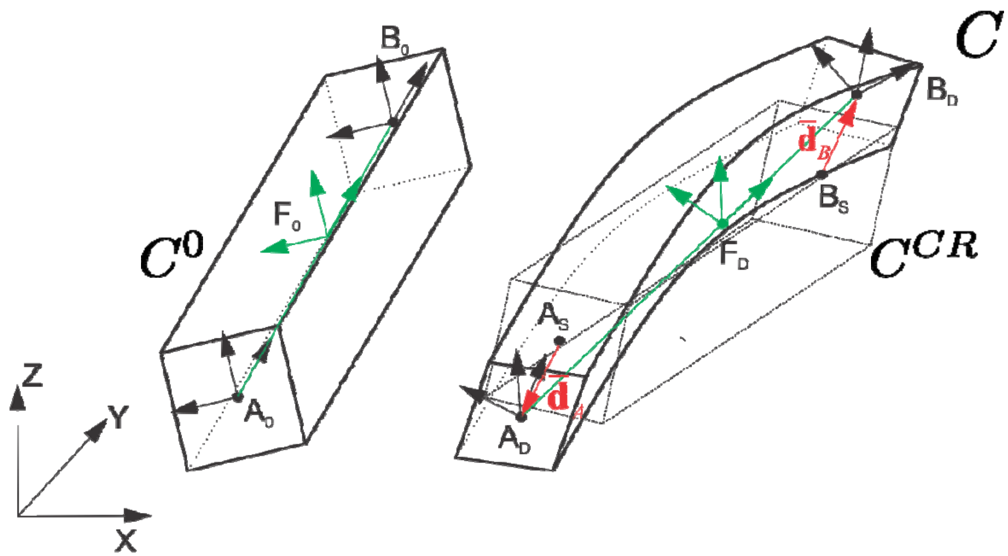
Chrono::FEA

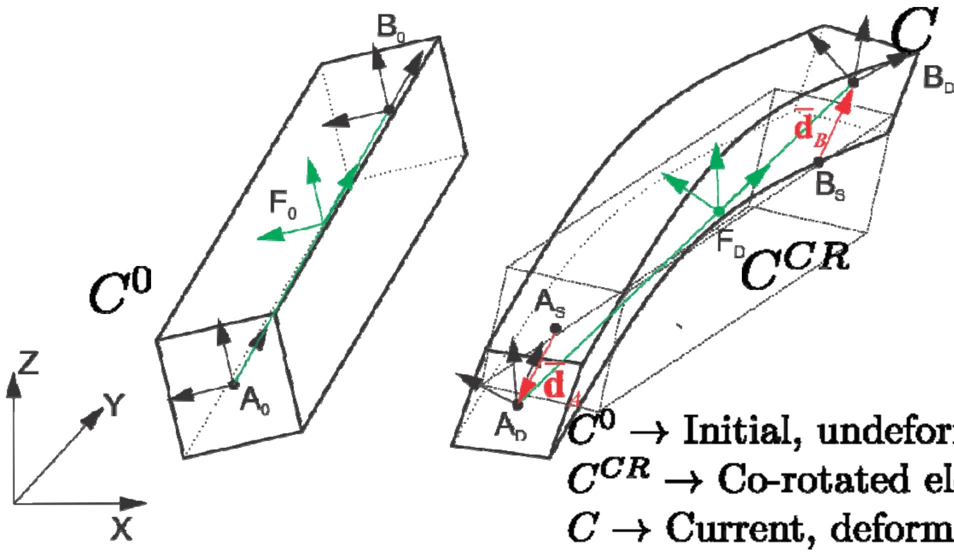
Co-rotational Formulation

Corotational Formulation

- In a nutshell:
  - Each finite element has a frame of reference associated with it
    - This frame describes base rotations and translations –rigid body-style
  - Based on linear finite elements –infinitesimal deformation
  - The element frame of reference absorbs rigid body motion and allows defining infinitesimal deformation with respect to the element
  - We will analyze this formulation on a per-element basis, only a beam elem.



