General Theoretical Concepts Related to Multibody Dynamics

Before Getting Started

- Material draws on two main sources
  - Ed Haug’s book, available online: http://sbel.wisc.edu/Courses/ME751/2010/bookHaugPointers.htm
  - Course notes, available at: http://sbel.wisc.edu/Courses/ME751/2016/

Looking Ahead

- Purpose of this segment:
  - Quick discussion of several theoretical concepts that come up time and again when using Chrono

- Concepts covered
  - Reference frames and changes of reference frames
  - Elements of the kinematics of a 3D body (position, velocity and acceleration of a body)
  - Kinematic constraints (joints)
  - Formulating the equations of motion
    - Newton-Euler equations of motion (via D’Alembert’s Principle)

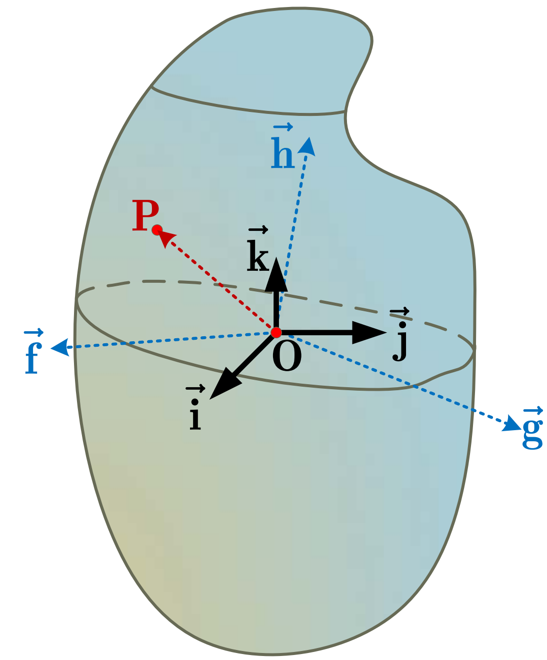
Reference Frames in 3D Kinematics. Problem Setup

- Global Reference Frame (G-RF) attached to ground at point O

- Imagine point P is fixed (red-pen mark) on the rigid body

- Rigid body has a reference frame attached (fixed) to it
  - Assume its origin is at O (same as G-RF)
  - Called Local Reference Frame (L-RF) – shown in blue
  - Axes: **f**, **g**, **h**

- Question of interest:
  - What is the relationship between the coordinates of point P in G-RF and L-RF?



More Formal Way of Posing the Question

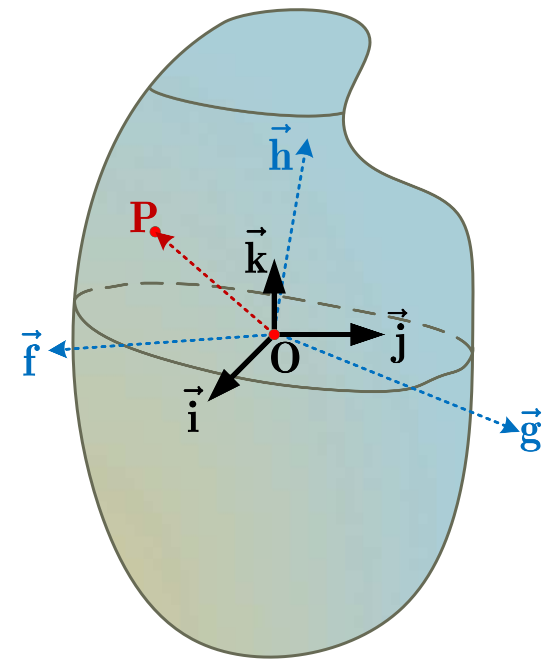
- Let $\vec{\mathbf{q}} = \overrightarrow{\mathbf{OP}}$ be a geometric vector (see figure)

- In the G-RF defined by $(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}})$, the geometric vector $\vec{\mathbf{q}}$ is represented as

$$\vec{\mathbf{q}} = q_x \vec{\mathbf{i}} + q_y \vec{\mathbf{j}} + q_z \vec{\mathbf{k}}$$

- In the L-RF defined by $(\vec{\mathbf{f}}, \vec{\mathbf{g}}, \vec{\mathbf{h}})$, the geometric vector $\vec{\mathbf{q}}$ is represented as

$$\vec{\mathbf{q}} = \bar{q}_x \vec{\mathbf{f}} + \bar{q}_y \vec{\mathbf{g}} + \bar{q}_z \vec{\mathbf{h}}$$



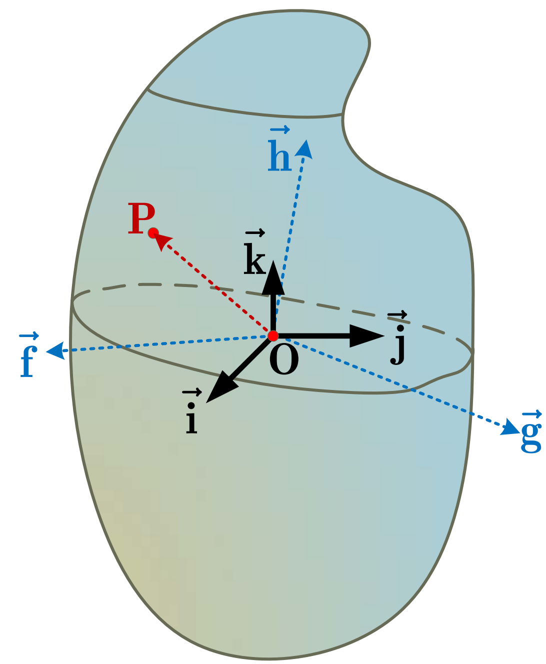
- QUESTION: how are (q_x, q_y, q_z) and $(\bar{q}_x, \bar{q}_y, \bar{q}_z)$ related?

Relationship Between L-RF Vectors and G-RF Vectors

$$\overrightarrow{\mathbf{f}} = a_{11}\,\overrightarrow{\mathbf{i}} + a_{21}\,\overrightarrow{\mathbf{j}} + a_{31}\,\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{g}} = a_{12}\,\overrightarrow{\mathbf{i}} + a_{22}\,\overrightarrow{\mathbf{j}} + a_{32}\,\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{h}} = a_{13}\,\overrightarrow{\mathbf{i}} + a_{23}\,\overrightarrow{\mathbf{j}} + a_{33}\,\overrightarrow{\mathbf{k}}$$



$$\mathbf{f} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$\boxed{\begin{aligned} a_{11} &= \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{f}}) \\ a_{12} &= \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{g}}) \\ a_{13} &= \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{h}}) \end{aligned}} \quad \boxed{\begin{aligned} a_{21} &= \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{f}}) \\ a_{22} &= \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{g}}) \\ a_{23} &= \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{h}}) \end{aligned}} \quad \boxed{\begin{aligned} a_{31} &= \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{f}}) \\ a_{32} &= \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{g}}) \\ a_{33} &= \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{h}}) \end{aligned}}$$

There is a good reason the values a_{ij} above are called "direction cosines".

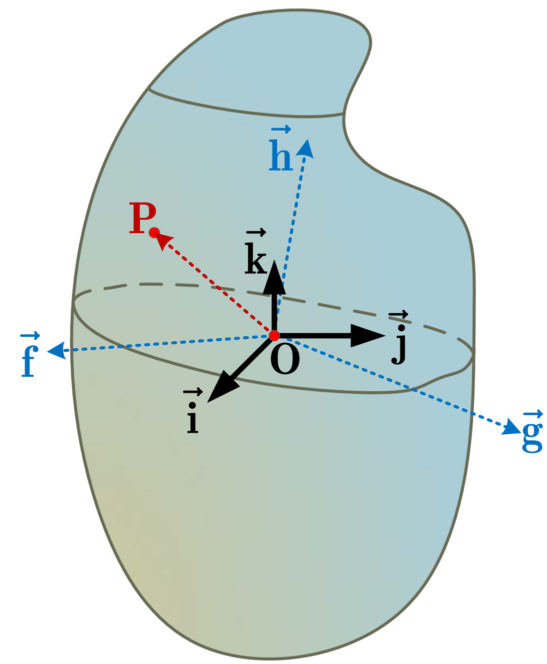
Punch Line, Change of Reference Frame (from “source” to “destination”)

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \bar{q}_x \\ \bar{q}_y \\ \bar{q}_z \end{bmatrix}$$

$$\boxed{\mathbf{q}_d = \mathbf{A}_{ds}\, \mathbf{q}_s}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [\mathbf{f} \quad \mathbf{g} \quad \mathbf{h}]$$

$$\mathbf{f} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$



The Bottom Line: Moving from RF to RF

- Representing the same geometric vector in two different RFs leads to the concept of "rotation matrix", or "transformation matrix" $\mathbf{A}_{ds}$:

  - Getting the new coordinates, that is, representation of the **same** geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix $\mathbf{A}_{ds}$:

$$\mathbf{q} = \mathbf{A}_{ds}\bar{\mathbf{q}}$$

- NOTE 1: what is changed is the RF used to represent the vector
  - We are talking about the *same* geometric vector, represented in two RFs

