Project: Secrets of the Andorian tombs

CHE 400 Mingxuan Li

1. Abstract.

The Andorian people was trapped in the wired tombs, I am a Engineer and I'm gonna solve the secret of this tomb and rescue these people. The only thing I need to do is to find the correct Temperature condition of inner surface so that I can unlock the tomb.

2. Problem statement

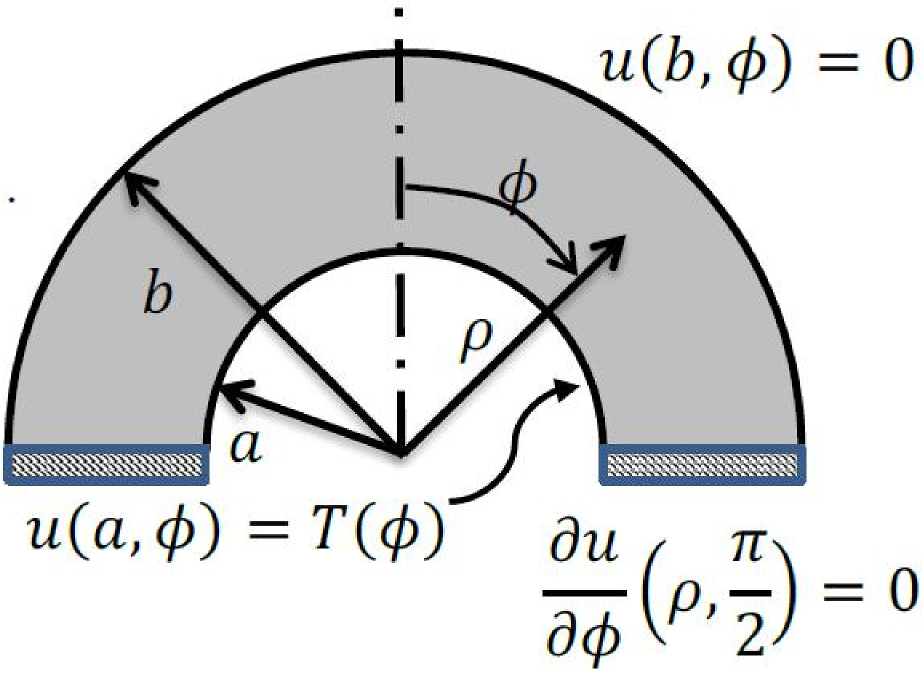

Fig 1. Problem statement

The tomb can be regarded as a hemisphere shell and if we assume the material of this tomb is homogeneous then we can write down the equation for this:

PDE: $\nabla^2 u(\rho,\varphi) = 0$, $a < \rho < b$, $0 < \varphi < \frac{\pi}{2}$

BC: $u(b,\varphi)=0$, $u(a,\varphi)=T(\varphi)$, $\frac{\partial u}{\partial \varphi}(\rho,\frac{\pi}{2})=0$

Constraints: $u(\frac{a+b}{2},0)=T_{cr}=12$, $u(\frac{a+b}{2},\frac{\pi}{4})=\frac{T_{cr}}{2}=6$

The problem is to solve for $T(\varphi)$

3. derivation

First write down the Laplace equation in spherical coordinate system:

$$\nabla^2 u(\rho,\varphi)=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2 Sin\varphi}\frac{\partial}{\partial\varphi}(Sin\varphi\frac{\partial u}{\partial\varphi})=0 \quad (1)$$

Then separate the variables by assuming $u(\rho,\varphi)=G(\varphi)F(\rho)$ (2)

To get $\frac{G}{\rho^2}\frac{d}{d\rho}(\rho^2\frac{d}{d\rho}F)+\frac{F}{\rho^2 Sin\varphi}\frac{d}{d\varphi}(Sin\varphi\frac{dG}{d\varphi})=0$ (3)

