Computer Assignment I.

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Our task has two main parts. First, in the **Returns** section, we work with historical monthly S&P-500 data to calculate and model returns. We estimate the mean and volatility, test whether the mean return differs from zero, and explore how much data is needed to pin down estimates precisely. We also simulate return paths to verify our results, study how parameter uncertainty affects investment decisions, and compare two different models for the index dynamics. Finally, we use these models to forecast the index, simulate its distribution, and value simple digital options.

In the **Binomial Trees** section, we shift focus to option pricing. We construct binomial trees for the S&P-500 step by step, ensuring that they match empirical variance and expected return. We then compute option prices both with the tree and with the Black-Scholes formula, compare the outcomes, and study how results converge as the number of steps increases. We also examine probability distributions under the risk-neutral measure, extend the framework to put options, verify put-call parity, and finally allow for early exercise to evaluate American options.

Together, these two parts train us to move from modeling returns to applying them in option pricing.

Data

Download historical data of S&P-500 index levels with a monthly frequency (for instance from FRED). The end date of your sample is August 29, 2025. The number of month ends to cover in your sample depends on your months of birth: $10(m1 + m2) - m1m2$, *with a minimum of 61 and a maximum of 120.*

The number of months in our case is 108, since we were born in February (2) and November (11). We downloaded the data, read it into a data frame, and selected the last 108 observations.

There are several ways to calculate returns from the level data. The **simple net return** between time t and $t+1$ is defined as:

$$R_{t:t+1} = \frac{S_{t+1}}{S_t} - 1,$$

where S_t stands for the stock price or index level at time t.

Sometimes researchers model the simple net return as:

$$R_{t:t+1} = \mu + \sigma \epsilon_{t:t+1}, \tag{1}$$

where

$$\epsilon_{t:t+1} \sim N(0, 1).$$

We calculate the simple net returns using the definition above and store them in a new column of our data frame. The last observation will be empty since we do not have S_{t+1} for the last month end in our sample. This makes the number of returns (i.e. the sample size) equal to 107.

Part I. – Returns

2.1 Question *(a)*

The sample mean and standard deviation of the simple net returns are $\mu = 0.0113$ and $\sigma = 0.0453$, respectively. These are rounded to four decimal places, but during calculations we use the full precision.

We test the null hypothesis $H_0 : \mu = 0$ against the alternative $H_1 : \mu \neq 0$ for the mean simple net return using a two-sided one-sample t-test at the 5% significance level.

The test statistic is

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}},$$

where $\bar{x}$ is the sample mean, s the sample standard deviation, and n the sample size.

The p-value is computed as

$$p = 2 \cdot P(T_{n-1} \geq |t|),$$

and we reject H_0 if $p < \alpha$.

In our case, $t = 2.5778$ and $p = 0.0113 < 0.05$. Thus, we reject H_0 and conclude that the mean simple net return is statistically different from zero.

2.2 Question *(b)*

For solving this task, we use the margin of error formula for a confidence interval for the mean when the population standard deviation is unknown:

$$ME = z_{\alpha/2} * (\sigma/\sqrt{n}),$$

where $z_{\alpha/2}$ is the critical value from the standard normal distribution for a two-sided confidence interval with confidence level $1 - \alpha$.

We want a 95% confidence interval for μ that equals $[0.45\%, 0.55\%]$. The margin of error is half the width of the confidence interval, i.e. $ME = (0.55\% - 0.45\%)/2 = 0.005$. We solve the margin of error formula for n:

$$n = (z_{\alpha/2} * (\sigma/ME))^2.$$

We use $\sigma = 0.0453$ from part (a) and $z_{0.025} = 1.96$ for a 95% confidence interval. This gives us $n = 31460$ months, which is approximately 2622 years. This is an absurdly long time period, which illustrates that the standard deviation of the simple net return is large relative to the mean. It means that we need a very large sample size to estimate the mean simple net return with such a small margin of error.

We used a two-sample t-test to test whether the mean of simple net return differs significantly from the mean of the returns generated by the equation. Using the parameters $\mu = 0.5\%$ and $\sigma = 0.0453$, we generated a sample of 107 returns (the same size as our original sample) and tested

$$H_0 : \mu_1 = \mu_2 \text{ vs. } H_1 : \mu_1 \neq \mu_2$$

at $\alpha = 0.05$, where μ_1 is the mean of the simple net returns from the historical data and μ_2 is the mean of the generated returns. The results showed that the means are not significantly different, indicating that the returns generated by the equation are a reasonable approximation of the actual returns.

