Modeling Friction and Contact in Chrono

Theoretical Background

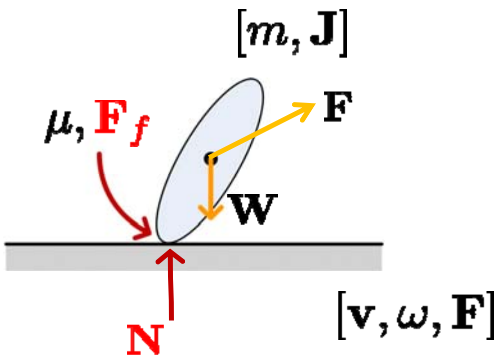
Things Covered

- Friction and contact, understanding the problem at hand
- The penalty approach
- The complementarity approach

Mass × Acceleration = Force

Mass × Acceleration = Force

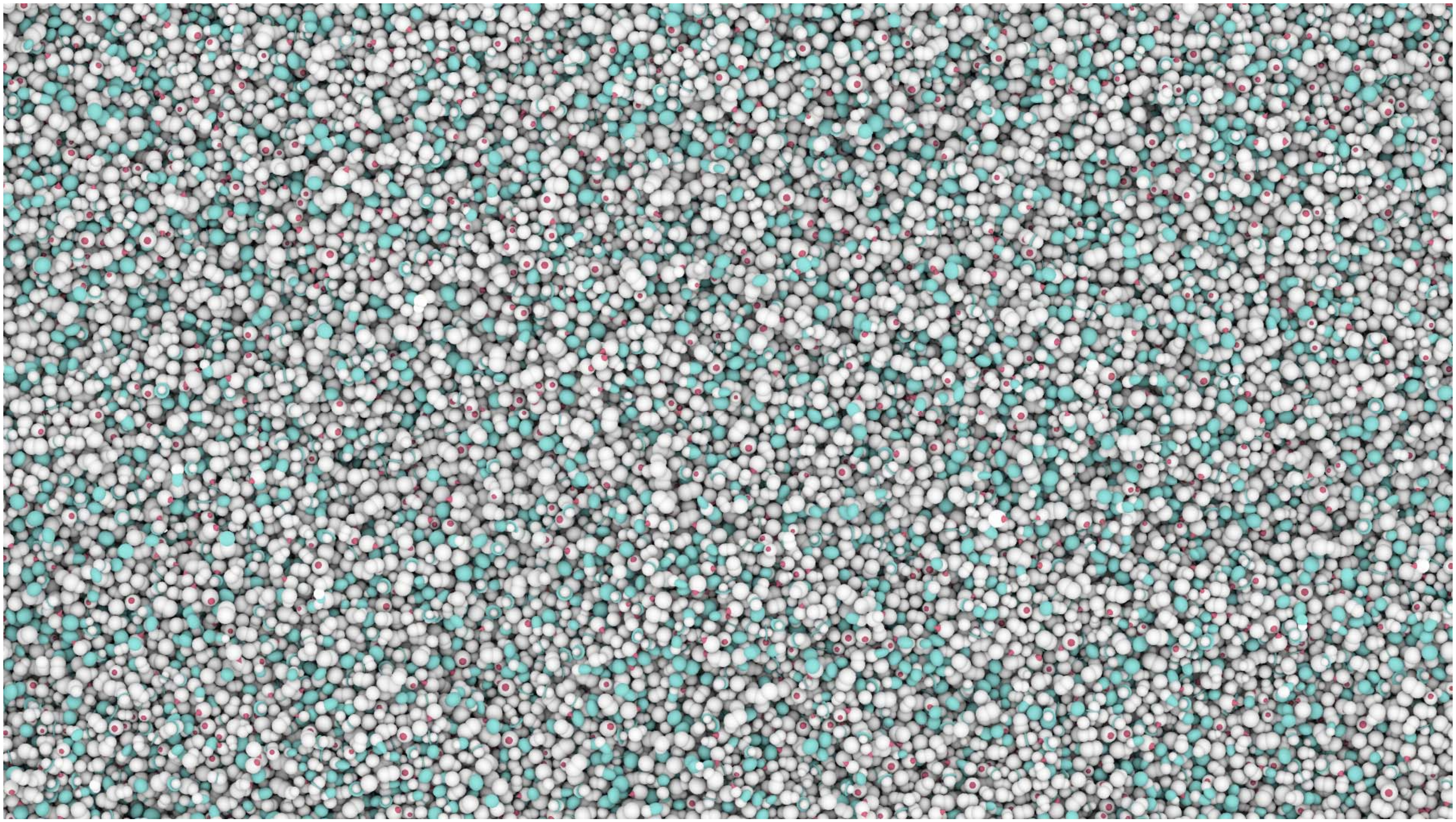
- Coulomb friction coefficient - μ



$$m\dot{\mathbf{v}} = \mathbf{W} + \mathbf{F} + \mathbf{F}_f + \mathbf{N}$$

$$||\mathbf{F}_f|| \leq \mu||\mathbf{N}||$$

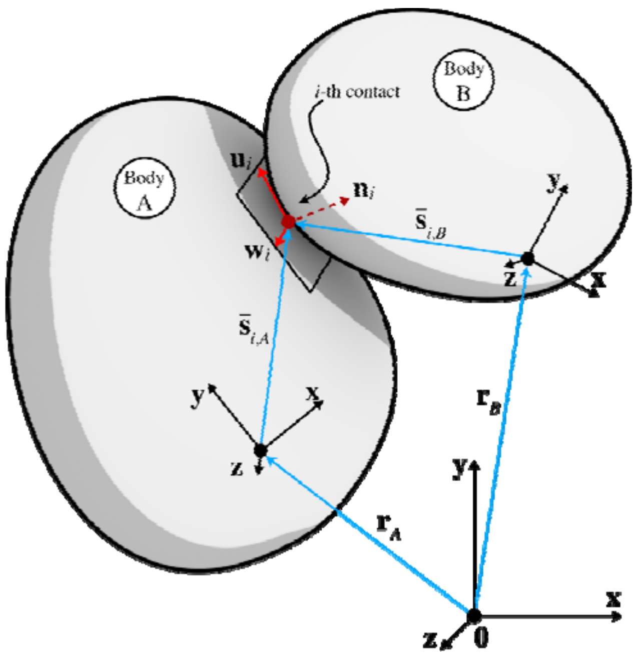
Reflect on this: friction force can assume a bunch of values (as long as they're smaller than $\mu \times \mathrm{N}$ though)



Additive Manufacturing (3D SLS Printing)

Courtesy of Professor Tim Osswald, Polymer Engineering Center, UW-Madison

Two main approaches: penalty & complementarity



Problem	Computational many-body dynamics Handling frictional contact	
Modelling approach	Penalty-based approach	Complementarity approach
Numerical techniques	Collision detection	Optimization techniques

General Comments, Penalty Approach

- Approach commonly used in handling granular material
  - Called “Discrete Element Method”

- The “Penalty” approach works well for sphere-to-sphere and sphere-to-plane scenarios
  - Deformable body mechanics used to characterize what happens under these scenarios
  - Standard reference: K. L. Johnson, Contact Mechanics, University Press, Cambridge, 1987.

- Methodology subsequently grafted to general dynamics problem of rigid bodies – arbitrary geometry
  - When they collide, a fictitious spring-damper element is placed between the two bodies
    - Sometimes spring & damping coefficient based on continuum theory mentioned above
    - Sometimes values are guessed (calibration) based on experimental data

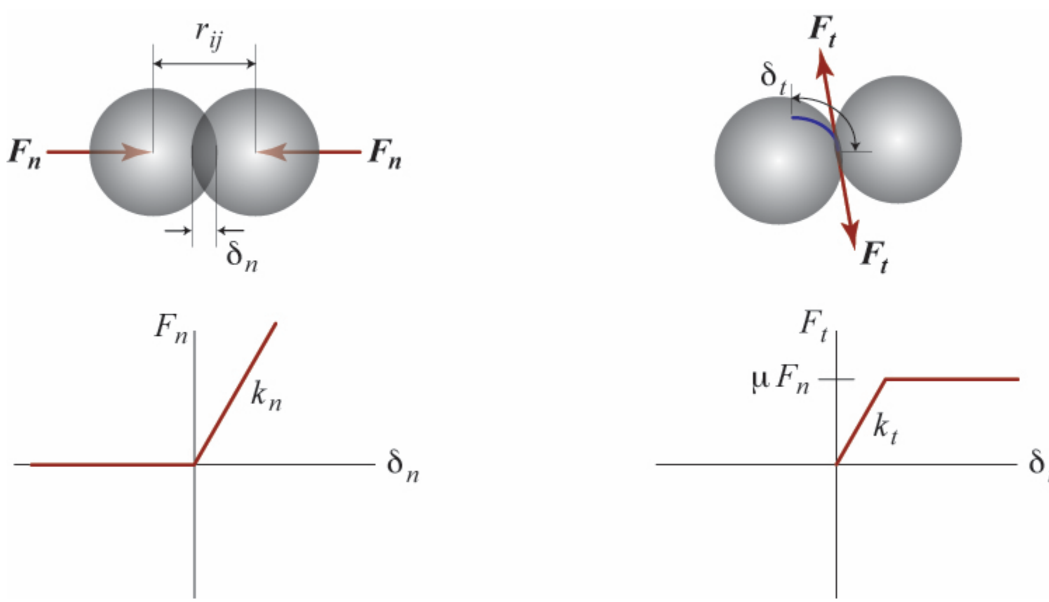
The Penalty Method, Taxonomy

- Depending on the normal relative velocity between bodies that experience a collision and their material properties, if there is no relative angular velocity, the collision is
  - Elastic, if the contact induced deformation is reversible and independent of displacement rate
  - Viscoelastic, if the contact induced deformation is irreversible, but the deformation is dependent on the displacement rate
  - Plastic, if collision leaves an involved body permanently deformed but the deformation of body is independent of the displacement rate
  - Viscoplastic, if impact is irreversible and similar to the viscoelastic contact but deformation depends on the displacement rate