Multiple by $\frac{\rho^2}{FG}$

then $\frac{1}{F}\frac{d}{d\rho}(\rho^2\frac{dF}{d\rho})=-\frac{1}{G\cdot Sin\varphi}\frac{d}{d\varphi}(Sin\varphi\frac{dG}{d\varphi})=\lambda$ (4)

Then we get two ODE

ODE#1:

$$\frac{d}{d\rho}(\rho^2\frac{d}{d\rho}F)-\lambda F=0 \quad (5)$$

ODE#2:

$$\frac{d}{d\varphi}(Sin\varphi\frac{d}{d\varphi}G)+\lambda Sin\varphi\cdot G=0 \quad (6)$$

If $\lambda=0$,

(6) becomes:

$$\frac{d}{d\varphi}(Sin\varphi\frac{d}{d\varphi}G)=0, \quad (7)$$

Two solution for G are $G=$ constant and $G=-\ln(Sin\varphi)$.

Both of them can not satisfy the BCs.

So consider $\lambda\neq 0$

For ODE#2, use $x=Cos\varphi$

Then we get $\frac{d}{dx}((1-x^2)\frac{dG}{dx})+\lambda G=0$ (8)

The solution to this equation is Legendre Polynomials: $P_n(x)$

To satisfy the BC:

$\frac{\partial u}{\partial\varphi}(\rho,\frac{\pi}{2})=0$, Only when n is even number are BC satisfied.

So we pick eigenfunctions $P_{2n}(x)$ and eigenvalues $\lambda_n=2n(2n+1)$

Where $n = 0,1,2,3\cdots$

Then the ODE#1 becomes:

$$\frac{d}{d\rho}(\rho^2 \frac{d}{d\rho}F) - 2n(2n+1)F = 0 \tag{9}$$

Simplify:

$$\rho^2 \frac{d^2F}{d\rho^2} + 2\rho\frac{dF}{d\rho} - 2n(2n+1)F = 0 \tag{10}$$

This is a Equidimensional equation.

Assume $F = C\cdot\rho^s$ (11)

Then the equation becomes

$$2Cs\rho^s + s(s-1)C\rho^s - 2n(n+1)C\rho = 0 \tag{12}$$

$$C\rho^s(2s + s(s-1) - 2n(2n+1)) = 0 \tag{13}$$

$$s^2 + s - 2n(2n+1) = 0 \tag{14}$$

$S_1 = 2n$ and $S_2 = -1-2n$ (15)

So $F = A\cdot\rho^{-1-2n} + B\cdot\rho^{2n}$ (16)

Use BC $u(b,\varphi) = 0$,

Then $A\cdot b^{-1-2n} + B\cdot b^{2n} = 0$ (17)

get $\frac{A}{B} = -b^{4n+1}$

set $B = 1$ then $A = -b^{4n+1}$

Then $F = -b^{4n+1}\cdot\rho^{-1-2n} + \rho^{2n}$ (18)

Combine the solution for F and G

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} An\cdot(-b^{4n+1}\cdot\rho^{-1-2n} + \rho^{2n})\cdot P_{2n}[Cos\varphi] \tag{19}$$

Use the non-homogeneous BC : $u(a,\varphi) = T(\varphi)$

to determine the coefficient:

$$An\cdot(-b^{4n+1}\cdot a^{-1-2n} + a^{2n})\cdot P_{2n}[Cos\varphi] = T[\varphi] \tag{20}$$

$$An = \frac{1}{a^{2n} - b^{4n+1}\cdot a^{-1-2n}} \frac{\int_0^{\frac{\pi}{2}} T[\varphi]\cdot P_{2n}[Cos\varphi]Sin\varphi\cdot d\varphi}{\int_0^{\frac{\pi}{2}} (P_{2n}[Cos\varphi])^2 Sin\varphi\cdot d\varphi} \tag{21}$$

