 + bx + c)^{k-1}}$$

A,B, $q(x)$ poiščemo tako da enačbo odvajamo.

Korenske funkcije

1. $\int f(\sqrt{ax+b})dx \qquad t = \sqrt{ax+b}$
2. $\int f(\sqrt{ax^2+bx+c})dx$

a $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ ga prevedemo na oblike:

• Če je $a < 0$: $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$

• Če je $a > 0$: $\int \frac{dx}{\sqrt{x^2+c}} = \ln \left| x + \sqrt{x^2+c} \right|$

- b $\int \frac{p(x)}{\sqrt{ax^2+bx+c}} = q(x)\sqrt{ax^2+bx+c} + A \int \frac{dx}{\sqrt{ax^2+bx+c}}$
 $st(p(x)) - 1 = st(q(x)) \quad A, q(x) \text{ poiščemo z odvanjanjem}$
3. $\int \sqrt{a^2-x^2}dx \quad x = a \sin t \quad dx = a \cos t dt \quad t = \arcsin \frac{x}{a}$
4. $\int \sqrt{a^2+x^2}dx \quad x = a \operatorname{sh} t \quad dx = a \operatorname{ch} t dt \quad t = \operatorname{arsh} \frac{x}{a}$

Kotne funkcije

1.
$$\begin{aligned} \int \sin(ax) \sin(bx)dx &= \int -\frac{1}{2} [\cos(a+b)x - \cos(a-b)x] dx = \\ &= -\frac{1}{2} \left[\frac{\sin(a-b)x}{(a-b)} - \frac{\sin(a+b)x}{(a+b)} \right] \\ \int \cos(ax) \cos(bx)dx &= \int \frac{1}{2} [\cos(a+b)x + \cos(a-b)x] dx \dots \\ \int \sin(ax) \cos(bx)dx &= \int \frac{1}{2} [\sin(a+b)x + \sin(a-b)x] dx \dots \end{aligned}$$
2. $\int \cos^m x \sin^n x dx$

(a) Eno od števil m,n je liho (npr. $m = 2k + 1$)

$$\int \cos^{2k} x \cos x \sin^n x dx = \int t^n (1-t^2)^k dt$$

$$t = \sin x \quad dt = \cos x dx$$

$$\cos^{2k} x = (\cos^2 x)^k = (1-t^2)^k$$

- (b) m, n sta oba soda, $m = 2m_1, n = 2n_1$

$$\begin{aligned} \int \cos^{2m_1} x \sin^{2n_1} x dx &= \int (\cos^2 x)^{m_1} (\sin^2 x)^{n_1} dx = \\ &= \int \left(\frac{1 + \cos 2x}{2} \right)^{m_1} \left(\frac{1 - \cos 2x}{2} \right)^{n_1} dx = \\ &= \text{vsota integralov oblike } \int \cos^k 2x dx \end{aligned}$$

kjer je $k \leq m_1 + n_1 = \frac{1}{2}(m + n) < m + 1$
Ce je k lih gremo po 1 točki
Ce je k sod ponovimo postopek

- (c) $\int R(\cos x, \sin x)dx$ ($R \dots$ racinonalni izraz)

$$t = \tan \frac{x}{2} \quad \cos x = \frac{1-t^2}{t^2+1}$$

$$\sin x = \frac{2t}{t^2+1} \quad dx = \frac{2}{t^2+1} dt$$

$$t = \tan x \quad \cos x = \frac{1}{\sqrt{t^2+1}}$$

$$\sin x = \frac{t}{\sqrt{t^2+1}} \quad dx = \frac{dt}{t^2+1}$$

Uporaba integralov

1. Ploščina ravnisnkih likov

$$s = \int_a^b \sqrt{1+f'(x)^2}dx$$

3. Prostornina vrtenine

$$V = \pi \int_a^b f(x)^2 dx$$

4. Povrsina vrtenine

$$S = 2\pi \int_a^b f(x) \sqrt{1+f(x)^2} dx$$

5. Tezisce ravninskih likov

$$M_x = \varphi g \int_a^b \frac{f(x)^2}{2} dx \quad \text{navor}$$

$$M_y = \varphi g \int_a^b x f(x) dx$$

$$x_T = \frac{\int_a^b x f(x) dx}{P} \quad \text{tezisce}$$

$$y_T = \frac{\int_a^b \frac{f(x)^2}{2} dx}{2P} = \frac{V}{2\pi P}$$

$$V = 2\pi y_T P \quad \text{Guldinovo pravilo}$$

$$2\pi y_T \dots \text{pot, ki jo pri vrtenju opiše tezisce} \quad P \dots \text{Ploščina}$$

