# Advanced Noise and Vibration Analysis (ANVA)

**Chapter 2**

- Forced vibration problems
- Frequency response
- Direct solution
- Modal superposition

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# Learning Objectives

**1. Forced Vibration Problems :**

- Understand the fundamental concepts of forced vibration in mechanical and structural systems.

**2. Frequency Response**

- Analyze the steady-state response of a system subjected to harmonic excitation.
- Define and interpret the frequency response function (FRF) and its significance in vibration analysis.

**3. Direct Solution Method**

- Apply the direct solution approach to solve the equations of motion under harmonic excitation.
- Derive the algebraic formulation of the forced vibration response in the frequency domain.
- Utilize numerical and analytical techniques to compute system responses.

**4. Modal Superposition Method**

- Explain the concept of mode shapes and natural frequencies in multi-degree-of-freedom (MDOF) systems.
- Decompose the system response into a sum of modal contributions.
- Solve forced vibration problems using modal superposition to simplify complex systems.

# Forced Vibrations (Undamped)

▶ **Forced vibrations** in an undamped system are governed by the equation:

$$[\mathbf{M}]\{\ddot{\mathbf{D}}(t)\} + [\mathbf{K}]\{\mathbf{D}(t)\} = \{\mathbf{f}(t)\}$$

initial conditions $\{\mathbf{D}(0)\} = \{\mathbf{D}\}_0, \quad \{\dot{\mathbf{D}}(0)\} = \{\mathbf{V}\}_0$

Vibrations are caused by non-zero initial conditions and/or non-zero forcing terms

## ▶ Full Solution of the Forced Vibration Problem

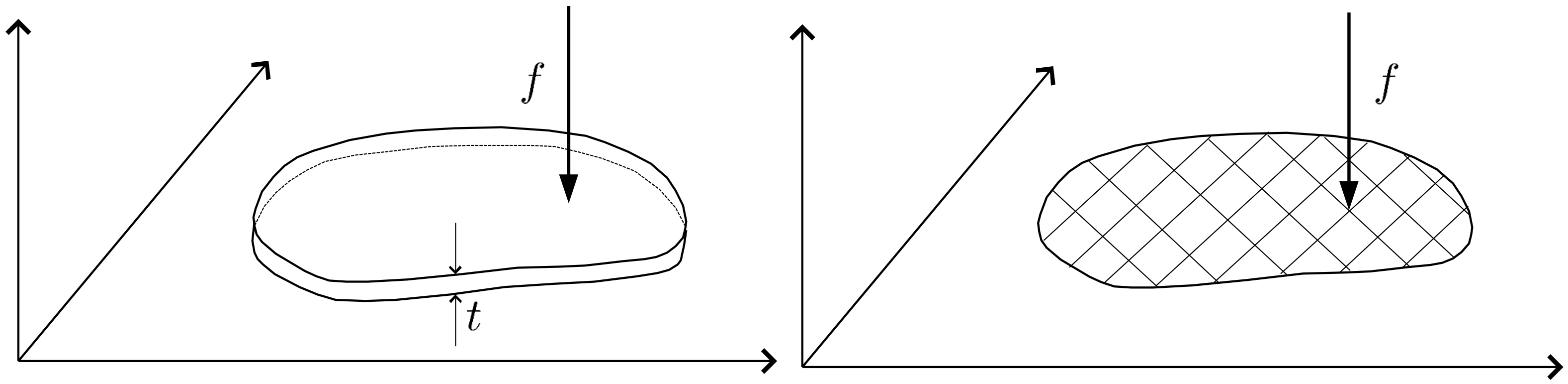
The **general solution** consists of two parts: Full solution = Transient response + Forced response

1. **Transient response** (also called the **homogeneous solution**):
- Solution to the **free vibration** problem ($\{\mathbf{f}\}(t) = 0$).
- Captures the system's natural oscillations.
- Decays over time if damping is introduced (but in this case, we assume no damping).

2. **Forced response** (**particular solution**):
- Accounts for **external forces** ($\{\mathbf{f}\}(t) \neq 0$).
- Depends on the frequency content of the applied force.
- Steady-state behavior remains even after transients decay.

# Forcing types – point loads

► **Point loading (for plates - in the vertical direction)**

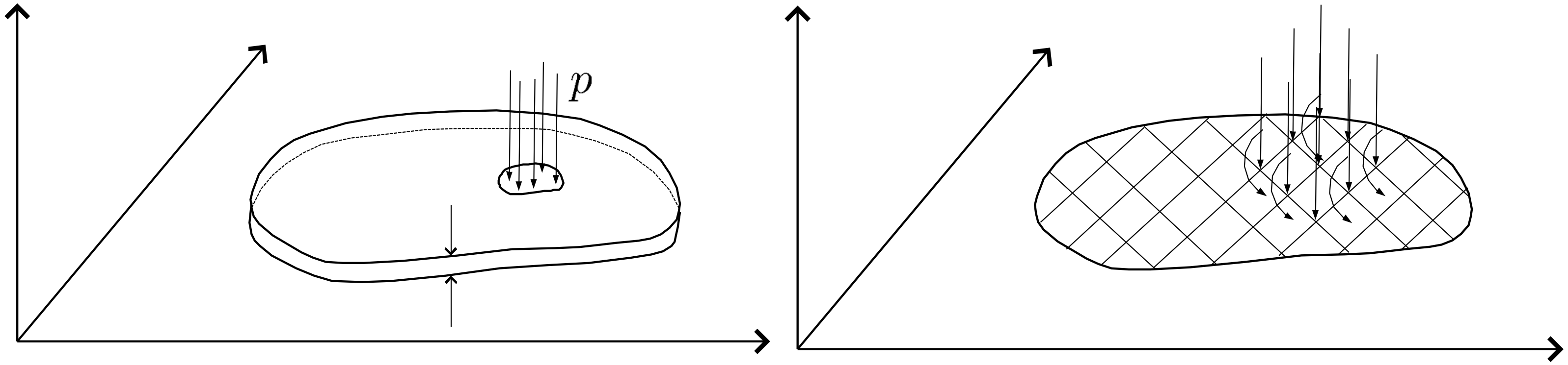



In the finite element formulation the force is included directly in the corresponding dof in the global force vector