From the book:

$$\int_{-1}^{1}(P_n[x])^2 dx = (n+\frac{1}{2})^{-1}$$

Then the coefficient can be simplify to:

$$An = \frac{(4n+1)\int_0^{\frac{\pi}{2}} T(\varphi)\cdot P_{2n}[Cos\varphi]\cdot Sin\varphi\cdot d\varphi}{a^{2n} - b^{4n+1}\cdot a^{-1-2n}} \tag{22}$$

4. Solution for U_{trial}

Let $T(\varphi) = T_{trial} = T_1$

Then

$$u(\rho,\varphi)_{trial} = \sum_{n=0}^{\infty} An_{trial} \cdot (-b^{4n+1} \cdot \rho^{-1-2n} + \rho^{2n}) \cdot P_{2n}[Cos\varphi] \quad (23)$$

Use mathematica to find that:

$$A_{0trial} = \frac{a\int_0^{\frac{\pi}{2}} T_1 \cdot Sin\varphi \cdot d\varphi}{a-b} \quad (24)$$

$An_{trial} = 0$ for $n = 1,2,3...$ (25)

Plug in:

$a = 1; b = 2$

To get:

$A_{0trial} = -T_1$

Then:

$$u(\rho,\varphi)_{trial} = -T_1(-b \cdot \rho^{-1} + 1) \quad (26)$$

Use the critical constraint to find T_1

$u(\frac{a+b}{2},0) = T_{cr} = 12$;

$$-T_1(-b \cdot (\frac{a+b}{2})^{-1} + 1) = 12 \quad (27)$$

Solve for $T_1 = 36$ °C

Plot in mathematica.