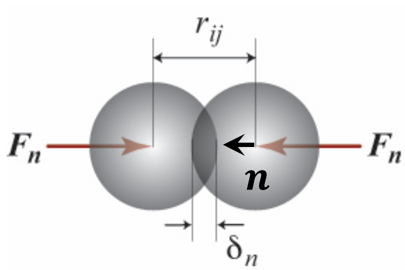
- According to the dependency of the normal force on the overlap and the displacement rate, the force schemes can be subdivided into
  - Continuous potential models (like Lennard-Jones, for instance)
  - Linear viscoelastic models (simple, used extensively, what we use here)
  - Non-linear viscoelastic models
  - Hysteretic models (see papers of L. Vu-Quoc, in “DEM Further Reading” slide)

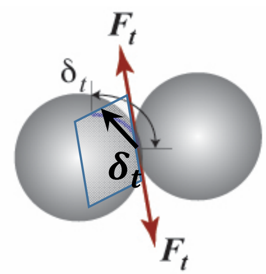
The Penalty Method in Chrono, Nuts and Bolts

- Method relies on a *record (history)* of tangential displacement $\boldsymbol{\delta_t}$ to model static friction (see figure at right)



The Penalty Method in Chrono, Nuts and Bolts





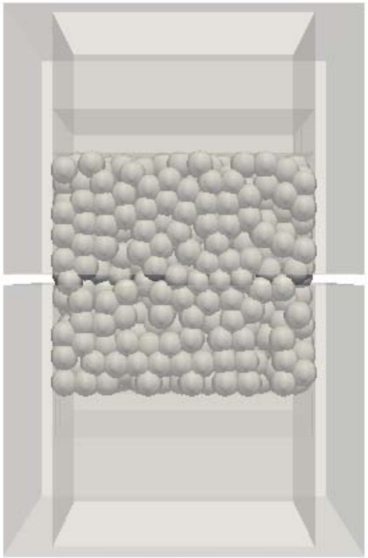
Visualize this δ_t as creep.

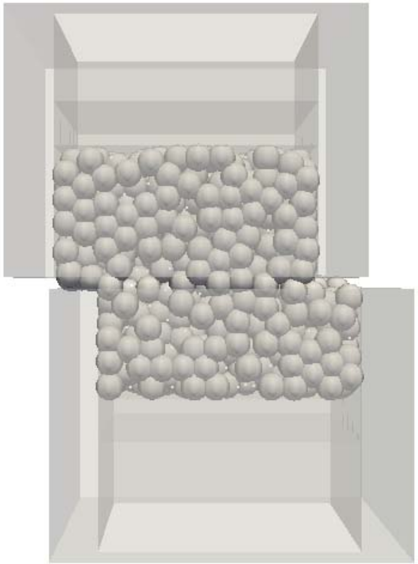
$$\boldsymbol{F_n} = f\left(\frac{\delta_n}{D_{\text{eff}}}\right)(k_n \delta_n \boldsymbol{n} - \gamma_n m_{\text{eff}} \boldsymbol{v_n}) \qquad \boldsymbol{F_t} = f\left(\frac{\delta_n}{D_{\text{eff}}}\right)(-k_t \boldsymbol{\delta_t} - \gamma_t m_{\text{eff}} \boldsymbol{v_t})$$

If $|\boldsymbol{F_t}| > \mu|\boldsymbol{F_n}|$ then scale $|\boldsymbol{\delta_t}|$ so that $|\boldsymbol{F_t}| = \mu|\boldsymbol{F_n}|$

Direct Shear Analysis via Granular Dynamics
[using LAMMPS/LIGGGHTS and Chrono]

- 1800 uniform spheres randomly packed
- Particle Diameter: D = 5 mm
- Shear Speed: 1 mm/s
- Inter-Particle Coulomb Friction Coefficient: μ = 0.5 (Quartz on Quartz)
- Void Ratio (dense packing): e = 0.4





[J. Fleischmann]→

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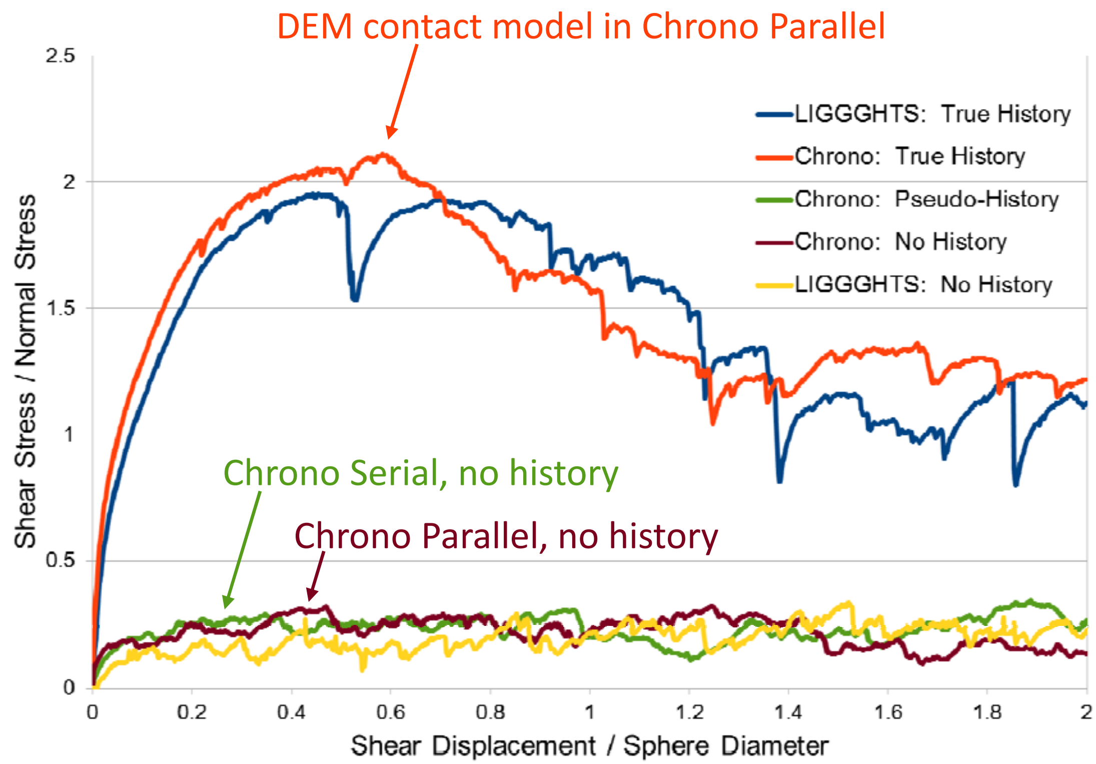
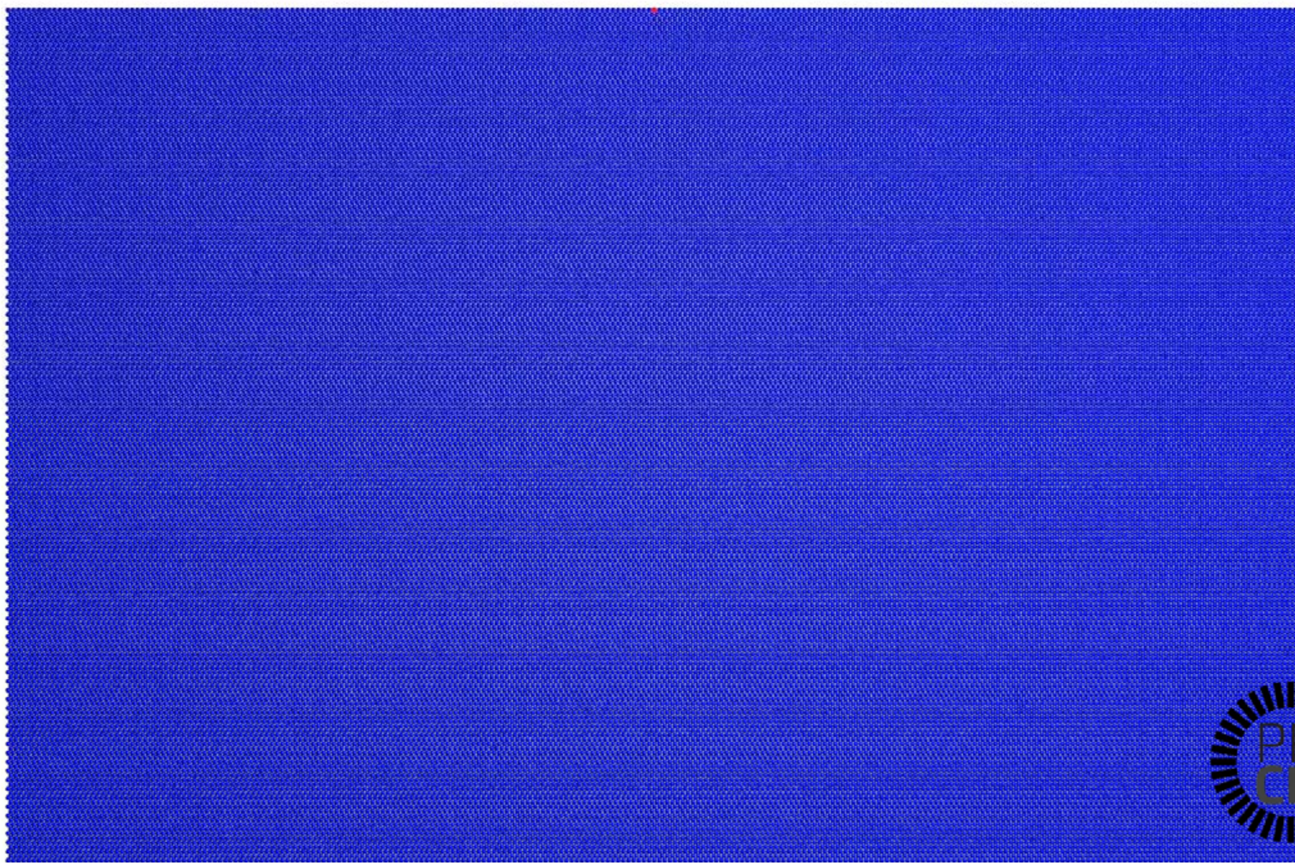
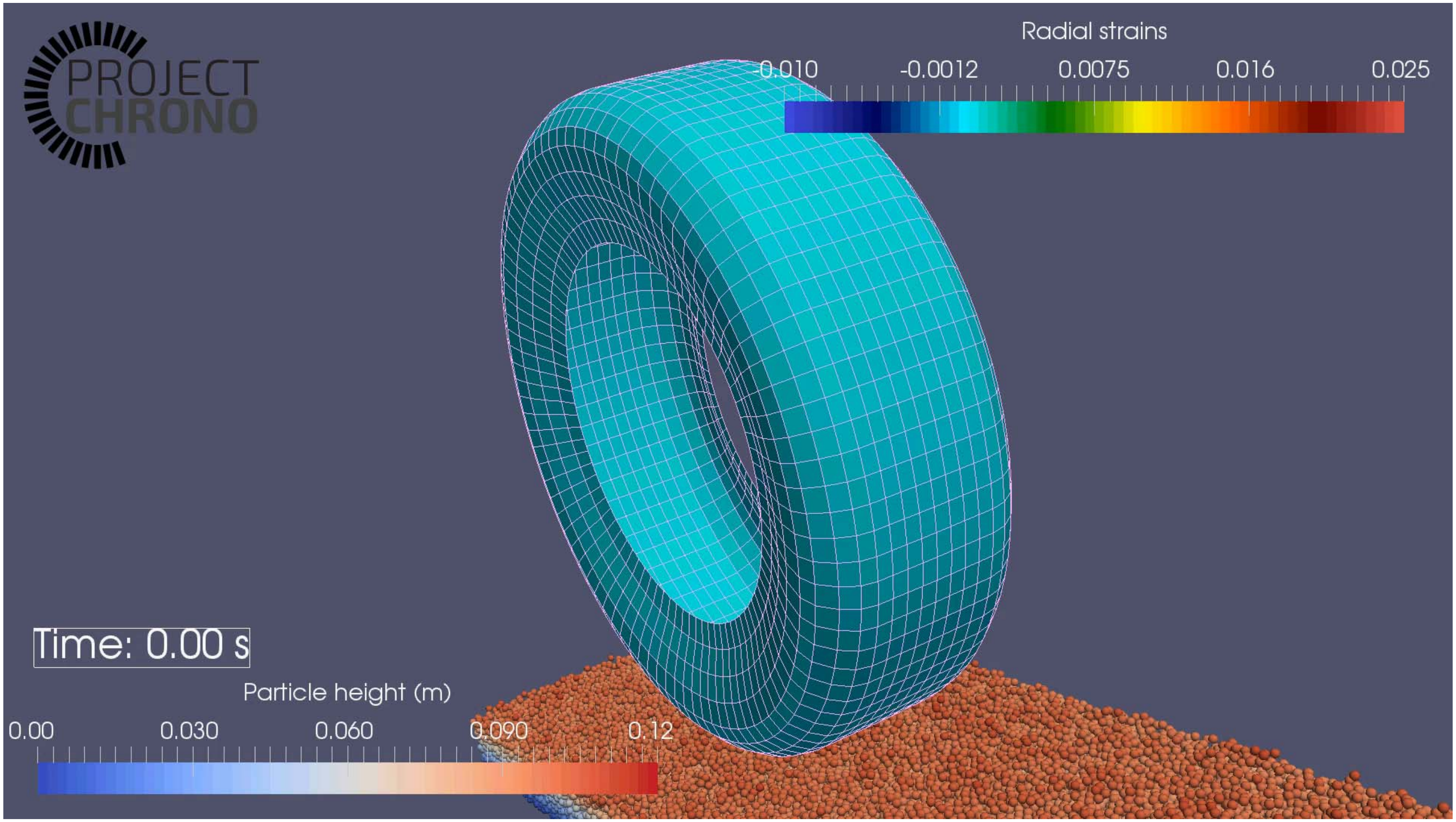