$$\{\mathbf{f}(t)\} = \{0 \ \ldots \ f \ \ldots \ 0\}^T$$

# Forcing types – distributed load

► **Distributed load (again shown for the plate model)**



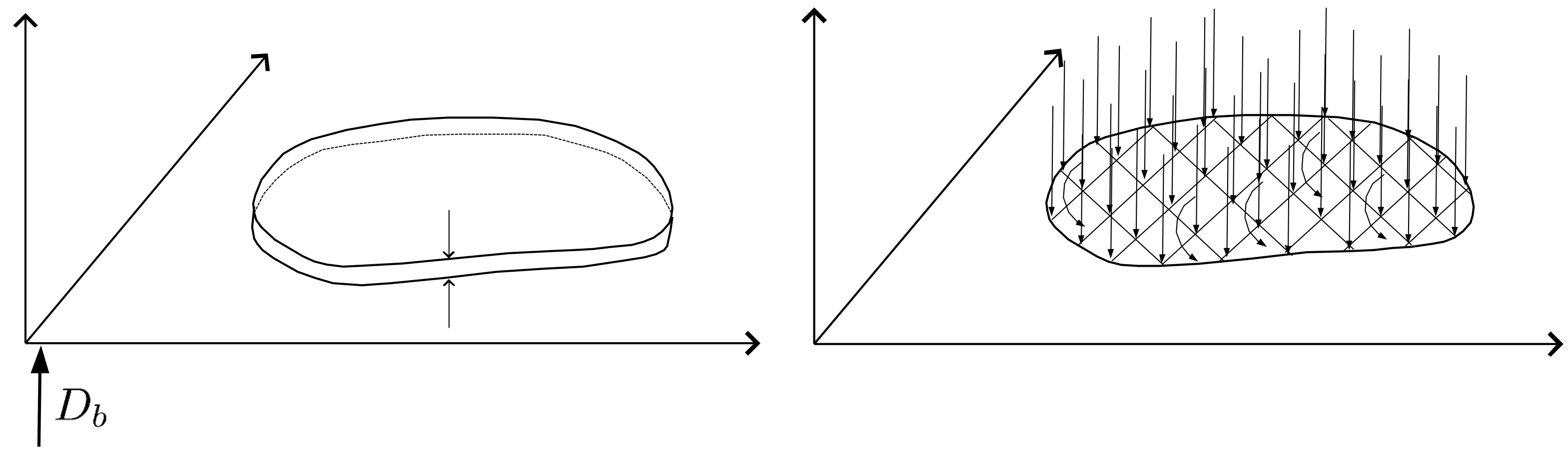

In the finite element formulation we build the consistent load vector by integrating over the element surface

$$\{\mathbf{f}(t)\} = \int_S [\mathbf{N}]^T p(t) dS = \{0 \ \ldots \ f(t) \ M_x(t) \ M_y(t) \ \ldots \ 0\}^T$$

Note that in the case of rectangular 12-dof Kirchhoff element (introduced in Chapter 1) the distributed pressure load gives rise to both forces and moments in the nodes

# Forcing types – support excitation

► **Support excitation**




We express the displacements in the accelerated coordinate system and must express the inertial forces using the total acceleration

$$[\mathbf{M}](\{\ddot{\mathbf{D}}\} + \{\ddot{\mathbf{D}}_b\}) + [\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{0}\}$$

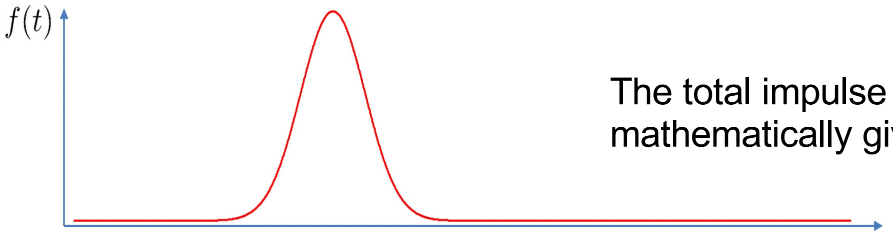
$$[\mathbf{M}]\{\ddot{\mathbf{D}}\} + [\mathbf{K}]\{\mathbf{D}\} = -[\mathbf{M}]\{\ddot{\mathbf{D}}_b\}$$

Support excitation results in a distributed excitation in the whole structure

# Different types of external excitations in vibration analysis - 1

## 1. Impulse Loading

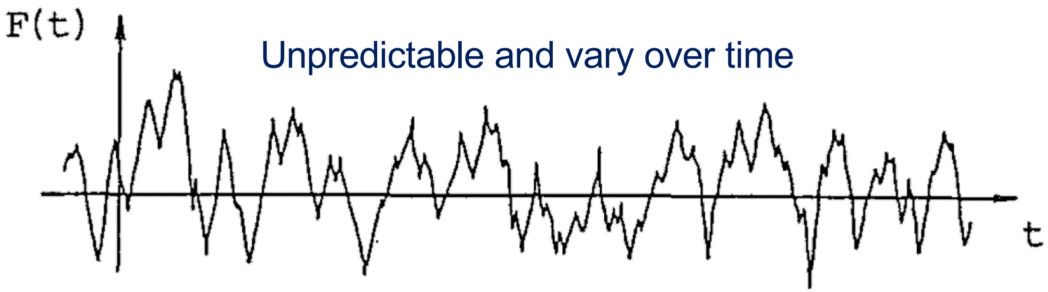
impulse force $f(t)$ , which is a short-duration, high-intensity force applied to a system.




The total impulse applied is mathematically given by: $I = \int_0^T f(t)dt$

- It can cause transient responses in structures, requiring both time-domain and frequency-domain analysis.
- An **earthquake**, where a sudden force is applied to buildings and infrastructure.

## 2. Random Excitation



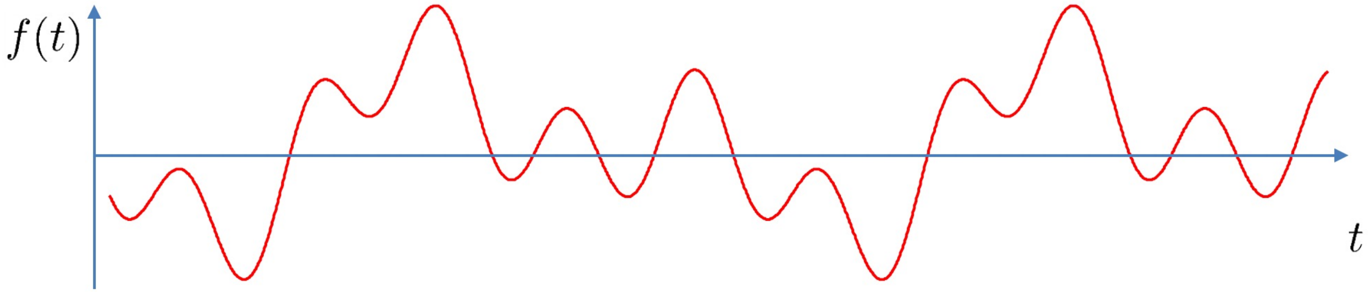

Unpredictable and vary over time

- These forces arise in environments with stochastic vibrations, such as:
  - Wind loads on tall buildings
  - Road-induced vibrations in vehicles
  - Turbulence effects on aerospace structures
- Because random excitation lacks periodicity, **statistical methods** like power spectral density (PSD) analysis or stochastic process modeling are necessary for analysis.

# Different types of external excitations in vibration analysis - 2

## 3. Periodic Excitation

**periodic forces**, which repeat at regular intervals over time.




- Typical sources of periodic excitation include:
  - **Engine vibrations** due to rotating or reciprocating parts
  - **Mobile phone vibrators**, where an unbalanced rotating mass generates a sinusoidal force
  - **Ship rudder vibrations**, where hydrodynamic forces cause oscillatory motions

- These types of excitations are typically analyzed using **Fourier series** or **harmonic response analysis** to determine steady-state behavior.

