Def: $(\mathcal{F}, \mathcal{P})$ is a *signature*, where $\mathcal{F}$ – set of *function symbols*, $\mathcal{P}$ – set of *predicate symbols.* Each $f \in \mathcal{F}$ and $P \in \mathcal{P}$ has an associated *arity* $\mathrm{Ar}(f)$ or $\mathrm{Ar}(P)$.
Syntax: A **term** t represents an element of a model. Any variable x is a term or for any $f \in \mathcal{F}, \mathrm{Ar}(f) = n : f(t_1, \dots, t_n)$ is a term.
Def: A *model* (or *structure*) is $\mathcal{M} = (M, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}}, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}})$, where $\mathcal{M}$ is the underlying set, for each $P \in \mathcal{P}, \mathrm{Ar}(P) = n :\ P^{\mathcal{M}} : M^n \to \{\text{true}, \text{false}\}$, for each $f \in \mathcal{F}, \mathrm{Ar}(f) = n :\ f^{\mathcal{M}} : M^n \to M$.
A term t is interpreted in $\mathcal{M}$ as $t^\rho \in M$, where $\rho : \text{Vars} \to M$ is an environment. $x^\rho = \rho(x)(f(t_1, \dots, t_n))^\rho = f^{\mathcal{M}}(t_1^\rho, \dots, t_n^\rho)$. A formula φ is either true or false in $\mathcal{M}$ under environment ρ.
$\mathcal{M}$ is a model of S ($\mathcal{M} \models S$) if $A \in S \implies \mathcal{M} \models A$. $\mathrm{Mod}(S)$ = set of all models of S.
T is a theory if $T \models A \implies A \in T$. $\mathrm{Th}(S)$ = mn. stavkov izpeljivih iz S. $\mathrm{Th}(\mathbf{M})$ = mn. stavkov, ki so izpeljivi iz vseh modelov v $\mathbf{M}$. Mimogrede: $Th(S) = \mathrm{Th}(Mod(S))$. Teorija je **konsistentna**, če se iz nje ne da izpeljati protislovja iff če ima model.
Interpretation: $\mathcal{M} \models P(t_1, \dots, t_n)^\rho \iff P^{\mathcal{M}}(t_1^\rho, \dots, t_n^\rho) = \text{true}$; $\mathcal{M} \models (t_1 = t_2)^\rho \iff t_1^\rho = t_2^\rho$; $\mathcal{M} \models (\varphi_1 \wedge \varphi_2)^\rho \iff \mathcal{M} \models \varphi_1^\rho$ and $\mathcal{M} \models$; φ_2^ρ; $\mathcal{M} \models (\neg\varphi)^\rho \iff$ it is not the case that $\mathcal{M} \models \varphi^\rho$; $\mathcal{M} \models (\exists x.\varphi)^\rho \iff$ exists $a \in M : \mathcal{M} \models \varphi^{\rho[x \mapsto a]}$; $\mathcal{M} \models (\forall x.\varphi)^\rho \iff$ for all $a \in M : \mathcal{M} \models \varphi^{\rho[x; \mapsto a]}$
Substitution: $u[t/x]^\rho_{\mathcal{M}} = u_{\mathcal{M}}^{\rho[t^\rho_{\mathcal{M}}/x]}$, and $M \models (A[t/x])^\rho$ iff $M \models A^{\rho[x \mapsto t^\rho_{\mathcal{M}}]}$.
Sufficient set of connectives: $\neg$ and at least one of $\wedge, \vee, \to$ and at least one of $\forall, \exists$
We can distribuate $\exists$ over $\vee$ and $\forall$ over $\wedge$, others NOT!!
Prenex normal form: all quantifiers at the beginning.
Negation normal form: brez kvantifikatorjev, le iz konjunkcij in disjunkcij literalov.
Conjunctive normal form: konjunkcija D_i, kjer so D_i disjunkcije literalov
Disjunctive normal form: disjunkcija C_i, kjer so C_i konjunkcije literalov
Quantifier elimination: brez vseh kvantifikatorjev. **Prop:** T has q.e. iff quantifiers are elimenable from every formula of the form $\exists x.(L_1 \wedge \dots \wedge L_k)$ where $x \in FV(L_i)$ and L_i from literal base. **Ex:** Theory of finite cardinal bounds and $\mathrm{Th}(\mathbb{R}, 0, 1, +, <)$ enjoy q.e.
Kako? PNF, $\forall \leftrightarrow \neg\exists\neg$, DNF na notranji formuli, potem pa korakoma odpravljaš kvantifikatorje na majhnih formulah $\exists x.A$