Corotational Formulation



$C^0 \rightarrow$ Initial, undeformed configuration
$C^{CR} \rightarrow$ Co-rotated element frame (rigid body motion)
$C \rightarrow$ Current, deformed configuration

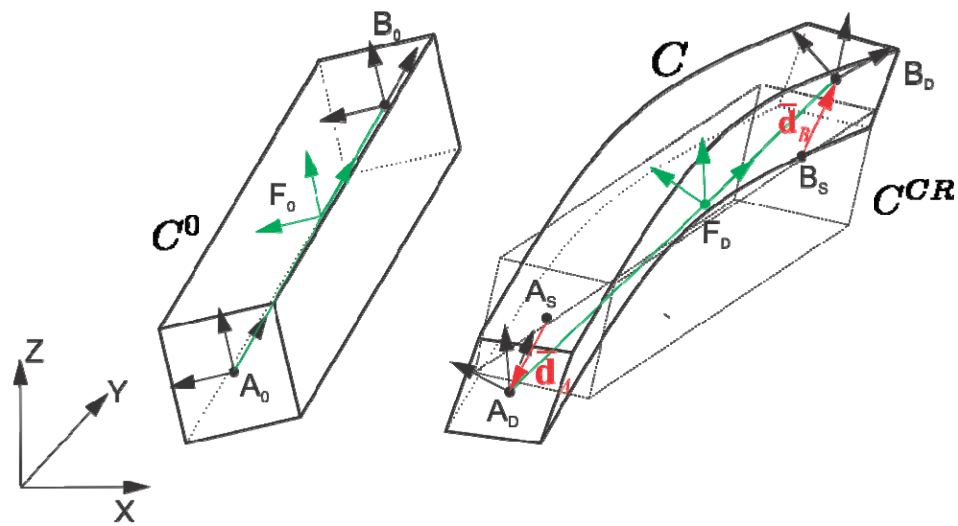
- The motion of a node i of a beam element is defined by the position vector $\boldsymbol{x}_i$ and a set of quaternions $\boldsymbol{\rho}_i$ of a reference frame.

- The vector $\boldsymbol{\rho}_i$ captures the rigid body rotation and deformation rotation of the beam cross section at node i. The state of a system with n nodes is, therefore, $\boldsymbol{s} = [\boldsymbol{q}, \boldsymbol{v}]$

- $\boxed{\boldsymbol{q} = [\boldsymbol{x}_1,\ \boldsymbol{\rho}_1, \boldsymbol{x}_2,\ \boldsymbol{\rho}_2, ..., \boldsymbol{x}_3,\ \boldsymbol{\rho}_3]} \in \mathbb{R}^{(3+4)n}$ and $\boldsymbol{v} = [\boldsymbol{v}_1,\ \bar{\boldsymbol{\omega}}_1, \boldsymbol{v}_2,\ \bar{\boldsymbol{\omega}}_2, ..., \boldsymbol{v}_3,\ \bar{\boldsymbol{\omega}}_3] \in \mathbb{R}^{(3+3)n}$.