- NOTE 2: rotation matrix $\mathbf{A}_{ds}$ sometimes called "orientation matrix"

Rotation Matrix is Orthogonal

- Recall that $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, and $\vec{\mathbf{h}}$ are mutually orthogonal
- Recall that $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, and $\vec{\mathbf{h}}$ are are unit vectors
- Therefore, the following holds:

$$\mathbf{f}^T\mathbf{f} = \mathbf{g}^T\mathbf{g} = \mathbf{h}^T\mathbf{h} = 1$$

$$\mathbf{f}^T\mathbf{g} = \mathbf{g}^T\mathbf{h} = \mathbf{h}^T\mathbf{f} = 0$$

- Consequently, the rotation matrix $\mathbf{A}$ is orthogonal

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}_{3\times 3}$$

Summarizing Key Points, Reference Frames

- Started with the representation $\mathbf{q}_s$ of a geometric vector $\vec{\mathbf{q}}$ in a "source" reference frame s
- The representation of the geometric vector $\vec{\mathbf{q}}$ in a "destination" reference frame d is given by

$$\mathbf{q}_d = \mathbf{A}_{ds}\mathbf{q}_s$$

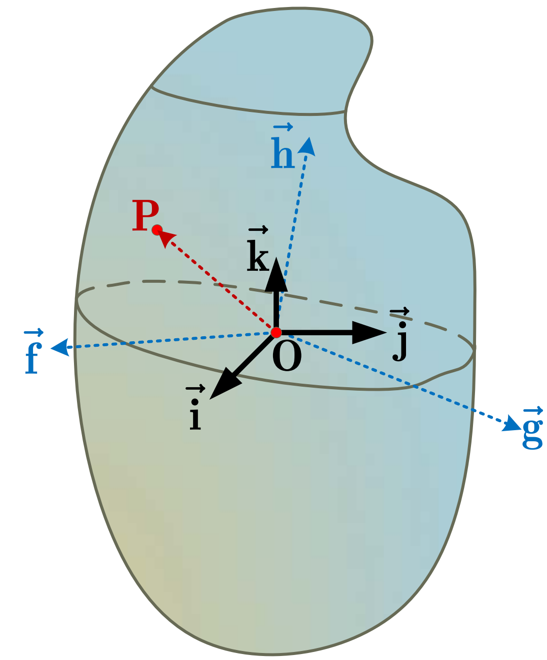
- Matrix $\mathbf{A}_{ds}$ called transformation, or rotation matrix (taking vector from the source RF s to the destination RF d)
- Because $\mathbf{A}_{ds}$ is orthogonal, one has that

$$\mathbf{q}_s = \mathbf{A}_{ds}^T\mathbf{q}_d \qquad \text{therefore} \qquad \mathbf{A}_{sd} = \mathbf{A}_{ds}^T$$

- Many times, the "destination" RF is the global reference frame (G-RF), which has ID "0"
  - In this case, we don't show "0" anymore, simply call $\mathbf{A}_s$ instead of $\mathbf{A}_{0s}$

New Topic:
Angular Velocity. 3D Problem Setup

- Global Reference Frame (G-RF) attached to ground at point O
- Imagine point P is fixed (red-pen mark) on the rigid body
- Rigid body has a reference frame attached to it
  - Assume its origin is at O (same as G-RF)
  - Local Reference Frame (L-RF) – shown in blue
  - Axes: **f**, **g**, **h**
- Question of interest:
  - How do we express rate of change of blue RF wrt global RF?



Angular Velocity, Getting There…

- Recall that $\mathbf{A}_i \mathbf{A}_i^T = \mathbf{I}_{3\times3}$. Taking a time derivative yields

$$\dot{\mathbf{A}}_i \mathbf{A}_i^T + \mathbf{A}_i \dot{\mathbf{A}}_i^T = \mathbf{0}_{3\times3} \qquad \Rightarrow \qquad \dot{\mathbf{A}}_i \mathbf{A}_i^T = -\mathbf{A}_i \dot{\mathbf{A}}_i^T$$

- Quick remarks
  - The matrix $\dot{\mathbf{A}}_i \mathbf{A}_i^T$ is a 3×3 matrix
  - The matrix $\dot{\mathbf{A}}_i \mathbf{A}_i^T$ is skew-symmetric

- CONCLUSION: there must be a vector, ω_i, whose cross product matrix is equal to the 3×3 skew symmetric matrix $\dot{\mathbf{A}}_i \mathbf{A}_i^T$:

$$\tilde{\omega}_i = \dot{\mathbf{A}}_i \mathbf{A}_i^T$$

- This vector ω_i is called the angular velocity of the L-RF with respect to the G-RF.

Angular Velocity: Represented in G-RF or in L-RF

- Since $\mathbf{A}_i$ is orthogonal, rate of change $\dot{\mathbf{A}}_i$ of orientation matrix is simply

$$\dot{\mathbf{A}}_i = \tilde{\omega}_i \mathbf{A}_i$$

- Angular velocity vector can be represented in the *local* reference frame. Skipping details,

$$\tilde{\bar{\omega}}_i = \mathbf{A}_i^T \dot{\mathbf{A}}_i$$

- Therefore, rate of change $\dot{\mathbf{A}}_i$ of orientation matrix can also be represented as

$$\dot{\mathbf{A}}_i = \mathbf{A}_i \tilde{\bar{\omega}}_i$$

- Notation convention: an over-bar placed on a vector (like $\bar{\omega}_i$ above) indicates that quantity is a representation of a geometric vector in a local reference frame

New Topic:
Using Euler Parameters to Define Rotation Matrix A

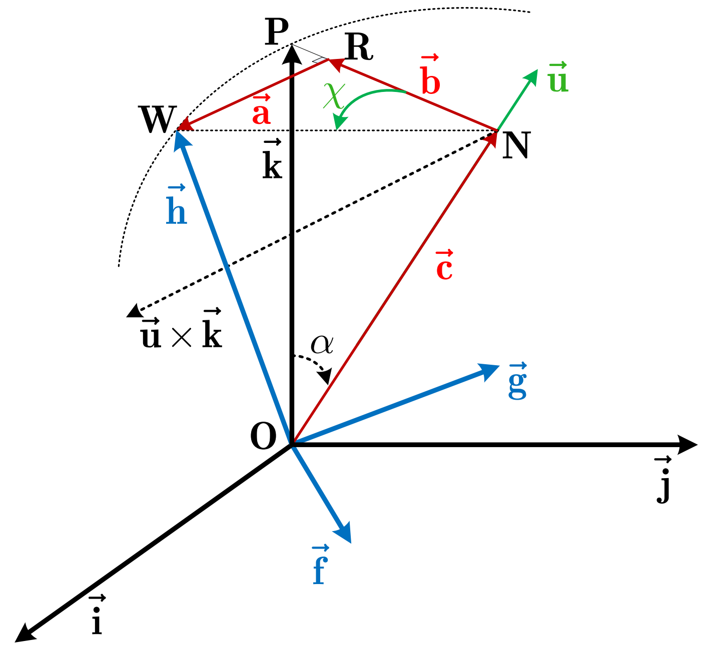
- Starting point: Euler's Theorem

  "If the origins of two right-hand Cartesian reference frames coincide, then the RFs may be brought into coincidence by a single rotation of a certain angle χ about a carefully chosen unit axis **u**"

- Euler's Theorem proved in the following references:
  - Wittenburg – Dynamics of Systems of Rigid Bodies (1977)
  - Goldstein – Classical Mechanics, 2nd edition, (1980)
  - Angeles – Fundamentals of Robotic Mechanical Systems (2003)

Warming up…

- Green color - used for quantities that define the Euler rotation: the axis of rotation defined by the **unit** vector $\vec{\mathbf{u}}$ and the angle χ
- Red color - used to indicate the vectors that need to be summed up to get axis $\vec{\mathbf{h}}$ of the L-RF
- Blue color - denotes the $\vec{\mathbf{f}} - \vec{\mathbf{g}} - \vec{\mathbf{h}}$ axes of the L-RF
- Black dotted line - support entities (helpers, don't play any role but only help with the derivation). The angle α measured between the axis of rotation $\vec{\mathbf{u}}$ and the $\vec{\mathbf{k}}$ unit vector.