2.3 Question (*c*)

After generating a large sample of returns using the parameters $\mu = 0.5\%$ and $\sigma = 0.0453$, with a sample size equal to the number of months calculated in part (*b*), we expect that the margin of error for a 95% confidence interval for the mean will be approximately equal to the previously defined 0.005. The margin of error of the simulated returns is roughly 0.00049959, which confirms our calculations in part (*b*).

2.4 Question (*d*)

Uncertainty about the true mean return and volatility in model (1) directly affects the distribution of future payoffs. If the mean return is lower than estimated or volatility higher, the actual risk-adjusted performance of the S&P-500 could be worse than expected. A risk-averse investor, who values downside protection, will treat this parameter uncertainty as an additional source of risk and therefore allocate less wealth to the index than if the parameters were known with certainty. In other words, estimation risk reduces the optimal risky investment share because the investor requires compensation not only for market risk but also for the possibility of model misspecification.

An alternative way to model the S&P-500 index is as follows:

$$S_{t+1} = S_t e^{(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2) + \tilde{\sigma} e_{t:t+1}} \tag{2}$$

2.5 Question (*e*)

Modellers might prefer model (1) over model (2) for several reasons:

- **Simplicity of estimation**: In model (1), returns are assumed normal with constant mean and variance, so estimation reduces to straightforward sample mean and variance of net returns.

- **Avoid log transformation**: Many empirical studies and portfolio applications work directly with simple net returns (as in (1)), making interpretation of μ more intuitive (it's the expected net return).
- **Approximation suffices**: For short horizons and small returns, the difference between log returns and net returns is minor, so the simpler linear model (1) is often accurate enough.
- **Analytical convenience**: In regressions, factor models, and asset pricing tests, model (1) aligns more naturally with linear frameworks, whereas model (2) is non-linear in S_t.

2.6 Question *(f)*

We derived the expectation of the S&P-500 index after 60 months using the properties of the geometric Brownian motion (GBM) model given in equation (2).

The derivation steps correspond to the following mathematics:

The log-return over one period:

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = \mu - \frac{1}{2}\sigma^2 + \sigma\epsilon$$

where μ is the drift, σ volatility, and ϵ is a standard normal random variable.

The overall log-return from $t = 0$ to $t = T$:
Summing through time gives

$$\ln\left(\frac{S_T}{S_0}\right) = T\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma\sum_{i=1}^{T}\epsilon_i$$

Or, exponentiating:

$$S_T = S_0 \cdot \exp\left(T\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma\sum_{i=1}^{T}\epsilon_i\right)$$

Expectation of Returns at Time T

The expected value of S_T (the price at time T), given the stochastic process above:

$$\mathbb{E}[S_T] = S_0 \cdot \mathbb{E}\left[\exp\left(T\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma\sum_{i=1}^{T}\epsilon_i\right)\right]$$

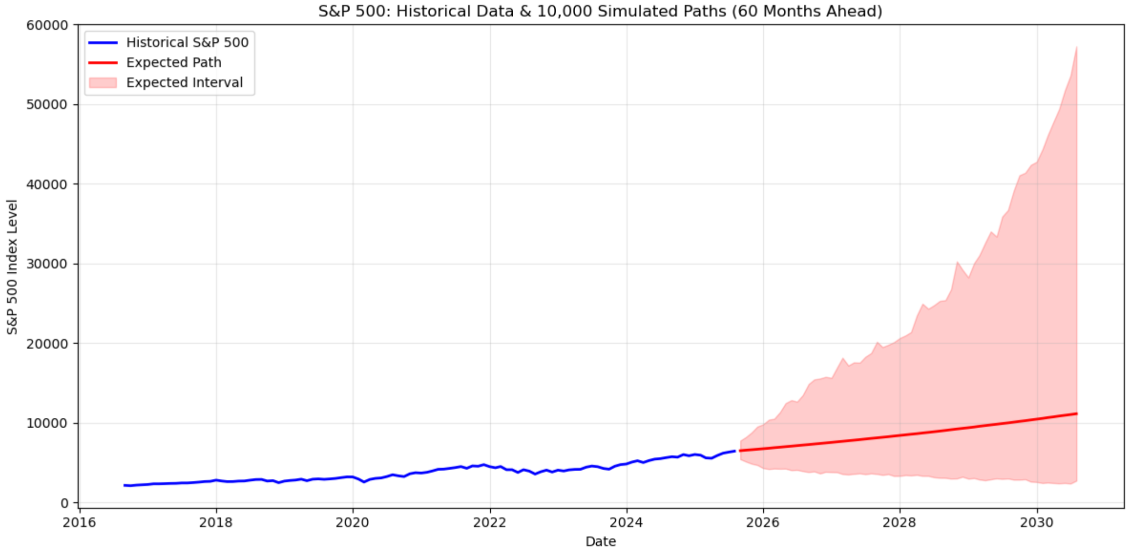
Since the sum of standard normal variables is itself normal ($\sum_{i=1}^{T}\epsilon_i \sim N(0,T)$), the expectation simplifies to:

$$\boxed{\mathbb{E}[S_T] = S_0 \cdot \exp\left(T\mu\right)}$$

This uses the property of log-normal expectation:
If $X \sim N(\mu_X, \sigma_X^2)$, then $E[e^X] = e^{\mu_X + \frac{1}{2}\sigma_X^2}$.