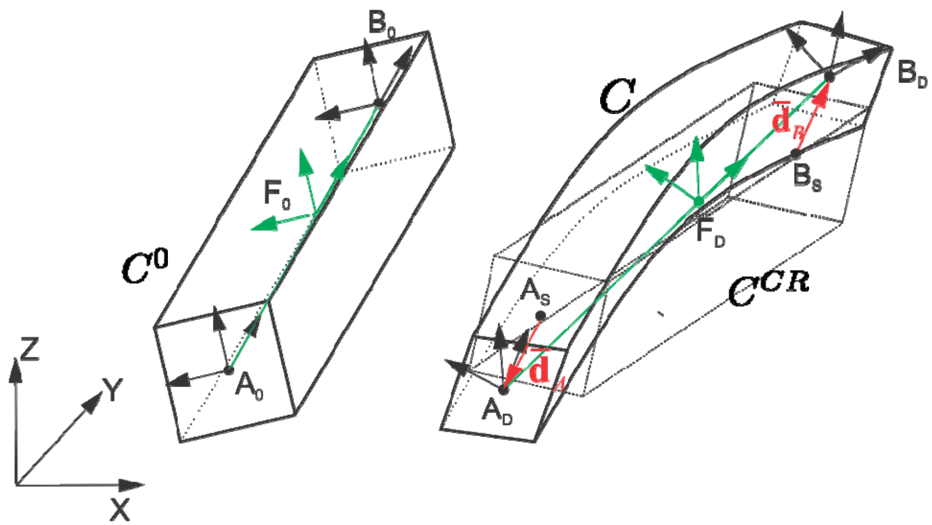
  Global coordinates

Corotational Formulation



- When an element j moves, the position and rotation of the floating frame, $\langle \mathbf{F} \rangle$, are updated. The origin of $\langle \mathbf{F} \rangle$ is placed at the element's midpoint $\boldsymbol{x}_F = 1/2\,(\boldsymbol{x}_B - \boldsymbol{x}_A)$.
- The floating (shadow) frame's longitudinal axis $\boldsymbol{X}$ is aligned with the vector $\boldsymbol{x}_B - \boldsymbol{x}_A$, whereas the $\boldsymbol{Y}$ and $\boldsymbol{Z}$ axes are obtained via a Gram-Schmidt orthogonalization.
- The rotation matrix and unit quaternion of $\langle \mathbf{F} \rangle$ will be denoted as $\boldsymbol{R}_F$ and $\boldsymbol{\rho}_F$, respectively.

Corotational Formulation



Local displacements

- One can compute the local displacements of a node i as $\boxed{\bar{\boldsymbol{d}}_i = \bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_{i_0} = \boldsymbol{R}_{\boldsymbol{F}}^{\mathrm{T}}(\boldsymbol{x}_{i_0} - \boldsymbol{x}_{\boldsymbol{F}_0}) - \boldsymbol{R}_{\boldsymbol{F}}^{\mathrm{T}}(\boldsymbol{x}_i - \boldsymbol{x}_{\boldsymbol{F}})}$, where the bar over displacement quantities describes locality, and the subscript 0 refers to the initial configuration.

- The local rotation of the nodes can be obtained in terms of rotation matrices for nodes A and B as $\bar{\mathbf{R}}_A = \mathbf{R}_F^T \mathbf{R}_A \mathbf{R}_{A0}^T$ and $\bar{\mathbf{R}}_B = \mathbf{R}_F^T \mathbf{R}_B \boxed{\mathbf{R}_{B0}^T}$

Infinitesimal rotations for elastic forces

Possible initial relative rotation between node and reference frame

- The local infinitesimal angles may be obtained as $\boxed{\bar{\boldsymbol{\theta}}_A} = \bar{\theta}_A \mathbf{u}_A$ and $\boxed{\bar{\boldsymbol{\theta}}_B} = \bar{\theta}_B \mathbf{u}_B$. We know that a quaternion can be written as $\boldsymbol{\rho} = [\cos(\theta/2), \sin(\theta/2)\mathbf{u}]$. To compute the local rotation vectors, $\bar{\theta}_A = 2\arccos(\Re(\bar{\boldsymbol{\rho}}_A))$, $\mathbf{u}_A = \frac{1}{\sin(\theta_a/2)}\Im(\bar{\boldsymbol{\rho}}_A)$

Corotational Formulation

Nodal quaternions can be used to describe linearized angles for the definition of local strains. A unit quaternion $(e_0, \mathbf{e})$, where $\mathbf{e} = (e_1, e_2, e_3)$, may be written as

$$\boldsymbol{\rho} = \left[\cos\left(\theta/2\right), \quad \mathbf{n}\sin\left(\theta/2\right)\right], \tag{1}$$

with the rotation angle θ being $\theta = \arccos\left(2e_0^2 - 1\right)$, and the vector defining the axis of rotation as