**Fourier Series Representation:**

$$\{\mathbf{f}(t)\} = \{\mathbf{f}\}_0 + \sum_{i=1}^{\infty}(\{\mathbf{f}\}_i \cos \Omega_i t + \{\mathbf{g}\}_i \sin \Omega_i t)$$

Since the system is **linear**, the response to a periodic excitation can be determined by **superposing** the individual responses to each sinusoidal component.

# Direct Solution in Harmonic Analysis

The governing equation for a dynamic system is given by:

$$[\mathbf{M}]\{\ddot{\mathbf{D}}\}_j + [\mathbf{K}]\{\mathbf{D}\}_j = \{\mathbf{f}\}_j \cos \Omega_j t$$

where:

- $[\mathbf{M}]$ is the mass matrix
- $[\mathbf{K}]$ is the stiffness matrix
- $\{\mathbf{D}\}$ is the displacement response
- $\{\mathbf{f}\}$ is the applied force
- $\Omega$ is the excitation frequency

The solution can be assumed to be **harmonic**: $\{\mathbf{D}(t)\} = \{\mathbf{D}\} \cos \Omega t$

which inserted into the model equations gives

$$-\Omega^2[\mathbf{M}]\{\mathbf{D}\} + [\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{f}\}$$

The **frequency-domain equation** describing the response.

✅ The system follows a time-harmonic response due to linearity.
✅ The equation transforms into an algebraic equation in the frequency domain.
✅ The solution provides insight into resonance and frequency response behavior.

# Undamped Forced Vibrations

The resulting equations:

$$(-\Omega^2[\mathbf{M}] + [\mathbf{K}])\{\mathbf{D}\} = \{\mathbf{f}\}$$

Defining of system matrix (**dynamic stiffness matrix**): $[\mathbf{S}] = -\Omega^2[\mathbf{M}] + [\mathbf{K}]$

The equation then simplifies to:

$$[\mathbf{S}]\{\mathbf{D}\} = \{\mathbf{f}\}$$

➢ This equation resembles a static finite element equation but with a frequency-dependent matrix $[\mathbf{S}]$.

Properties:

✅ **Real Matrix**: $[\mathbf{S}]$ remains real in undamped cases but changes when damping is introduced.
✅ **Symmetric**: If both $[\mathbf{M}]$ and $[\mathbf{K}]$ are symmetric, then $[\mathbf{S}]$ is also symmetric.
✅ **Not Singular (in most cases)**: Unlike $[\mathbf{K}]$, $[\mathbf{S}]$ is not necessarily singular unless boundary conditions dictate otherwise.
✅ **Not Positive Definite for Higher Frequencies**:

- **For** $\Omega < \omega_1$: $[\mathbf{S}]$ is positive definite.
- **For** $\Omega > \omega_1$: $[\mathbf{S}]$ is not positive definite, unlike the mass $[\mathbf{M}]$ and stiffness $[\mathbf{K}]$ matrices, which are usually positive definite.

Consequences:

🔷 Efficient solvers for positive definite matrices cannot be used –Cholesky factorization (and also efficient iterative solvers for very large scale problems)

🔷 The vibration response of an unsupported structure makes sense and can be computed

# When the system matrix $[\mathbf{S}]$ is positive definite

## ▶ Numerical Meaning:

1. **Unique Solution Exists:**
   - The equation $[\mathbf{S}]\{\mathbf{D}\} = \{\mathbf{f}\}$ always has a unique solution.
   - Since $[\mathbf{S}]$ is positive definite, it is **invertible**, meaning that $\{\mathbf{D}\} = [\mathbf{S}]^{-1}\{\mathbf{f}\}$ exists and is well-defined.
2. **Well-Conditioned System:**
   - A positive definite matrix generally leads to a **well-conditioned** system, meaning numerical methods (e.g., $\mathbf{LU}$ decomposition, Cholesky factorization) will be stable and efficient for solving the equation.
3. **Cholesky Decomposition is Possible:**
   - Since [S] is symmetric and positive definite, it can be factorized as:

     $$[\mathbf{S}] = [\mathbf{L}][\mathbf{L}]^{\mathrm{T}}$$

     where $[\mathbf{L}]$ is a lower triangular matrix.
   - This allows for efficient numerical solutions.

## ▶ Physical Meaning:

1. **Stable and Stiff System:**
   - In **structural mechanics**, $[\mathbf{S}]$ often represents the **stiffness matrix** [K] .
   - If $[\mathbf{K}]$ is positive definite, the structure is **stable** (i.e., no rigid body modes), and all deformations require positive energy.
2. **Dynamic Systems and Natural Frequencies:**
   - If $[\mathbf{S}]$ represents the **dynamic stiffness matrix** (e.g., $[\mathbf{K}] - \Omega^2[\mathbf{M}]$), then **all natural frequencies** $\Omega$ are **real and positive**.
   - This ensures that the system does not have **unstable or imaginary frequencies**, which would indicate non-physical responses.

## Solving the Equations in Forced Vibrations

The system equation for forced vibrations at frequency $\Omega$

$$[\mathbf{S}(\Omega)]\{\mathbf{D}\} = \{\mathbf{f}\}$$

where:

- $[\mathbf{S}(\Omega)] = -\Omega^2[\mathbf{M}] + [\mathbf{K}]$is the **dynamic stiffness matrix**.
- $\{\mathbf{D}\}$ is the **steady-state displacement vector**.
- $\{\mathbf{f}\}$ is the **forcing vector**.