- **Other notation used:** $||\vec{\mathbf{a}}|| = a \qquad ||\vec{\mathbf{b}}|| = b \qquad ||\vec{\mathbf{c}}|| = c$

How Euler Parameters Come to Be

- Using as input χ and $\mathbf{u}$, one can express the vectors $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, and $\vec{\mathbf{h}}$ in the global reference frame as

$$\begin{aligned}
\mathbf{f} &= \mathbf{i}(2\cos^2\tfrac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T\mathbf{i})\sin^2\tfrac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{i}\sin\tfrac{\chi}{2}\cos\tfrac{\chi}{2} \\
\mathbf{g} &= \mathbf{j}(2\cos^2\tfrac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T\mathbf{j})\sin^2\tfrac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{j}\sin\tfrac{\chi}{2}\cos\tfrac{\chi}{2} \\
\mathbf{h} &= \mathbf{k}(2\cos^2\tfrac{\chi}{2} - 1) + 2\mathbf{u}(\mathbf{u}^T\mathbf{k})\sin^2\tfrac{\chi}{2} + 2\tilde{\mathbf{u}}\mathbf{k}\sin\tfrac{\chi}{2}\cos\tfrac{\chi}{2}
\end{aligned}$$

- The expression of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ justifies the introduction of the following generalized coordinates (the "Euler Parameters"):

$$\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad \text{where} \qquad e_0 = \cos\frac{\chi}{2} \quad \text{and} \quad \mathbf{e} \equiv \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \mathbf{u}\sin\frac{\chi}{2}$$

- Note: $\mathbf{u}$ unit vector $\Rightarrow$ values of e_0, e_1, e_2, and e_3 must satisfy the normalization condition

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = e_0^2 + \mathbf{e}^T\mathbf{e} = 1$$

Orientation Matrix, Based on Euler Parameters

- Based on definition of e_0, e_1, e_2, and e_3,

$$
\begin{aligned}
\mathbf{f} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{i} \\
\mathbf{g} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{j} \\
\mathbf{h} &= [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]\mathbf{k}
\end{aligned}
$$

- Recall that $\mathbf{A} = [\mathbf{f} \quad \mathbf{g} \quad \mathbf{h}]$
- Therefore,

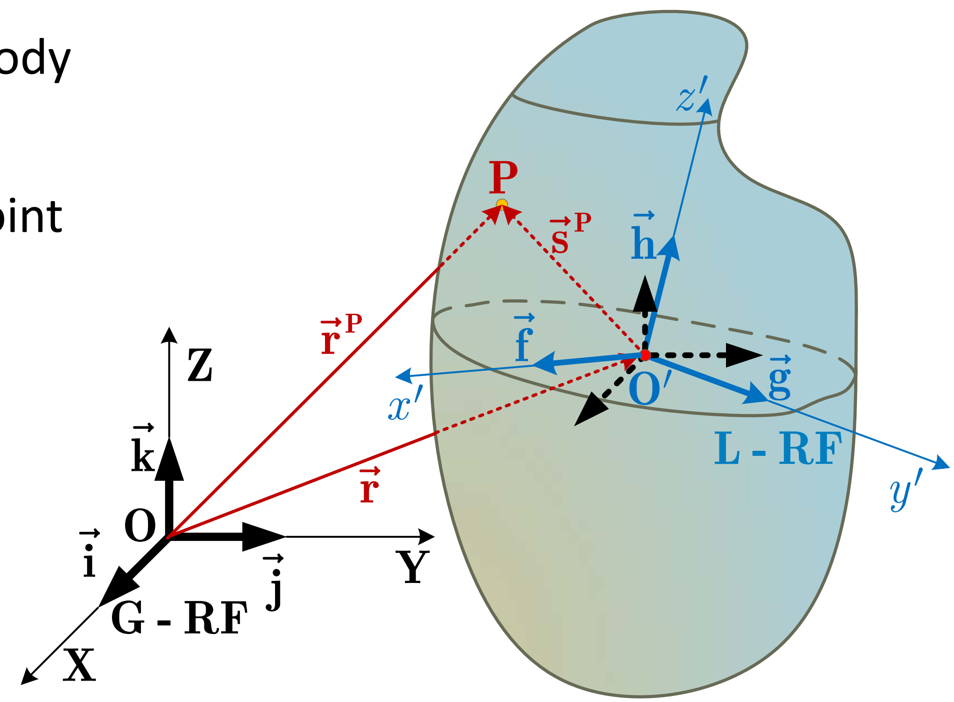
$$
\mathbf{A} = [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})]
$$

- Equivalently,

$$
\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}
$$

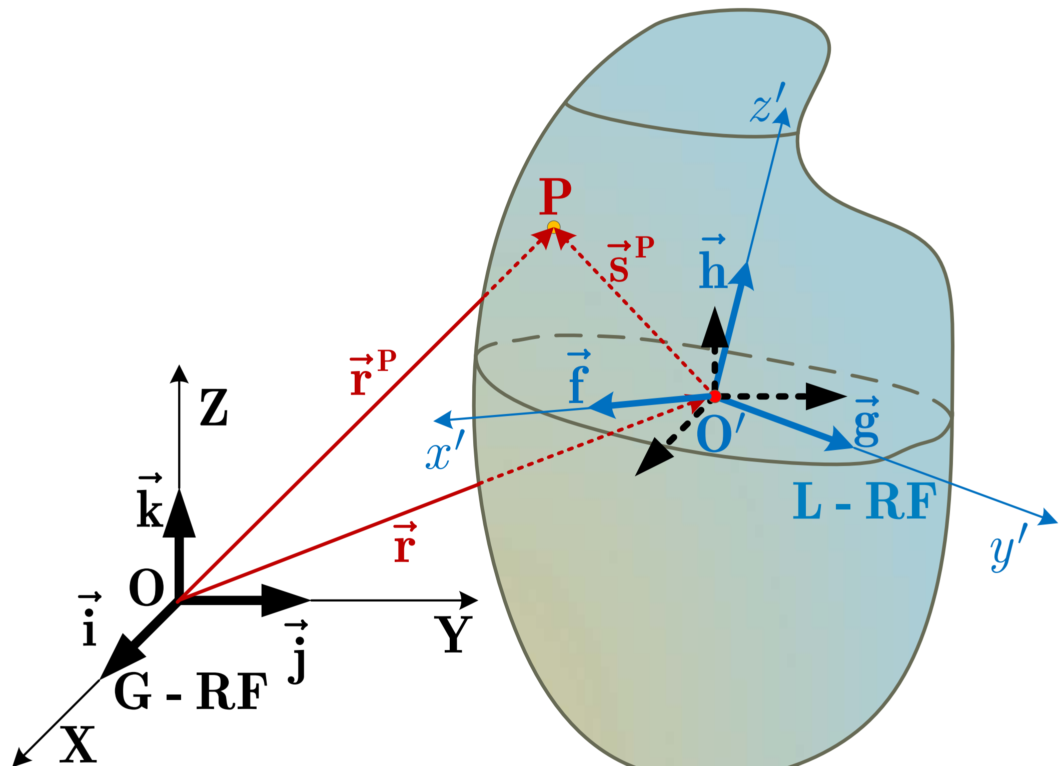
New Topic:
Beyond Rotations – Full 3D Kinematics of Rigid Bodies

- So far, focus was only on the rotation of a rigid body
- Body connected to ground through a spherical joint
  - Body experienced an arbitrary rotation
- Yet bodies are experiencing both translation and rotation



3D Kinematics of Rigid Body: Problem Backdrop

- Framework and Notation Conventions:
  - A L-RF is attached to the rigid body at some location denoted by O'
  - Relative to the G-RF, point O' is located by vector $\vec{\mathbf{r}}$
  - L-RF defined by vectors $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, $\vec{\mathbf{h}}$
  - An arbitrary point P of the rigid body is considered. Its location relative to the L-RF is provided through the vector $\vec{\mathbf{s}}^P$



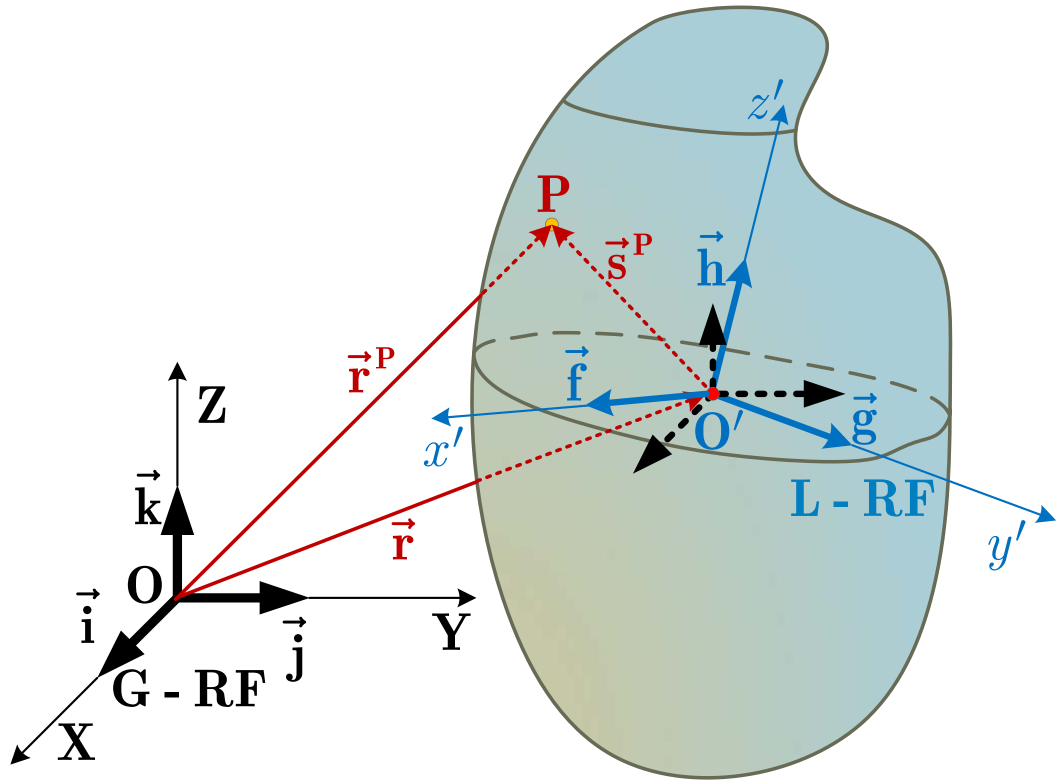
3D Rigid Body Kinematics: Position of an Arbitrary Point P