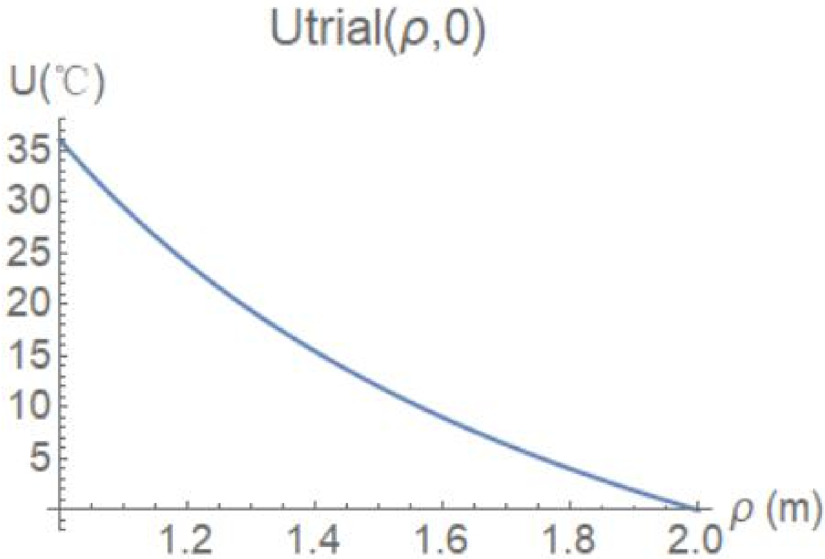

Fig 2. Temperature distribution of $u_{trial}(\rho,0)$

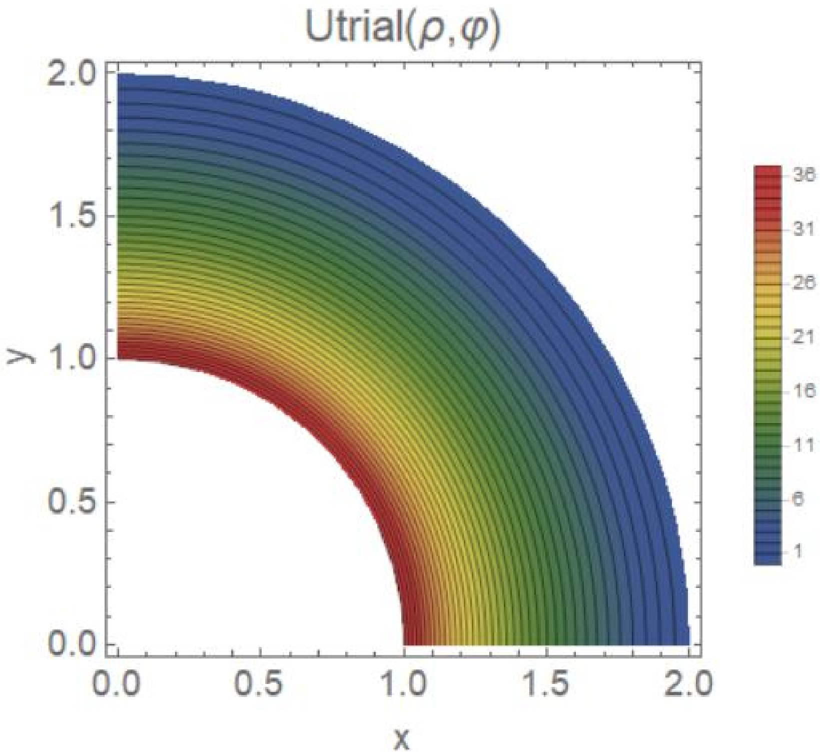

Fig 3. Temperature distribution of $u_{trial}(\rho,\varphi)$

5.Solution for U_{final}

First I need to find a good $T_{final}(\varphi)$.

To simplify the calculation, if we can use finite term in the $u(\rho,\varphi)$, then we can set equation for the coefficients of these term by using the constraints. Since we have two constraints so we can set two equation, Using first two term of $u(\rho,\varphi)$ is a good idea. Then the only thing I need to do is find some kind of $T_{final}(\varphi)$ to make the coefficients in $u(\rho,\varphi)$, $An = 0$ for all $n \neq 0$ and $n \neq 1$.

A little observation of the structure of the equation for :

$$An = \frac{(4n+1)\int_0^{\frac{\pi}{2}} T(\varphi)\cdot P_{2n}[Cos\varphi]\cdot Sin\varphi \cdot d\varphi}{a^{2n} - b^{4n+1}\cdot a^{-1-2n}}$$

Easily to imagine that if we choose $T_{final}(\varphi) = c \times (Cos\varphi)^2 + d$ (28)

We can satisfy the requirement mentioned above.

Then, define the first two term we use are A_0 and A_1

Use constrains to set equation:

$$A_0\cdot(\frac{a+b}{2} - b\cdot(\frac{a+b}{2})^{-1}) + A_1\cdot(-b^5\cdot(\frac{a+b}{2})^{-3} + (\frac{a+b}{2})^2)\cdot P_2[Cos0] = 12 \qquad (29)$$

$$A_0\cdot(\frac{a+b}{2} - b\cdot(\frac{a+b}{2})^{-1}) + A_1\cdot(-b^5\cdot(\frac{a+b}{2})^{-3} + (\frac{a+b}{2})^2)\cdot P_2[Cos\frac{\pi}{4}] = 6 \qquad (30)$$

We can solve that:

$$A_0 = -12 \qquad (31)$$

and

$$A_1 = -\frac{864}{781} \tag{32}$$

Then, c and d can be determined by A_0 and A_1.

Plug (28) into (21), we get:

$$A_0 = -\frac{1}{3}(c+3d) \tag{33}$$

$$A_1 = -\frac{2c}{93} \tag{34}$$

Then we ca get:

$$c = \frac{40176}{781} \tag{35}$$

$$d = -\frac{4020}{781} \tag{36}$$

So the $T_{final}(\varphi) = \frac{40176}{781} \cdot (Cos\varphi)^2 - \frac{4020}{781}$

And solution for u_{final} is:

$$u_{final}(\rho,\varphi) = -12 \cdot (\rho^0 - b \cdot \rho^{-1}) - \frac{864}{781} \cdot (-b^5 \cdot \rho^{-3} + \rho^2) \cdot P_2[Cos\varphi] \tag{37}$$

At two critical points, u_{final} of course satisfy the constraints.

Plot in the mathmetica.