Proof calculi

Hilbert's system: (A1) $A \to (B \to A)$, (A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, (A3) $(\neg B \to \neg A) \to (A \to B)$, (A4) $\forall x.A \to A[t/x]$, (A5) $\forall x.(A \to B) \to (A \to \forall x.B)$ $(x \notin FV(A)))$, (A6) $\forall x.x = x$, (A7) $x = y \to (A \to A[y/x])$ (y ni v A)
(MP) če velja $A \to B$ in če velja A, potem velja B. (IMP) Če $\Gamma \vdash A \to B$ in $\Gamma \vdash A$, potem $\Gamma \vdash B$.
(Gen) Če $\Gamma \vdash A$, potem $\Gamma \vdash \forall x.A$, kjer x ne nastopa v Γ. (ID) $\Gamma, A \vdash B \iff \Gamma \vdash A \to B$ (kulsko!)
(IPR) če $\Gamma \vdash A_1, \dots, \Gamma \vdash A_n$ in $\Gamma, A_1, \dots, A_n \vdash B$, potem $\Gamma \vdash B$, (DN) $\vdash \neg\neg A \to A$ in $\vdash A \to \neg\neg A$
Sequent calculus: Pomemben trik: pri $\exists$R, moraš pogosto podvojiti sekvent.
$\Gamma \vdash \Delta \iff \exists$ finite $\Gamma' \subseteq \Gamma, \Delta \subseteq \Delta$ such that $\Gamma' \implies \Delta'$ is derivable.
$\Gamma \models \Delta \iff \forall \mathcal{M}, \forall \rho :$ if $\forall A \in \Gamma.\mathcal{M} \models A^\rho$ then $\exists B \in \Delta.\mathcal{M} \models B^\rho$.
Soundness and completeness: $\Gamma \vdash \Delta \iff \Gamma \models \Delta$ (desno je soundness, levo je completeness)
Prop: If $\Gamma \not\models \Delta$ then $\exists \mathcal{M}, \exists \rho$ s.t. $\forall A \in \Gamma.\mathcal{M} \models A^\rho$ and $\forall B \in \Delta.\mathcal{M} \not\models B^\rho$.

Decidability

Decidability for theory T there is an algorithm for: given input Φ, returning true if $T \models \Phi$ and false if $T \not\models \Phi$.
Thm: The theory of $(0, 1, +, \times)$ over $\mathbb{R}$ is decidable, but over $\mathbb{N}$ it is not.
Thm: Theory of groups is undecidable, but theroy of abelian groups is decidable.
Thm: Theory of finite cardinal bounds (and hence also $L_\emptyset$) is decidable.
Thm: Theory $\mathrm{Th}(\mathbb{R}, 0, 1, +, q., <)$ (and hence also $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$ is decidable.
$S \subseteq \mathbb{N}$ is **decidable/computable/recursive** if there exists as algorithm that given any $n \in \mathbb{N}$ as input, eventually halts and returns true if $n \in S$ and false if $n \notin S$. This alg. is called a decision procedure.
$S \subseteq \mathbb{N}$ is **semidecidable/computably enumerable/recursively enumerable** if there exists as algorithm that given any $n \in \mathbb{N}$ as input either eventually halts with true id $n \in S$ or runs forever if $n \notin S$. This alg. is called a semi-decision procedure.
Prop:

1. Every finite subset of $\mathbb{N}$ is decidable.

2. There also exist infinite decidable sets. (eg. $\mathbb{N}, 2\mathbb{N}, \ldots$)
3. Every decidable set is semidecidable.
4. There exists semidecidable sets that are not decidable.
5. There also exists sets that are not semidecidable (cardinality – uncount. sets, count. alg.)
6. If S is decidable, then so is $\overline{S} = S^C$.
7. If S and $\overline{S}$ are semidecidable, then S is decidable (en korak enega, en korak drugega alg.).