2.7 Question (*g*)

Figure 2.1: Simulated S&P-500 Index Distribution After 5 Years



We created 10,000 simulated paths of the S&P-500 index level over a 5-year horizon (60 months) using the estimated parameters from model (2). The simulation was performed using the geometric Brownian motion formula, which incorporates both the drift and volatility components.

After this, we plotted the historical index values along with the simulated paths after 5 years. We used an expected value line to indicate the mean of the simulated index levels, and an interval representing the simulated values to illustrate the spread. We think that is better than plotting every simulated path, which would create a cluttered and unreadable graph.

As for the shape of the distribution, we expected it to be right-skewed, which is typical for stock prices modeled by geometric Brownian motion. This is because the exponential function used in the model tends to produce a long right tail, reflecting the possibility of large upward movements in stock prices, while downward movements are bounded by zero.

Initially, we attempted to use the Shapiro-Wilk test for normality, but it was not suitable for our large sample size of 10,000. Instead, we opted for the Kolmogorov-Smirnov test, which is more appropriate for larger datasets. This test compares the empirical distribution function of the sample with the cumulative distribution function of the normal distribution.

The Kolmogorov–Smirnov test shows that while the S&P-500 index distribution after 1 month looks roughly normal, deviations from normality become statistically significant from month 2 onward. By month 3 and later, the test strongly rejects normality ($p \approx 0$), which is consistent with theory: index levels follow a lognormal, not a normal, distribution, and the longer the horizon, the more the skewed lognormal shape dominates.

2.8 Question *(h)*

To determine the no-arbitrage price of the 5-year digital put option, we simulate the terminal S&P-500 index level after 60 months under the risk-neutral dynamics of model (2). The option payoff equals 1 if the simulated index level is below the strike price of 6,300, and 0 otherwise. We then take the average of these simulated payoffs to approximate the risk-neutral expectation of the option payoff. Since the risk-free rate is assumed to be 0%, no discounting is needed. The resulting mean payoff therefore directly gives the model-implied value of the digital put option.

The Monte Carlo simulation under the risk-neutral measure yields an estimated probability of

$$\mathbb{Q}(S_T < 6300) \approx 0.0719.$$

This means that, in the risk-neutral world, there is about a 7.21% chance that the S&P-500 index will fall below the strike level of 6,300 after 5 years. Since the digital put pays 1 in that event and 0 otherwise, the no-arbitrage price of the option is 0.0721 (with zero risk-free rate, no discounting is applied).

2.9 Question *(i)*

Derivation of the analytical formula

Model (2) specifies the dynamics of the index:

$$S_{t+1} = S_t \exp\big(\tilde{\mu} - \tfrac{1}{2}\tilde{\sigma}^2 + \tilde{\sigma}\epsilon_{t+1}\big), \quad \epsilon_{t+1} \sim N(0,1).$$

Iterating over T periods gives:

$$S_T = S_0 \exp\Big((\tilde{\mu} - \tfrac{1}{2}\tilde{\sigma}^2)T + \tilde{\sigma}\sqrt{T}\,\epsilon\Big), \quad \epsilon \sim N(0,1).$$

Thus,

$$\ln\frac{S_T}{S_0} \sim \mathcal{N}\big((\tilde{\mu} - \tfrac{1}{2}\tilde{\sigma}^2)T,\ \ \tilde{\sigma}^2 T\big).$$

The **digital put payoff** is:

$$\text{Payoff} = \mathbf{1}\{S_T < K\}.$$

Therefore, the option price under measure $\mathbb{Q}$ is the risk-neutral probability:

$$\Pi_0 = \mathbb{Q}(S_T < K) = \mathbb{Q}(\ln S_T < \ln K).$$

Substituting the distribution:

$$\Pi_0 = \Phi\left(\frac{\ln(K/S_0) - \left(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2\right)T}{\tilde{\sigma}\sqrt{T}}\right),$$

where $\Phi(\cdot)$ is the standard normal CDF.

The **Analytical formula (with price of 0.5415)** was derived under the **risk-neutral measure** $\mathbb{Q}$ using the risk-free rate $r = 0$. That produces the *risk-neutral probability* that $S_T < K$.