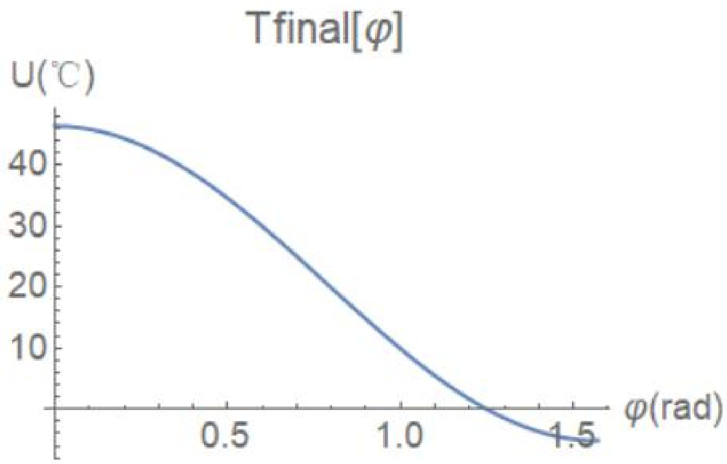

Fig 4. The curve of $T_{final}(\varphi)$

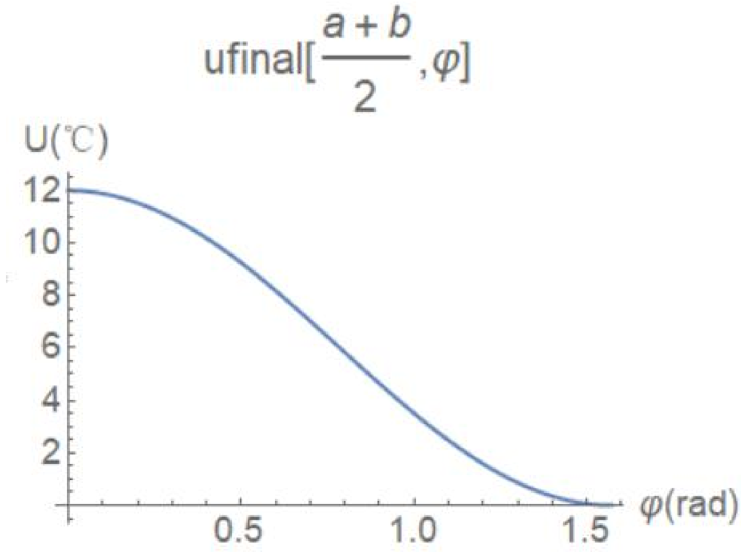

Fig 5. Temperature of $u_{final}(\frac{a+b}{2},\varphi)$

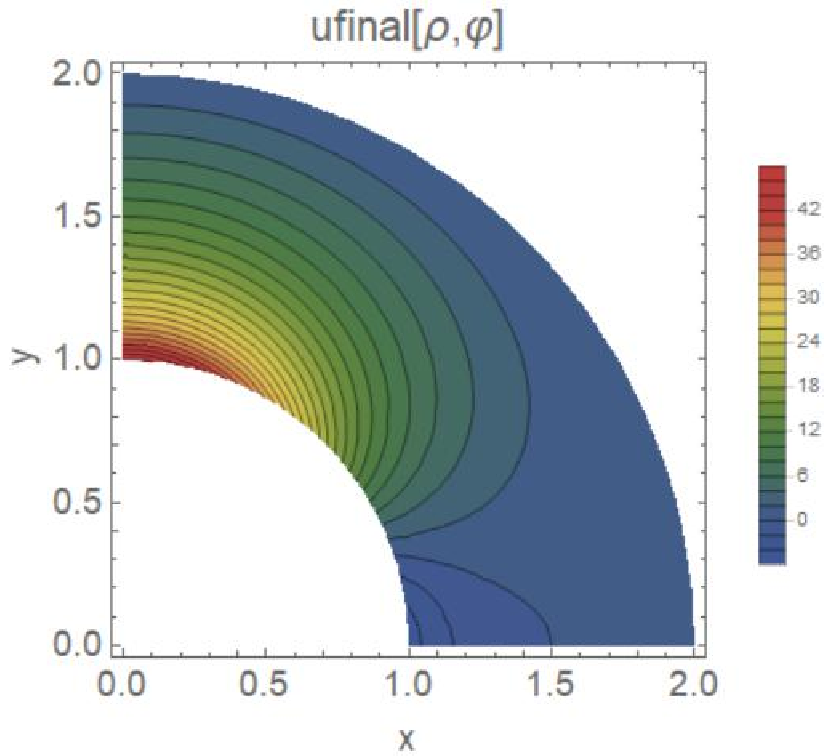

Fig 6.Temperature distribution of $u_{final}(\rho,\varphi)$

5. Extension

I want to do some extension by adding a extra constraint:

$u(\frac{a+b}{2},\frac{\pi}{8})=9$

Since we get three independent constraints, we need at least 3 parameter in $u(\rho,\varphi)$

Familiar to above, this time we pick first 3 term of $u(\rho,\varphi)$ and to find specific $T_{extension}(\varphi)$
Which can satisfy that other terms in $u(\rho,\varphi)$ is 0.
Define the first 3 terms are $B_0\ B_1\ B_2$.
Then, use the constraints we get 3 equations:

$$B_0\cdot(\frac{a+b}{2}-b\cdot(\frac{a+b}{2})^{-1})+B_1\cdot(-b^5\cdot(\frac{a+b}{2})^{-3}+(\frac{a+b}{2})^2)\cdot P_2[Cos0]+B_2\cdot(-b^{17}\cdot(\frac{a+b}{2})^{-9}+(\frac{a+b}{2})^8)\cdot P_4[Cos0]=12 \quad (38)$$

