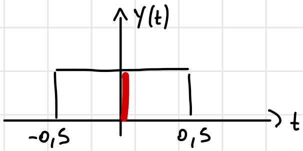
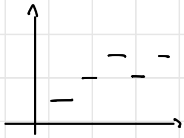
SPLINES

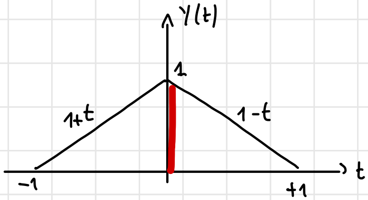
HAAR



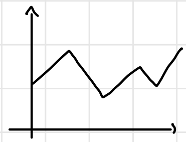
$$s(t) = \begin{cases} \bar{s}, & t \in [-0.5, 0.5] \\ 0, & \text{elsewhere} \end{cases}$$



LINEAR

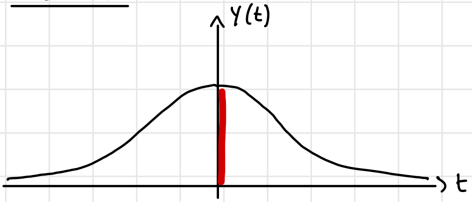


$$s(t) = \begin{cases} 1-t, & t \in [0,1] \\ 1+t, & t \in [-1,0] \\ 0, & \text{elsewhere} \end{cases}$$

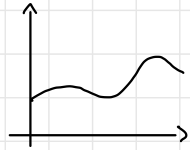


$y(t) = \sum_i a_i \cdot s(t-t_i)$

CUBIC



$$s(t) = \int s(\tau) \cdot s(t-\tau)\, d\tau$$



Then, the 2D splines are obtained in this way:

$$s_i(x,y) = s_i(x) \cdot s_i(y)$$

When in presence of a hole/gap in the data such that the two nearest splines do not interact with each others, it is mandatory to introduce a regularization parameter (TyKonov principle).
The least squares are used for the estimation of the spline's parameters, which enforces the solution to pass as close as possible to the data.
This, when a gap in the data is present, will force the solution to go to zero inside the gap, which doesn't make much sense.

Therefore, to overcome this limitation, it is enforced the regularization principle, which corresponds to the minimization of the energy of the solution (i.e., reducing the slope of the estimated curve).
So, the objective function (the one we want to minimize for the solution) becomes:

this gives i pseudo-observations

$$\phi_b = \underbrace{\sum_i \left(y_i - f_i(\underline{x})\right)^2}_{\text{from LS}} + \underbrace{\lambda \sum_i x_i^2}_{\text{to regularize}} \quad , \quad \lambda \in [0, +\infty)$$

$\Rightarrow$ the regularization is controlled by λ (regularization parameter).

$\lambda = 0 \rightarrow$ no regularization

$\lambda = \infty \rightarrow$ only regularization $\rightarrow$ solution is the mean of data (=min slope)

A good estimate of λ can be obtained as ratio between amplitude of noise and signal.

$$\lambda \sim \frac{\sigma_n^2}{\sigma_s^2}$$

But, of course, it depends on the data you have and what you expect to be the signal like.

Delta discretization

- 1 delta.

$$f'(t_i) = \frac{f(t_{i+1}) - f(t_i)}{\Delta}$$

- 2 delta.

$$f'(t_i) = \frac{f(t_{i+1}) - f(t_{i-1})}{2\Delta} \quad)$$

explaination:

$$f(t_{i+1}) = f(t_i) + \overbrace{\frac{\partial f}{\partial t}\Big|_{t_i}}^{f'(t_i)} (\overbrace{t_{i+1} - t_i}^{\Delta}) + \overbrace{\frac{\partial^2 f}{\partial t^2}\Big|_{t_i}}^{f''(t_i)} \frac{(\overbrace{t_{i+1} - t_i}^{\Delta})^2}{2} + \sigma(t^3)$$

$$f(t_{i-1}) = f(t_i) + \frac{\partial f}{\partial t}\Big|_{t_i} (\underbrace{t_{i-1} - t_i}_{-\Delta}) + \frac{\partial^2 f}{\partial t^2}\Big|_{t_i} \frac{(\overbrace{t_{i-1} - t_i}^{\Delta})^2}{2} + \sigma(t^3)$$

$$\Rightarrow f(t_{i+1}) - f(t_{i-1}) = f'(t_i) \cdot 2\Delta + \sigma(t^3)$$

$$\rightarrow f'(t_i) = \frac{f(t_{i+1}) - f(t_{i-1})}{2\Delta}$$