Gauss's Method for Orbit Determination

Given three observations of an asteroid at times t_1, t_2, and t_3, we want to determine the geocentric position vector $\vec{r}_2$ of the object at time t_2. The method assumes:

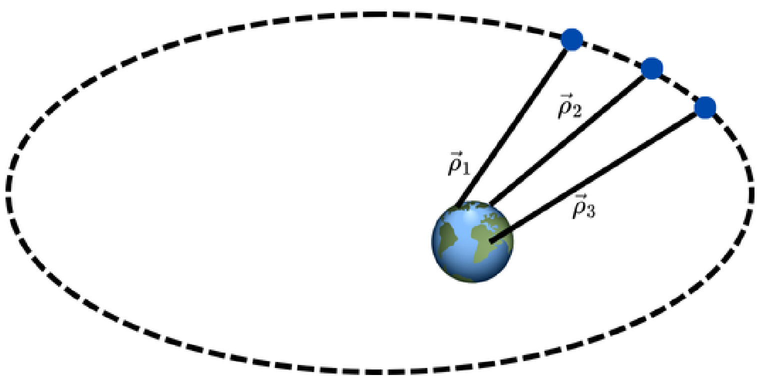

Figure 1: Asteroid observations from 3 different observatories, where the blue dots represent the asteroid position at the three different times.

- The observer is located on the Earth's surface at known position vectors $\vec{R}_1$, $\vec{R}_2$, $\vec{R}_3$.
- The observed line-of-sight unit vectors $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$ are known.
- The asteroid lies in a Keplerian orbit; and has, the three radius vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ that are coplanar.

We express the geocentric position of the asteroid at each time as:

$$\vec{r}_i = \vec{R}_i + \rho_i \hat{\rho}_i \quad \text{for } i = 1, 2, 3.$$

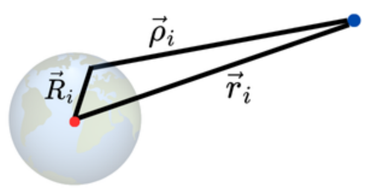

Figure 2: Vector addition.

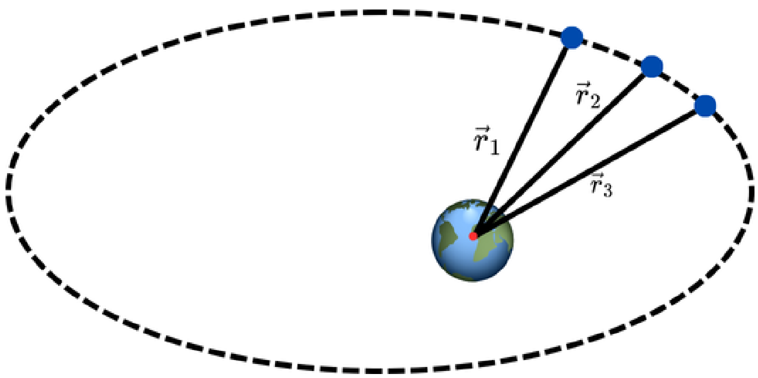

Figure 3: position of the asteroid with respect to the center of the earth.

Since they are coplanar vectors, we Assume linear dependence:

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = 0,$$

We solve for c_1 and c_3, when $c_2 = -1$, using the vector cross products:

$$c_1 = \frac{(\vec{r}_2 \times \vec{r}_3)}{(\vec{r}_1 \times \vec{r}_3)},$$
$$c_3 = \frac{(\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_3)}.$$

Using the f and g function we can define $\vec{r}_1 = f_1\vec{r}_2 + g_1\vec{v}_2$, and $\vec{r}_3 = f_3\vec{r}_2 + g_3\vec{v}_2$ using the second order approximation of :

$$f_i = 1 - \frac{\sigma}{2}(\Delta t_i)^2,$$
$$g_i = \Delta t_i - \frac{\sigma}{6}(\Delta t_i)^3, \qquad \text{where } \sigma = \frac{\mu}{r_2^3}.$$