- **The Geometric View:**

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$$
$$\Downarrow$$
$$\vec{\mathbf{r}}^P = \vec{\mathbf{r}} + \vec{\mathbf{s}}^P$$

- **The Algebraic Representation:**

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\bar{\mathbf{s}}^P$$

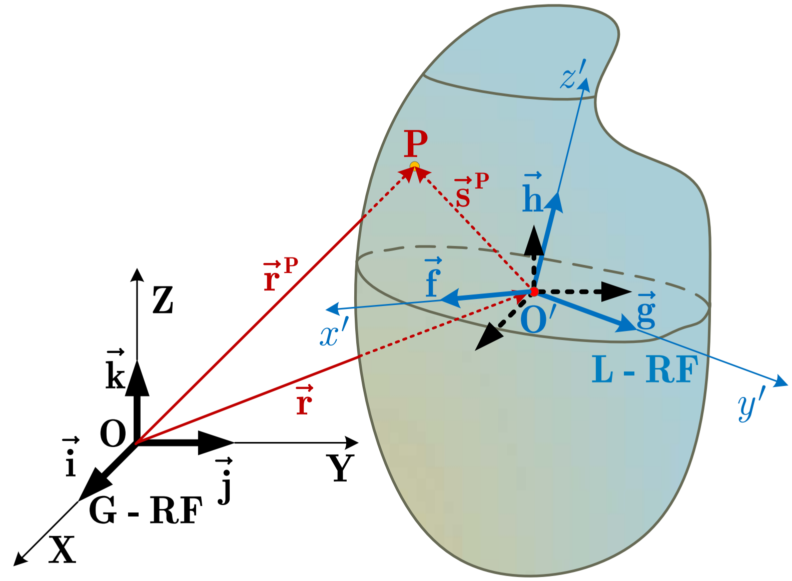


- Important observation:
  - The vector $\bar{\mathbf{s}}^P$ that provides the location of P in the L-RF is a constant vector
    - True because the body is assumed to be rigid

3D Rigid Body Kinematics: Velocity of Arbitrary Point P

- In the Geometric Vector world:

$$\vec{\mathbf{v}}^P = \frac{d\vec{\mathbf{r}}^P}{dt} = \dot{\vec{\mathbf{r}}} + \dot{\vec{\mathbf{s}}}^P = \dot{\vec{\mathbf{r}}} + \vec{\omega} \times \vec{\mathbf{s}}^P$$



- Using the Algebraic Vector representation (Chrono):

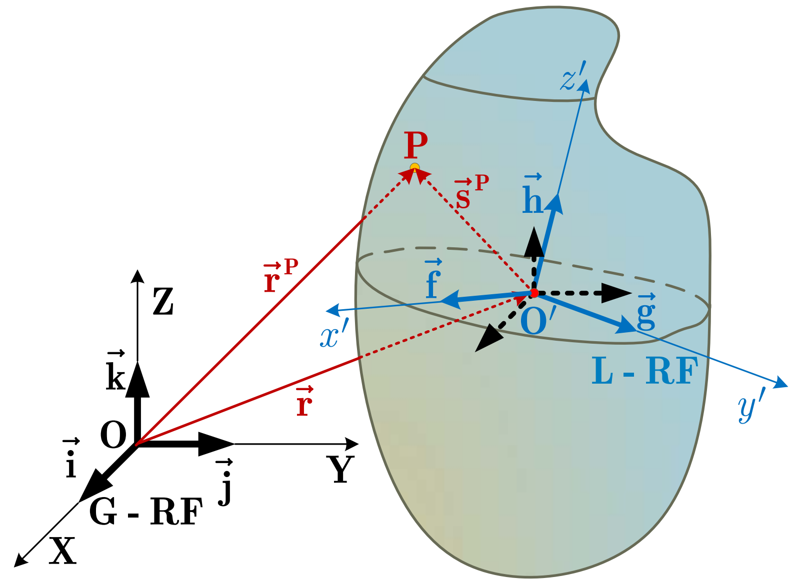
$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{s}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\bar{\mathbf{s}}^P = \dot{\mathbf{r}} + \tilde{\omega}\mathbf{A}\bar{\mathbf{s}}^P = \dot{\mathbf{r}} + \tilde{\omega}\mathbf{s}^P$$

- In plain words: the velocity $\dot{\mathbf{r}}^P$ of a point P is equal to the sum of the velocity $\dot{\mathbf{r}}$ of the point where the L-RF is located and the velocity $\tilde{\omega}\mathbf{s}^P$ due to the rotation with angular velocity ω of the rigid body

3D Rigid Body Kinematics: Acceleration of Arbitrary Point P

- In the Geometric Vector world, by definition:

$$\vec{\mathbf{a}}^P \equiv \frac{d^2\vec{\mathbf{r}}^P}{dt^2} = \ddot{\vec{\mathbf{r}}} + \vec{\omega} \times \vec{\omega} \times \vec{\mathbf{s}}^P + \dot{\vec{\omega}} \times \vec{\mathbf{s}}^P$$



- Using the Algebraic Vector representation (Chrono):

$$\mathbf{a}^P \equiv \ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\mathbf{s}}^P = \ddot{\mathbf{r}} + \tilde{\omega}\tilde{\omega}\mathbf{A}\bar{\mathbf{s}}^P + \tilde{\dot{\omega}}\mathbf{A}\bar{\mathbf{s}}^P = \ddot{\mathbf{r}} + \tilde{\omega}\tilde{\omega}\mathbf{s}^P + \tilde{\dot{\omega}}\mathbf{s}^P$$

Putting Things in Perspective: What We've Covered so Far

- Discussed how to get the expression of a geometric vector in a "destination" reference frame knowing its expression in a "source" reference frame
  - Done via rotation matrix A

- Euler Parameters: a way of computing the A matrix when knowing the axis of rotation and angle of rotation

- Rate of change of the orientation matrix A $\rightarrow$ led to the concept of angular velocity

- Position, velocity and acceleration of a point P attached to a rigid body

Looking Ahead

- Kinematic constraints; i.e., joints
- Formulating the equations of motion

New Topic:
Kinematic Constraints

- Geometric Constraint (GCon): a real world geometric attribute of the motion of the mechanical system
  - Examples:
    - Particle moves around point (1,2,3) on a sphere of radius 2.0
    - A unit vector $\mathbf{u}_6$ on body 6 is perpendicular on a certain unit vector $\mathbf{u}_9$ on body 9
    - The y coordinate of point Q on body 8 is 14.5

- Algebraic Constraint Equations (ACEs): in the virtual world, a collection of one or more algebraic constraints, involving the generalized coordinates of the mechanism and possibly time t, that capture the geometry of the motion as induced by a certain Geometric Constraint
  - Examples:
    - $(x-1)^2 + (y-2)^2 + (z-3)^2 - 4 = 0$
    - $\mathbf{u}_6^T \cdot \mathbf{u}_9 = 0$
    - $[0\ 1\ 0] \cdot \mathbf{r}_8^Q - 14.5 = 0$

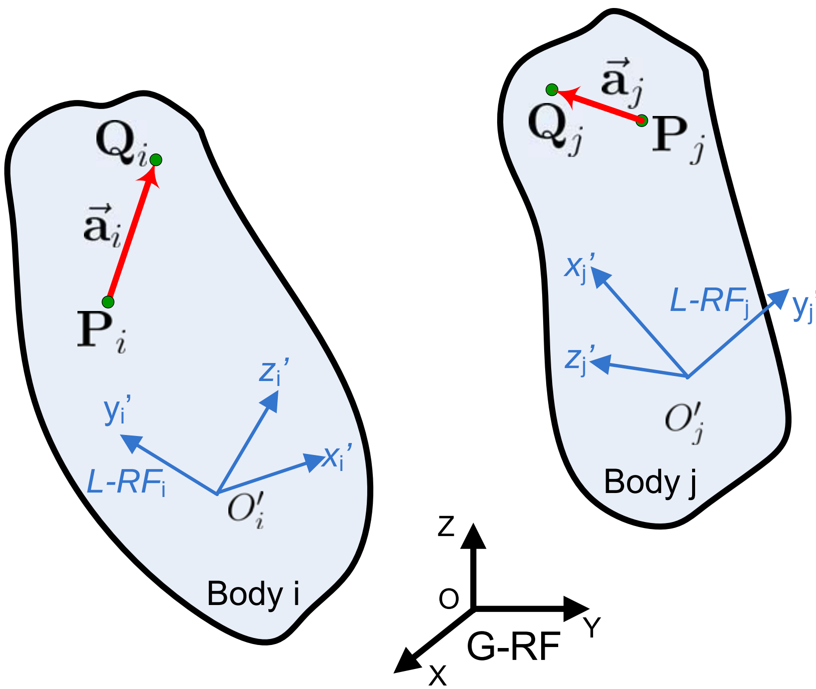
- Modeling: the process that starts with the idealization of the real world to yield a GCon and continues with the GCon abstracting into a set of ACEs