### Solution Using the Receptance Matrix

$$\{\mathbf{D}\} = [\mathbf{H}(\Omega)]\{\mathbf{f}\}$$

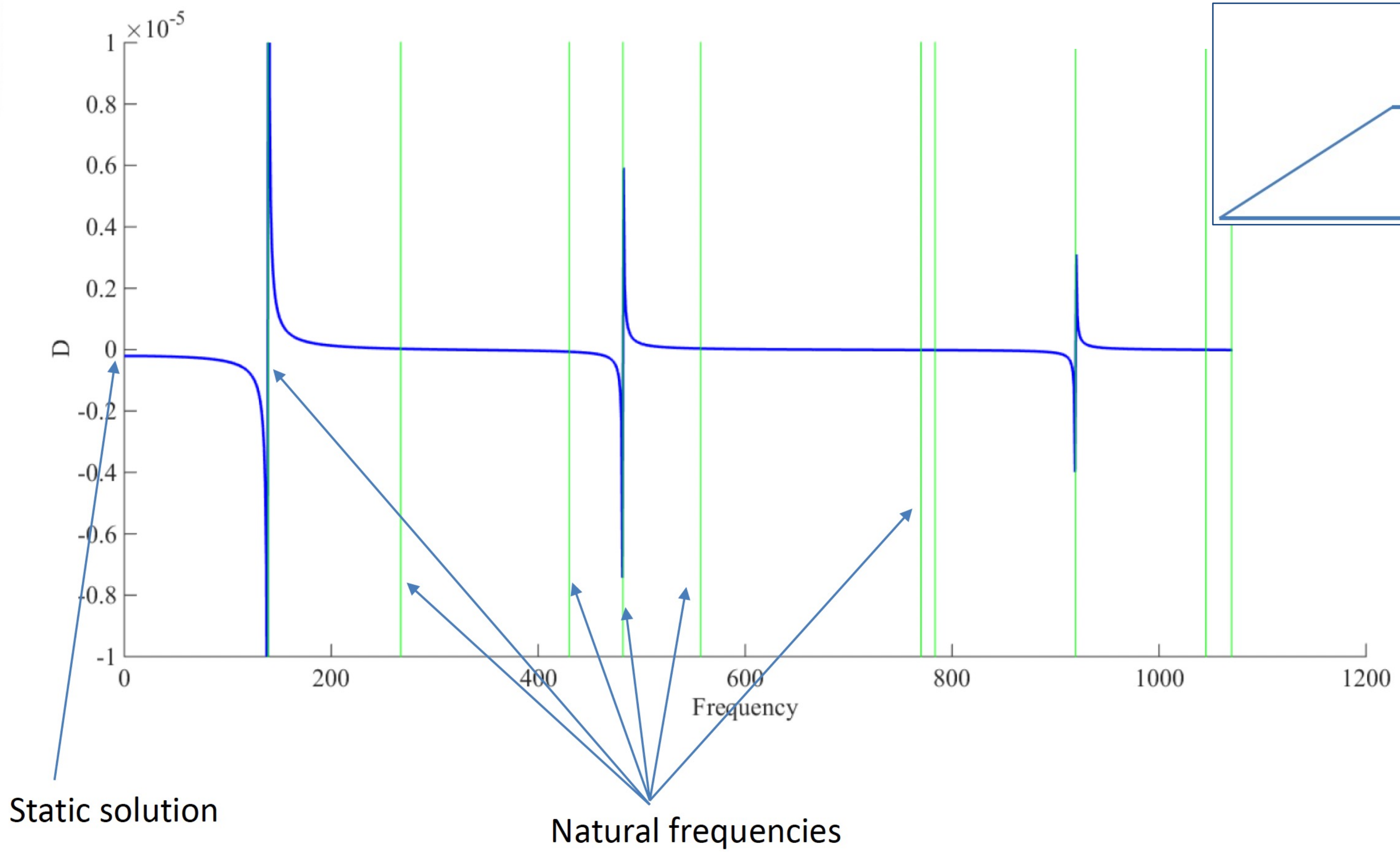
The **receptance matrix** (also called the frequency response function, FRF) is defined as:

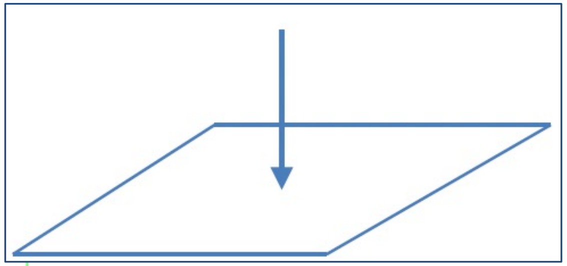
$$[\mathbf{H}(\Omega)] = [\mathbf{S}(\Omega)]^{-1}$$

**Computational Considerations**

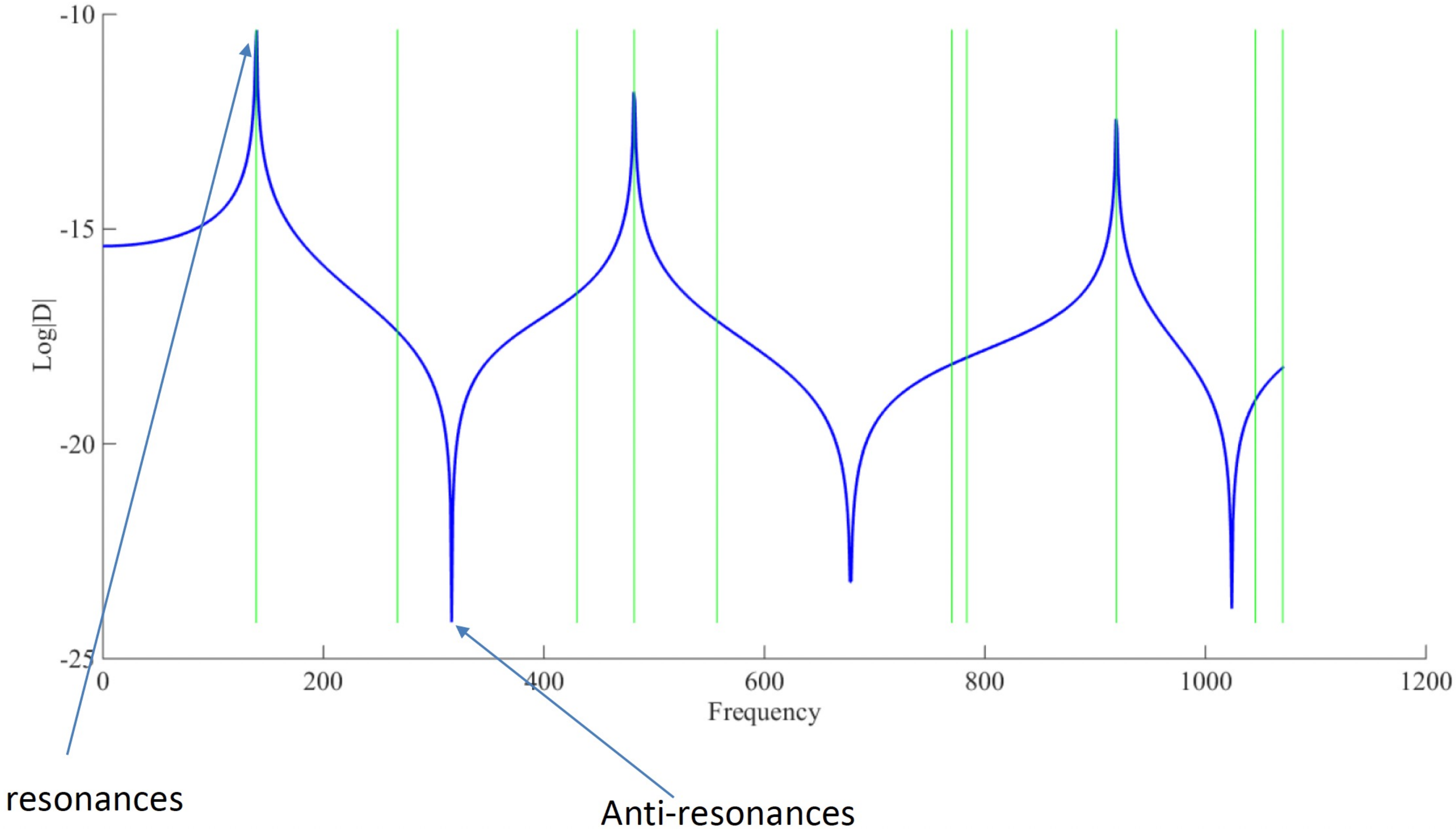
- **Important: $[\mathbf{H}]$ is not explicitly formed** because computing the full inverse matrix $[\mathbf{S}(\Omega)]^{-1}$is computationally expensive, especially for large systems.
- Instead, **direct solvers** or **iterative methods** (such as $\mathbf{LU}$ factorization or Krylov subspace methods) are used to solve for $\{\mathbf{D}\}$ without explicitly forming $[\mathbf{H}]$.