6. Tezisce ravninskih krivulj

$$M_x = \varphi g \int_a^b f(x) \sqrt{1 + f(x)^2} dx = g \varphi s y_T$$

$$x_T = \frac{\int_a^b x \sqrt{1 + f'(x)^2} dx}{\int_a^b \sqrt{1 + f'(x)^2} dx}$$

$$y_T = \frac{\int_a^b f(x) \sqrt{1 + f'(x)^2} dx}{\int_a^b \sqrt{1 + f'(x)^2} dx} = \frac{1}{s} \int_a^b f(x) \sqrt{1 + f'(x)^2} dx = \frac{1}{s} \frac{S}{2\pi}$$

$$S = 2\pi y_T s \quad \text{Guldinovo pravilo}$$

Numericna integracija

1. Trapezna metoda

$$I = \sum_{i=0}^{n-1} \frac{y_i + y_{i+1}}{2} \frac{b-a}{4} = \frac{b-a}{2n} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$$

$$|R_n| \leq \frac{(b-a)^3}{12n^2} \max_{x \in [a,b]} |f''(x)| \quad \text{napaka}$$

2. Simpsonova metoda

Izberemo $2n$ delilnih tock in na vsakem intervalu ploscino aproks. z kvad. parabolo.

$$x_i = a + i \frac{b-a}{2n}$$

$$I = \int_a^b f(x) dx \doteq \frac{b-a}{6n} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 4y_{2n-1} + y_{2n})$$

$$|R_n| \leq \frac{(b-a)^5}{2880n^4} \max_{x \in [a,b]} |f^{(4)}(x)| \quad \text{napaka}$$

Izlimitirani integrali

1. f ni omejena v krajiscih a ALI b

$$\int_a^b f(x) dx = \lim_{c \rightarrow a} \int_c^b f(x) dx \quad \text{enako za b}$$

2. f ni omejena v krajjscih a IN b

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx = \lim_{d \rightarrow a} \int_d^c f(x) dx + \lim_{e \rightarrow b} \int_c^e f(x) dx$$

3. f ni omejena v krajjscu $c \in [a,b]$

$$\int_a^b f(x) dx = \int_c^a f(x) dx + \int_b^c f(x) dx = \lim_{d \rightarrow -c} \int_d^a f(x) dx + \lim_{e \rightarrow c} \int_b^e f(x) dx$$

4. Ena od mej gre v neskoncnost

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

$$\int_{-\infty}^\infty = \int_{-\infty}^0 + \int_0^\infty$$

Konvergenca integralov

Integralski kriterij

$$f \geq 0 \quad a \in \mathbb{R} : \quad \int_a^\infty \frac{1}{f(x)} dx \quad \text{obstaja} \iff \sum_{n=a}^\infty \frac{1}{f(n)}$$

$$\int_a^b \frac{f(x)}{(x-a)^s} dx \quad f(x) \text{je zvezdna na} [a,b]$$

- ce je $s < 1 \Rightarrow$ integral konvergira
- ce je $s \geq 1$ in $f(a) = 0 \Rightarrow$ integral divergira

$$\int_a^\infty \frac{f(x)}{x^s} dx \quad f(x) \text{je zvezdna in omejena na} [a,b]$$

- ce je $s > 1 \Rightarrow$ integral konvergira
- ce je $s \leq 1$ in $|f(a)| \geq m > 0$ za vse x od nekje naprej \Rightarrow integral divergira

Parametricno podane funkcije

$$pl = \frac{1}{2} \left| \int_{t_1}^{t_2} (x\dot{y} - \dot{x}y) dt \right|$$

$$pl = \left| \int_{t_1}^{t_2} x \dot{y} dt \right| = \left| \int_{t_1}^{t_2} \dot{x} y dt \right| \quad \text{enostavna sklenjena krivulja}$$

$$pl = \frac{1}{2} \int_{\varphi_1}^{\varphi_2} r^2 d\varphi \quad \text{v polarnih kordinatah}$$

$$l = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2} dt \quad \text{dolzina krivulje}$$

$$V = \int_{z_1}^{z_2} S(z) dz \quad \text{Volumen vrtenine}$$

Pazi

$$\int \text{soda funkcija} = \text{liha funkcija} \quad \int \text{liha funkcija} = \text{soda funkcija}$$

$$\int_a^b f(x) dx = \int_\beta^\alpha f(g(t)) g'(t) dx$$

$$x = g(t) \quad \text{za monotone funkcije} \quad \alpha = g(a) \quad \beta = g(b)$$

Parcialni odvodi

$$f_{x_i} \frac{\partial f(x)}{\partial x_1} (a_1, \dots, a_n) = \lim_{h \rightarrow 0} \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_n)}{h}$$

$$\text{grad} f = \left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right) \quad \text{Smer največjega narascanje}$$

$$f_{\vec{u}}(\vec{a}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h} = \langle \text{grad} f(\vec{a}), \vec{u} \rangle$$

Kotne funkcije

Adicijski izreki

$$\sin x \pm y = \sin x \cos y \pm \sin y \cos x$$

$$\cos x \mp y = \cos x \cos y \pm \sin x \sin y$$

$$\tan x \pm y = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Faktorizacija

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \sin \frac{x-y}{2} \cos \frac{x+y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x-y}{2} \sin \frac{x+y}{2}$$

Razclenevanje

$$\sin x \sin y = -\frac{1}{2} (\cos (x+y) - \cos (x-y))$$

$$\cos x \cos y = \frac{1}{2} (\cos (x+y) + \cos (x-y))$$

$$\sin x \cos y = \frac{1}{2} (\sin (x+y) + \sin (x-y))$$

$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$