Basic Geometric Constraints (GCons)

- We have four basic GCons:
  - DP1: the dot product of two vectors on two bodies is specified
  - DP2: the dot product of a vector of on a body and a vector between two bodies is specified
  - D: the distance between two points on two different bodies is specified
  - CD: the difference between the coordinates of two bodies is specified

- Note:
  - DP1 stands for Dot Product 1
  - DP2 stands for Dot Product 2
  - D stands for distance
  - CD stands for coordinate difference

Basic GCon: DP1



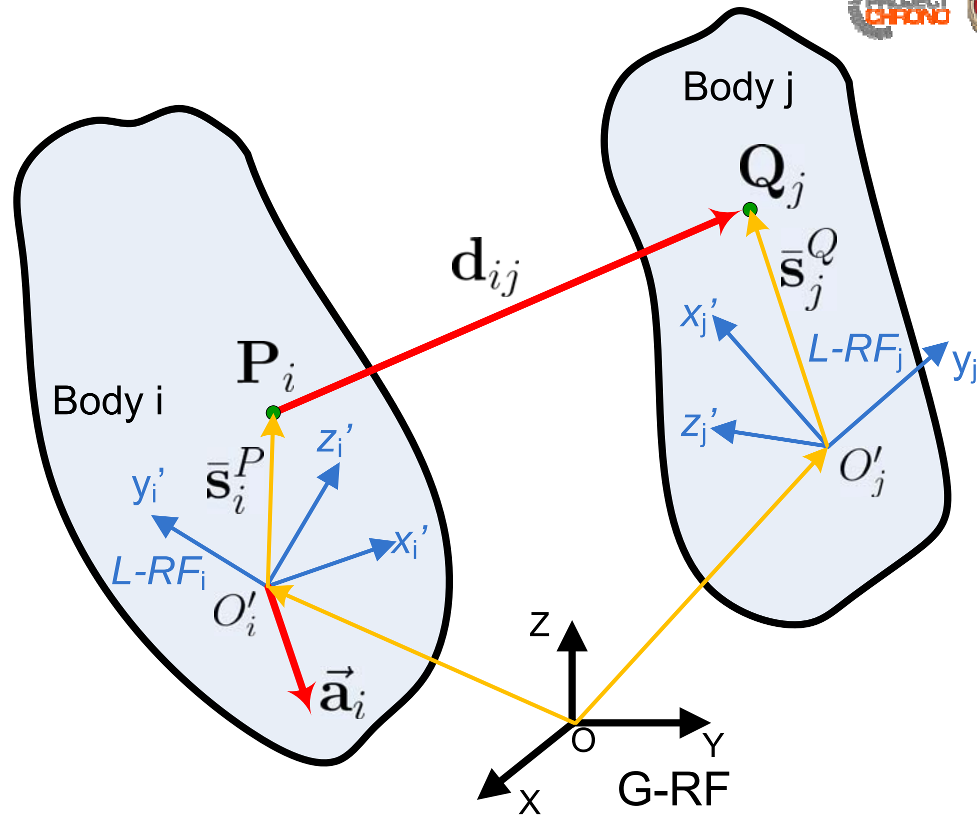
- Geometrically:

$$\vec{\mathbf{a}}_i \cdot \vec{\mathbf{a}}_j - f(t) = 0$$

- Algebraically (matrix-vector notation):

$$\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = 0$$

Basic GCon: DP2



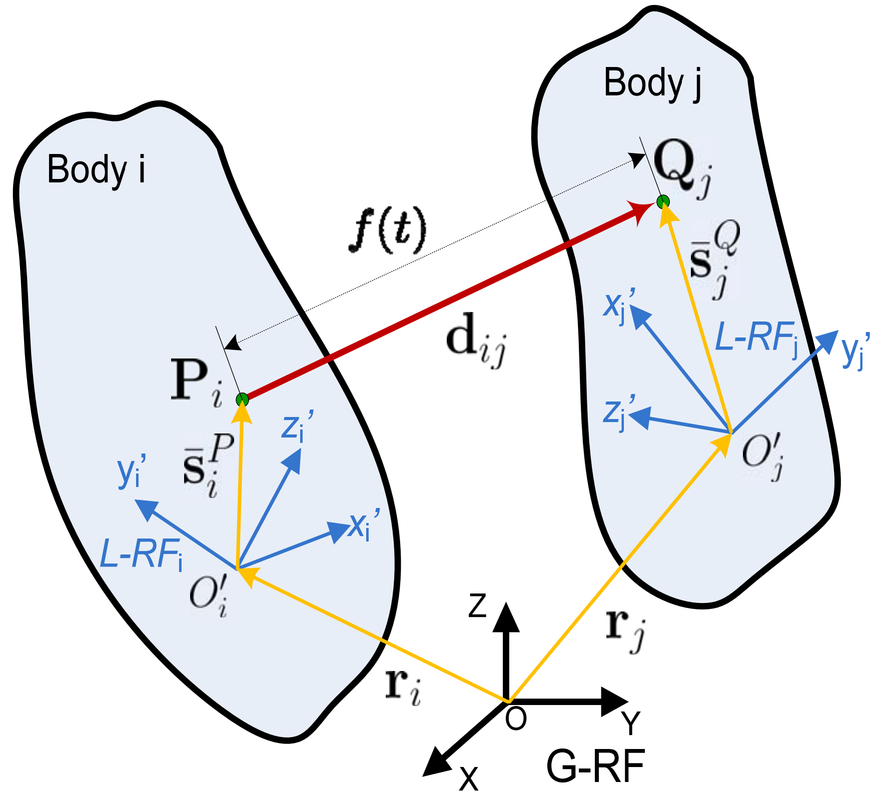
- Geometrically:

$$\vec{\mathbf{a}}_i \cdot \vec{\mathbf{d}}_{ij} - f(t) = 0$$

- Algebraically (matrix-vector notation):

$$\begin{aligned}\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) &= \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) \\ &= \bar{\mathbf{a}}_i^T \mathbf{A}_i^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0\end{aligned}$$

Basic GCon: D



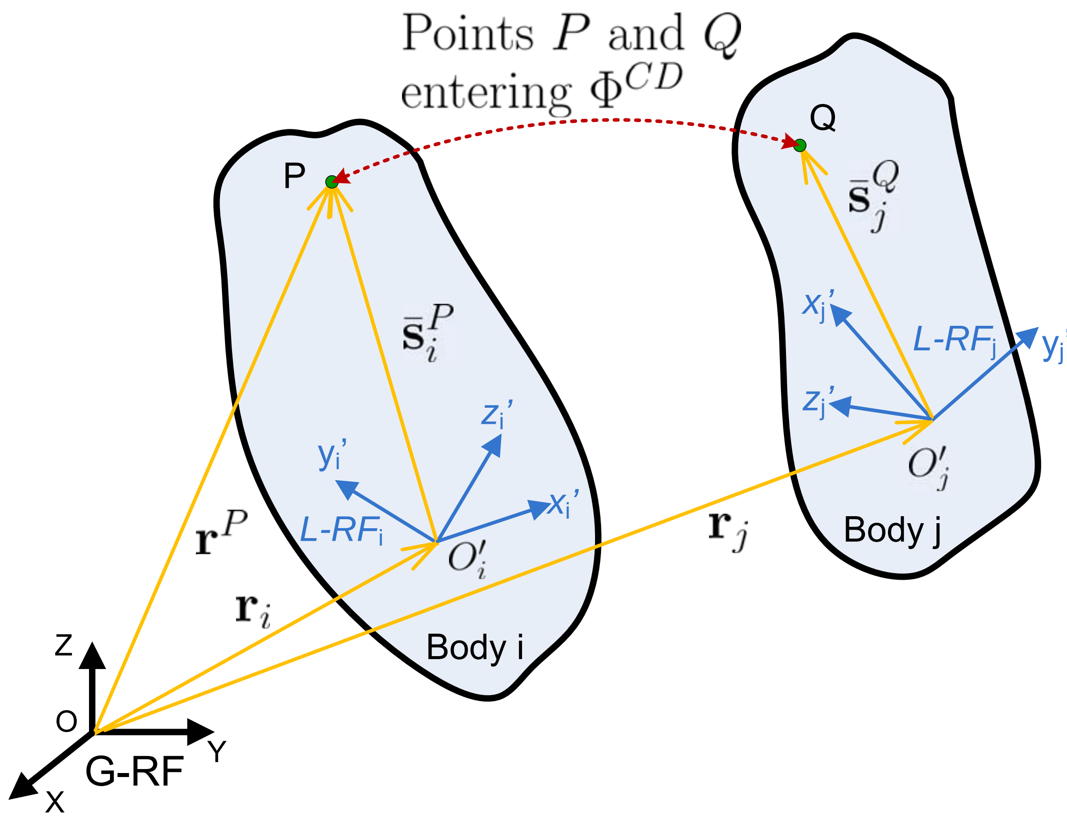
- Geometrically:

$$\vec{\mathbf{d}}_{ij} \cdot \vec{\mathbf{d}}_{ij} - f^2(t) = 0$$

- Algebraically (matrix-vector notation):

$$\begin{aligned}\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) &= \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f^2(t) \\ &= (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P)^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f^2(t) = 0\end{aligned}$$