## Frequency response

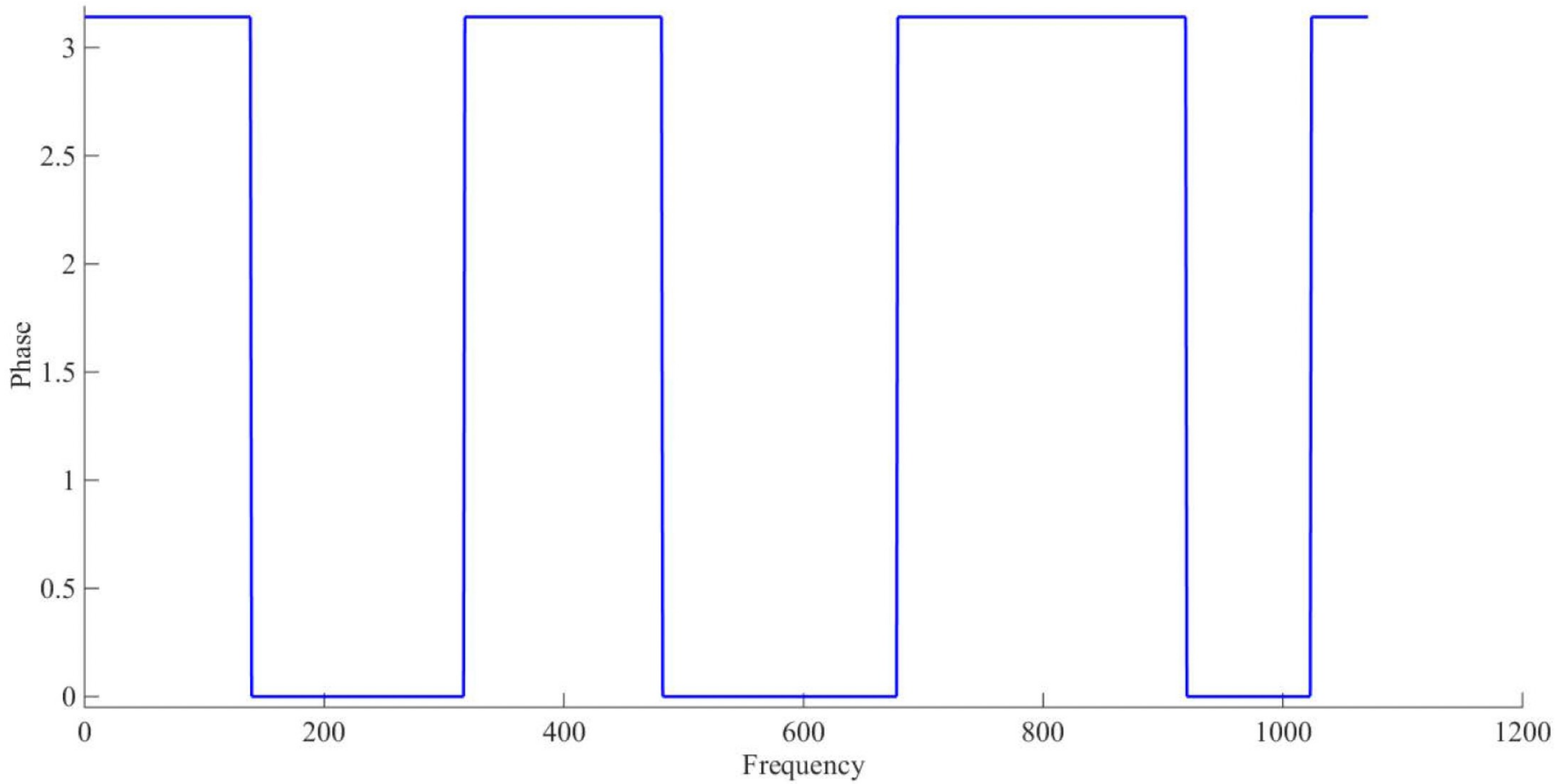


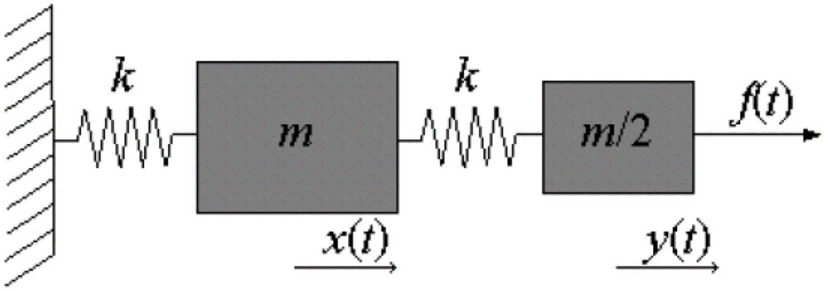
# Frequency response (log(|D|))



# Phase



## 2-Degree-of-Freedom (2DOF) System: Equation of Motion




Equation for Mass $m$ $$m\ddot{x} = -kx - k(x-y)$$

Equation for Mass $m/2$ $$\frac{m}{2}\ddot{y} + ky - kx = f(t)$$

Writing in Matrix Form

$$\begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F\}$$

Mass Matrix:

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}$$

Stiffness Matrix:

$$[K] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

Displacement Vector:

$$\{q\} = \begin{Bmatrix} x \\ y \end{Bmatrix}$$

Forcing Vector:

$$\{F\} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

# Eigenvalue Problem for the 2DOF System

The equation of motion for the **undamped free vibration** of the system is:

$$[M]\{\ddot{q}\} + [K]\{q\} = 0$$

Substituting the **mass matrix** and **stiffness matrix**:

$$\begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming **harmonic motion**:

$$\{q(t)\} = \{Q\}e^{j\omega t}, \quad \{\ddot{q}(t)\} = -\omega^2\{Q\}e^{j\omega t}$$

Substituting into the equation:

$$\left([K] - \omega^2[M]\right)\{Q\} = 0 \qquad \left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}\right) \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For a non-trivial solution, the determinant of the coefficient matrix must be **zero**:

$$\det\left(\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & k - \frac{1}{2}\omega^2 m \end{bmatrix}\right) = 0$$

# Eigenvalue Problem for the 2DOF System

Solving for Natural Frequencies $$(2k-\omega^2 m)\left(k-\frac{1}{2}\omega^2 m\right)-k^2=0$$

Rearranging into a quadratic equation in $\omega^2$ $$\frac{1}{2}\omega^4 m^2-2\omega^2 mk+k^2=0$$ Solving for $\omega^2$ using the **quadratic formula**

Thus, the **natural frequencies** are $$\omega_1=\sqrt{\frac{(2-\sqrt{2})k}{m}},\quad \omega_2=\sqrt{\frac{(2+\sqrt{2})k}{m}}$$

► Finding the Mode Shapes

For each ω, we solve for **relative displacement** Φ.

$$\begin{bmatrix}(2-(2-\sqrt{2}))k & -k \\ -k & ((1-\frac{1}{2}(2-\sqrt{2}))k\end{bmatrix}\begin{Bmatrix}X\\Y\end{Bmatrix}=0$$

For $\omega_1^2=\frac{(2-\sqrt{2})k}{m}$: $\{\Phi_1\}=\begin{Bmatrix}\frac{\sqrt{2}}{2\sqrt{m}}\\ \frac{1}{\sqrt{m}}\end{Bmatrix}$

✅ **First mode $\Phi_1$**

- The two masses move **in phase** (same direction but with different amplitudes).