$$\mathbf{n} = 2\mathbf{e}e_0/\sin\theta. \tag{2}$$

Note that Equations (1)-(2) link Euler parameters to Euler's Rotation Theorem, which expresses any three-dimensional rotation as a finite rotation about a single axis $\mathbf{n}$. At this point, we can compute the linearized angles as $\bar{\boldsymbol{\theta}}_i = \theta_i \mathbf{n}_i$. For nodes A and B of a beam element, the vector of local deformations is $\bar{\mathbf{d}}_{12\times 1} = \left[\bar{\mathbf{d}}_A, \bar{\boldsymbol{\theta}}_A, \bar{\mathbf{d}}_B, \bar{\boldsymbol{\theta}}_B\right]$. The local stiffness matrix, $\bar{\mathbf{K}}_{12\times 12}(\bar{\mathbf{d}})$, and the local internal force vector, $\bar{\mathbf{f}}_{\text{in}} = \bar{\mathbf{K}}(\bar{\mathbf{d}})$, are then mapped onto the global frame of reference by introducing new matrices and building projectors.

Corotational Formulation

The dimensions of the stiffness matrix and internal force vector are dependent upon the number of coordinates used to describe deformation and, therefore, they are dependent on the element. For the beam element described in these notes, $\bar{\mathbf{K}}$ is a 12-by-12 matrix, and $\bar{\mathbf{d}}$ is a 12-by-1 vectors.

$$
\bar{\mathbf{K}} = \begin{bmatrix}
\frac{EA}{L} & & & & & & & & & & & \\
0 & \frac{12EI_z}{L^3} & & & & & & & & & & \\
0 & 0 & \frac{12EI_y}{L^3} & & & & & & & \text{Symmetric} & & \\
0 & 0 & 0 & \frac{GI}{L} & & & & & & & & \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & & & & & & & \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_y}{L} & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & & & & & \\
\frac{-EA}{L} & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & & & & \\
0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & & & \\
0 & 0 & 0 & \frac{-GI}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI}{L} & & \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
$$

Generalized Elastic Forces

The elastic forces are obtained, in the local (co-rotated) frame, using infinitesimal coordinates. These forces need to be mapped into the global, generalized coordinates. Internal forces of an element, defined as

$\bar{\mathbf{f}}_{\text{in}} = \bar{\mathbf{K}}\bar{\mathbf{d}}$

To establish the relation between the two sets of coordinates: Virtual work of internal forces

$\bar{\mathbf{f}}^{T}_{\text{in},\bar{\mathbf{u}}}\delta\bar{\mathbf{d}}_{\bar{\mathbf{u}}} + \bar{\mathbf{f}}^{T}_{\text{in},\bar{\boldsymbol{\theta}}}\delta\bar{\mathbf{d}}_{\bar{\boldsymbol{\theta}}} = \mathbf{f}^{T}_{\text{in},\mathbf{x}}\delta\mathbf{x} + \mathbf{f}^{T}_{\text{in},\boldsymbol{\rho}}\delta\boldsymbol{\rho}$

This transformation may be defined by a Jacobian in the following form $\begin{bmatrix}\delta\bar{\mathbf{u}}\\ \delta\bar{\boldsymbol{\theta}}\end{bmatrix} = \mathbf{J}_{CR}\begin{bmatrix}\delta\mathbf{x}\\ \delta\boldsymbol{\rho}\end{bmatrix}$, $\mathbf{J}_{CR} = \begin{bmatrix}\frac{\partial\bar{\mathbf{u}}}{\partial\mathbf{x}} & \frac{\partial\bar{\mathbf{u}}}{\partial\boldsymbol{\rho}}\\ \frac{\partial\bar{\boldsymbol{\theta}}}{\partial\mathbf{x}} & \frac{\partial\bar{\boldsymbol{\theta}}}{\partial\boldsymbol{\rho}}\end{bmatrix}$