Basic GCon: CD



- Geometrically ($\mathbf{c}$ is a constant vector):

$$\vec{\mathbf{c}} \cdot (\vec{\mathbf{a}}_j - \vec{\mathbf{a}}_i) - f(t) = 0$$

- Algebraically (matrix-vector notation):

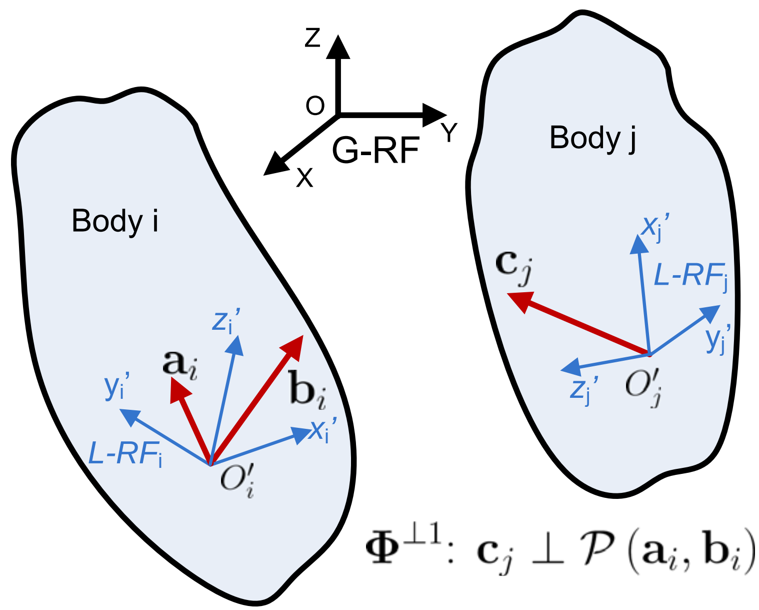
$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = \mathbf{c}^T(\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0$$

Intermediate GCons

- Two Intermediate GCons:
  - $\perp 1$: a vector is perpendicular on a plane belonging to a different body
  - $\perp 2$: a vector between two bodies is perpendicular to a plane belonging to the different body

Intermediate GCon: ⊥1 (Perpendicular Type 1)

- Geometrically, the motion is such that a vector $\mathbf{c}_j$ on body j is perpendicular to a plane of body i that is defined by $\mathbf{a}_i$ and $\mathbf{b}_i$

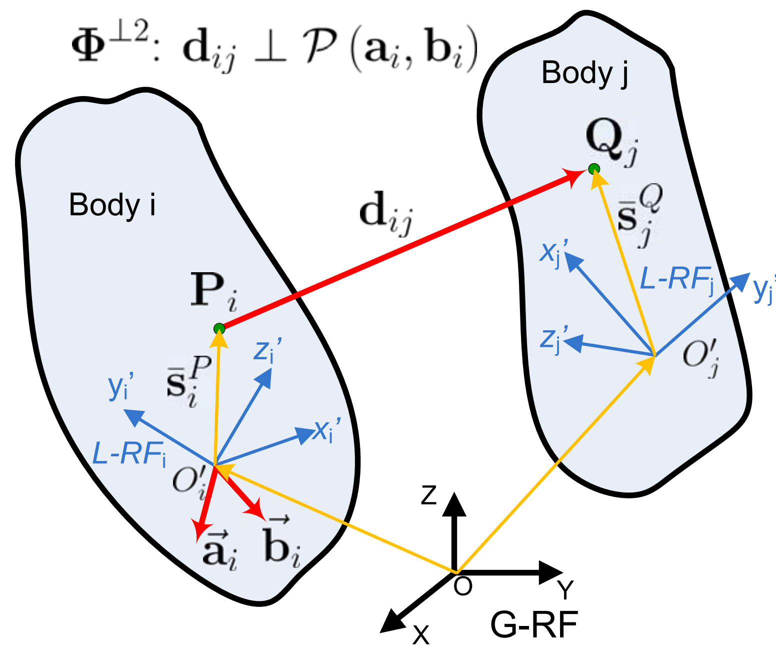


- Algebraically (matrix-vector notation):

$$\Phi^{\perp 1}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j) = \begin{bmatrix} \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{c}}_j, 0) \\ \Phi^{DP1}(i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j, 0) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{c}}_j \\ \bar{\mathbf{b}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{c}}_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Intermediate GCon: ⊥2 (Perpendicular Type 2)

- Geometrically, a vector $\overrightarrow{P_iQ_j}$ from body i to body j remains perpendicular to a plane defined by two vectors $\vec{\mathbf{a}}_i$ and $\vec{\mathbf{b}}_i$



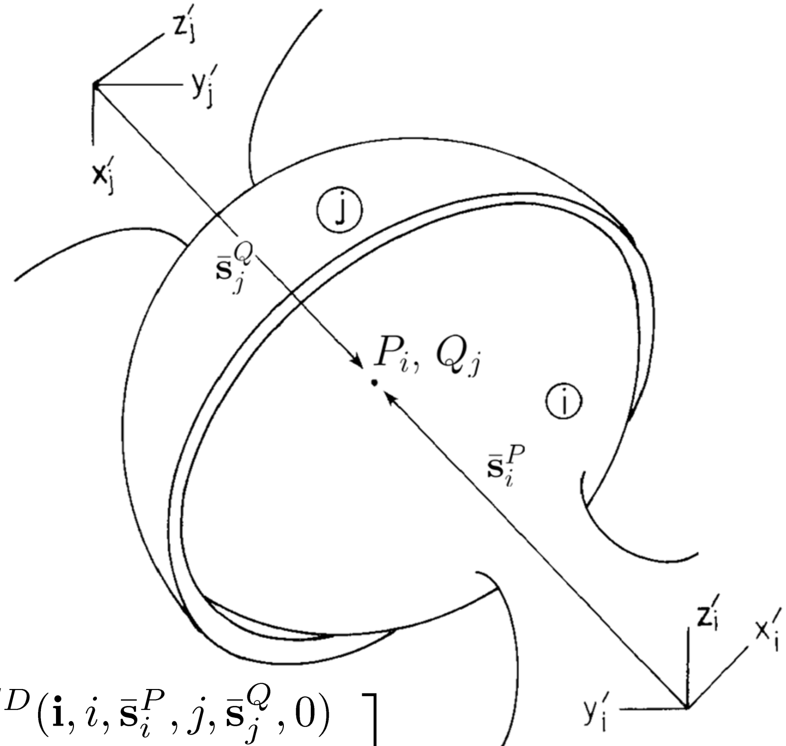
- Algebraically (matrix-vector notation):

$$\Phi^{\perp 2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q) = \begin{bmatrix} \Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{DP2}(i, \bar{\mathbf{b}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} \\ \bar{\mathbf{b}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} \end{bmatrix} = 0$$

High Level GCons

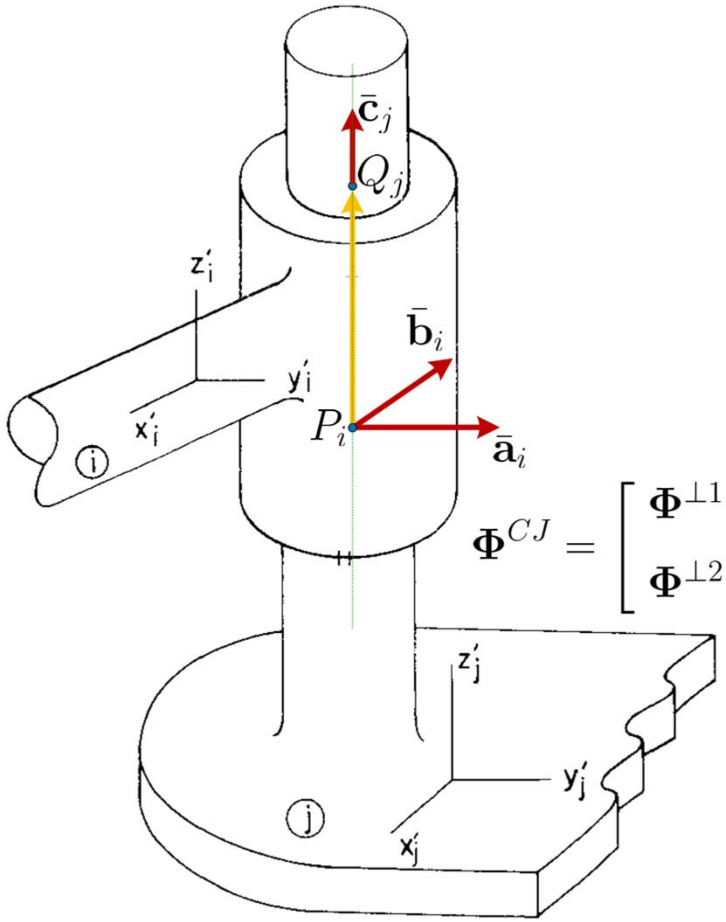
- High Level GCons also called joints:
  - Spherical Joint (SJ)
  - Universal Joint (UJ)
  - Cylindrical Joint (CJ)
  - Revolute Joint (RJ)
  - Translational Joint (TJ)
  - Other composite joints (spherical-spherical, translational-revolute, etc.)