For $\omega_2^2=\frac{(2+\sqrt{2})k}{m}$: $\{\Phi_2\}=\begin{Bmatrix}-\frac{\sqrt{2}}{2\sqrt{m}}\\ \frac{1}{\sqrt{m}}\end{Bmatrix}$

✅ **Second mode $\Phi_2$**

- The two masses move **out of phase** (opposite directions).

# Receptance Matrix for the 2DOF System

The **dynamic stiffness matrix** (also called the system matrix) for the **2-degree-of-freedom (2DOF) system** is:

$$[\mathbf{S}] = \begin{bmatrix} 2k - \Omega^2 m & -k \\ -k & k - \frac{1}{2}\Omega^2 m \end{bmatrix}$$

## ► Definition of the Receptance Matrix

The **receptance matrix** (also called the **frequency response function** matrix) is the **inverse of the system matrix**:

$$[\mathbf{H}] = [\mathbf{S}]^{-1} \qquad [\mathbf{S}]^{-1} = \frac{1}{\det([\mathbf{S}])}\mathrm{adj}([\mathbf{S}])$$

where:

- $\mathrm{Det}([\mathrm{S}])$ is the determinant of $[\mathrm{S}]$,
- $\mathrm{adj}([\mathrm{S}])$ is the adjugate (cofactor matrix transpose) of $[\mathrm{S}]$.

$$\det([\mathbf{S}]) = m^2\Omega^4 - 4mk\Omega^2 + 2k^2 \quad \text{or} \quad \det([\mathbf{S}]) = m(\Omega^2 - \omega_1^2)(\Omega^2 - \omega_2^2)$$

$$\mathrm{adj}([\mathbf{S}]) = \begin{bmatrix} k - \frac{1}{2}\Omega^2 m & k \\ k & 2k - \Omega^2 m \end{bmatrix}$$

$$[\mathbf{H}] = \frac{1}{m(\Omega^2 - \omega_1^2)(\Omega^2 - \omega_2^2)} \begin{bmatrix} \frac{2k}{m} - \Omega^2 & \frac{2k}{m} \\ \frac{2k}{m} & 2\left(\frac{2k}{m} - \Omega^2\right) \end{bmatrix}$$

# Solution to the Vibration Problem for the 2DOF System

For a **linear system** subjected to harmonic excitation, the displacement response can be expressed as:

$$[\mathbf{S}]\{\mathbf{D}\} = \{\mathbf{f}\} \qquad\qquad [\mathbf{H}] = [\mathbf{S}]^{-1}$$

$$\{\mathbf{D}\} = [\mathbf{H}]\{\mathbf{f}\} \qquad\qquad [\mathbf{H}] = \frac{1}{m(\Omega^2-\omega_1^2)(\Omega^2-\omega_2^2)}\begin{bmatrix} \frac{2k}{m}-\Omega^2 & \frac{2k}{m} \\ \frac{2k}{m} & 2\left(\frac{2k}{m}-\Omega^2\right) \end{bmatrix}$$

For our **2DOF system**, this equation simplifies to:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

where:
- $H_{ij}$ are the elements of the **receptance matrix** $[\mathbf{H}]$,
- $f_x$ and $f_y$ are the external forces applied to the two masses,
- $x$ and $y$ are the displacement responses of the masses.

Expanding: $x = H_{11}f_x + H_{12}f_y$

$$y = H_{21}f_x + H_{22}f_y$$

## Resonance Condition

### ► What Happens at Natural Frequencies?

$$[\mathbf{H}] = \frac{1}{m(\Omega^2 - \omega_1^2)(\Omega^2 - \omega_2^2)} \begin{bmatrix} \frac{2k}{m} - \Omega^2 & \frac{2k}{m} \\ \frac{2k}{m} & 2\left(\frac{2k}{m} - \Omega^2\right) \end{bmatrix}$$

- The **denominator** of $[\mathbf{H}]$ contains the product $(\Omega^2 - \omega_1^2)(\Omega^2 - \omega_2^2)$
- When the excitation frequency **matches a natural frequency** $(\Omega = \omega_1 \; or \; \Omega = \omega_2)$, the denominator **approaches zero**, causing:

$$H_{11}, H_{12}, H_{21}, H_{22} \rightarrow \infty$$

- This means that the displacement **tends to infinity**, leading to **resonance**

### ► Key Observations

✅ **Resonance is Global**: It affects **all degrees of freedom** ($x$ and $y$).

✅ **Coupled Behavior**: Due to the **off-diagonal terms** $(H_{12}, H_{21})$, even if a force is applied to **one mass**, both masses respond.

✅ **Practical Implications**:

- Structures and machines should **avoid operating** near natural frequencies.
- **Damping** is introduced in real-world systems to **reduce resonance effects**.

# Anti-Resonances in a 2DOF System

## ► Definition of Anti-Resonance

- The **receptance matrix** $[\mathbf{H}]$ describes how the system responds to external forces at different frequencies.
- When a diagonal element of $[\mathbf{H}]$ becomes **zero**, the system does not respond to forces applied at that degree of freedom.
- These frequencies are called **anti-resonance frequencies**.

$$H_{11} = 0 \quad \Rightarrow \quad \Omega = \sqrt{\frac{2k}{m}} \qquad\qquad H_{22} = 0 \quad \Rightarrow \quad \Omega = \sqrt{\frac{2k}{m}}$$

## ► What Does This Mean?

- At **anti-resonance frequencies**, a force applied at a given mass **does not contribute to its vibration**.
- Unlike resonance, where the system vibrates excessively, **anti-resonance results in no vibration** at that degree of freedom.

## ► How Does Anti-Resonance Occur?

- Anti-resonances arise due to **destructive interference** in the dynamic system.
- The motion of one mass cancels out the effect of an external force at that frequency.