Generalized Elastic Forces

The Jacobian of the transformation allows to go from local coordinates to our global, generalized coordinates

$$\begin{bmatrix} \mathbf{f}_{\text{in},\mathbf{x}} \\ \mathbf{f}_{\text{in},\boldsymbol{\rho}} \end{bmatrix} = \mathbf{J}_{CR}^{T} \begin{bmatrix} \bar{\mathbf{f}}_{\text{in},\bar{\mathbf{u}}} \\ \bar{\mathbf{f}}_{\text{in},\bar{\boldsymbol{\theta}}} \end{bmatrix} \Leftrightarrow \mathbf{f}_{\text{in}} = \mathbf{J}_{CR}^{T} \bar{\mathbf{f}}_{\text{in}}$$

where the Jacobian has the form...

$$\mathbf{J} = \underbrace{\bar{\mathbf{H}}}_{\substack{\text{Rotation to} \\ \text{spin Jacobian}}} \quad \underbrace{\bar{\mathbf{P}}}_{\substack{\text{in CR:From displacements/spin} \\ \text{to deformation displacements/spin}}} \quad \underbrace{\mathbf{T}}_{\substack{\text{Global to} \\ \text{CR rotator}}}$$

- We are not getting into more details

Stiffness Matrix

The stiffness matrix looks like...

$$\mathbf{K} = \mathbf{R}_\alpha \left(\underbrace{\bar{\mathbf{P}}^{\mathrm{T}}\bar{\mathbf{H}}^{\mathrm{T}}\bar{\mathbf{K}}\bar{\mathbf{H}}\bar{\mathbf{P}}}_{\mathbf{K}_{GM}} - \underbrace{\bar{\mathbf{F}}_{nm}\bar{\mathbf{G}}}_{\mathbf{K}_{GR}} - \underbrace{\bar{\mathbf{G}}^{\mathrm{T}}\mathbf{F}_n^{\mathrm{T}}\bar{\mathbf{P}}}_{\mathbf{K}_{GP}} + \underbrace{\bar{\mathbf{P}}^{\mathrm{T}}\mathbf{L}_H\bar{\mathbf{P}}}_{\mathbf{K}_{GH}} \right) \mathbf{R}_\alpha^{\mathrm{T}} \tag{1}$$

where the terms $\mathbf{K}_{GR}$ (rotational), $\mathbf{K}_{GP}$ (equilibrium projection), and $\mathbf{K}_{GH}$ (moment-correction) capture geometric stiffness; $\mathbf{K}_{GM}$ is the material stiffness.

The following notes apply to Chrono's implementation:

- The term $\mathbf{K}_{GH}$ is not computed in the implementation because its influence was found to be negligible
- $\mathbf{K}_{GM}$ is a symmetric matrix for Euler-Bernoulli beams; however, the overall global stiffness matrix $\mathbf{K}$ is non-symmetric due to the influence of geometric stiffness terms
- Under reasonable assumptions, the global stiffness matrix can be symmetrized without affecting convergence of the Newton-Raphson method:

$$\mathbf{K} = \mathbf{R}_\alpha \left(\bar{\mathbf{P}}^{\mathrm{T}}\bar{\mathbf{H}}^{\mathrm{T}}\bar{\mathbf{K}}\bar{\mathbf{H}}\bar{\mathbf{P}} - \bar{\mathbf{F}}_{sy}\bar{\mathbf{G}} - \bar{\mathbf{G}}^{\mathrm{T}}\bar{\mathbf{F}}_{sy}^{\mathrm{T}}\bar{\mathbf{P}} \right) \mathbf{R}_\alpha^{\mathrm{T}} ,$$

where $\bar{\mathbf{F}}_{sy} = \frac{1}{2}\left(\bar{\mathbf{F}}_{nm} + \bar{\mathbf{F}}_n\right)$.

Generalized Loads in CRF

Since the coordinates of a system modeled using the co-rotational formulation are global translations and rotations, the generalized force of a concentrated loads looks like that of a rigid body. But...

- For loads not applied at the nodes, shape function evaluation is necessary
- Pressure and volumetric loads also require numerical integration

Procedure

- CRF assumes the use of finite element –small def.
- Each node is characterized by translational and rotation coordinates
- Coordinates describe rigid body motion and deformation
- Small deformation is treated linearly
- Elastic forces must be mapped back into global coordinates
- Generalized stiffness matrices become "ugly" (e.g. see white papers on projectchrono website)