An Elementary Study of Sandpile

Xu Han,2013301020084

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Abstract

I show that the certain sandpile dynamical systems naturally evolve into a critical state, with the distribution of avalanche sizes and lengths displaying a power law behavior. Furthermore, I check that this power law relation is invariant under kinds of disruption added inside the system.

1 Introduction

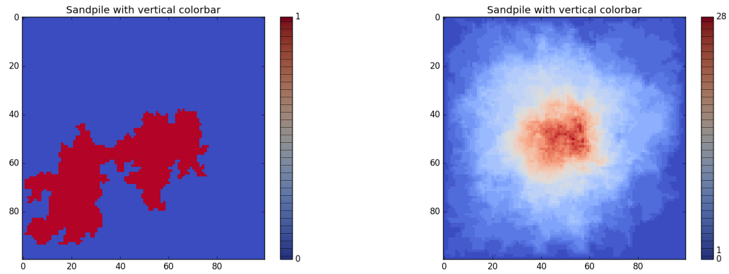
Cellular automata provide a fascinating class of dynamical systems based on very simple rules of evolution yet capable of displaying highly complex behavior. These include simplified models for many phenomena seen in nature. Among other things, they provide insight into self-organized criticality, wherein dissipative systems naturally drive themselves to a critical state with important phenomena occurring over a wide range of length and time scales.

I, in the first place, would love to do the simulation of "The Game of Life" since I read some article discussing it when I was a high school student. However, when you think about it, you may find this is not a good idea for the sack of practicing the cellular automata method. It still has interesting physics inside the life evolving system, so I will save it for future practice with the help of *Liu Wentao's MiniGUI*. In this article, I discussed the sandpile question which was originally studied by Bak, Tang, Wiesenfeld back in 1987[1]. They showed that the avalanche frequency f caused by adding a single sand can described by a power-law relation with the scale this avalanche influenced s and a distribution of system sizes leads to a distribution of fluctuation lifetimes. I do have some trouble with understanding concepts in their two articles, but I'll try to mimic their work in elementary level and show how fascinating the sandpile truly is.

2 Sandpile Quest

2.1 dynamic and boundary condition

For this, each site of a two dimensional lattice has a state represented by a positive integer $z(i, j)$, This integer can be thought of as representing the



(a) avalanche caused by one drop of sand in a 100×100 lattice

(b) avalanche accumulated in 5,000 add-evolve process in a 100×100 lattice

Figure 1: demonstration of avalanche. Right bar shows how many times avalanche comes up at a specific site on the left, colored according to the collapse frequency at this site

amount of sand at that location, or in another sense it represents the slope of the sandpile at that point. Neither of these analogies is fully accurate, the model has aspects of each. I hope its does capture many features of a sandpile's instability [3].
The dynamic is as follow, where $z_c = 3$:

$$z(i,j) \rightarrow z(i,j) + 1, z(i,j) \leq z_c \tag{1}$$

The equation above represent the process of adding a sand in some location while the one below shows how the avalanche happens.

$$z(i,j) \rightarrow z(i,j) - 4, \tag{2}$$
$$z(i \pm 1,j) \rightarrow z(i \pm 1,j) + 1, \tag{3}$$
$$z(i,j \pm 1) \rightarrow z(i,j \pm 1) + 1, z(i,j) > z_c \tag{4}$$

In this article I choose the closed boundary conditions. In this case, $z(0,y) = z(L,y) = z(x,0) = z(x,L) = 0$. The initial state condition could choose to be far from equilibrium or to be exact threshold value. In either condition, we must allow system to evolve to the state that all sites have value below z_c, before give a perturbation. I'll call the perturbation after evolve to stable subscribed above as an add-evolve calculation
One might expect the minimally stable state is every $z(i,j)$ all assume around the threshold value z_c, while Figure. 1(a) convinces us that it's not. When we increase one site by 1, this will render the surrounding sites unstable and the perturbation eventually propagates throughout the lattice just like researchers in [1] said. The minimally stable state is thus unstable with respect to small fluctuations and cannot represent an attracting fixed point for the dynamics.