Partial enumeration algorithm for S is an alg. that takes $i \in \mathbb{N}$ as input and either: runs forever or returns some number l_i as output, where $S = \{l_i\ ;\ \text{the alg. returns a value on input } i\}$.
Total 1-1 enumeration algorithm is a partial enumeration algorithm that moreover produces a value l_i for every $i \in \mathbb{N}$ and also $l_i \neq l_j$ for $i \neq j$. (only possible for infinite S)
Total ordered enumeration algorithm is a total 1-1 enum. alg. which also has $l_i < l_j$ for $i < j$.
Prop: S is semidecidable iff it has partial enumeration algorithm.
Prop: For an infinite S: S is computably enumerable iff it has a total 1-1 enumeration alg.; S is computable iff it has a total ordered enumeration alg.
Decidable theory: encode sentences as $\mathbb{N}$, and see if the set is decidable.
A theory T is **computably axiomatisable** if there exists a computable set S s.t. for all sentences A, $A \in T (T \vdash A) \iff S \vdash A$.
Prop: T is computably axiomatisable iff it is computably enumerable. (For ($\Leftarrow$) we need Craig's trick: define $S = \{(A \vee \bot) \wedge \top_1 \wedge \ldots \wedge \top_N\ ;\ A \in T$ and N number of computation steps taken by semidecision procedure for T on input A to return true$\}$)

A consistent theory T is **complete** if for every sentence A either $A \in T$ or $\neg A \in T$. Eg. $\mathrm{Th}(\mathcal{M})$ is always complete.
Theorem (Janicak): If a theory T is complete then T is decidable iff computably axiomatisable. (($\Rightarrow$) $S = T$, ($\Leftarrow$) semidecision procedure form T and the complement of T, then use upper prop.)
A theory T is **ω-categorical** if any two countably-infinite models of T are isomorphic.
Thm: If T has no finite models and is ω-categorical, then T is complete. (we see that uncountable model is elem.equivalent to countable model by downwards L-S. Any two countable are iso., so elem.equiv. As then all models are elem.equiv., $T = \mathrm{Th}(\mathcal{M})$ for any model.)
Ex (Cantor): Any two countable models of dense $<$ linear order are isomorfic. Hence it is complete.

T satisfies the **finite model property** if every sentence A that is satisfied by some model of T is satisfied by some finite model.
Theorem (Harrop): IF T is finitely axiomatised and has finite model property, then T is decidable. (fin.axiom. implies comp.axiom. implies semidecidable. Then also show $\overline{T}$ semidecidable (go through all finite models until $\mathcal{M} \vDash S, \neg A$))
Ex: Theory of valid sentences over L_{MOD} is decidable. (no function simbols, unary relation simbols $P_1, P_2, \ldots$).

Elementary model theory

Theorem: (Compactness) A set of sentences S is satisfiable iff every finite subset of S is satisfiable. (If an infinite set of sentences is not satisfiable, then there is a finite subset that is not satisfiable) **Corr:** If S has arbitrarily large finite models, it has an infinite model.
Theorem: (Löwenheim-Skolem) Let S be a set of sentences over a countable signature. If S has a model, it has a countable model. If S has an infinite model, then it has a countably infinite model.
Def: A homomorphism ϑ from an L-structure $\mathcal{M}$ to an L-structure $\mathcal{M}'$ is a function $\vartheta\colon \mathcal{M} \to \mathcal{M}'$, such that $P_{\mathcal{M}}(d_1, \ldots, d_n) \implies P_{\mathcal{M}'}(\vartheta(d_1), \ldots, \vartheta(d_n))$, and $\vartheta(f_{\mathcal{M}}(d_1, \ldots, d_n)) = f_{\mathcal{M}'}(\vartheta(d_1), \ldots, \vartheta(d_n))$. **Prop:** Equivalently, ϑ is a homomorphism iff for every atomic formula A and $\mathcal{M}$-env ρ: $M \models A^{\rho} \implies \mathcal{M}' \models A^{\vartheta \circ \rho}$.
Def: A homomorphism $\vartheta\colon \mathcal{M} \to \mathcal{M}'$ is an embedding from M to $\mathcal{M}'$ if it is injective, this impiles $P_{\mathcal{M}'}(\vartheta(d_1), \ldots, \vartheta(d_n)) \implies P_{\mathcal{M}}(d_1, \ldots, d_n)$.
Prop: Equivalently, ϑ is an embedding iff for every quantifier free formula A and $\mathcal{M}$-env ρ: $M \models A^{\rho} \implies \mathcal{M}' \models A^{\vartheta \circ \rho}$.
Def: An embedding $\vartheta\colon \mathcal{M} \to \mathcal{M}'$ is an elementary embedding from M to $\mathcal{M}'$ if for every formula A and every $\mathcal{M}$ env ρ: $\mathcal{M} \models_{\rho} A \implies \mathcal{M}' \models A^{\vartheta \circ \rho}$.
Def: A fn ϑ is an isomorphism if it is a bijection and ϑ^{-1} is an isomorphism. Every isomorphism is an

elementary embedding.