Substitute these definitions into $c_1 = \frac{(\vec{r}_2 \times \vec{r}_3)}{(\vec{r}_1 \times \vec{r}_3)}$ and $c_3 = \frac{(\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_3)}$, to result with:

$$c_1 = \frac{g_3}{f_1 g_3 - f_3 g_1}, \quad c_3 = \frac{-g_1}{f_1 g_3 - f_3 g_1}.$$

When we substitute the f and g functions, we get:

$$c_1 = \underbrace{\frac{\Delta t_3}{\Delta t_3 - \Delta t_1}}_{a_1} + \underbrace{\frac{\Delta t_3\left((\Delta t_3 - \Delta t_1)^2 - (\Delta t_3)^2\right)}{6(\Delta t_3 - \Delta t_1)}}_{b_1}\sigma \quad \text{Thus } c_1 = a_1 + b_1\sigma$$

And

$$c_3 = \underbrace{\frac{-\Delta t_1}{\Delta t_3 - \Delta t_1}}_{a_3} + \underbrace{\frac{-\Delta t_1\left((\Delta t_3 - \Delta t_1)^2 - (\Delta t_1)^2\right)}{6(\Delta t_3 - \Delta t_1)}}_{b_3}\sigma \quad \text{Thus } c_3 = a_3 + b_3\sigma$$

Then using scalar products, we construct a system of equations:

$$\begin{bmatrix}\hat{\rho}_1 & \hat{\rho}_2 & \hat{\rho}_3\end{bmatrix}\begin{bmatrix}c_1\rho_1\\ c_2\rho_2\\ c_3\rho_3\end{bmatrix} = \begin{bmatrix}\vec{R}_1 & \vec{R}_2 & \vec{R}_3\end{bmatrix}\begin{bmatrix}-c_1\\ -c_2\\ -c_3\end{bmatrix}$$

Let A be the matrix of the unit vector for ρ_i and B the matrix of known observation vectors without C's included:

$$\begin{bmatrix}c_1\rho_1\\ c_2\rho_2\\ c_3\rho_3\end{bmatrix} = \underbrace{\left(\begin{bmatrix}\hat{\rho}_1 & \hat{\rho}_2 & \hat{\rho}_3\end{bmatrix}^{-1}\cdot\begin{bmatrix}\vec{R}_1 & \vec{R}_2 & \vec{R}_3\end{bmatrix}\right)}_{B}\begin{bmatrix}-c_1\\ -c_2\\ -c_3\end{bmatrix}.$$

From this matrix, and previous stated definitions of c_1 and c_3 we solve for ρ_2, assuming $c_2 = -1$.

$$|\rho_2| = \underbrace{B_{21}a_1 - B_{22} + B_{23}a_3}_{d_1} + \underbrace{(B_{21}b_1 + B_{23}b_3)}_{d_2}\sigma$$

Now we will define $\vec{r}_i$ with

$$\|\vec{r}_i\| = \|\vec{\rho}_i + \vec{R}_i\| = \sqrt{\|\rho_i\|^2 + \|\vec{R}_i\|^2 + 2\|\rho_i\|\hat{\rho}_i\cdot\vec{R}_i}$$

Making the appropiate substituions to this equation we result with the following eight degree equation:

$$(r_2)^8 = ((d_1)^2 + \|\vec{R}_2\|^2 + 2d_1\hat{\rho}_2\cdot\vec{R}_2)(r_2)^6 + 2(d_1d_2 + d_2\hat{\rho}_2\cdot\vec{R}_2)(r_2)^3\mu + (d_2)^2\mu^2$$

1 Appendix

1. Initial Setup and Vector Addition

With vector addition we know:

$$\vec{r}_i = \vec{R}_i + \vec{\rho}_i$$

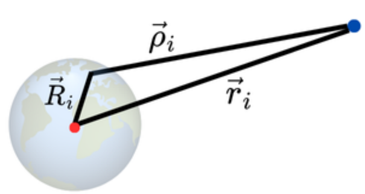

Figure 4: Vector addition.

for any time t_i.