To demonstrate the avalanche happened in our system, I did 5,000 times of add-evolve process in a square located in the center of a minimally

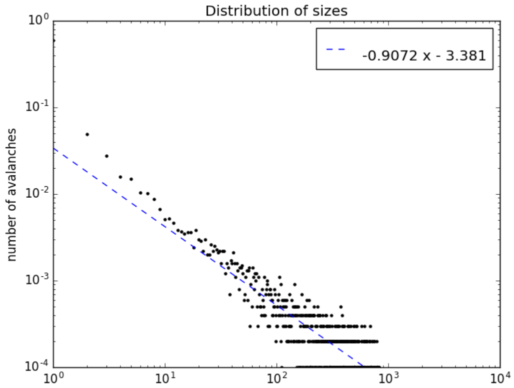


Figure 2: frequency of avalanche in a 100×100 lattice as their size based on just one "10,000 add-evolve simulation" from the initial state with cells with average-random values among $z = 2$ and $z = 3$. The line is fit to (5) which gives $\tau = -0.91$

stable lattice and record the avalanche occurred. Figure. 1(b) shows an analogy between our model and the real sandpile.

2.2 power law phenomenon

P.Bak et.cl. showed that avalanche frequency depends on their size by a power-law ,

$$f(s) = \frac{n(s)}{N_t} = s^\tau \tag{5}$$

where $f(s)$ is the frequency of avalanche of size s(the size is measured by unit size range). $n(s)$ is the number of avalanche of size s and N_t is the total amount of add-evolve calculation. I followed the works in [4] and recorded the evolve in 10,000 add-evolve calculation in a 100×100 lattice. As we can see in Figure. 2, the slope of fitted curve is close to the result $\tau \approx -1.0, \text{for} D = 2$ in the article by P. Bak et.cl., while fitting data from $s = 10^0$ to $s = 10^3$ shows a more sloppy($\tau = -1.14$) curve.

Amongst the interval from $s = 10^2$ to $s = 10^3$, we see the frequency dispersion along vertical direction which means that by sufficient simulation times, we may get an line cross the middle of that area and simultaneously adjust other results. Therefore, if you want more similar result to their work, you can use my program to calculate enough time and save the record in the same amount of files; then read the data in a circulation to do average and finally draw a perfect line.

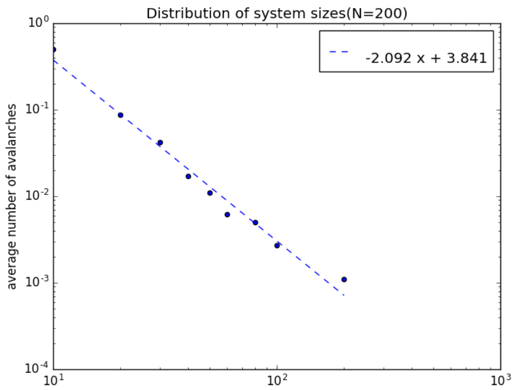


Figure 3: reverse of average avalanche happened in a changeable size lattice as their size based on "200 add-evolve simulation" from the minimally stable state

2.3 lattice size and special boundaries

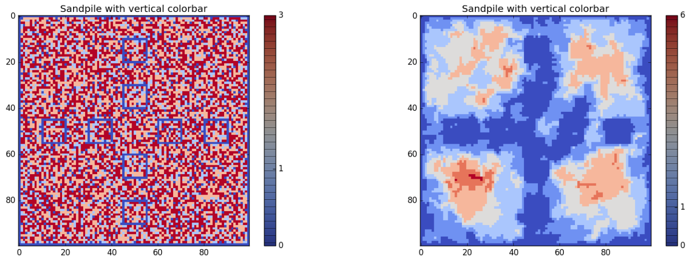
After got an initial scope of the power-law relation, I change the lattice size from 10×10 to 200×200 each step increased by 10. Recode the average number, N, of sites where have an avalanche led by an add-evolve action. Fit these number with its lattice size, L, by an power-law relation. In Figure. 3, we have the equation between N and L:

$$N \approx L^2 \equiv N_{site} \tag{6}$$

where N_{site} is the total sites number in our lattice. This is quite an reasonable result, because the total number of sites in the system is equal to L^2. Such a party pooper.
You could try out repeating the experiment on the previous section when the lattice size, L, changes. Then, fit curves to the experimental data to find out the power-law described previously is independent with the lattice size. This will need like one billion times of calculation?