Def: Two structures $\mathcal{M}$ and $\mathcal{M}'$ are elementary equivalent iff $Th(\mathcal{M}) = Th(\mathcal{M}')$, or equivalently, there exists an elementary embedding $\vartheta\colon \mathcal{M} \to \mathcal{M}'$.

Def: $\mathcal{M}$ is a substructure of $\mathcal{M}'$ if inclusion from M to M' is an embedding. Also, M' is an extension of M. The substructure / extension is elementary if ϑ is an elementary embedding.

Thm: (Proper elementary extension) Every infinite structure $\mathcal{M}$ has a proper elementaryy extension $\mathcal{M}' \succeq \mathcal{M}$. Also, if $\mathcal{M}$ and L are countable, $\mathcal{M}'$ is also countable.

Thm: (Countable elementary substructure) $\mathcal{M}$ is an infinite structure over countable L. Then, for any countable subset $X \subseteq M$, there exists a countably infinite elementary substructure $\mathcal{M}' \preceq \mathcal{M}$ with $X \subseteq M'$.

Language of arithmetic and computability

Δ_0 formulas: $t = u, t < u, t \leq u$, closed under logical connectives and bounded quantifiers.

Σ_1 formulas: $\exists x_1 \ldots \exists x_k.A$, A in Δ_0

Π_1 formulas: $\forall x_1 \ldots \forall x_k.A$, A in Δ_0

Relation R is represented by a formula A if $\forall n_1, \ldots, n_k \in \mathbb{N}.R(n_1, \ldots, n_k) \iff \mathbb{N} \models A(n_1, \ldots, n_k)$

Thm: R relation. R is Σ_1-representable $\iff$ R is computably enumerable.

Graph of partial function f on domain D is relation $graph(f)(n_1, \ldots, n_k, m) \iff (n_1, \ldots, n_k) \in D, f(n_1, \ldots, n_k) = m$ (ekviv: za vsako k terico imamo največ en m s to lastnostjo)

Thm: f partial function. $gr(f)$ is Σ_1-representable $\iff$ f is computable partial function.

Set of computable partial functions: naprej na listu.

Primitive recursive function: 1-3 from above.

Relation R is $\star$ if there exists a $\star$ function s.t. $R(\ldots) \iff f(\ldots) = 0$; $\star$ is prim.recursive / computable / comp. enumerable.

Prop: If a k-ary relation R is Δ_0-representable, then R is primitive recursive.

β-function lemma: There exists Δ_0-representable binary function β s.t. for every $k \geq 0$ and $n_0, \ldots, n_{k-1} \in \mathbb{N}$ there exists $s \in \mathbb{N}$ s.t. $\beta(s, i) = n_i$.

Theories of arithmetic

(A1) $\forall x \forall y. x + 1 = y + 1 \to x = y$	(A2) $\forall x. x + 1 \neq 0$
(A3) $\forall x. x + 0 = x$	(A4) $\forall x \forall y. x + (y + 1) = (x + y) + 1$
(A5) $\forall x. x \cdot 0 = 0$	(A6) $\forall x \forall y. x \cdot (y + 1) = x \cdot y + x$
(A7) $\forall x. x = 0 \vee \exists y.(x = y + 1))$	(A-Ind) $\forall \vec{y}.(A(0, \vec{y}) \wedge (\forall x.A(x, \vec{y}) \to A(x+1, \vec{y})) \to \forall x.A(x, \vec{y}))$
PA = (A1)–(A6) + (A-Ind) $\forall$ frmls A	IΣ_1 = (A1)–(A6) + (A-Ind) for all Σ_1 frmls A
Q = (A1)–(A7)	IΔ_0 = (A1)–(A6) + (A-Ind) for all Δ_0 frmls A

$Q \subsetneqq I\Delta_0 \subsetneqq I\Sigma_1 \subsetneqq PA \subsetneqq \mathrm{Th}(\mathbb{N})$

Lemma: For all $n \in \mathbb{N}$: $Q \vdash \forall x.(x \leq \overline{n}) \leftrightarrow (x = \overline{0} \vee x = \overline{1} \vee \ldots \vee x = \overline{n})$. Proof by induction.