[J. Fleischmann]→

Wave propagation in ordered granular material



[Arman]→



[Antonio Recuero]⟶

Penalty Method – the Pros

- Backed by large body of literature and numerous validation studies
- No increase in the size of the problem
  - This is unlike the "complementarity" approach, discussed next
- Can accommodate shock wave propagation
  - Can't do w/ "complementarity" approach since it's a pure "rigid body" solution
- Easy to implement
  - Entire numerical solution decoupled
    - Easy to scale up to large problems
    - Parallel-computing friendly – run in parallel on per contact basis
      - Memory communication intensive

Penalty Method – Cons

1. Numerical stability requires small integration time steps
   - Long simulation times

2. Choice of integration time step strongly influences results

3. Sensitive wrt information provided by the collision detection engine

4. There is some hand-waving when it comes to arbitrary shapes and the fact that the friction force is a multi-valued function

DEM, Further Reading

[1] D. Ertas, G. Grest, T. Halsey, D. Levine and L. Silbert, *Gravity-driven dense granular flows*, EPL (Europhysics Letters), 56 (2001), pp. 214-220.

[2] H. Kruggel-Emden, E. Simsek, S. Rickelt, S. Wirtz and V. Scherer, *Review and extension of normal force models for the Discrete Element Method*, Powder Technology, 171 (2007), pp. 157-173.

[3] H. Kruggel-Emden, S. Wirtz and V. Scherer, *A study on tangential force laws applicable to the discrete element method (DEM) for materials with viscoelastic or plastic behavior*, Chemical Engineering Science (2007).

[4] D. C. Rapaport, *Radial and axial segregation of granular matter in a rotating cylinder: A simulation study*, Physical Review E, 75 (2007), pp. 031301.

[5] L. Silbert, D. Ertas, G. Grest, T. Halsey, D. Levine and S. Plimpton, *Granular flow down an inclined plane: Bagnold scaling and rheology*, Physical Review E, 64 (2001), pp. 51302.

[6] L. Vu-Quoc, L. Lesburg and X. Zhang, *An accurate tangential force–displacement model for granular-flow simulations: Contacting spheres with plastic deformation, force-driven formulation*, Journal of Computational Physics, 196 (2004), pp. 298-326.

[7] L. Vu-Quoc, X. Zhang and L. Lesburg, *A normal force-displacement model for contacting spheres accounting for plastic deformation: force-driven formulation*, Journal of Applied Mechanics, 67 (2000), pp. 363.

The "Complementarity" Approach
aka
Differential Variational Inequality (DVI) Method

Two Shapes, and the Distance [Gap Function]

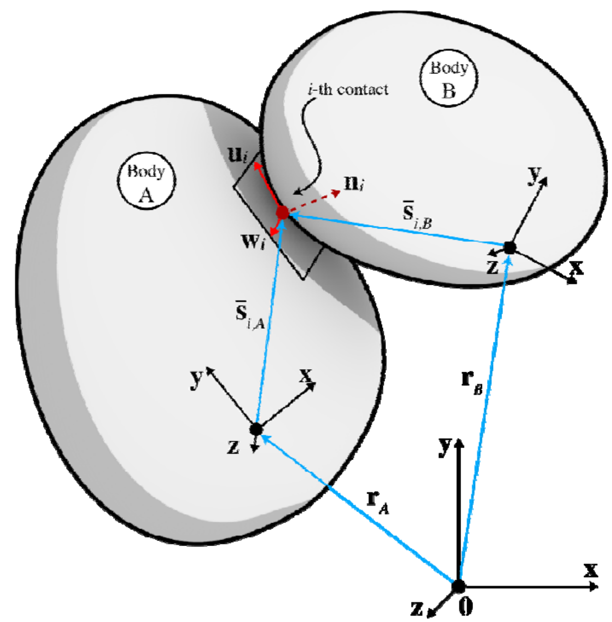
- Notation: ∂A represents set of points making up the boundary of body A
- Shape body A: collection of points S with $\mathbf{r}_A^S = \mathbf{r}_A + \mathbf{A}_A \bar{\mathbf{s}}_A^S$, $\quad \bar{\mathbf{s}}_A^S \in \partial A$
- Shape body B: collection of points S with $\mathbf{r}_B^S = \mathbf{r}_B + \mathbf{A}_B \bar{\mathbf{s}}_B^S$, $\quad \bar{\mathbf{s}}_B^S \in \partial B$

- Signed distance function in a given configuration $\boldsymbol{q}_A$ and $\boldsymbol{q}_B$

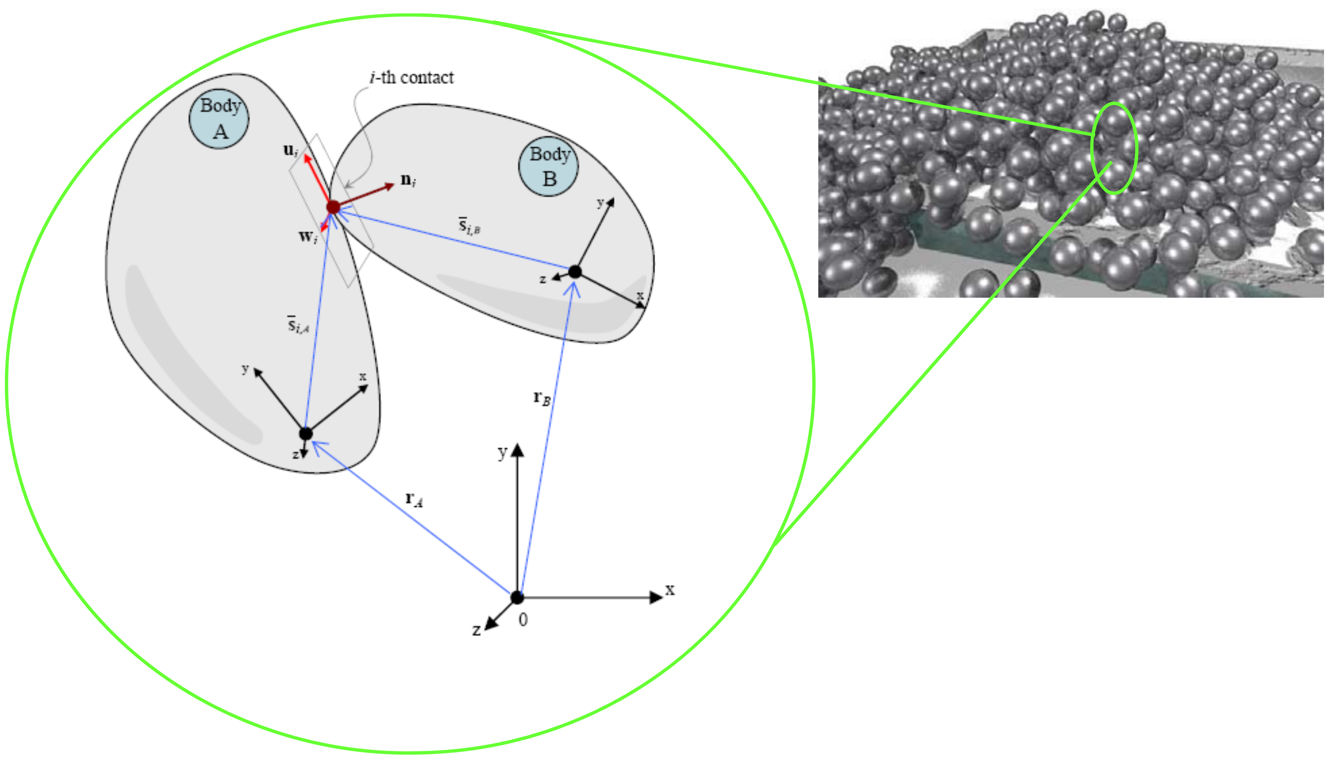
$$\Phi(\mathbf{q}_A(t), \mathbf{q}_B(t)) \equiv \min_{\bar{\mathbf{s}}_A^S \in \partial A,\, \bar{\mathbf{s}}_B^S \in \partial B} ||\mathbf{r}_A^S - \mathbf{r}_B^S||_2$$