Because of the nature of an orbit, we know our 3 known points are linearly dependent, so that

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = 0$$

This can also be proven by:

$$(\vec{r}_2 \times \vec{r}_3) \cdot \vec{r}_1 = 0$$

$$\vec{r}_1 \cdot (\vec{r}_2 \times \vec{r}_3) = 0$$

Thus $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ lie on the same plane.

Now we shall assume $c_2 = -1$ and we will solve for c_1 and c_3 as follows:

$$0 = (c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3) \times \vec{r}_3 = c_1(\vec{r}_1 \times \vec{r}_3) + c_2(\vec{r}_2 \times \vec{r}_3) + c_3(\vec{r}_3 \times \vec{r}_3)$$

$$= c_1(\vec{r}_1 \times \vec{r}_3) + c_2(\vec{r}_2 \times \vec{r}_3) + c_3 \cdot 0$$

$$\Rightarrow c_1(\vec{r}_1 \times \vec{r}_3) + c_2(\vec{r}_2 \times \vec{r}_3) = 0 \qquad \Rightarrow c_1(\vec{r}_1 \times \vec{r}_3) = -c_2(\vec{r}_2 \times \vec{r}_3)$$

$$c_1 = \frac{-c_2(\vec{r}_2 \times \vec{r}_3)}{(\vec{r}_1 \times \vec{r}_3)} = \frac{(\vec{r}_2 \times \vec{r}_3)}{(\vec{r}_1 \times \vec{r}_3)}$$

And for c_3:

$$0 = \vec{r}_1 \times (c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3) = c_1(\vec{r}_1 \times \vec{r}_1) + c_2(\vec{r}_1 \times \vec{r}_2) + c_3(\vec{r}_1 \times \vec{r}_3)$$

$$= c_1 \cdot 0 + c_2(\vec{r}_1 \times \vec{r}_2) + c_3(\vec{r}_1 \times \vec{r}_3)$$

$$\Rightarrow c_3(\vec{r}_1 \times \vec{r}_3) = -c_2(\vec{r}_1 \times \vec{r}_2)$$

$$c_3 = \frac{-c_2(\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_3)} = \frac{(\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_3)}$$

2. Integrating f and g functions

Now we know we can express any orbital radius vector using f and g functions (as a function of a vector and vector velocity of another vector of the same orbit). So we will write $\vec{r}_1$ and $\vec{r}_3$ in terms of f and g functions and vector $\vec{r}_2$ and $\vec{v}_2$:

$$\vec{r}_1 = f_1\vec{r}_2 + g_1\vec{v}_2 \qquad \vec{r}_3 = f_3\vec{r}_2 + g_3\vec{v}_2$$

Now using the previous equations we solved for c_1 and c_3, in terms of this f and g functions:

$$c_1 = \frac{\vec{r}_2 \times \vec{r}_3}{\vec{r}_1 \times \vec{r}_3} = \frac{\vec{r}_2 \times (f_3\vec{r}_2 + g_3\vec{v}_2)}{(f_1\vec{r}_2 + g_1\vec{v}_2) \times (f_3\vec{r}_2 + g_3\vec{v}_2)}$$

$$= \frac{f_3(\vec{r}_2 \times \vec{r}_2) + g_3(\vec{r}_2 \times \vec{v}_2)}{f_1f_3(\vec{r}_2 \times \vec{r}_2) + f_1g_3(\vec{r}_2 \times \vec{v}_2) + f_3g_1(\vec{v}_2 \times \vec{r}_2) + g_1g_3(\vec{v}_2 \times \vec{v}_2)} = \frac{g_3(\vec{r}_2 \times \vec{v}_2)}{f_1g_3(\vec{r}_2 \times \vec{v}_2) + f_3g_1(\vec{v}_2 \times \vec{r}_2)}$$

$$\frac{g_3(\vec{r}_2 \times \vec{v}_2)}{f_1g_3(\vec{r}_2 \times \vec{v}_2) - f_3g_1(\vec{r}_2 \times \vec{v}_2)} = \frac{g_3}{f_1g_3 - f_3g_1}$$