High Level GCon: SJ [Spherical Joint]



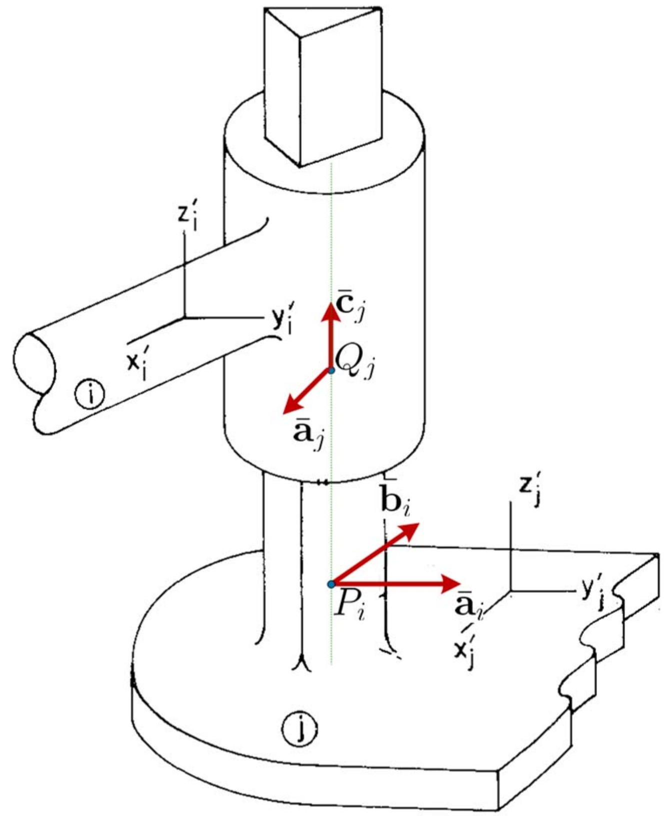
$$\mathbf{\Phi}^{SJ} = \begin{bmatrix} \Phi^{CD}(\mathbf{i}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{j}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{k}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \end{bmatrix}$$

High Level GCon: CJ [Cylindrical Joint]



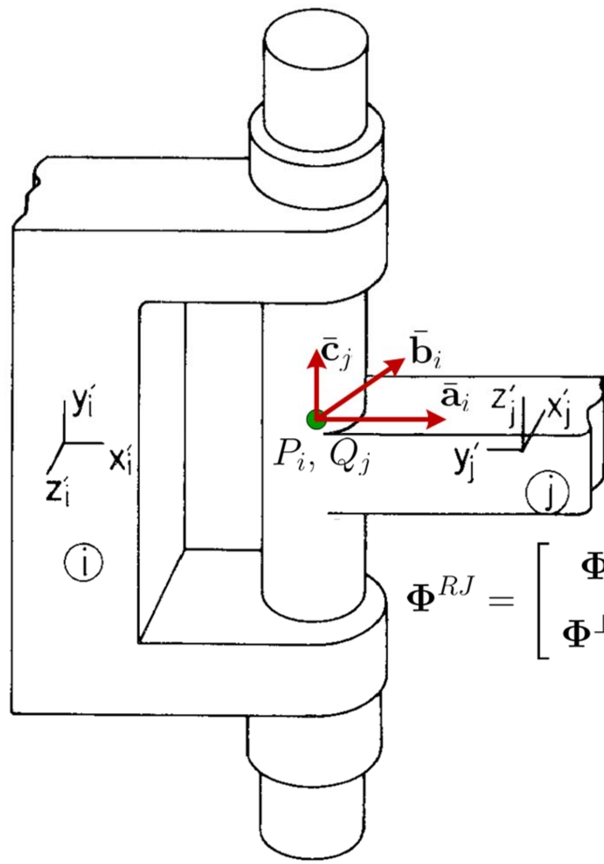
$$\mathbf{\Phi}^{CJ} = \begin{bmatrix} \mathbf{\Phi}^{\perp 1}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j) \\ \mathbf{\Phi}^{\perp 2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q) \end{bmatrix}$$

High Level GCon: TJ [Translational Joint]



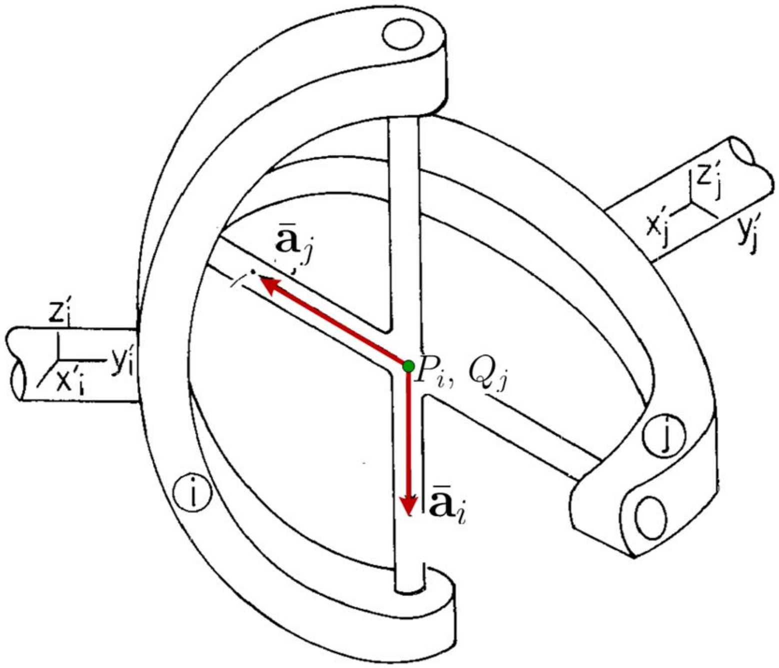
$$\mathbf{\Phi}^{TJ} = \begin{bmatrix} \mathbf{\Phi}^{CJ}(i, \bar{\mathbf{s}}_i^P, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{s}}_j^Q, \bar{\mathbf{c}}_j) \\ \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, \text{const.}) \end{bmatrix}$$

High Level GCon: RJ [Revolute Joint]



$$\mathbf{\Phi}^{RJ} = \begin{bmatrix} \mathbf{\Phi}^{SJ}(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q) \\ \mathbf{\Phi}^{\perp 1}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j) \end{bmatrix}$$

High Level GCon: UJ [Universal Joint]



$$\mathbf{\Phi}^{UJ} = \left[\begin{array}{c} \mathbf{\Phi}^{SJ}(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q) \\ \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, 0) \end{array} \right]$$

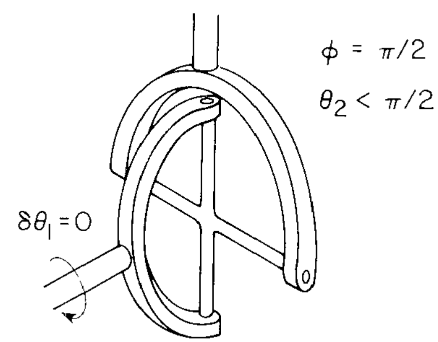


Figure 9.4.15 Singular behavior of universal joint.

Connection Between Basic and Intermediate/High Level GCons

	$DP1$	$DP2$	D	CD
$\perp 1$	$\times\times$			
$\perp 2$		$\times\times$		
SJ				$\times\ \times\ \times$
UJ	$\times$			$[\times\ \times\ \times]_{SJ}$
CJ	$[\times\times]_{\perp 1}$	$[\times\times]_{\perp 2}$		
RJ	$[\times\times]_{\perp 1}$			$[\times\ \times\ \times]_{SJ}$
TJ	$\times\,[\times\times]_{\perp 1}$	$[\times\times]_{\perp 2}$		

- Note that there are other GCons that are used, but they see less mileage

Constraints Supported in Chrono

New Topic:
Formulating the Equations of Motion

- Road map, full derivation of constrained equations of motion
  - Step 1: Introduce the types of force acting on one body present in a mechanical system
    - Distributed
    - Concentrated
  - Step 2: Express the virtual work produced by each of these forces acting on *one body*
  - Step 3: Evaluate the virtual work for the *entire mechanical system*
  - Step 4: Apply principle of virtual work (via D'Alembert's principle) to obtain the EOM

Generic Forces/Torques Acting on a Mechanical System

- Distributed forces
  - Inertia forces
  - Volume/Mass distributed force (like gravity, electromagnetic, etc.)
  - Internal forces

- Concentrated forces/torques
  - Reaction forces/torques (induces by the presence of kinematic constraints)
  - Externally applied forces and torques (me pushing a cart)

Virtual Work for One Body, Side Trip

- Quick example below only shows virtual work produced by the **inertial force**
  - Same recipe applied for all other forces, distributed or concentrated

- Starting point: consider point P of body i associated with infinitesimal mass element $dm_i(P)$
- Expression of the force:

$$-\ddot{\mathbf{r}}_i^P \, dm_i(P)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot [-\ddot{\mathbf{r}}_i^P \, dm_i(P)]$$