- Contact when distance function is zero

$$\Phi(\mathbf{q}_A(t^\star), \mathbf{q}_B(t^\star)) = 0$$

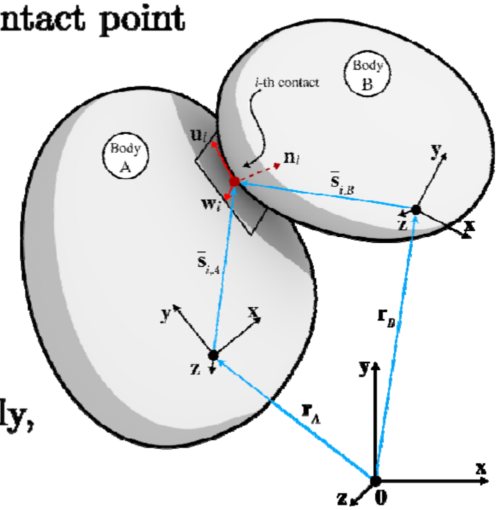


Body A – Body B Contact Scenario



Defining the Normal and Tangential Forces

- When a contact occurs: point of contact and local reference frame identified. Latter defined as follows:
  - $\mathbf{u}_i$ and $\mathbf{w}_i$ are two mutually perpendicular unit vectors in the tangent plan at the contact point
  - Unit vector $\mathbf{n}_i$ defines the normal direction in the local reference frame



- A normal force appears along the direction normal to the plane of contact
  - Magnitude of the force is $\widehat{\gamma}_{i,n}$. Specifically,

$$\mathbf{F}_{i,N} = \widehat{\gamma}_{i,n}\mathbf{n}_i$$

- A friction force appears in the tangent plane
  - Has two components along the axes $\mathbf{u}_i$ and $\mathbf{w}_i$: $\widehat{\gamma}_{i,u}$ and $\widehat{\gamma}_{i,w}$, respectively. Specifically,

$$\mathbf{F}_{i,T} = \widehat{\gamma}_{i,u}\mathbf{u}_i + \widehat{\gamma}_{i,w}\mathbf{w}_i$$

- NOTE: The point of contact, $\mathbf{n}_i$, $\mathbf{u}_i$, and $\mathbf{w}_i$ are obtained at the end of the collision detection task, which is executed at the beginning of each time step

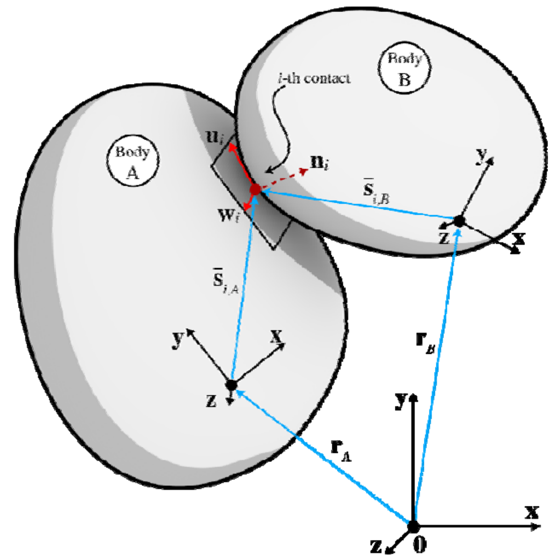
DVI-Based Methods: The Contact Model

- A contact is modeled by one inequality constraints, which states that either the distance between two bodies is greater than zero $\Phi_i(\mathbf{q}) > 0$, in which case the normal force is zero $\hat{\gamma}_{i,n} = 0$, or vice-versa; i.e., if the distance is zero, the contact force is nonzero.
  - Condition above captured in the following complementarity condition:

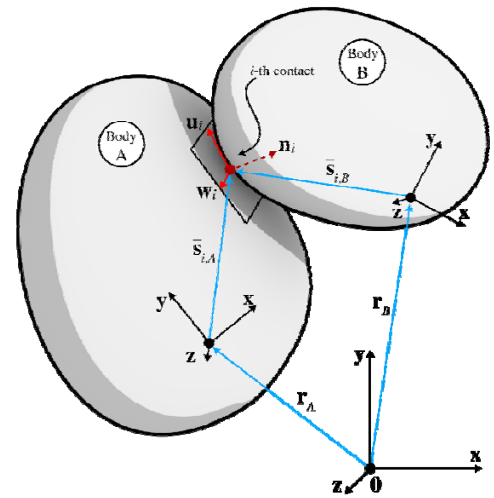
$$\widehat{\gamma}_{i,n} \geq 0, \quad \Phi_i(\mathbf{q}) \geq 0, \quad \Phi_i(\mathbf{q})\widehat{\gamma}_{i,n} = 0,$$

  - Another way to state the complementarity condition:

$$0 \leq \widehat{\gamma}_{i,n} \quad \perp \quad \Phi_i(\mathbf{q}) \geq 0$$



DVI-Based Methods: The Friction Model



- The friction model considered is Coulomb's:

$$\mu_i \widehat{\gamma}_{i,n} \geq \sqrt{\widehat{\gamma}_{i,u}^2 + \widehat{\gamma}_{i,w}^2}$$

$$\mathbf{F}_{i,T}^T \cdot \mathbf{v}_{i,T} = -\|\mathbf{F}_{i,T}\| \ \|\mathbf{v}_{i,T}\|$$

$$\|\mathbf{v}_{i,T}\| \left(\mu_i \widehat{\gamma}_{i,n} - \sqrt{\widehat{\gamma}_{i,u}^2 + \widehat{\gamma}_{i,w}^2}\right) = 0$$

  - First condition: friction force is within the friction cone
  - Second condition: friction force and tangential velocity between two bodies at point of contact are collinear and of opposite direction
  - The third condition captures the stick-slip condition. If the velocity is greater than zero, it means that the friction force saturated; i.e., $\mu_i \widehat{\gamma}_{i,n} = \sqrt{\widehat{\gamma}_{i,u}^2 + \widehat{\gamma}_{i,w}^2}$; this is the sliding scenario. Conversely, if the bodies stick to each other, then the relative tangential velocity is zero, $\mathbf{v}_{i,T} = \mathbf{0}_3$, and the friction force is not saturated $\mu_i \widehat{\gamma}_{i,n} > \sqrt{\widehat{\gamma}_{i,u}^2 + \widehat{\gamma}_{i,w}^2}$.

Coulomb's Model Posed as the Solution of an Optimization Problem

- Assume that $\widehat{\gamma}_{i,n}$ and $\mathbf{v}_{i,T}$ are given and you pose the following optimization problem in variables x and y:
  - Minimize the function $\mathbf{v}_{i,T}^T (x\mathbf{u}_i + y\mathbf{w}_i)$ subject to the constraint $\sqrt{x^2 + y^2} \leq \mu_i \widehat{\gamma}_{i,n}$

- If you pose the first order Karush-Kuhn-Tucker optimality conditions for this optimization problem you end up precisely with the set of three conditions that define the Coulomb friction model

- It follows that there is an interplay between $\widehat{\gamma}_{i,n}$, $\widehat{\gamma}_{i,u}$, $\widehat{\gamma}_{i,w}$, and $\mathbf{v}_{i,T}$. Using math notation

$$(\widehat{\gamma}_{i,u}, \widehat{\gamma}_{i,w}) = \underset{\sqrt{x^2+y^2} \leq \mu_i \widehat{\gamma}_{i,n}}{\operatorname{argmin}} \quad \mathbf{v}_{i,T}^T (x\mathbf{u}_i + y\mathbf{w}_i) \,.$$

The DVI Problem: The EOM, in Fine-Granularity Form

- Time evolution of the dynamical system is the solution of the following DVI problem:

$$
\begin{aligned}
B = 1,\ldots,nb \quad &: \quad m_B \ddot{\mathbf{r}}_B = \sum_{i\in\mathcal{B}(B)} \left[\mathbf{\Psi}^{(i)}_{\mathbf{r}_B}\right]^T \widehat{\gamma}_{i,b} + \mathbf{f}_B(t,\mathbf{q},\mathbf{v}) + \sum_{i\in\mathcal{A}(B)} \left(\widehat{\gamma}_{i,n}\,\mathbf{n}_i + \widehat{\gamma}_{i,u}\,\mathbf{u}_i + \widehat{\gamma}_{i,w}\,\mathbf{w}_i\right) \\
B = 1,\ldots,nb \quad &: \quad \bar{\mathbf{J}}_B \dot{\bar{\omega}}_B = \sum_{i\in\mathcal{B}(B)} \bar{\mathbf{\Pi}}^T_B(\mathbf{\Psi}^{(i)}) \widehat{\gamma}_{i,b} + \tau_B(t,\mathbf{q},\mathbf{v}) + \sum_{i\in\mathcal{A}(B)} \tilde{\bar{\mathbf{s}}}_{i,B} \mathbf{A}^T_B \left(\widehat{\gamma}_{i,n}\,\mathbf{n}_i + \widehat{\gamma}_{i,u}\,\mathbf{u}_i + \widehat{\gamma}_{i,w}\,\mathbf{w}_i\right) \\
B = 1,\ldots,nb \quad &: \quad \dot{\mathbf{p}}_B = \tfrac{1}{2}\mathbf{G}^T(\mathbf{p}_B)\bar{\omega}_B \\
i \in \mathcal{B} \quad &: \quad \mathbf{\Psi}_i(\mathbf{q},t) = 0 \\
i \in \mathcal{A} \quad &: \quad 0 \le \widehat{\gamma}_{i,n} \perp \Phi_i(\mathbf{q}) \ge 0, \\
i \in \mathcal{A} \quad &: \quad (\widehat{\gamma}_{i,u}, \widehat{\gamma}_{i,w}) = \underset{\sqrt{x^2+y^2} \le \mu_i \widehat{\gamma}_{i,n}}{\operatorname{argmin}} \quad \mathbf{v}^T \left(x\,\mathbf{D}_{i,u} + y\,\mathbf{D}_{i,w}\right)
\end{aligned}
$$