The first one in two more experiment is to put an close boundary in our lattice where, in a way, the sand runoff. These boundaries in general, prevent the avalanche spreading or exaggerating. Thus, I supposed they both would lead the critical property shift. Increase the inside square from 10×10 to 80×80 each step increased by 10. Measure the average number N as the square size l. Fitted curve is shown in Figure. 4(a).
The last experiment is to put more 10×10 squares in our lattice. Measure the average avalanche number N as the number of squares in the lattice. Figure. 5 shows a demonstration and fitted result is in Figure. 4(b).

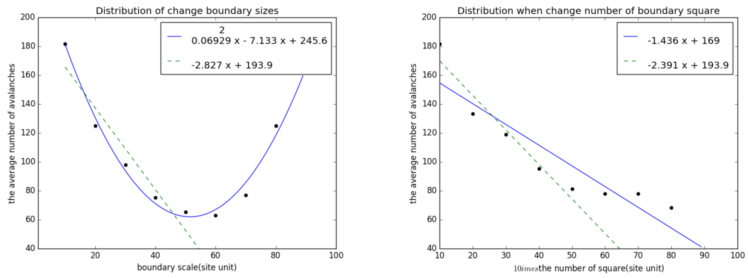
Meanwhile, I checked the power-law relation in this system when I chose the boundary structure witch had the most obvious impact on the avalanche scope according to Figure. 4. The result in Figure. 6 is quite astonish-



(a) initial state of sandpile

(b) the avalanche distribution after "400 add-evolve simulation"

Figure 4: demonstration of avalanche happened in a 100×100 size lattice with special boundary condition



(a) number of average avalanche as the single square boundary size

(b) number of average avalanche as the number of 10×10 square inside the lattice

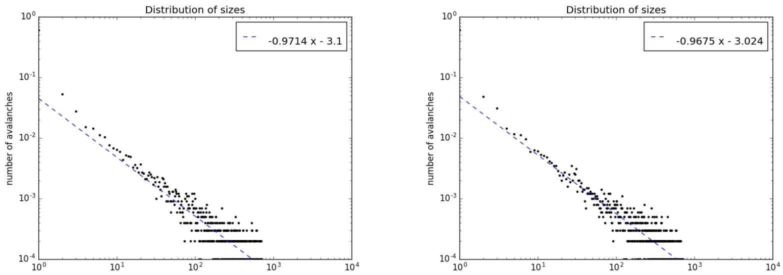
Figure 5: fitted curve of avalanche happened in a 100×100 size lattice with special boundary conditions

ing because it's clearly these two kind of boundaries only led a minimal change of the power law in (5).

3 Conclusion

In Section2.1, I presents the minimally stale state is an analogy with critical phenomenon which is unstable under a small disturbance. In Section2.2, we found an power-law relation similar to reference [1] and [4]. The numerical simulations showed that dissipative dynamical systems can evolve towards a self-organized critical state, with spatial power-law scaling behavior. I studied the influence on the critical phenomenon from the special boundary condition that was discussed in Section2.3.

From Figure. 5, relation between the average number of sites involved in one avalanche caused by one drop of sand as the boundary changes is



(a) on the condition that there has eight 10×10 squares in the lattice, the line is fit to (5) which gives $\tau \approx -0.97$

(b) on the condition that there has one 50×50 squares in the lattice, the line is fit to (5) which gives $\tau \approx -0.97$

Figure 6: frequency of avalanche in a 100×100 lattice as their size based on just one "10,000 add-evolve simulation" with special boundary conditions

obscure. Fortunately, the fitted curve in Figure. 6 shows that this system has an invariant quality when we change the system size or use boundaries to cutoff the avalanche's path. Even the sufficient amount of boundary would not disrupt the self-organized process in different levels.
Though I did not carry out the work as precise and comprehensive as P. Bak et.cl.'s, I did get a framework of program to do it. However, I missed the important part of showing comprehensively whether the boundaries cause the power-law changed.

References

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