

CS726, Fall 2016

Homework 1 (posted 9/9/16; due Friday 9/23/16 at start of class)

Please submit your answers in the order listed below.

All Exercise numbers refer to the course text, *Numerical Optimization* (second edition, 2006).

Some of these questions require knowledge of Appendix A of *Numerical Optimization*, and the other background material posted on piazza.

1. Say which of the following sets are polyhedra. If they are polyhedral, express them in the form $\{x \mid Ax \geq b, Fx = g\}$.
 - (a) $S = \{x \in \mathbb{R}^n \mid x \geq 0, \sum_{i=1}^n x_i = 1, \sum_{i=1}^n a_i x_i = b, \sum_{i=1}^n a_i^2 x_i = c\}$, where $a_1, a_2, \dots, a_n, b, c \in \mathbb{R}$.
 - (b) $S = \{x \in \mathbb{R}^n \mid x \geq 0, x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$.
 - (c) $S = \{x \in \mathbb{R}^n \mid x \geq 0, x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_1 = 1\}$.
2. Exercise 2.6: Prove that all isolated minima are strict. (Hint: One way to do this is to prove that “not strict” \Rightarrow “not isolated”.)
3.
 - (a) Give an example of a matrix that is *not* positive definite despite having all positive entries.
 - (b) If A is a positive definite matrix, must its diagonal elements all be positive? Explain.
4. Exercise 2.1.
5. Exercise 2.9.
6. Exercise 2.10.
7. For each value of the scalar β , find the set of all stationary points (that is, the points x for which $\nabla f(x) = 0$) of the following function:

$$f(x) = x_1^2 + x_2^2 + \beta x_1 x_2 + x_1 + 2x_2.$$

Which of these points is a global minimizer?

8. Let f be a twice continuously differentiable function whose domain is an open set Ω . Prove that f is convex if and only if Ω is convex and $\nabla^2 f(x) \succeq 0$ for all $x \in \Omega$. Hint: The following characterization of convexity of a smooth convex function may be helpful: $f(y) \geq f(x) + \nabla f(x)^T(y - x)$ for all $x, y \in \Omega$. Also Taylor's theorem.
9. For the following functions of two variables, justify your answers fully using optimality conditions.
 - (a) Show that $f(x_1, x_2) := (x_1^2 - 4)^2 + x_2^2$ has two global minima and one saddle point.
 - (b) Find all local minima of $f(x_1, x_2) = \frac{1}{2}x_1^2 + x_1 \cos x_2$.
 - (c) Show that $f(x_1, x_2) := (x_2 - x_1^2)^2 - x_1^2$ has only one stationary point, which is a saddle point.