$$B_0\cdot(\frac{a+b}{2}-b\cdot(\frac{a+b}{2})^{-1})+B_1\cdot(-b^5\cdot(\frac{a+b}{2})^{-3}+(\frac{a+b}{2})^2)\cdot P_2[Cos\frac{\pi}{8}]+B_2\cdot(-b^{17}\cdot(\frac{a+b}{2})^{-9}+(\frac{a+b}{2})^8)\cdot P_4[Cos\frac{\pi}{8}]=9 \quad (39)$$

$$B_0\cdot(\frac{a+b}{2}-b\cdot(\frac{a+b}{2})^{-1})+B_1\cdot(-b^5\cdot(\frac{a+b}{2})^{-3}+(\frac{a+b}{2})^2)\cdot P_2[Cos\frac{\pi}{4}]+B_2\cdot(-b^{17}\cdot(\frac{a+b}{2})^{-9}+(\frac{a+b}{2})^8)\cdot P_4[Cos\frac{\pi}{4}]=6 \quad (40)$$

We can solve that:

$$B_0=-\frac{6\left(19-42\,Cos\left[\frac{\pi}{8}\right]^2+20\,Cos\left[\frac{\pi}{8}\right]^4\right)}{5\left(1-3\,Cos\left[\frac{\pi}{8}\right]^2+2\,Cos\left[\frac{\pi}{8}\right]^4\right)} \quad (41)$$

$$B_1=-\frac{216\left(1-48\,Cos\left[\frac{\pi}{8}\right]^2+56\,Cos\left[\frac{\pi}{8}\right]^4\right)}{5467\left(1-3\,Cos\left[\frac{\pi}{8}\right]^2+2\,Cos\left[\frac{\pi}{8}\right]^4\right)} \quad (42)$$

$$B_2=\frac{241\,864\,704\left(-3+4\,Cos\left[\frac{\pi}{8}\right]^2\right)}{596\,775\,515\,735\left(1-3\,Cos\left[\frac{\pi}{8}\right]^2+2\,Cos\left[\frac{\pi}{8}\right]^4\right)} \quad (43)$$