Now a similar process for c_3:

$$\begin{aligned}
c_3 &= \frac{(\vec{r}_1 \times \vec{r}_2)}{(\vec{r}_1 \times \vec{r}_3)} = \frac{(f_1\vec{r}_2 + g_1\vec{v}_2) \times \vec{r}_2}{(f_1\vec{r}_2 + g_1\vec{v}_2) \times (f_3\vec{r}_2 + g_3\vec{v}_2)} \\
&= \frac{f_1(\vec{r}_2 \times \vec{r}_2) + g_1(\vec{v}_2 \times \vec{r}_2)}{f_1f_3(\vec{r}_2 \times \vec{r}_2) + f_1g_3(\vec{r}_2 \times \vec{v}_2) + g_1f_3(\vec{v}_2 \times \vec{r}_2) + g_1g_3(\vec{v}_2 \times \vec{v}_2)} \\
&= \frac{g_1(\vec{v}_2 \times \vec{r}_2)}{f_1g_3(\vec{r}_2 \times \vec{v}_2) + f_3g_1(\vec{v}_2 \times \vec{r}_2)} \\
&= \frac{g_1(\vec{v}_2 \times \vec{r}_2)}{-f_1g_3(\vec{v}_2 \times \vec{r}_2) + f_3g_1(\vec{v}_2 \times \vec{r}_2)} = \frac{g_1}{f_3g_1 - f_1g_3}
\end{aligned}$$

Now we will define $\sigma = \frac{\mu}{r^3}$, where $\mu =$ gravitational parameter and

$$\Delta t_1 = t_1 - t_2 \qquad\qquad \Delta t_3 = t_3 - t_2$$

where t_i denotes the time of observation Now we will define the f and g functions as series using σ:

$$f_i = 1 - \frac{\sigma}{2}(\Delta t_i)^2 \dots \quad \text{and} \quad g_i = \Delta t_i - \frac{\sigma}{6}(\Delta t_i)^3 \dots$$

The purpose of this these f and g functions is as are initial guesses to help us estimate r_i not looking for precision. Therefore, this method targets small time differences, made for cases of successive observations that cover a small arc of the overall orbit.

Now we solve for c_1 and c_3 in terms of Δt_i and σ:

$$c_1 = \frac{g_3}{f_1g_3 - f_3g_1}$$

$$\begin{aligned}
f_1g_3 &= \left(1 - \frac{\sigma}{2}(\Delta t_1)^2\right)\left(\Delta t_3 - \frac{\sigma}{6}(\Delta t_3)^3\right) \\
&= \Delta t_3 - \frac{\sigma}{2}(\Delta t_1)^2\Delta t_3 - \frac{\sigma}{6}(\Delta t_3)^3 + \frac{\sigma^2}{12}(\Delta t_1)^2(\Delta t_3)^3
\end{aligned}$$

$$\sigma^2 = \left(\frac{\mu}{r^3}\right)^2 \quad \text{is a very small number} \Rightarrow \sigma^2 \approx 0$$

$$
\begin{aligned}
f_3 g_1 &= \left(1-\frac{\sigma}{2}(\Delta t_3)^2\right)\left(\Delta t_1-\frac{\sigma}{6}(\Delta t_1)^3\right) \\
&= \Delta t_1 - \frac{\sigma}{2}(\Delta t_3)^2\Delta t_1 - \frac{\sigma}{6}(\Delta t_1)^3 + \frac{\sigma^2}{12}(\Delta t_3)^2(\Delta t_1)^3
\end{aligned}
$$

