# Odvodi

| $f(x)$ | $f'(x)$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $a^x$ | $a^x \ln(a)$ |
| $e^x$ | $e^x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\frac{1}{\cos^2 x}$ |
| $\cot x$ | $-\frac{1}{\sin^2 x}$ |
| $\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\arctan x$ | $\frac{1}{1+x^2}$ |
| $\operatorname{sh} x$ | $\operatorname{ch} x$ |
| $\operatorname{ch} x$ | $-\operatorname{sh} x$ |
| $\operatorname{th} x$ | $\frac{1}{\operatorname{ch}^2 x}$ |
| $\operatorname{cth} x$ | $-\frac{1}{\operatorname{sh}^2 x}$ |
| $\operatorname{arsh} x$ | $\frac{1}{\sqrt{x^2+1}}$ |
| $\operatorname{arth} x$ | $\frac{1}{1-x^2}$ |

# Integrali

| $f(x)$ | $\int f(x)dx$ |
|---|---|
| $x^n$ | $\frac{x^{n+1}}{n+1}(n \neq -1)$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert$ |
| $e^x$ | $e^x$ |
| $\sin x$ | $-\cos x$ |
| $\cos x$ | $\sin x$ |
| $\frac{1}{\cos^2 x}$ | $\tan x$ |
| $\frac{1}{\sin^2 x}$ | $-\cot x$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin x$ |
| $\frac{1}{\sqrt{1+x^2}}$ | $\operatorname{arsh} x = \ln\left\lvert x+\sqrt{x^2+1}\right\rvert$ |
| $\frac{1}{1+x^2}$ | $\arctan x$ |

# Per Partes

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$
$$\int udv = uv - \int vdu$$

# Racionalne funkcije

$\int \frac{p(x)}{q(x)}dx, \qquad p(x), q(x)$ sta polinoma

1. Če je $st(p(x)) <= st(p(x))$ polinoma delimo
2. $q(x)$ razdelimo na linearne in kvadratne faktorje
3. Izraz pod integralom razcepimo na parcialne ulomke
   $\frac{p(x)}{q(x)} = \left[\frac{A_1}{x-a_1} + \cdots + \frac{A_{n_1}}{(x-a_1)^{n_1}}\right] + \cdots + \left[\frac{Z_1}{x-a_k} + \cdots + \frac{Z_{n_k}}{(x-a_k)^{n_k}}\right] + \left[\frac{\alpha_1 x+\beta_1}{x^2+b_1x+c_1} + \cdots + \frac{\alpha_{m_1}x+\beta_{m_1}}{(x^2+b_1x+c_1)^{m_1}}\right] + \cdots + \left[\frac{\varphi_1 x+\omega_1}{x^2+b_lx+c_l} + \cdots + \frac{\varphi_{m_l}x+\omega_{m_l}}{(x^2+b_lx+c_l)^{m_l}}\right]$
4. Integriramo vsakega zase

$$k \geq 2 \qquad st(p(x)) \leq 2k-1$$
$$st(q(x)) \leq 2k-3 \qquad (ax^2+bx+c) \qquad \text{nerazcepen v } \mathbb{R}$$
$$I = \int \frac{p(x)}{(ax^2+bx+c)^k} = \int \frac{Ax+B}{ax^2+bx+c} + \frac{q(x)}{(ax^2+bx+c)^{k-1}}$$

A,B, $q(x)$ poiščemo tako da enačbo odvajamo.

# Korenske funkcije

1. $\int f(\sqrt{ax+b})dx \qquad t = \sqrt{ax+b}$
2. $\int f(\sqrt{ax^2+bx+c})dx$

   a $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ ga prevedemo na oblike:
   - Če je $a < 0 : \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$
   - Če je $a > 0 : \int \frac{dx}{\sqrt{x^2+c}} = \ln\left\lvert x+\sqrt{x^2+c}\right\rvert$

   b $\int \frac{p(x)}{\sqrt{ax^2+bx+c}} = q(x)\sqrt{ax^2+bx+c} + A\int \frac{dx}{\sqrt{ax^2+bx+c}}$
   $st(p(x)) - 1 = st(q(x))$ A, q(x) poiščemo z odvanjanjem
3. $\int \sqrt{a^2-x^2}dx \quad x = a\sin t \quad dx = a\cos t dt \quad t = \arcsin\frac{x}{a}$
4. $\int \sqrt{a^2+x^2}dx \quad x = a\operatorname{sh} t \quad dx = a\operatorname{ch} t dt \quad t = \operatorname{arsh}\frac{x}{a}$