Frictional Contact: The Matrix-Vector Form

- Problem on previous slide reformulated using matrix-vector notation, assumes form

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{L}(\mathbf{q})\mathbf{v} \\
\mathbf{M}\dot{\mathbf{v}} &= \mathbf{f}(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in \mathcal{B}} \widehat{\gamma}_{i,b} \nabla \Psi_i + \sum_{i \in \mathcal{A}} \left(\widehat{\gamma}_{i,n} \mathbf{D}_{i,n} + \widehat{\gamma}_{i,u} \mathbf{D}_{i,u} + \widehat{\gamma}_{i,w} \mathbf{D}_{i,w} \right) \\
i \in \mathcal{B} &: \quad \Psi_i(\mathbf{q}, t) = 0 \\
i \in \mathcal{A} &: \quad 0 \le \widehat{\gamma}_{i,n} \perp \Phi_i(\mathbf{q}) \ge 0, \\
(\widehat{\gamma}_{i,u}, \widehat{\gamma}_{i,w}) &= \underset{\sqrt{x^2+y^2} \le \mu_i \widehat{\gamma}_{i,n}}{\operatorname{argmin}} \quad \mathbf{v}^T \left(x\, \mathbf{D}_{i,u} + y\, \mathbf{D}_{i,w} \right)
\end{aligned}
$$

The Discretization Process

- For straight index-3 DAE solution (like ADAMS), one uses the Newton-Euler form of the equations of motion in conjunction with the level zero constraints (the position constraint equations)
- The DVI solution relies on the level one constraints (velocity level constraints)
- Implications:
  - Since the level zero constraints are not enforced, there will be drift in the solution.
  - Stabilization terms, that penalize the violation of the level zero constraints, are added to the level one bilateral and unilateral constraints
  - Bilateral and unilateral constraints massaged into the following (superscript (l) denotes the time step):

$$
\begin{aligned}
i \in \mathcal{B} &: \quad \frac{1}{h}\Psi_i(\mathbf{q}^{(l)}, t) + \nabla\Psi_i^T \mathbf{v}^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0 \\
i \in \mathcal{A} &: \quad 0 \le \gamma_{i,n} \quad \perp \quad \frac{1}{h}\Phi_i(\mathbf{q}^{(l)}) + \mathbf{D}_{i,n}^T \mathbf{v}^{(l+1)} \ge 0 \,.
\end{aligned}
$$

    - Reminiscent of a Baumgarte stabilization scheme

The Discretization Process

- The discretized form of the DVI problem:

$$
\begin{aligned}
\mathbf{M}(\mathbf{v}^{(l+1)} - \mathbf{v}^{(l)}) &= h\mathbf{f}(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)}) + \sum_{i \in \mathcal{B}} \gamma_{i,b} \nabla \Psi_i + \sum_{i \in \mathcal{A}} (\gamma_{i,n}\, \mathbf{D}_{i,n} + \gamma_{i,u}\, \mathbf{D}_{i,u} + \gamma_{i,w}\, \mathbf{D}_{i,w}) \\
i \in \mathcal{B} &: \; \tfrac{1}{h}\Psi_i(\mathbf{q}^{(l)}, t) + \nabla \Psi_i^T \mathbf{v}^{(l+1)} + \tfrac{\partial \Psi_i}{\partial t} = 0 \\
i \in \mathcal{A} &: \; 0 \le \gamma_{i,n} \perp \tfrac{1}{h}\Phi_i(\mathbf{q}^{(l)}) + \mathbf{D}_{i,n}^T \mathbf{v}^{(l+1)} \ge 0 \\
(\gamma_{i,u}, \gamma_{i,w}) &= \underset{\mu_i \gamma_{i,n} \ge \sqrt{x^2+y^2}}{\operatorname{argmin}} \; \mathbf{v}^T (x\, \mathbf{D}_{i,u} + y\, \mathbf{D}_{i,w}) \\
\mathbf{q}^{(l+1)} &= \mathbf{q}^{(l)} + h\mathbf{L}(\mathbf{q}^{(l)})\mathbf{v}^{(l+1)}.
\end{aligned}
$$

- The first four of the equations above together combine for an optimization problem with equilibrium constraints
- Why an optimization problem?
  - Because the way the Coulomb friction model is posed
- What type of optimization problem?
  - This represents a nonlinear optimization problem
  - Can be linearized if the friction cone is discretized and represented as a multifaceted pyramid (problem size increases & anisotropy creeps in)
- What are the 'equilibrium constraints' ?
  - Your typical optimization problem might display algebraic equality or inequality constraints
  - Above, we are solving an optimization problem for which the constraints represent the discretization of a set of differential equations

The NCP → CCP Metamorphosis

- Dealing with some generic nonlinear optimization problem like the one above is daunting
- Trick used to recast it as a simpler optimization problem for which
  (i) We are guaranteed that a solution exists (ideally, it would be unique, in some sense), and
  (ii) There are tailored algorithms that we can use to efficiently find the solution
- Trick (coming from the left field): introduce a relaxation of the complementarity constraints

Instead of working with this:

$$i \in \mathcal{A} : 0 \leq \gamma_{i,n} \quad \perp \quad \frac{1}{h}\Phi_i(\mathbf{q}^{(l)}) + \mathbf{D}_{i,n}^T \mathbf{v}^{(l+1)} \geq 0$$

Work with this:

$$i \in \mathcal{A} : 0 \leq \gamma_{i,n} \quad \perp \quad \frac{1}{h}\Phi_i(\mathbf{q}^{(l)}) + \mathbf{D}_{i,n}^T \mathbf{v}^{(l+1)} - \mu_i \sqrt{(\mathbf{v}^T \mathbf{D}_{i,u})^2 + (\mathbf{v}^T \mathbf{D}_{i,w})^2} \geq 0$$

- Owing to this relaxation, the NCP problem becomes a cone complementarity problem (CCP)

The Cone Complementarity Problem

- The relaxed problem we have to deal with now looks like this

$$
\begin{aligned}
\mathbf{M}(\mathbf{v}^{(l+1)} - \mathbf{v}^{(l)}) &= h\mathbf{f}(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)}) + \sum_{i \in \mathcal{B}} \gamma_{i,b} \nabla\Psi_i + \sum_{i \in \mathcal{A}} (\gamma_{i,n}\,\mathbf{D}_{i,n} + \gamma_{i,u}\,\mathbf{D}_{i,u} + \gamma_{i,w}\,\mathbf{D}_{i,w}) \\
i \in \mathcal{B} &: \quad \frac{1}{h}\Psi_i(\mathbf{q}^{(l)}, t) + \nabla\Psi_i^T \mathbf{v}^{(l+1)} + \frac{\partial\Psi_i}{\partial t} = 0 \\
i \in \mathcal{A} &: \quad 0 \le \gamma_{i,n} \perp \frac{1}{h}\Phi_i(\mathbf{q}^{(l)}) + \mathbf{D}_{i,n}^T \mathbf{v}^{(l+1)} - \mu_i \sqrt{(\mathbf{v}^T \mathbf{D}_{i,u})^2 + (\mathbf{v}^T \mathbf{D}_{i,w})^2} \ge 0 \\
(\gamma_{i,u}, \gamma_{i,w}) &= \underset{\sqrt{x^2+y^2} \le \mu_i \gamma_{i,n}}{\operatorname{argmin}} \quad \mathbf{v}^T (x\,\mathbf{D}_{i,u} + y\,\mathbf{D}_{i,w}) \\
\mathbf{q}^{(l+1)} &= \mathbf{q}^{(l)} + h\mathbf{L}(\mathbf{q}^{(l)})\mathbf{v}^{(l+1)}.
\end{aligned}
$$

Cone Complementarity Problem (CCP)

- After some algebraic massaging, the equations on the previous slide combine to lead to the following CCP:
  - Introduce the convex hypercone...

$$\Upsilon = \left(\bigoplus_{i \in \mathcal{A}(\mathbf{q}^{(l)})} \mathcal{FC}^i \right) \oplus \left(\bigoplus_{i \in \mathcal{B}(\mathbf{q}^{(l)})} \mathcal{BC}^i \right) \qquad \text{where} \quad \begin{cases} \mathcal{FC}^i & \text{is the } i\text{-th friction cone} \\ \mathcal{BC}^i & \text{is } \mathbb{R} \end{cases}$$

  - ... and its polar hypercone

$$\Upsilon^\circ = \left(\bigoplus_{i \in \mathcal{A}(\mathbf{q}^{(l)})} \mathcal{FC}^{i\circ} \right) \oplus \left(\bigoplus_{i \in \mathcal{B}(\mathbf{q}^{(l)})} \mathcal{BC}^{i\circ} \right)$$

  - The CCP that needs to be solved at each time step is as follows:
    * Find the Lagrange hyper-multiplier γ that satisfies:

$$\Upsilon \ni \gamma \quad \perp \quad -(\mathbf{N}\gamma + \mathbf{r}) \in \Upsilon^\circ$$

    * The matrix $\mathbf{N}$ and vector $\mathbf{r}$ are given, computed based on state information at time-step $t^{(l)}$