$$
f_1 g_3 - f_3 g_1 = \Delta t_3 - \frac{\sigma}{2}(\Delta t_1)^2\Delta t_3 - \frac{\sigma}{6}(\Delta t_3)^3 - \Delta t_1 + \frac{\sigma}{2}(\Delta t_3)^2\Delta t_1 + \frac{\sigma}{6}(\Delta t_1)^3
$$

$$
= \Delta t_3 - \Delta t_1 - \frac{\sigma}{6}((\Delta t_3)^3 - (\Delta t_1)^3) + \frac{\sigma}{2}((\Delta t_3)^2\Delta t_1 - (\Delta t_1)^2\Delta t_3)
$$

$$
= \Delta t_3 - \Delta t_1 - \frac{\sigma}{6}(\Delta t_1)^3 + \frac{\sigma}{6}(\Delta t_3)^3 - \frac{\sigma}{2}(\Delta t_1)^2\Delta t_3 + \frac{\sigma}{2}(\Delta t_3)^2\Delta t_1
$$

$$
= (\Delta t_3 - \Delta t_1) - \frac{\sigma}{6}\left[(\Delta t_1)^3 - (\Delta t_3)^3\right] - \frac{\sigma}{2}\left[(\Delta t_1)^2\Delta t_3 - (\Delta t_3)^2\Delta t_1\right]
$$

$$
c_1 = \underbrace{(\Delta t_3 - \frac{\sigma}{6}(\Delta t_3)^3)}_{g_3} \cdot \underbrace{(\Delta t_3 - \Delta t_1 - \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^3)}_{f_1 g_3 - f_3 g_1}{}^{-1}
$$

simplify as:

$$
c_3 = \frac{\Delta t_3}{\Delta t_3 - \Delta t_1} \cdot (1 - \frac{\sigma}{6}(\Delta t_3)^3) \cdot \underbrace{(1 - \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^3)}_{\text{binomial expansion}}{}^{-1}
$$

Binomial expansion:

$$
(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad \text{(to second order)}
$$

$$
(1 - \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2)^{-1} = 1 - \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2(-1) + \frac{(-1)(1)}{2!}(-\frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2)^2 + \dots
$$

$$
= 1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2 + \dots
$$

$$
\therefore \quad c_1 = \frac{\Delta t_3}{\Delta t_3 - \Delta t_1}\left(1 - \frac{\sigma}{6}(\Delta t_3)^2\right)\left(1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2\right)
$$

With the binomial expansion included $(f_1 g_3 - f_3 g_1)^{-1}$ is:

$$
(f_1 g_3 - f_3 g_1)^{-1} = \frac{1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2}{\Delta t_3 - \Delta t_1}
$$

Now we will simplify c_1

$$c_1 = \frac{\Delta t_3}{\Delta t_3 - \Delta t_1}\left(1 - \frac{\sigma}{6}(\Delta t_3)^2\right)\left(1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2\right)$$

$$(1 - \frac{\sigma}{6}(\Delta t_3)^2)(1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2) = 1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2 - \frac{\sigma}{6}(\Delta t_3)^2 - \frac{\sigma^2}{36}(\Delta t_3)^2(\Delta t_3 - \Delta t_1)^2$$

$$= 1 + \frac{\sigma}{6}\left[(\Delta t_3 - \Delta t_1)^2 - (\Delta t_3)^2\right]$$

$$\Rightarrow c_1 = \frac{\Delta t_3}{\Delta t_3 - \Delta t_1}(1 + \frac{\sigma}{6}[(\Delta t_3 - \Delta t_1)^2 - (\Delta t_3)^2])$$

$$c_1 = \frac{\Delta t_3}{\Delta t_3 - \Delta t_1} + \frac{\Delta t_3\left((\Delta t_3 - \Delta t_1)^2 - (\Delta t_3)^2\right)\sigma}{6(\Delta t_3 - \Delta t_1)}$$