**Key Observations**

✅ **Anti-resonance frequencies are system-dependent**, determined by the mass and stiffness properties.
✅ **Each mass has its own anti-resonance behavior**, but they may coincide in some cases.
✅ **Practical Implications**:

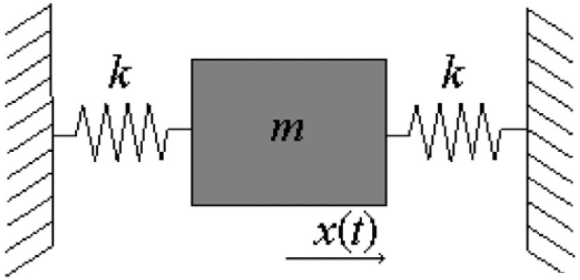
- Anti-resonance is **useful for vibration isolation**, where systems can be designed to **avoid transmission of forces**.

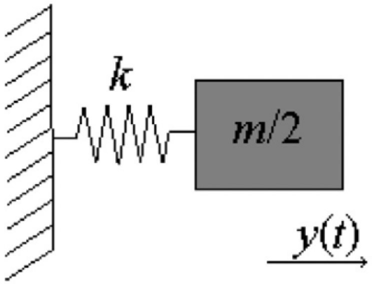
# Anti-Resonances in a 2DOF System

## ► Anti-Resonance vs. Resonance

- **Resonance occurs when** the excitation frequency $\Omega$ matches a **natural frequency** $\omega_j$, causing **maximum response**.
- **Anti-resonance occurs when** the excitation frequency matches a frequency where the response at a particular mass is **completely canceled**.

The **anti-resonance frequency** for a given mass corresponds to the **resonance frequency of a modified system where that mass is fixed**.




$$(\omega_1^n)_a = (\omega_2^n)_a = \sqrt{\frac{2k}{m}}$$

Thus, the **anti-resonance frequency of the original 2DOF system** is the **natural frequency of the simplified system with one mass fixed**.

# Modal Superposition Method - 1

The **modal superposition method** is a powerful approach for solving the **equations of motion** for multi-degree-of-freedom (MDOF) systems by utilizing **mode shapes and natural frequencies**.

The general equation of motion for a system subjected to **harmonic excitation** is:

$$[\mathbf{M}]\{\ddot{\mathbf{D}}\} + [\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{f}\}\cos\Omega t$$

## ► Modal Expansion of the Solution

- Using **modal superposition**, we expand the displacement vector **in terms of mode shapes**:

$$\{\mathbf{D}\} = \sum_{j=1}^{n}\{\mathbf{\Phi}\}_j\varphi_j = [\mathbf{P}]\{\varphi\}$$

where:

- $\{\mathbf{\Phi}\}_j$ are the **mode shape vectors** (eigenvectors),
- $\varphi_j$are the **generalized coordinates** (modal displacements),
- $[\mathbf{P}]$ is the **modal matrix**, composed of the eigenvectors:

$$[\mathbf{P}] = \left[\{\mathbf{\Phi}\}_1 \quad \{\mathbf{\Phi}\}_2 \quad \ldots \quad \{\mathbf{\Phi}\}_n\right]$$

Since the mode shapes are **mass-normalized**, they satisfy:

$$[\mathbf{P}]^T[\mathbf{M}][\mathbf{P}] = [\mathbf{I}]$$

$$[\mathbf{P}]^T[\mathbf{K}][\mathbf{P}] = [\mathbf{\Lambda}]$$

where $[\mathbf{\Lambda}]$ **is the diagonal matrix of eigenvalues**.

# Modal Superposition Method - 2

Substituting **modal expansion** into the original equation:

$$[\mathbf{P}]^T[\mathbf{M}][\mathbf{P}] = [\mathbf{I}]$$

$$[\mathbf{M}][\mathbf{P}]\{\ddot{\varphi}\} + [\mathbf{K}][\mathbf{P}]\{\varphi\} = \{\mathbf{f}\}\cos\Omega t$$

$$[\mathbf{P}]^T[\mathbf{K}][\mathbf{P}] = [\mathbf{\Lambda}]$$

Multiplying by $[\mathbf{P}]^{\mathrm{T}}$ from the left:

$$[\mathbf{I}]\{\ddot{\varphi}\} + [\mathbf{\Lambda}]\{\varphi\} = [\mathbf{P}]^T\{\mathbf{f}\}\cos\Omega t$$

$$[\mathbf{P}]^T[\mathbf{M}][\mathbf{P}] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad [\mathbf{P}]^T[\mathbf{K}][\mathbf{P}] = \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 \end{bmatrix} \quad [\mathbf{P}]^T\{\mathbf{f}\} = \begin{Bmatrix} \{\mathbf{\Phi}\}_1^T\{\mathbf{f}\} \\ \{\mathbf{\Phi}\}_2^T\{\mathbf{f}\} \\ \vdots \\ \{\mathbf{\Phi}\}_n^T\{\mathbf{f}\} \end{Bmatrix}$$

Since $[\mathbf{I}]$ is the **identity matrix**, we get **n independent single-degree-of-freedom (SDOF) equations**:

$$\ddot{\varphi}_j + \lambda_j\varphi_j = [\mathbf{P}]_j^T\{\mathbf{f}\}\cos\Omega t$$

where:

- $\lambda_{\mathrm{j}} = \omega_j^2$ is the **eigenvalue corresponding to mode *j***.

Each equation can be solved **independently** using standard **SDOF vibration solutions**.

# Decoupled Modal Equations

The **modal superposition method** simplifies the equations of motion for **multi-degree-of-freedom (MDOF) systems** by transforming them into a set of **independent equations** for each mode

$$\ddot{\varphi}_1 + \omega_1^2 \varphi_1 = g_1 \cos \Omega t$$

$$\ddot{\varphi}_2 + \omega_2^2 \varphi_2 = g_2 \cos \Omega t \qquad \text{where: } g_i = \{\mathbf{\Phi}\}_i^T \{\mathbf{f}\}$$

$$\ddot{\varphi}_n + \omega_n^2 \varphi_n = g_n \cos \Omega t$$

## ► Physical Interpretation

### Excitation of Modes

- **The response in each mode is independent** and follows the equation of motion for a **forced vibration system**.
- The **modal force** $g_i$ determines the **participation of each mode** in the response.
- **Not all modes are excited by a given force distribution!** If the force vector $\{\mathbf{f}\}$ is **orthogonal** to a mode shape $\{\mathbf{\Phi}\}$, then:

  $$g_i = \{\mathbf{\Phi}\}_i^T \{\mathbf{f}\} = 0$$

  meaning that **mode $i$ will not be excited**.

### Nodal Points and Symmetric Loading

- If an external force **acts at a nodal point** of a mode, that mode will **not be excited**.
- **Symmetric loads cannot excite asymmetric modes**, and vice versa.

# Solving the Modal Equations in Forced Vibration

Each **decoupled modal equation** is of the form:

$$\ddot{\varphi}_j + \omega_j^2 \varphi_j = g_j \cos \Omega t$$

The steady-state **particular solution** of this equation is:

$$\varphi_j(t) = \frac{g_j}{\omega_j^2 - \Omega^2} \cos \Omega t$$

✓ This shows that **each mode responds independently** to the forcing function.