The Optimization Angle

- CCP represents first order optimality condition (KKT conditions) for a quadratic problem with conic constraints

$$\min_{\gamma} \frac{1}{2}\gamma^T \mathbf{N}\gamma + \mathbf{r}^T\gamma$$
$$\text{subject to } \gamma_i \in \Upsilon_i \text{ for } i = 1, 2, \ldots, N_c \,.$$

  - $\mathbf{N} \in \mathbb{R}^{3N_c \times 3N_c}$ is symmetric and positive semi-definite
  - $\mathbf{N}$ and $\mathbf{r} \in \mathbb{R}^{3N_c}$ do not depend on γ. They are computed once at the beginning of each time step
  - The problem is convex, therefore it has a global solution
  - Problem does not have a unique solution (since $\mathbf{N}$ is not positive-definite)

Wrapping it Up, Complementarity Approach

- Everything straightforward once frictional contact forces are available
  - The velocity $\mathbf{v}^{(l+1)}$ is computed via a matrix-vector multiplication
  - Once velocity available, generalized positions $\mathbf{q}^{(l+1)}$ computed as

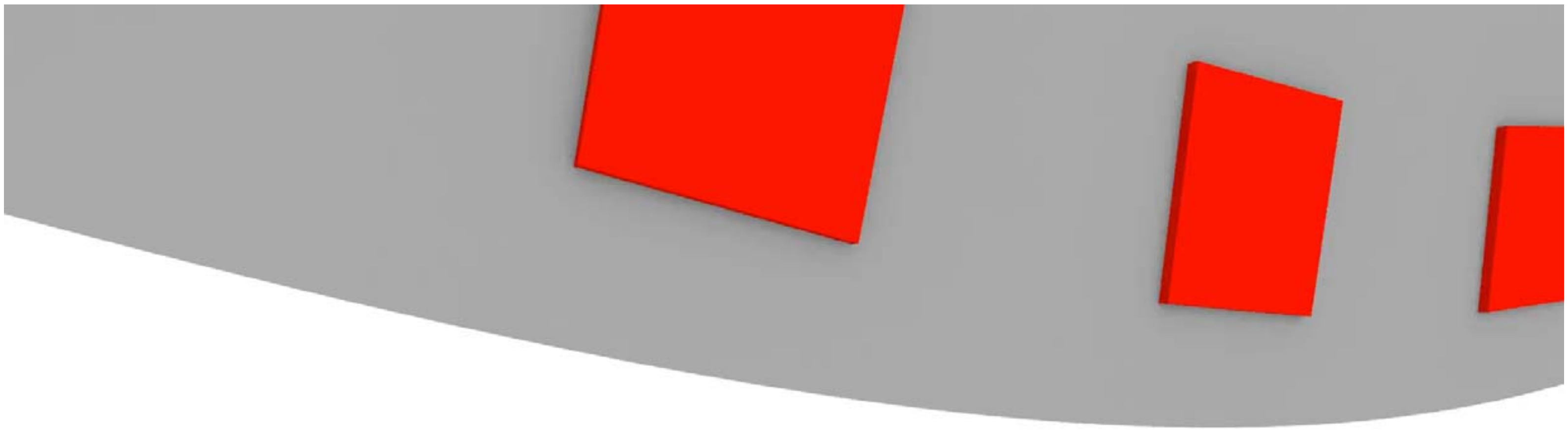
$$\mathbf{q}^{(l+1)} = \mathbf{q}^{(l)} + h\mathbf{L}(\mathbf{q}^{(l)})\mathbf{v}^{(l+1)}$$

Complementarity Approach: Putting Things in Perspective

- Perform collision detection
- Formulate equations of motion; i.e., pose DVI problem
- DVI discretized to lead to nonlinear complementarity problem (NCP)
- Relax NCP to get CCP
- Equivalently, solve QP with conic constraints to compute γ
- Once friction and contact forces available, velocity available
- Once velocity available, positions are available (numerical integration)

Additive Manufacturing (3D SLS Printing)

Courtesy of Professor Tim Osswald, Polymer Engineering Center, UW-Madison



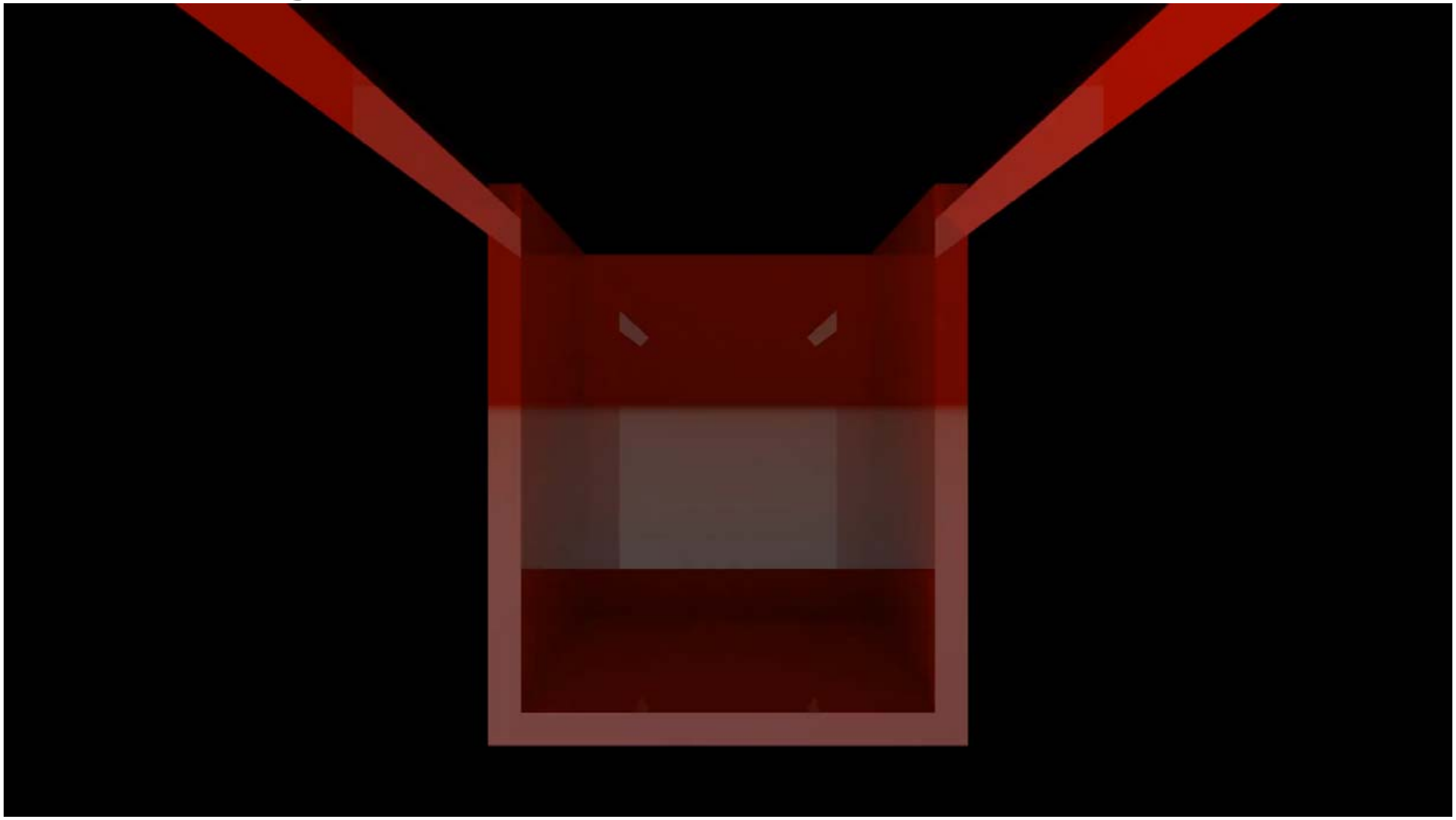
[Hammad]→

Selective Laser Sintering (SLS) Layering

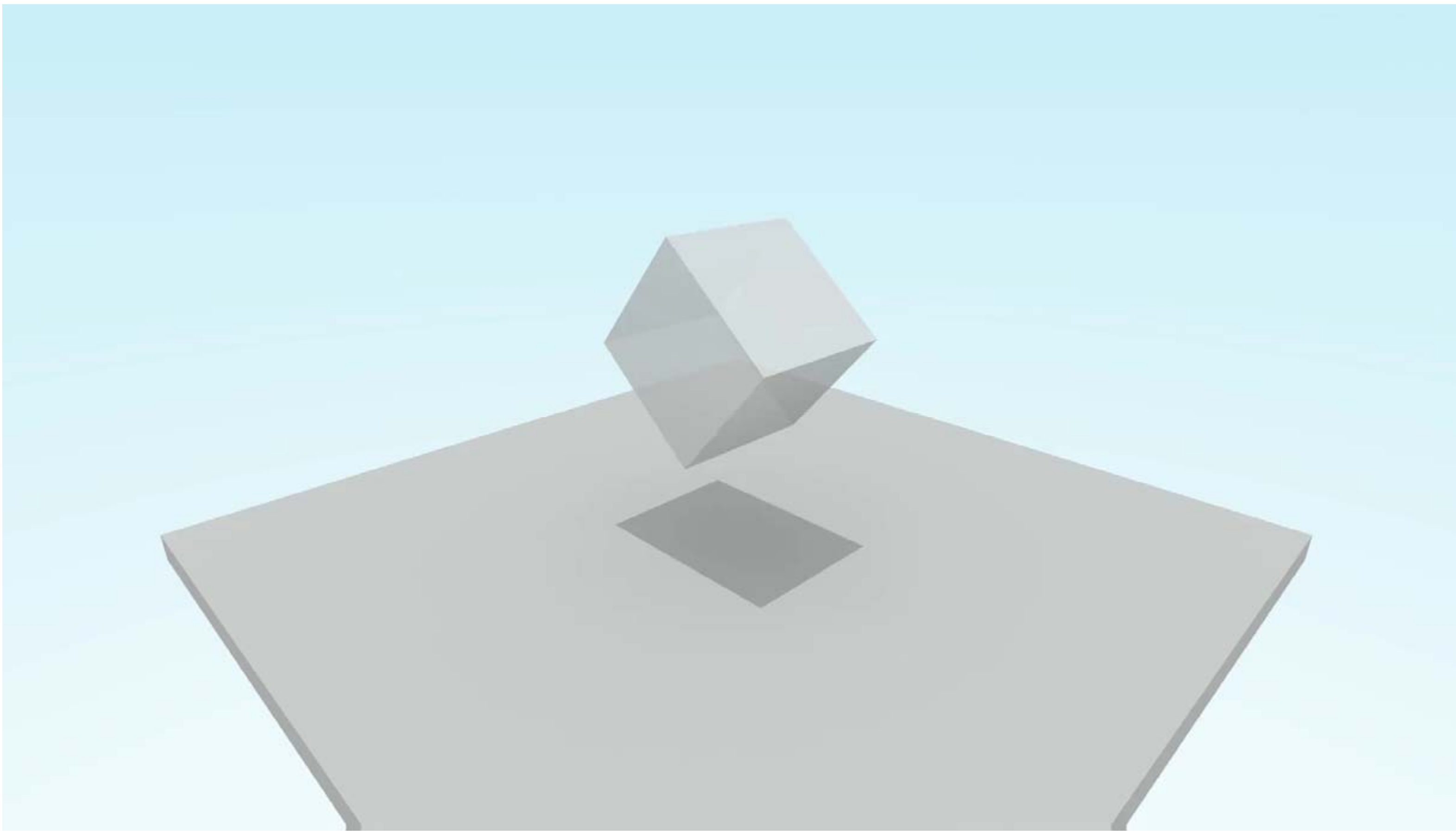
Granular Material	
N	1 186 185
ρ	$930\ [kg/m^3]$
$r(mean)$	$0.029\ [mm]$
$r(\sigma)$	$0.0075\ [mm]$
Simulation	
Simulation Length:	$20\ [s]$
Δt	$5 \times 10^{-5}\ [s]$
Run Time	49 Hours

