EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.)

CAN THE MEASURE OF $\bigcup_1^n [a_i, b_i]$ BE COMPUTED IN LESS THAN $O(n \log n)$ STEPS?

VICTOR KLEE

Problem statement. Given n intervals $[a_1, b_1], \ldots, [a_n, b_n]$ in the real line, it is desired to find the measure of their union. How efficiently can that be done? The list of endpoints can be sorted with $O(n \log n)$ comparisons, and then $O(n)$ additions suffice to determine the measure of $\bigcup_1^n [a_i, b_i]$. However, the problem does not, *a priori*, require sorting, and it would be of interest to know whether there is a solution involving less than $O(n \log n)$ computational steps.

Reason for interest. In addition to its intrinsic appeal, the problem is of interest because it arises naturally in approximating the areas of certain subsets of the cartesian plane. Suppose that $[u_1, v_1], \ldots, [u_n, v_n]$ are intervals in the real line, f_i and g_i are continuous real functions on $[u_i, v_i]$, and

$$P_i = \{(x, y) \colon u_i \leqq x \leqq v_i, f_i(x) \leqq y \leqq g_i(x)\}.$$

Even when the areas of the individual P_i's are easily computed, it may be difficult to compute the area of the union $P = \bigcup_1^n P_i$ if the P_i's have a complicated intersection-pattern. However, when the functions are Lipschitzian the area of P is closely approximated by sums of the form

$$(*) \qquad \sum_{h=1}^{m} (\text{measure of } L_h \cap P)\,(s_h - s_{h-1}),$$

where $s_0 < s_1 < \ldots < s_m$ is an appropriate (sufficiently dense) sequence in $\bigcup_1^n [u_i, v_i]$ and L_h is the line of abscissa $(s_{h-1} + s_h)/2$. Since

$$L_h \cap P = \bigcup_{i=1}^{n} L_h \cap P_i,$$

computing (*) involves m problems of the sort mentioned above.

Solution by sorting. Suppose that each left endpoint a_i is tagged with 1 and each right endpoint b_i is tagged with -1. By means of various procedures requiring $O(n \log n)$ comparisons, the sequence of all $2n$ endpoints can be arranged in increasing order $e_1 \leqq e_2 \leqq \ldots \leqq e_{2n}$ such that if $t_1, \ldots, t_{2n}$ is the corresponding permutation of the tags, then $t_i \geqq t_{i+1}$ whenever $e_i = e_{i+1}$; that is, the sorting does not place a right endpoint before a left endpoint of the same numerical value. With EXCESS denoting the excess of the number of left endpoints over the number of right endpoints, the following program constructs the sequence $[c_1, d_1], \ldots, [c_m, d_m]$ of components of $\bigcup_1^n [a_i, b_i]$ and then computes the measure of $\bigcup_1^n [a_i, b_i]$.

begin

MEASURE $\leftarrow 0$; $m \leftarrow$ EXCESS $\leftarrow 0$;

for $i \leftarrow 1$ until $2n$ do

```
    begin
      EXCESS ← EXCESS + t_i;
      if EXCESS = 1 then begin m ← m + 1; c_m ← e_i end;
      if EXCESS = 0 then d_m ← e_i
    end;
  for i ← 1 until m do MEASURE ← MEASURE + (d_i − c_i)
end
```

Problems in d-space. For d-dimensional intervals $I_1, \ldots, I_n$ in cartesian d-space, each given as the product of d real intervals, we state the following three problems and in each case ask how efficiently the problem can be solved.

PARTITION PROBLEM: *Find a sequence $H_1, \ldots, H_m$ of intervals such that $\bigcup_1^m H_k = \bigcup_1^n I_k$ and no two of the H_k's have a common interior point.*

MINIMUM PARTITION PROBLEM: *Find $H_1, \ldots, H_m$ as described such that m is minimum.*

MEASURE PROBLEM: *Find the d-measure of $\bigcup_1^n I_k$.*

When $d = 1$ the above algorithm solves all three problems in $O(n \log n)$ steps, but it is not obvious that the three problems are actually of the same computational complexity. It would be of special interest to know whether the measure problem can be solved in a number of steps that is bounded by a polynomial in d and n.

The books [1] and [2] are recommended as general references on sorting and computational complexity, while the methods of [3] can probably be applied to the partition problems.

References

1. A. V. Aho, J. E. Hopcroft and J. D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Mass., 1974.
2. D. E. Knuth, The Art of Computer Programming, Vol. 3: Sorting and Searching. Addison-Wesley, Reading, Mass., 1973.
3. H. T. Kung, F. Luccio and F. P. Preparata, On finding the maxima of a set of vectors, J. Assoc. Comp. Mach., 22 (1975) 469–476.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195.

CLASSROOM NOTES

EDITED BY RICHARD A. BRUALDI

Material for this Department should be sent to Richard A. Brualdi, Laboratoire Calcul des Probabilités, Université de Paris, T.56, 4 Place Jussieu, 75–230 Paris, France.

GROUP THEORY AND THE DESIGN OF A LETTER FACING MACHINE

JOSEPH A. GALLIAN

In this note we consider a number of problems relating to the optimum design of a letter facing machine for square letters. The author uses the content of this paper for the first lecture of a beginning abstract algebra course. The verifications of the solutions presented are left to the students in the course as exercises. Also, the students are asked to undertake a similar analysis for objects of other