Thm: For every Σ_1 sentence A: $\mathbb{N} \models A \implies Q \vdash A$. Same for any $Q \subseteq T$.

T is 1-consistent if for every Σ_1 sentence A: $T \vdash A \implies \mathbb{N} \models A$. 1-consistent implies consistent.

Thm: If T is 1-consistent extension of Q (eg. Q, PA, Th($\mathbb{N}$), then T is undecidable.

Corr: Validity in first-order logic is undecidable. (If we could decide any sentence, we could decide $\vdash$ (conjunction of Q) $\to B$, so $Q \vdash B$, contradiction)

Corr: Any computably axiomatised (1-)consistent extension T of Q is incomplete (ie. exists sentence A, s.t. $T \nvdash A$ and $T \nvdash \neg A$) (Janicak: any complete computably axiomatisable theory is decidable)

For Th($\mathbb{N}$) there is no computable axiomatisation!! (as 1-consistent, undecidable, complete)

Gödel's incompleteness theorems

Gödel's diagonal lemma: For any formula $A(x)$ there exists a sentence B s.t. $Q \vdash B \leftrightarrow A(\overline{\ulcorner B \urcorner})$. ($B$ says: I enjoy property A.)

Strong Q representability: Suppose $R(x_1, \ldots, x_k, y)$ is a computable relation s.t. for any $n_1, \ldots, n_k$ there is at most one m s.t. $R(n_1, \ldots, n_k, m)$. Then there exists Σ_1-formila $A_R(x_1, \ldots, x_k, y)$ s.t. $Q \vdash A_R(\overline{n_1}, \ldots, \overline{n_k}, \overline{m}) \implies R(n_1, \ldots, n_k, m)$ and $R(n_1, \ldots, n_k, m) \implies Q \vdash \forall y.(A_R(\overline{n_1}, \ldots, \overline{n_k}, y) \leftrightarrow y = \overline{m})$.

Tarski Thm: If $T \supseteq Q$ is consistent, then there is no formula $Tr(x)$ s.t. for all sentences B $T \vdash Tr(\ulcorner B \urcorner) \leftrightarrow B$.

Corr: There is no formula $Tr(x)$ s.t. for all sentences B $\mathbb{N} \models Tr(\ulcorner B \urcorner) \iff \mathbb{N} \models B$.

$\mathsf{Prf}_T(m,n) \iff \exists$ formula $A, \exists$ proof p of A, s.t. $n = \ulcorner A \urcorner, m = \ulcorner p \urcorner$ (aka. "m is proof of n")

$\mathsf{Prov}_T(x) \iff \exists y.\mathsf{Prf}_T(x,y)$ (aka. "x is provable")
$\mathsf{Con}_T \iff \neg\mathsf{Prov}_T(\ulcorner\bot\urcorner)$ (aka. "T is consistent")
Gödel sentence: By diagonal lemma find G_T s.t. $Q \vdash G_T \leftrightarrow \neg\mathsf{Prov}_T(\ulcorner G_T\urcorner)$
First incompleteness Thm: Let T be a computably axiomatised consistent extension of Q. Then: 1. $T \nvdash G_T$, 2. If T is 1-consistent, then $T \nvdash \neg G_T$ and 3. $\mathbb{N} \vDash G_T$.
Second incompleteness Thm: Let T be a computably axiomatised consistent extension of $I\Sigma_1$. Then: 1. $T \nvdash \mathsf{Con}_T$, 2. If T is 1-consistent, then $T \nvdash \neg\mathsf{Con}_T$, and 3. $\mathbb{N} \vDash \mathsf{Con}_T$
Löb's derivability conditions: A, B sentences, T comp. axiom. extension of $I\Sigma_1$.