# Kotne funkcije

1.
$$\int \sin(ax)\sin(bx)dx = \int -\frac{1}{2}\left[\cos(a+b)x - \cos(a-b)x\right]dx =$$
$$= -\frac{1}{2}\left[\frac{sin(a-b)x}{(a-b)} - \frac{sin(a+b)x}{(a+b)}\right]$$
$$\int \cos(ax)\cos(bx)dx = \int \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right]dx \ldots$$
$$\int \sin(ax)\cos(bx)dx = \int \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right]dx \ldots$$

2. $\int \cos^m x \sin^n x dx$

   (a) Eno od stevil $m$,$n$ je liho (npr. $m = 2k+1$)
   $$\int \cos^{2k} x \cos x \sin^n x dx = \int t^n(1-t^2)^k dt$$
   $$t = \sin x \quad dt = \cos x dx$$
   $$cos^{2k}x = (cos^2 x)^k = (1-t^2)^k$$

   (b) $m$, $n$ sta oba soda, $m = 2m_1$,$n = 2n_1$
   $$\int \cos^{2m_1} x \sin^{2n_1} x dx = \int (\cos^2 x)^{m_1}(\sin^2 x)^{n_1}dx =$$
   $$= \int \left(\frac{1+\cos 2x}{2}\right)^{m_1}\left(\frac{1-\cos 2x}{2}\right)^{n_1} =$$
   $$= \text{vsota integralov oblike} \int \cos^k 2x dx$$
   kjer je $k \leq m_1 + n_1 = \frac{1}{2}(m+n) < m+1$
   Ce je $k$ lih gremo po 1 tocki
   Ce je $k$ sod ponovimo postopek

   (c) $\int R(\cos x, \sin x)dx$ ($R\ldots$ racinonalni izraz)
   $$t = \tan\frac{x}{2} \quad \cos x = \frac{1-t^2}{t^2+1}$$
   $$\sin x = \frac{2t}{t^2+1} \quad dx = \frac{2}{t^2+1}dt$$
   $$t = \tan x \quad \cos x = \frac{1}{\sqrt{t^2+1}}$$
   $$\sin x = \frac{t}{\sqrt{t^2+1}} \quad dx = \frac{dt}{t^2+1}$$

# Uporaba integralov

1. Ploscina ravnisnkih likov
2. Dolzina krivulj
   $$s = \int_a^b \sqrt{1+f'(x)^2}dx$$
3. Prostornina vrtenine
   $$V = \pi\int_a^b f(x)^2 dx$$
4. Povrsina vrtenine
   $$S = 2\pi\int_a^b f(x)\sqrt{1+f(x)^2}dx$$
5. Tezisce ravninskih likov
   $$M_x = \varphi g\int_a^b \frac{f(x)^2}{2}dx \quad \text{navor}$$
   $$M_y = \varphi g\int_a^b xf(x)dx$$
   $$x_T = \frac{\int_a^b xf(x)dx}{P} \quad \text{tezisce}$$
   $$y_T = \frac{\int_a^b \frac{f(x)^2}{2}dx}{2P} = \frac{V}{2\pi P}$$
   $$V = 2\pi y_T P \quad \text{Guldinovo pravilo}$$
   $2\pi y_T\ldots$pot, ki jo pri vrtenju opise tezisce $\quad P\ldots$Ploscina

6. Tezisce ravninskih krivulj

$$M_x = \varphi g \int_a^b f(x)\sqrt{1+f(x)^2}dx = g\varphi s y_T$$
$$x_T = \frac{\int_a^b x\sqrt{1+f'(x)^2}dx}{\int_a^b \sqrt{1+f'(x)^2}dx}$$
$$y_T = \frac{\int_a^b f(x)\sqrt{1+f'(x)^2}dx}{\int_a^b \sqrt{1+f'(x)^2}dx} = \frac{1}{s}\int_a^b f(x)\sqrt{1+f'(x)^2}dx = \frac{1}{s}\frac{S}{2\pi}$$
$$S = 2\pi y_T s \quad \text{Guldinovo pravilo}$$

## Numericna integracija

1. Trapezna metoda

$$I = \sum_{i=0}^{n-1} \frac{y_i + y_{i+1}}{2}\frac{b-a}{4} = \frac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$
$$|R_n| \le \frac{(b-a)^3}{12n^2} \max_{x\in[a,b]} \left|f''(x)\right| \quad \text{napaka}$$