Since the total displacement $\{\mathbf{D}(t)\}$ is obtained by summing the contributions from all modes:

$$\{\mathbf{D}(t)\} = \sum_{j=1}^{n} \{\boldsymbol{\Phi}\}_j \varphi_j(t)$$

$$[\mathbf{P}] = \left[ \{\boldsymbol{\Phi}\}_1 \quad \{\boldsymbol{\Phi}\}_2 \quad \dots \quad \{\boldsymbol{\Phi}\}_n \right]$$

Expressing this in **matrix form**:

$$\{\mathbf{D}(t)\} = [\mathbf{P}] \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{Bmatrix} = [\mathbf{P}] \begin{Bmatrix} \frac{g_1}{\omega_1^2 - \Omega^2} \\ \frac{g_2}{\omega_2^2 - \Omega^2} \\ \vdots \\ \frac{g_n}{\omega_n^2 - \Omega^2} \end{Bmatrix} \cos \Omega t$$

# Solving the Modal Equations in Forced Vibration

$$\{\mathbf{D}(t)\} = [\mathbf{P}] \left\{ \frac{g_1}{\omega_1^2 - \Omega^2} \quad \frac{g_2}{\omega_2^2 - \Omega^2} \quad \cdots \quad \frac{g_n}{\omega_n^2 - \Omega^2} \right\}^T \cos \Omega t$$

The **denominators** $\omega_j^2 - \Omega^2$ indicate how each mode responds to excitation.

### ▶ Resonance Condition

- If the excitation frequency $\Omega$ is **close to a natural frequency** $\omega_j$:

$$\omega_j^2 - \Omega^2 \approx 0$$

then the displacement $\{\mathbf{D}(t)\}$ **diverges** (i.e., becomes very large), leading to **resonance**.

### ▶ Mode Participation

- The contribution of each mode **depends on the modal force** $g_i$.
- If $g_i$ =0 for a certain mode, that mode **does not participate** in the response.
- The total response is a **linear combination of modal contributions**, weighted by the **force distribution and frequency response**.

# The Modal Expansion and Truncation in Forced Vibration Analysis

## ► Final Expression for the Amplitude Vector

Using **modal superposition**, the displacement vector is given by:

$$\{\mathbf{D}\} = [\mathbf{P}] \begin{Bmatrix} \frac{g_1}{\omega_1^2 - \Omega^2} \\ \frac{g_2}{\omega_2^2 - \Omega^2} \\ \vdots \\ \frac{g_n}{\omega_n^2 - \Omega^2} \end{Bmatrix} = \sum_{j=1}^{n} \frac{g_j}{\omega_j^2 - \Omega^2} \{\mathbf{\Phi}\}_j$$

$$g_i = \{\mathbf{\Phi}\}_i^T \{\mathbf{f}\}$$

$$[\mathbf{P}] = \begin{bmatrix} \{\mathbf{\Phi}\}_1 & \{\mathbf{\Phi}\}_2 & \dots & \{\mathbf{\Phi}\}_n \end{bmatrix}$$

This expression shows that **each mode contributes to the total displacement**, weighted by its **modal force** and frequency response function.

## ► Truncation of Modal Expansion

In practice, using **all modes** ($n$) is not always necessary. Instead, we approximate the displacement response using only the **first *N* dominant modes**:

$$\{\mathbf{D}\} \approx \{\mathbf{D}\}_N = \sum_{j=1}^{N} \frac{g_j}{\omega_j^2 - \Omega^2} \{\mathbf{\Phi}\}_j$$

# The Modal Expansion and Truncation in Forced Vibration Analysis

## ► How to Choose $N$?

The number of modes $N$ should be chosen based on the following factors:

$$\{\mathbf{D}\} \approx \{\mathbf{D}\}_N = \sum_{j=1}^{N} \frac{g_j}{\omega_j^2 - \Omega^2}\{\mathbf{\Phi}\}_j$$

✅ **Low-Frequency Dominance**:

- In most **structural vibration problems**, **lower-order modes** contribute the most to the response.
- Higher-frequency modes are **less excited** unless the excitation frequency is high.

✅ **Frequency Range of Interest**:

- If the **excitation frequency** $\Omega$ is **close to a specific mode** $\omega_j$, that mode must be included.

✅ **Energy Contribution of Modes**:

- Higher modes may have very **small modal forces** ($g_i \approx 0$), meaning they contribute little to the response.

✅ **Engineering Approximation**:

- For **practical applications**, a small number of modes (e.g., **first 5–10 modes**) is often **sufficient**

## ► Practical Engineering Considerations

- If $\Omega \approx \omega_j$ for a certain mode, that mode **dominates the response**, and it must be included in the truncation.
- Higher-order modes are often **important in acoustic, aerospace, and impact problems**, where high-frequency effects cannot be ignored.

# Modal Acceleration Method (Static Correction)

## ► Motivation for Static Correction

- When using **truncated modal expansion**, higher-order modes are **neglected**, which can lead to **errors in displacement estimation**.
- A **static correction** accounts for the **missing contributions** of the truncated modes, improving accuracy **without computing additional high-frequency modes**.

## ► Static Correction Formula

- The **corrected displacement** is given by:

$$\{\mathbf{D}\}_a = \{\mathbf{D}\}_N + \{\mathbf{D}\}_{sc}$$

where:
- $\{\mathbf{D}\}_a$ a is the **accurate displacement including static correction**.
- $\{\mathbf{D}\}_N$ is the **truncated modal expansion** (sum of NNN modal contributions).
- $\{\mathbf{D}\}_{sc}$ is the **static correction displacement**.

## ► Static Correction as a Static Problem

The static correction $\{\mathbf{D}\}_{sc}$ is obtained by solving a **static equilibrium equation**:

$$[\mathbf{K}]\{\mathbf{D}\}_{sc} = \{\mathbf{f}\}_{sc}$$

where:
- $[\mathbf{K}]$ is the **stiffness matrix**,
- $\{\mathbf{f}\}_{\mathrm{sc}}$ is the **static correction load**.

# Modal Acceleration Method (Static Correction) - 2

## ► Computation of the Static Correction Load

The static correction load is given by:

$$\{\mathbf{f}\}_{sc} = \{\mathbf{f}\} - [\mathbf{M}][\mathbf{P}]_N\{\mathbf{g}\}_N$$

which can be expanded as:

$$\{\mathbf{f}\}_{sc} = \{\mathbf{f}\} - \sum_{j=1}^{N} g_j[\mathbf{M}]\{\mathbf{\Phi}\}_j$$

where:

- $\{\mathbf{f}\}$ is the **original force vector**,
- $g_j = \{\mathbf{\Phi}\}_j^T\{\mathbf{f}\}$ is the **modal force for mode** *j*
- $[\mathbf{M}]$ is the **mass matrix**,
- $\{\mathbf{\Phi}\}_j$ are the **mode shape vectors** for the included modes.

## ► Practical Importance

✅ **Reduces error from modal truncation** by approximating the effect of **neglected high-frequency modes**.

✅ **Improves accuracy of displacement calculations**, especially for **low-frequency approximations**.

✅ **Commonly used in structural and vibration analysis**, particularly in **FEA applications** where computing all modes is impractical.