(D1) $T \vdash A \implies T \vdash \mathsf{Prov}_T(\ulcorner A\urcorner)$
(D2) $T \vdash (\mathsf{Prov}_T(\ulcorner A \to B\urcorner) \wedge \mathsf{Prov}_T(\ulcorner A\urcorner) \to \mathsf{Prov}_T(\ulcorner B\urcorner))$ or
$T \vdash \mathsf{Prov}_T(\ulcorner A \to B\urcorner) \to (\mathsf{Prov}_T(\ulcorner A\urcorner) \to \mathsf{Prov}_T(\ulcorner B\urcorner))$
(D3) $T \vdash \mathsf{Prov}_T(\ulcorner A\urcorner) \to \mathsf{Prov}_T(\ulcorner \mathsf{Prov}_T(\ulcorner A\urcorner)\urcorner)$

Also holds: $\mathrm{I}\Sigma_1 \vdash \mathsf{Prov}_{T+B}(\ulcorner C\urcorner) \to \mathsf{Prov}_T(\ulcorner B \to C\urcorner)$ for every sentence B, C
Löb's Thm: T consistent computably axiomatised extension of $I\Sigma_1$. Za poljuben stavek A: $T \vdash A \iff T \vdash \mathsf{Prov}_T(\ulcorner A\urcorner) \to A$

Rules of sequent calculus:

$$\frac{}{\Gamma,\, A \Rightarrow A,\, \Delta}\,(\mathsf{Ax})$$

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\,(\bot\mathsf{L}) \qquad \frac{}{\Gamma \Rightarrow \top,\, \Delta}\,(\top\mathsf{R})$$

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\,(\neg\mathsf{L}) \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\,(\neg\mathsf{R})$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\,(\wedge\mathsf{L}) \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\,(\wedge\mathsf{R})$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\,(\vee\mathsf{L}) \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\,(\vee\mathsf{R})$$

$$\frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}\,(\to\mathsf{L}) \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\,(\to\mathsf{R})$$

$$\frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta}\,(\forall\mathsf{L}) \qquad \frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}\,(\forall\mathsf{R})^*$$

$$\frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x\, A \Rightarrow \Delta}\,(\exists\mathsf{L})^* \qquad \frac{\Gamma \Rightarrow A[t/x], \Delta}{\Gamma \Rightarrow \exists x\, A, \Delta}\,(\exists\mathsf{R})$$

$$\frac{\Gamma[t,u] \Rightarrow \Delta[t,u]}{\Gamma[u,t],\, t = u \Rightarrow \Delta[u,t]}\,(=\mathsf{L}) \qquad \frac{}{\Gamma \Rightarrow t = t,\, \Delta}\,(=\mathsf{R})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\,(\mathsf{Weak}) \qquad \frac{\Gamma,\, A \Rightarrow \Delta \quad \Gamma \Rightarrow A,\, \Delta}{\Gamma \Rightarrow \Delta}\,(\mathsf{Cut})$$

Computable partial functions:
The set of computable partial functions X is the smallest set, such that
1. X contains Z, s and u_i^k, defined by $Z(n) = 0$, $s(n) = n+1$, $u_i^k(n_1, \dots, n_k) = n_i$
2. If f of arity k and $g_1, \dots, g_k$ or arity ℓ are in X, then so is the $f \circ (g_1, \dots, g_k)$, defined by $f \circ (g_1, \dots, g_k)(n_1, \dots, n_\ell) = f(g_1(n_1, \dots, n_\ell), \dots, g_k(n_1, \dots, n_\ell))$.
3. If f of arity k and g of arity $k+2$ are in X, then so is R_{fg} of arity $k+1$, defined recursively by $R_{fg}(n_1, \dots, n_k, 0) = f(n_1, \dots, n_k)$ and $R_{fg}(n_1, \dots, n_k, n+1) = g(n_1, \dots, n_k, n, R_{fg}(n_1, \dots, n_k))$.
4. If f of arity $k+1$ is in X then so is μf of arity k, defined by $\mu f(n_1, \dots, n_k)$ = the least such n that $f(n_1, \dots, n_k, n) = 0$ and all $f(n_1, \dots, n_k, i)$ are defined for $i < n$, if such n exists. Otherwise undefined.
Bullets $1-3$ are the definition of a computable function (not partial). From exercises we know that: $+, \text{konst}, \dot{-}, \max, \min, == 0, != 0, \leq, ==, \sum_{n \leq n_0}, \prod_{n \leq n_0}$, partitional cases, are all computable.

Avtorji: Vesna Iršič, Jure Slak, Anja Petković, Žiga Lukšič