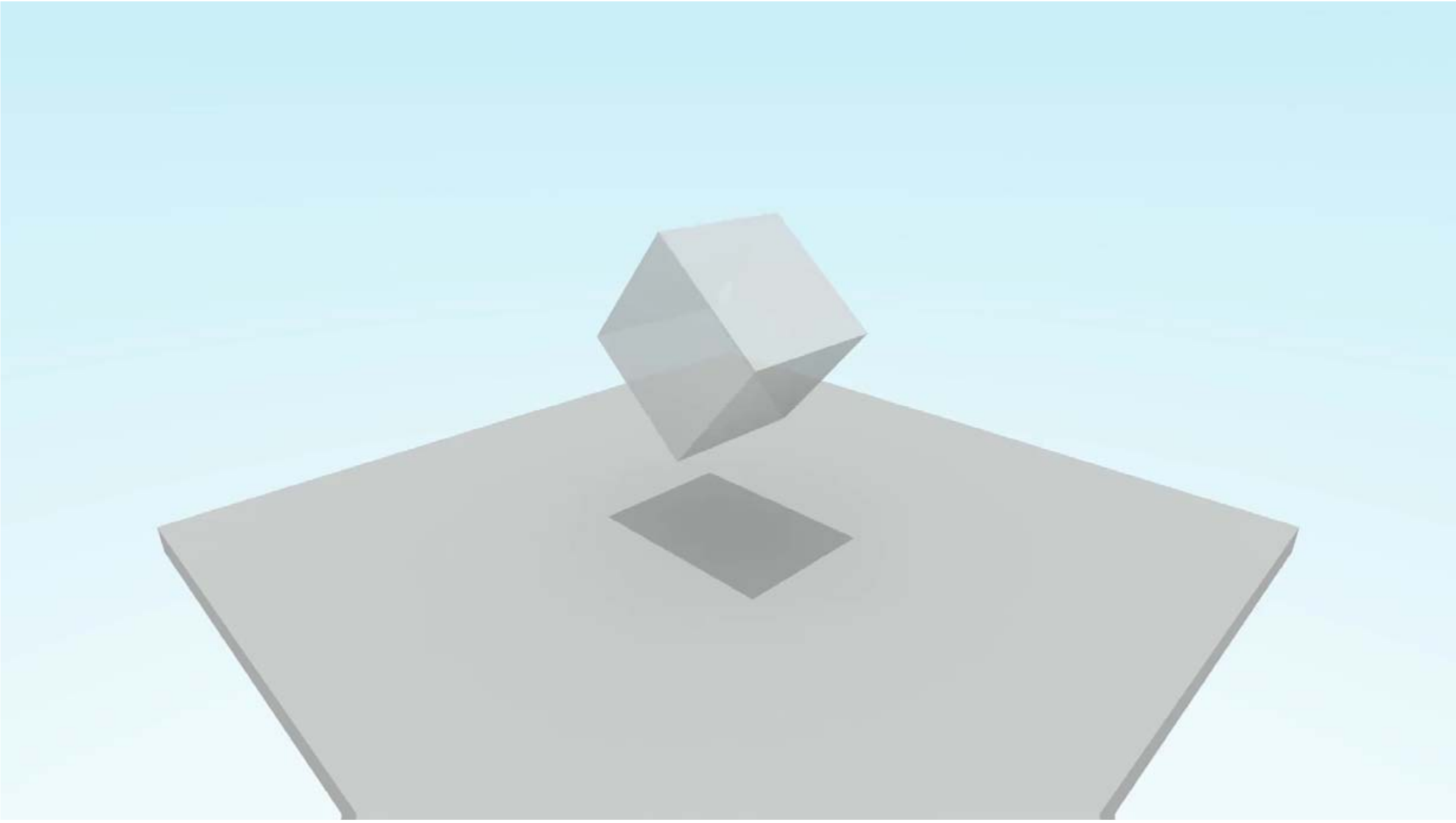
[Hammad]→

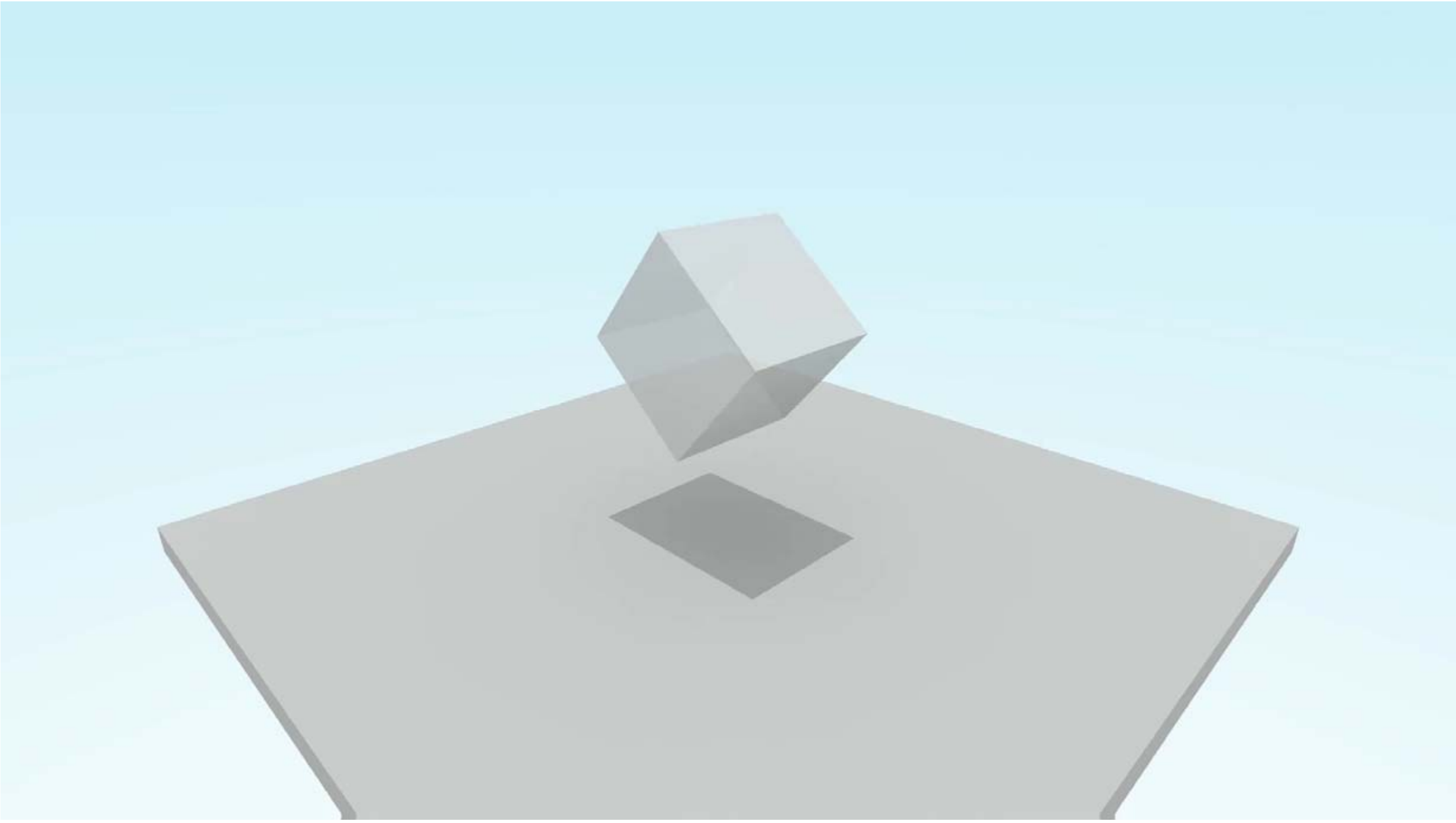
Dress 3D Printing Problem

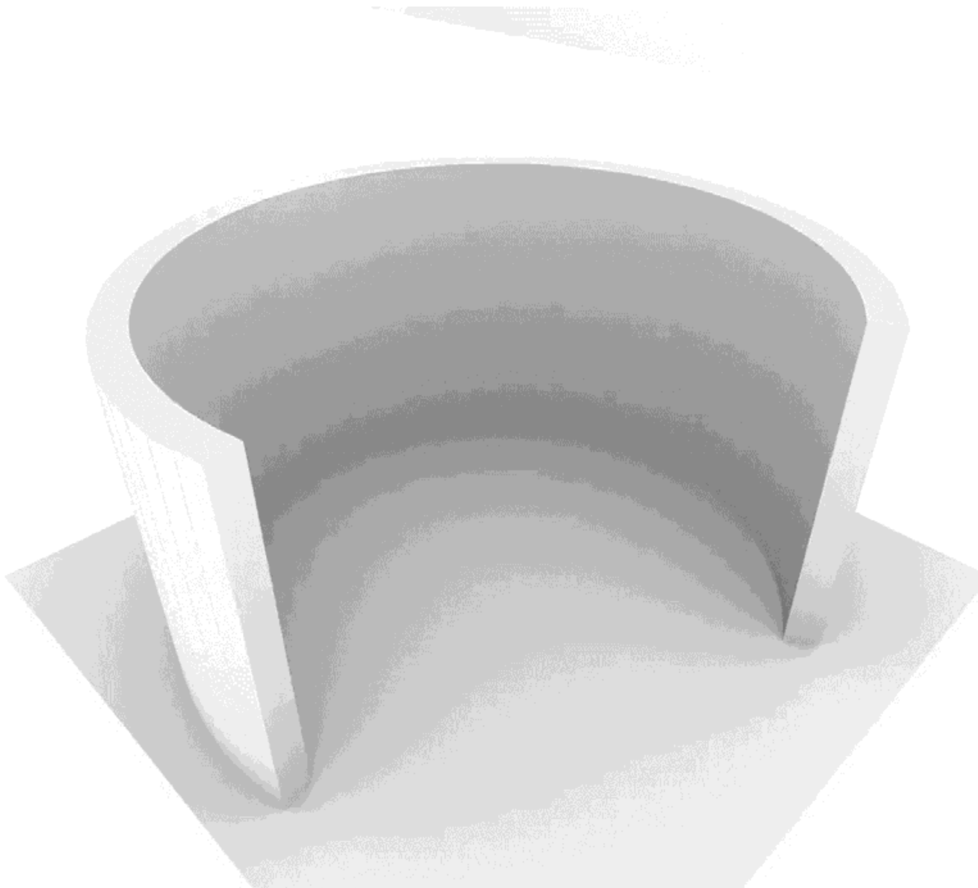


Using Simulation in 3D Printing of Clothes









Pros and Cons, Complementarity Approach

- Pros
  - Allows for large integration step sizes since it doesn't have to deal with contact stiffness
  - Reduced number of model parameters one can fiddle with
  - It looks at the entire problem, it doesn't artificially decouples the problem

- Cons
  - Requires a global solution, which means that large systems lead to large coupled problems
  - Our implementation has numerical artifacts owing to the relaxation of the non-penetration condition
  - Challenging to model coefficient of restitution (currently uses an inelastic model)
  - Stuck w/ a rigid body dynamics take on the problem (can't propagate shock waves)

Reference, DVI Literature

- Lab technical report:
  - TR-2016-12: “Posing Multibody Dynamics with Friction and Contact as a Differential Algebraic Inclusion Problem” D. Negrut, R. Serban: http://sbel.wisc.edu/documents/TR-2016-12.pdf

- D. E. Stewart and J. C. Trinkle, An implicit time-stepping scheme for rigid-body dynamics with inelastic collisions and Coulomb friction, International Journal for Numerical Methods in Engineering, 39 (1996), pp. 2673-2691.

- D. E. Stewart, Rigid-body dynamics with friction and impact, SIAM Review, 42 (2000), pp. 3-39

- M. Anitescu and G. D. Hart, A constraint-stabilized time-stepping approach for rigid multibody dynamics with joints, contact and friction, International Journal for Numerical Methods in Engineering, 60 (2004), pp. 2335-2371.

- M. Anitescu and A. Tasora, A matrix-free cone complementarity approach for solving large-scale, nonsmooth, rigid body dynamics, Comput. Methods Appl. Mech. Engrg. 200 (2011) 439–453

Closing Remarks
[Applies both for Penalty and DVI approaches]

- There is some hand waving when it comes to handling friction and contact
  - Both in Penalty and DVI

- Handling frictional contact is equally art and science
  - To get something to run robustly requires tweaking
  - Takes some time to understand strong/weak points of each approach

- Continues to be area of active research

Supplemental Slides

Notation Conventions
[1/2]

- To stick with the presentation in the paper of Anitescu and Tasora, we'll use the following notation
  - Point 1: Instead of sticking with transpose of Jacobians, we'll use gradients, which are defined precisely as the transpose of the Jacobians. Specifically,

$$\nabla_q \Psi_i = \Psi_{i,\mathbf{q}}^T = [\partial \Psi_i / \partial \mathbf{q}]^T \qquad \text{and} \qquad \nabla_q \Phi_i = \Phi_{i,\mathbf{q}}^T = [\partial \Phi_i / \partial \mathbf{q}]^T$$

  - Point 2: We'll use the transformation matrix $\mathbf{L}(\mathbf{q})$ to link the time derivative of the level zero unknowns in the $\mathbf{r}-\mathbf{p}$ formulation to the level one unknowns in the $\mathbf{r}-\omega$ formulation:

$$\dot{\mathbf{q}} = \mathbf{L}(\mathbf{q})\mathbf{v}$$

  - Point 3: To keep the notation simpler (and probably confuse you), we'll group the translational and rotational equations of motion in one big matrix-vector equation (nothing changed, except the notation) in order to have less symbols and equations to deal with
  - Point 4: We'll use the following notation (h is the integration step-size)

$$\gamma_{i,n} = h\widehat{\gamma}_{i,n} \qquad \gamma_{i,u} = h\widehat{\gamma}_{i,u} \qquad \gamma_{i,w} = h\widehat{\gamma}_{i,w} \qquad \gamma_{i,w} = h\widehat{\gamma}_{i,w} \qquad \gamma_{i,b} = h\widehat{\gamma}_{i,b}$$

    * Recall that time $\times$ force (like in $\gamma_{i,n} = h\widehat{\gamma}_{i,n}$) is impulse, and it's impulse that changes the momentum of a body