Similarly for c_3:

$$c_3 = \frac{g_1}{f_3 g_1 - f_1 g_3} = \frac{-g_1}{f_1 g_3 - f_3 g_1}$$

From solving c_1 we know:

$$(f_1 g_3 - f_3 g_1)^{-1} = \frac{1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2}{\Delta t_3 - \Delta t_1}$$

$$\text{and } g_1 = \Delta t_1 - \frac{\sigma}{6}(\Delta t_1)^3 = \Delta t_1\left(1 - \frac{\sigma}{6}(\Delta t_1)^2\right)$$

$$\Rightarrow c_3 = \frac{-g_1}{f_1 g_3 - f_3 g_1} = \frac{-\Delta t_1(1 - \frac{\sigma}{6}(\Delta t_1)^3)(1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2)}{\Delta t_3 - \Delta t_1}$$

$$(1 - \frac{\sigma}{6}(\Delta t_1)^2)\left(1 + \frac{\sigma}{6}(\Delta t_3 - \Delta t_1)^2\right) = 1 + \frac{\sigma}{6}\left((\Delta t_3 - \Delta t_1)^2 - (\Delta t_1)^2\right) - \frac{\sigma^2}{36}(\Delta t_1)^2(\Delta t_3 - \Delta t_1)^2$$

$$c_3 = \frac{-\Delta t_1}{\Delta t_3 - \Delta t_1}(1 + \frac{\sigma}{6}\left((\Delta t_3 - \Delta t_1)^2 - (\Delta t_1)^2\right))$$

$$c_3 = \frac{-\Delta t_1}{\Delta t_3 - \Delta t_1} - \frac{\Delta t_1((\Delta t_3 - \Delta t_1)^2 - (\Delta t_1)^2)\sigma}{6(\Delta t_3 - \Delta t_1)}$$

3. Solving for $\vec{r}_2$

Now we will look again at our original equation:

$$\vec{r_i} = \vec{\rho_i} + \vec{R_i}$$

$$c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3 = (c_1\vec{\rho_1} + c_2\vec{\rho_2} + c_3\vec{\rho_3}) + (c_1\vec{R}_1 + c_2\vec{R}_2 + c_3\vec{R}_3)$$

We know from theory that $0 = c_1\vec{r}_1 + c_2\vec{r}_2 + c_3\vec{r}_3$, thus we can say:

$$0 = (c_1\vec{\rho_1} + c_2\vec{\rho_2} + c_3\vec{\rho_3}) + (c_1\vec{R}_1 + c_2\vec{R}_2 + c_3\vec{R}_3)$$

$$(c_1\vec{\rho_1} + c_2\vec{\rho_2} + c_3\vec{\rho_3}) = -(c_1\vec{R}_1 + c_2\vec{R}_2 + c_3\vec{R}_3)$$

This as matrix can be written as

$$\underbrace{\begin{bmatrix}\hat{\rho}_1 & \hat{\rho}_2 & \hat{\rho}_3\end{bmatrix}}_{A \quad 3\times 3} \underbrace{\begin{bmatrix}c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3\end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix}\vec{R}_1 & \vec{R}_2 & \vec{R}_3\end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix}-c_1 \\ -c_2 \\ -c_3\end{bmatrix}}_{3\times 1}$$

$$\begin{bmatrix}c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3\end{bmatrix} = \underbrace{\left[A^{-1} \cdot \begin{bmatrix}\vec{R}_1 & \vec{R}_2 & \vec{R}_3\end{bmatrix}\right]}_{B} \begin{bmatrix}-c_1 \\ -c_2 \\ -c_3\end{bmatrix}$$