- Body virtual work obtained by summing over all points P of body i:

$$\delta \mathcal{W} = \int_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \, dm_i(P)$$

- Upon expressing virtual displacement of P and its acceleration $\ddot{\mathbf{r}}_i^P$:

$$\delta \mathcal{W} = \int_{m_i} [\delta \mathbf{r}_i^T + \delta \bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T] \cdot \left[\ddot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P + \mathbf{A}_i \tilde{\dot{\bar{\omega}}}_i \bar{\mathbf{s}}_i^P\right] \, dm_i(P) = \delta \mathbf{r}_i^T m_i \ddot{\mathbf{r}}_i + \delta \bar{\pi}_i^T \left[\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i + \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i\right]$$

Final Form, Expression of Virtual Work

- When all said and done, the expression of the virtual work assumes the form:

$$\begin{aligned}\delta\mathcal{W} &= \sum_{i=1}^{nb}\left[-\delta\mathbf{r}_i^T m_i\ddot{\mathbf{r}}_i - \delta\bar{\pi}_i^T\tilde{\bar{\omega}}_i\bar{\mathbf{J}}_i\bar{\omega}_i - \delta\bar{\pi}_i^T\bar{\mathbf{J}}_i\dot{\bar{\omega}}_i + \delta\mathbf{r}_i^T\cdot\mathbf{F}_i^m + \delta\bar{\pi}_i^T\cdot\bar{\mathbf{n}}_i^m\right.\\ &+ \left.\delta\mathbf{r}_i^T\mathbf{F}_i^a + \delta\bar{\pi}_i^T\bar{\mathbf{n}}_i^a + \delta\mathbf{r}_i^T\mathbf{F}_i^r + \delta\bar{\pi}_i^T\bar{\mathbf{n}}_i^r\ \right] = 0\end{aligned}$$

- Alternatively,

$$\delta\mathcal{W} = \sum_{i=1}^{nb}\left[\ \delta\mathbf{r}_i^T\left(-m_i\ddot{\mathbf{r}}_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r\right) + \delta\bar{\pi}_i^T\left(-\tilde{\bar{\omega}}_i\bar{\mathbf{J}}_i\bar{\omega}_i - \bar{\mathbf{J}}_i\dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r\right)\ \right] = 0$$

Moving from One Body to a Mechanical System

- Total virtual work, for the entire system, assumes the form:

$$\begin{aligned}\delta W &= \sum_{i=1}^{nb} \left[-\delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i m_i - \delta \bar{\pi}_i^T \tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \delta \bar{\pi}_i^T \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta \bar{\pi}_i^T \cdot \bar{\mathbf{n}}_i^m \right. \\ &+ \left. \delta \mathbf{r}_i^T \mathbf{F}_i^a + \delta \bar{\pi}_i^T \bar{\mathbf{n}}_i^a + \delta \mathbf{r}_i^T \mathbf{F}_i^r + \delta \bar{\pi}_i^T \bar{\mathbf{n}}_i^r \ \right] = 0\end{aligned}$$

- Alternatively,

$$\delta W = \sum_{i=1}^{nb} \left[\ \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \right) + \delta \bar{\pi}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r \right) \ \right] = 0$$

- Recall that for each body i, virtual translations $\delta \mathbf{r}_i$ and virtual rotations $\delta \bar{\pi}_i$ are arbitrary

Equations of Motion (EOM) for A System of Rigid Bodies

- Since equation on previous slide should hold for *any* set of virtual displacements $(\delta\mathbf{r}_1, \delta\bar{\pi}_1)$, $(\delta\mathbf{r}_2, \delta\bar{\pi}_2)$,..., $(\delta\mathbf{r}_{nb}, \delta\bar{\pi}_{nb})$, then we necessarily have that for $i = 1, \ldots, nb$:

$$\begin{aligned} -m_i\ddot{\mathbf{r}}_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r &= \mathbf{0}_3 \\ -\tilde{\bar{\omega}}_i\bar{\mathbf{J}}_i\bar{\omega}_i - \bar{\mathbf{J}}_i\dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r &= \mathbf{0}_3 \end{aligned}$$

- Equivalently, for $i = 1, \ldots, nb$

$$\begin{aligned} m_i\ddot{\mathbf{r}}_i &= \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \\ \bar{\mathbf{J}}_i\dot{\bar{\omega}}_i &= \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r - \tilde{\bar{\omega}}_i\bar{\mathbf{J}}_i\bar{\omega}_i \end{aligned}$$

- The set of equations above represent the EOM for the system of nb rigid bodies.

The Joints (Kinematic Constraints) Lead to Reaction Forces

- The collection of all nc kinematic and driving constraints – stack them together:

$$\mathbf{\Phi}(\mathbf{q}, t) = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \end{bmatrix} = \mathbf{0}_{nc}$$

- Recall that any one of the constraints in $\mathbf{\Phi}$ is one of the four basic GCons introduced earlier
- The variation of $\mathbf{\Phi}$: stack together the variation of each of the GCons that enters in $\mathbf{\Phi}$
- A virtual displacement of the bodies in the system will lead to a virtual variation $\delta\mathbf{\Phi}$ that depends on the position and orientation of the bodies:

$$\delta\mathbf{\Phi} = \mathbf{\Phi}_{\mathbf{r}}\delta\mathbf{r} + \bar{\mathbf{\Pi}}(\mathbf{\Phi})\delta\bar{\pi} = \mathbf{0}_{nc}$$

- In matrix form, we can express the above relations as

$$\delta\mathbf{\Phi}(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\mathbf{\Phi}) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \mathbf{0}_{nc}$$

- $\mathbf{\Phi}_{\mathbf{r}}$ and $\bar{\mathbf{\Pi}}(\mathbf{\Phi})$: the key ingredients needed to express the reaction forces induced by the constraints $\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0}_{nc}$

Switching to Matrix-Vector Notation

- Notation used to simplify expression of EOM:
  - $\mathbf{I}_3$ is the identity matrix of dimension 3
  - $\mathbf{F}_i^a$ and $\mathbf{F}_i^m$ – applied and mass-distributed force, body i
  - $\bar{\mathbf{n}}_i^a$ and $\bar{\mathbf{n}}_i^m$ – applied and mass-distributed torque, body i
  - m_i and $\bar{\mathbf{J}}_i$ – mass and mass moment of inertia, body i

- Matrix-vector notation:

$$\mathbf{M} = \begin{bmatrix} m_1\mathbf{I}_3 & \mathbf{0}_{3\times3} & \dots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & m_2\mathbf{I}_3 & \dots & \mathbf{0}_{3\times3} \\ \dots & \dots & \dots & \dots \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \dots & m_{nb}\mathbf{I}_3 \end{bmatrix} \qquad \bar{\mathbf{J}} = \begin{bmatrix} \bar{\mathbf{J}}_1 & \mathbf{0}_{3\times3} & \dots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \bar{\mathbf{J}}_2 & \dots & \mathbf{0}_{3\times3} \\ \dots & \dots & \dots & \dots \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \dots & \bar{\mathbf{J}}_{nb} \end{bmatrix}$$

$$\ddot{\mathbf{r}} = \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \vdots \\ \ddot{\mathbf{r}}_{nb} \end{bmatrix}_{3nb} \qquad \dot{\bar{\omega}} = \begin{bmatrix} \dot{\bar{\omega}}_1 \\ \vdots \\ \dot{\bar{\omega}}_{nb} \end{bmatrix}_{3nb} \qquad \mathbf{F} = \begin{bmatrix} \mathbf{F}_1^a + \mathbf{F}_1^m \\ \vdots \\ \mathbf{F}_{nb}^a + \mathbf{F}_{nb}^m \end{bmatrix}_{3nb} \qquad \tau = \begin{bmatrix} \bar{\mathbf{n}}_1^a + \bar{\mathbf{n}}_1^m - \tilde{\bar{\omega}}_1\bar{\mathbf{J}}_1\bar{\omega}_1 \\ \vdots \\ \bar{\mathbf{n}}_{nb}^a + \bar{\mathbf{n}}_{nb}^m - \tilde{\bar{\omega}}_{nb}\bar{\mathbf{J}}_{nb}\bar{\omega}_{nb} \end{bmatrix}_{3nb}$$

EOM: the Newton-Euler Form

- According to Lagrange Multiplier theorem, there exists a vector of Lagrange Multipliers, $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{nc} \end{bmatrix}$, so that

$$\begin{bmatrix} \mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} \\ \bar{\mathbf{J}}\dot{\bar{\omega}} - \bar{\tau} \end{bmatrix} + \begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}}^T \\ \bar{\mathbf{\Pi}}^T(\mathbf{\Phi}) \end{bmatrix} \boldsymbol{\lambda} = \mathbf{0}_{6nb}$$

- Expression above: Newton-Euler form of the EOM. Equivalently expressed as:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^T \boldsymbol{\lambda} = \mathbf{F} \\ \bar{\mathbf{J}}\dot{\bar{\omega}} + \bar{\mathbf{\Pi}}^T(\mathbf{\Phi})\boldsymbol{\lambda} = \bar{\tau} \end{cases}$$