Define the $T_{extension}(\varphi)=p\times(Cos\varphi)^4+q\times(Cos\varphi)^2+s$
Then similarly, calculate the coefficient by p,q,s to get:

$$B_0=-\frac{2p}{5}-\frac{2q}{3}-2s \quad (44)$$

$$B_1=-\frac{4}{651}(6p+7q) \quad (45)$$

$$B_2=-\frac{16p}{17\,885} \quad (46)$$

Use these equation to determine p,q,s:

$$p=\frac{15\,850\,724\,308\,992}{17\,050\,729\,021} \quad (47)$$

$$q=-\frac{10\,806\,650\,612\,890\,272}{13\,316\,619\,365\,401} \quad (48)$$

$$s=\frac{1\,477\,892\,485\,145\,460}{13\,316\,619\,365\,401} \quad (49)$$

And finally, $u(\rho,\varphi)$ is given by:

$$\text{uextension}[\rho\_, \varphi\_] := -\frac{6\left(19 - 42\,\text{Cos}\left[\frac{\pi}{8}\right]^2 + 20\,\text{Cos}\left[\frac{\pi}{8}\right]^4\right)}{5\left(1 - 3\,\text{Cos}\left[\frac{\pi}{8}\right]^2 + 2\,\text{Cos}\left[\frac{\pi}{8}\right]^4\right)} * \left(\rho^0 - b^1 * \rho^{-1}\right) - \frac{216\left(1 - 48\,\text{Cos}\left[\frac{\pi}{8}\right]^2 + 56\,\text{Cos}\left[\frac{\pi}{8}\right]^4\right)}{5467\left(1 - 3\,\text{Cos}\left[\frac{\pi}{8}\right]^2 + 2\,\text{Cos}\left[\frac{\pi}{8}\right]^4\right)} * \text{LegendreP}[2, \text{Cos}[\varphi]] * \left(-b^5 * \rho^{-3} + \rho^2\right) + \frac{241864704\left(-3 + 4\,\text{Cos}\left[\frac{\pi}{8}\right]^2\right)}{596775515735\left(1 - 3\,\text{Cos}\left[\frac{\pi}{8}\right]^2 + 2\,\text{Cos}\left[\frac{\pi}{8}\right]^4\right)}\left(-b^{17} * \rho^{-9} + \rho^8\right) * \text{LegendreP}[4, \text{Cos}[\varphi]] \qquad (50)$$

And $T_{extension}(\varphi) = p \times (Cos\varphi)^4 + q \times (Cos\varphi)^2 + s$

Where p,q,s are given by (47),(48),(49)

I will also plot them and check them in Mathematica.