Now we can calculate $|\rho_2|$ with this matrices, under the assumption that $c_2 = -1$
We will say

$$c_1 = \underbrace{\frac{\Delta t_3}{\Delta t_3 - \Delta t_1}}_{a_1} + \underbrace{\frac{\Delta t_3\left((\Delta t_3 - \Delta t_1)^2 - (\Delta t_3)^2\right)}{6(\Delta t_3 - \Delta t_1)}}_{b_1}\sigma$$

$$\Rightarrow c_1 = a_1 + b_1\sigma$$

Then

$$c_3 = \underbrace{\frac{-\Delta t_1}{\Delta t_3 - \Delta t_1}}_{a_3} + \underbrace{\frac{-\Delta t_1\left((\Delta t_3 - \Delta t_1)^2 - (\Delta t_1)^2\right)}{6(\Delta t_3 - \Delta t_1)}}_{b_3}\sigma$$

$$\Rightarrow c_3 = a_3 + b_3\sigma$$

Looking at our matrix and with our known values, we can establish:

$$|\rho_2| = \underbrace{B_{21}a_1 - B_{22} + B_{23}a_3}_{d_1} + \underbrace{(B_{21}b_1 + B_{23}b_3)}_{d_2}\sigma$$

$$\text{Thus } |\rho_2| = d_1 + d_2\sigma$$

Now we will look at:

$$\|\vec{r}_i\| = \|\vec{\rho}_i + \vec{R}_i\| = \sqrt{\|\rho_i\|^2 + \|\vec{R}_i\|^2 + 2\|\rho_i\|\hat{\rho}_i \cdot \vec{R}_i}$$

Now we will use this formulas and our variables to solve for r_2

$$\|\vec{r}_2\|^2 = \|\rho_i\|^2 + \|\vec{R}_i\|^2 + 2\|\rho_i\|\hat{\rho}_i \cdot \vec{R}_i$$

$$(r_2)^2 = (d_1 + d_2\sigma)^2 + \|\vec{R}_2\|^2 + 2(d_1 + d_2\sigma)\hat{\rho}_2 \cdot \vec{R}_2$$

$$(d_1 + d_2\sigma)^2 = (d_1)^2 + 2d_1d_2\sigma + (d_2)^2\sigma^2$$

$$(r_2)^2 = (d_1)^2 + 2d_1d_2\sigma + (d_2)^2\sigma^2 + \|\vec{R}_2\|^2 + 2d_1\hat{\rho}_2 \cdot \vec{R}_2 + 2d_2\sigma\hat{\rho}_2 \cdot \vec{r}_{0/G}(t_2)$$

To simplify the math we will divide this result into 3 variables

$$(r_2)^2 = \underbrace{(d_1)^2 + \|\vec{R}_2\|^2 + 2d_1\hat{\rho}_2 \cdot \vec{R}_2}_{e} + \underbrace{(2d_1d_2 + 2d_2\hat{\rho}_2 \cdot \vec{R}_2)}_{h}\sigma + \underbrace{(d_2)^2}_{k}\sigma^2$$

so that:

$$\Rightarrow (r_2)^2 = e + h\sigma + k\sigma^2 \qquad \text{We know: } \sigma = \frac{\mu}{(r_2)^3}$$

$$\Rightarrow r_2^2 = e + h\frac{\mu}{(r_2)^3} + k\frac{\mu^2}{(r_2)^6}$$

$$\text{Multiplying by } (r_2)^6 \text{ we get: } (r_2)^8 = e(r_2)^6 + h(\mu)(r_2)^3 + k\mu^2$$

substituting back e, h and k we get:

$$(r_2)^8 = ((d_1)^2 + \|\vec{R}_2\|^2 + 2d_1\hat{\rho}_2 \cdot \vec{R}_2)(r_2)^6 + 2(d_1d_2 + d_2\hat{\rho}_2 \cdot \vec{R}_2)(r_2)^3\mu + (d_2)^2\mu^2$$