- Define the transformation matrix $\mathbf{A}_i$ that given the representation of a geometric vector in the contact reference frame associated with contact i is used to generate its representation in the GRF:

$$\mathbf{A}_{i\to G} = \begin{bmatrix} \mathbf{n}_i & \mathbf{u}_i & \mathbf{w}_i \end{bmatrix}$$

  - Note that the frictional contact force at contact i as felt by body A is simply

$$\mathbf{F}^{fc}_{i,A} = \mathbf{n}_i \widehat{\gamma}_{i,n} + \mathbf{u}_i \widehat{\gamma}_{i,u} + \mathbf{w}_i \widehat{\gamma}_{i,w} = \begin{bmatrix} \mathbf{n}_i & \mathbf{u}_i & \mathbf{w}_i \end{bmatrix} \begin{bmatrix} \widehat{\gamma}_{i,n} \\ \widehat{\gamma}_{i,u} \\ \widehat{\gamma}_{i,w} \end{bmatrix} = \mathbf{A}_{i\to G} \cdot \widehat{\gamma}_i \quad \text{where} \quad \widehat{\gamma}_i \equiv \begin{bmatrix} \widehat{\gamma}_{i,n} \\ \widehat{\gamma}_{i,u} \\ \widehat{\gamma}_{i,w} \end{bmatrix}$$

- A projection matrix $\mathbf{D}_i$ is defined for each contact $i \in \mathcal{A}$ to project the contact forces onto the equations of motion, both for translation and rotation. If we assume that contact i acts between body A and body B,

$$\mathbf{D}_i \equiv \begin{bmatrix} \mathbf{0} \\ \dots \\ \mathbf{A}_{i\to G} \\ \tilde{\bar{\mathbf{s}}}_{i,A} \mathbf{A}_A^T \mathbf{A}_{i\to G} \\ \mathbf{0} \\ \dots \\ \mathbf{0} \\ -\mathbf{A}_{i\to G} \\ -\tilde{\bar{\mathbf{s}}}_{i,B} \mathbf{A}_B^T \mathbf{A}_{i\to G} \\ \mathbf{0} \end{bmatrix}_{6nb\times 3}$$

- Notation used in expression of $\mathbf{D}_i$: the vectors $\tilde{\bar{\mathbf{s}}}_{i,A}$ and $\tilde{\bar{\mathbf{s}}}_{i,B}$ represent the location of the contact point in the local reference frame of body A and B, respectively

- The columns of $\mathbf{D}_i$ are denoted by $\mathbf{D}_{i,n}$, $\mathbf{D}_{i,u}$, $\mathbf{D}_{i,w}$ and are each vectors of dimension $6nb$:

$$\mathbf{D}_i = \begin{bmatrix} \mathbf{D}_{i,n} & \mathbf{D}_{i,u} & \mathbf{D}_{i,w} \end{bmatrix}_{6nb\times 3}$$

General Comments, DVI

- Differential Variational Inequality (DVI): a set of differential equations that hold in conjunction with a collection of constraints

  - Classical equations of motion: Newton-Euler EOMs, govern time evolutions of constrained MBS

  - Kinematic constraints coming from joints
    - These constraints are called bilateral constraints

  - When dealing with contacts, the non-penetration condition captured as a unilateral constraint
    - At point of contact, relative to body 1, body 2 can move outwards, but not inwards

  - The variational attribute stems from the optimization problem posing the Coulomb friction model

[Nomenclature]

Bilateral vs. Unilateral Constraints

- Nomenclature: classical MBD uses kinematic constraints, which we'll call bilateral constraints. In DVI we also have non-penetration constraints, which are unilateral constraints and assume the form of inequalities.
- Notation: We'll call $\mathcal{A}$ the set of all *active unilateral* constraints present in the system. Think of these as active contacts. They'll be denoted by

$$\Phi_i(\mathbf{q}) \qquad i \in \mathcal{A}$$

  - Note that the nonpenetration condition is expressed as (the distance between two bodies should also be positive)

$$\Phi_i(\mathbf{q}) \geq 0, \qquad i \in \mathcal{A}$$

- Notation: We'll call $\mathcal{B}$ the set of all *bilateral* constraints present in the system. These expression of these constraints will be denoted by $\Psi(\mathbf{q}, t)$. Just like before we have that

$$\Psi_i(\mathbf{q}, t) = 0, \qquad i \in \mathcal{B}$$

- Remark: While the bilateral constraints typically don't change in time (a spherical joint stays a spherical joint throughout the simulation), the unilateral constraints appear and disappear; i.e., contacts are made and then broken. In other words, $\mathcal{A}$ depends on the state $\mathbf{q}$ of the system