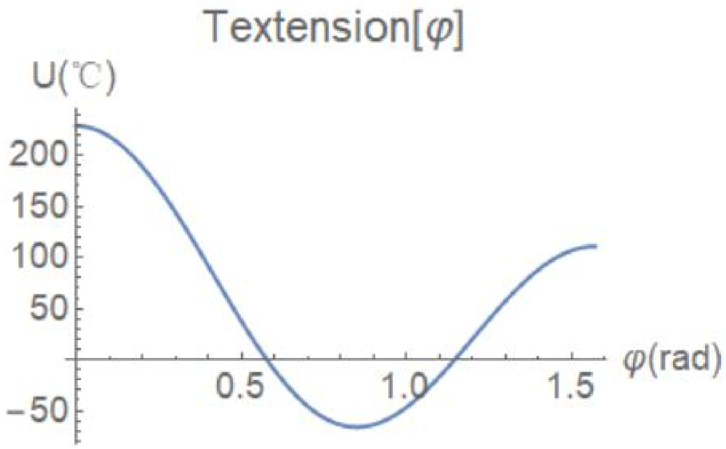

Fig 7. curve of $T_{extension}(\varphi)$

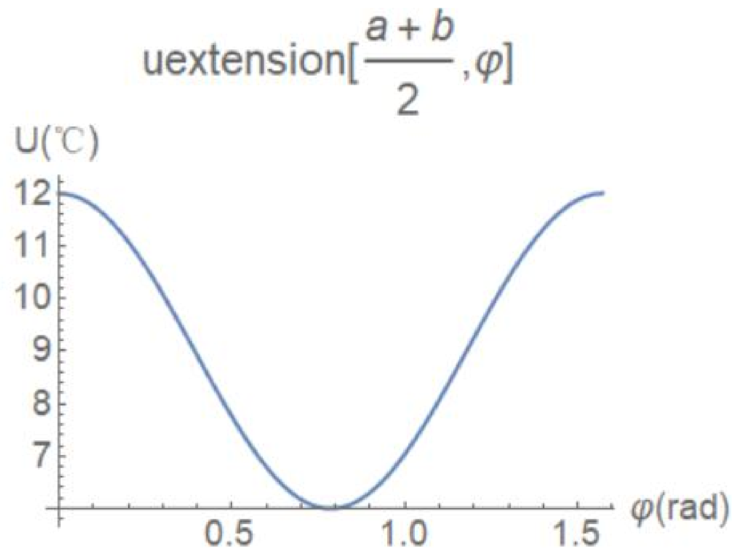

Fig 8. curve of $u_{extension}(\frac{a+b}{2},\varphi)$

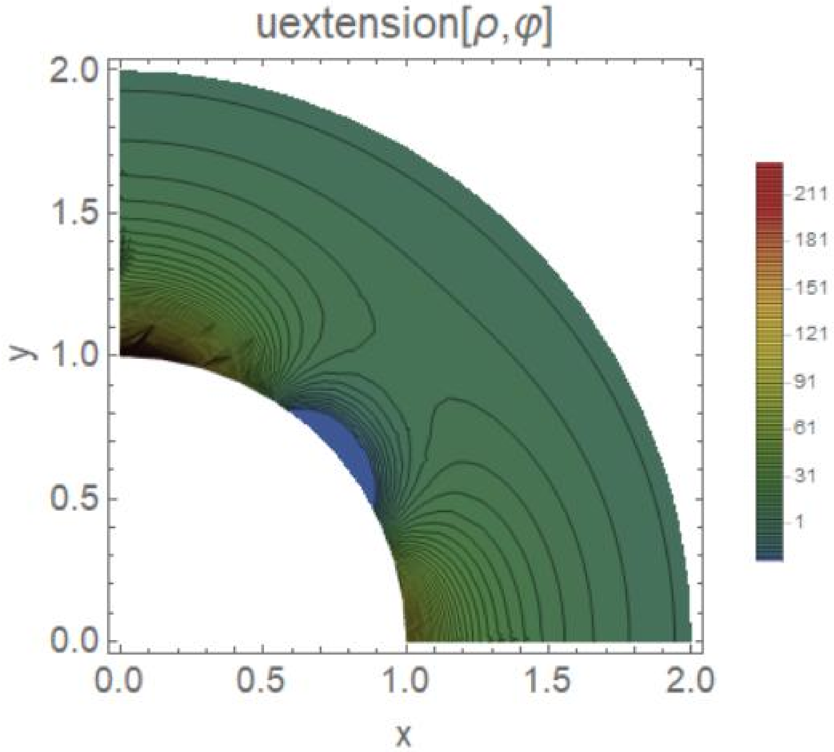

Fig 9. Temperature distribution of $u_{extension}(\rho,\varphi)$

6. Discussion

I checked all the plot by checking their BCs and all the Critical point, besides, I compared $T(\varphi)$ And $u(a,\varphi)$ to make sure they are the same.

Finding proper Temperature file is not easy, I start with assuming first several term of $u(\rho,\varphi)$ are counted, and besides of these terms, the other terms are equal to 0, this is a confining to the temperature file, which means the temperature file must orthogonality to other term, by matching term, we can assuming a structure of $T(\varphi)$,then we can use the constraints get a set of equation for coefficients of these terms and finally using these coefficients to determine the temperature profile.