2. Simpsonova metoda
Izberemo $2n$ delilnih tock in na vsakem intervalu ploscino aproks. z kvad. parabolo.

$$x_i = a + i\frac{b-a}{2n}$$
$$I = \int_a^b f(x)dx \doteq \frac{b-a}{6n}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{2n-1} + y_{2n})$$
$$|R_n| \le \frac{(b-a)^5}{2880n^4} \max_{x\in[a,b]} \left|f^{(4)}(x)\right| \quad \text{napaka}$$

## Izlimitirani integrali

1. $f$ ni omejena v krajiscih $a$ ALI $b$

$$\int_a^b f(x)dx = \lim_{c\to a}\int_c^b f(x)dx \quad \text{enako za b}$$

2. $f$ ni omejena v krajiscih $a$ IN $b$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{d\to a}\int_d^c f(x)dx + \lim_{e\to b}\int_c^e f(x)dx$$

3. $f$ ni omejena v krajiscu $c \in [a,b]$

$$\int_a^b f(x)dx = \int_c^a f(x)dx + \int_b^c f(x)dx = \lim_{d\to -c}\int_d^a f(x)dx + \lim_{e\to c}\int_b^e f(x)dx$$

4. Ena od mej gre v neskoncnost

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$
$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$
$$\int_{-\infty}^\infty = \int_{-\infty}^0 + \int_0^\infty$$

## Konvergenca integralov

Integralski kriterij

$$f \ge 0 \quad a \in \mathbb{R}: \quad \int_a^\infty \frac{1}{f(x)}dx \quad \text{obstaja} \Longleftrightarrow \sum_{n=a}^\infty \frac{1}{f(n)}$$

$$\int_a^b \frac{f(x)}{(x-a)^s}dx \quad f(x)\text{je zvezdna na}[a,b]$$

- ce je $s < 1$ => integral konvergira
- ce je $s \ge 1$ in $f(a) = 0$ => integral divergira

$$\int_a^\infty \frac{f(x)}{x^s}dx \quad f(x)\text{je zvezdna in omejena na}[a,b]$$

- ce je $s > 1$ => integral konvergira
- ce je $s \le 1$ in $|f(a)| \ge m > 0$ za vse x od nekje naprej => integral divergira

## Parametricno podane funkcije

$$pl = \frac{1}{2}\left|\int_{t_1}^{t_2}(x\dot{y} - \dot{x}y)dt\right|$$
$$pl = \left|\int_{t_1}^{t_2} x\dot{y}dt\right| = \left|\int_{t_1}^{t_2} \dot{x}ydt\right| \quad \text{enostavna sklenjena krivulja}$$
$$pl = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2 d\varphi \quad \text{v polarnih kordinatah}$$
$$l = \int_{t_1}^{t_2}\sqrt{\dot{x}^2 + \dot{y}^2}dt \quad \text{dolzina krivulje}$$
$$V = \int_{z_1}^{z_2} S(z)dz \quad \text{Volumen vrtenine}$$

## Pazi

$$\int \text{soda funkcija} = \text{liha funkcija} \qquad \int \text{liha funkcija} = \text{soda funkcija}$$
$$\int_a^b f(x)dx = \int_\beta^\alpha f(g(t))g'(t)dx$$
$$x = g(t) \quad \text{za monotone funkcije} \quad \alpha = g(a) \quad \beta = g(b)$$

## Parcialni odvodi

$$f_{x_i}\frac{\partial f(x)}{\partial x_1}(a_1,\ldots,a_n) = \lim_{h\to 0}\frac{f(a_1,\ldots,a_i+h,\ldots,a_n) - f(a_1,\ldots,a_n)}{h}$$
$$\text{grad} f = \left(\frac{\partial f(x)}{\partial x_1},\ldots,\frac{\partial f(x)}{\partial x_n}\right) \quad \text{Smer najvecjega narascanje}$$
$$f_{\vec{u}}(\vec{a}) = \lim_{h\to 0}\frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h} =< \text{grad} f(\vec{a}), \vec{u} >$$

## Kotne funkcije

### Adicijski izreki

$$\sin x \pm y = \sin x\cos y \pm \sin y\cos x$$
$$\cos x \mp y = \cos x\cos y \mp \sin x\sin y$$
$$\tan x \pm y = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

### Faktorizacija

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = 2\sin\frac{x-y}{2}\sin\frac{x+y}{2}$$

### Razclenevanje

$$\sin x\sin y = -\frac{1}{2}\left(\cos(x+y) - \cos(x-y)\right)$$
$$\cos x\cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$
$$\sin x\cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$

$$\cos^2 x = \frac{\cos 2x + 1}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$