The **Simulation result (with price of 0.0721)** was derived under the **real-world measure** $\mathbb{P}$ using $\tilde{\mu}$ estimated from data. That produces the *physical probability* that $S_T < K$.

In risk-neutral pricing, expected returns of the underlying are adjusted down to the risk-free rate. Since historically the S&P-500 has a positive drift, the real-world distribution has a much higher probability mass below the strike compared to the risk-neutral distribution. That's why the analytical probability ($\approx 54\%$) is much higher than the simulated option price under risk-neutral dynamics ($\approx 7\%$).

2.10 Question (j)

Sum the no-arbitrage prices of the digital put option (as calculated in (h)) and the digital call option and explain why the result makes sense.

The digital put and call exhaust all possibilities: either $S_T < K$ or $S_T > K$. Under the risk-neutral measure, the two events are **mutually exclusive and collectively exhaustive**, so their probabilities must sum to 1.

Thus, the no-arbitrage prices of the digital put and digital call add up to 1 (with $r = 0$, no discounting). This consistency check confirms both calculations.

Part II. – Binomial Trees

A commonly used approach to compute the price of an option is the so-called binomial tree method. In this approach, option prices are computed through a well-known backward induction scheme, which was explained in one of the first lectures and can be found in all option pricing literature.

Consider a European call option on a non-dividend-paying stock with a maturity of 3 months. Suppose that the underlying of this option is the S&P-500 index. The strike price of the call option equals 6,500. Let the 3-months per annum interest rate be equal to 3% (with quarterly compounding). Remark: 3% per annum with quarterly compounding means that the interest over 3 months is 0.75%.

The first step in applying the binomial tree approach for option pricing is to construct the nodes of the tree. In this exercise we will first use a step size of one month. Since we want to price an option with a maturity of 3 months, we need to build a 3-step binomial tree for the S&P-500 index. Starting value of the tree is the S&P-500 index level of August 29, 2025.

The tree will have two nodes after the first time step. We want to choose these nodes in such a way that the variance of the 1-month simple net return in the tree equals the sample variance of simple net returns on the S&P-500 index (using the time series of the previous exercise). The expected monthly simple net return in the tree should be equal to 0.50%. You may assume that the real-world probability p of an upward movement equals 55%. Construct your tree without the use of a pre-programmed package. You can of course confirm your results with the results from a package.

Expected (Risk-Neutral) Return:

$$\mathbb{E}^{\mathbb{Q}}[R_{01}] = \mathbb{E}^{\mathbb{Q}}\left(\frac{S_1 - S_0}{S_0}\right) = \mathbb{E}^{\mathbb{Q}}(G-1)$$

Let $G = S_1/S_0$. Then,

$$\mathbb{E}[S_1] = \mathbb{E}[G] \cdot S_0 \Leftrightarrow \mathbb{E}[S_1] = (r+1)S_0$$

Expectation and Variance:

$$\mathbb{E}[S_1] = puS_0 + (1-p)dS_0$$

$$\mathbb{E}[G] = pu + (1-p)d$$

$$\mathbb{E}[G^2] = pu^2 + (1-p)d^2$$

Variance is sample variance of simple net returns:

$$\mathrm{var}(R) = 0.002047$$

$$\mathrm{var}(R) = \mathbb{E}(R^2) - (\mathbb{E}(R))^2$$

Let $R = G - 1$:

$$\mathbb{E}(R^2) = \mathbb{E}\left((G-1)^2\right) = \mathbb{E}(G^2) - 2\mathbb{E}(G) + 1$$

Thus,

$$\mathrm{var}(R) = \mathbb{E}(G^2) - (\mathbb{E}(G))^2$$

Dicomposing Variance onto the difference of expectations :

$$\mathbb{E}[G] = pu + (1-p)d$$

$$\mathbb{E}[G^2] = pu^2 + (1-p)d^2$$

$$\text{var}(R) = pu^2 + (1-p)d^2 - (pu + (1-p)d)^2$$

$$\text{var}(R) = p(1-p)(u-d)^2$$

Variance Condition and Up/Down Factors:

$$0.55 \cdot 0.45 \cdot (u-d)^2 = 0.002047$$

$$(u-d)^2 = 0.00827, \text{ so } u - d = 0.090943$$

Mean Condition:

$$0.55(d + 0.0909) + 0.45d = 1.005$$

$$d = 0.95498 \text{ and } u = 1.045925$$

Risk-Neutral Probability:

$$q = \frac{1 + r - d}{u - d}$$

$$q = \frac{1 + 0.002466 - 0.954981}{1.045925 - 0.954981}$$

$$q = \frac{0.047485}{0.090943} = 0.522137$$

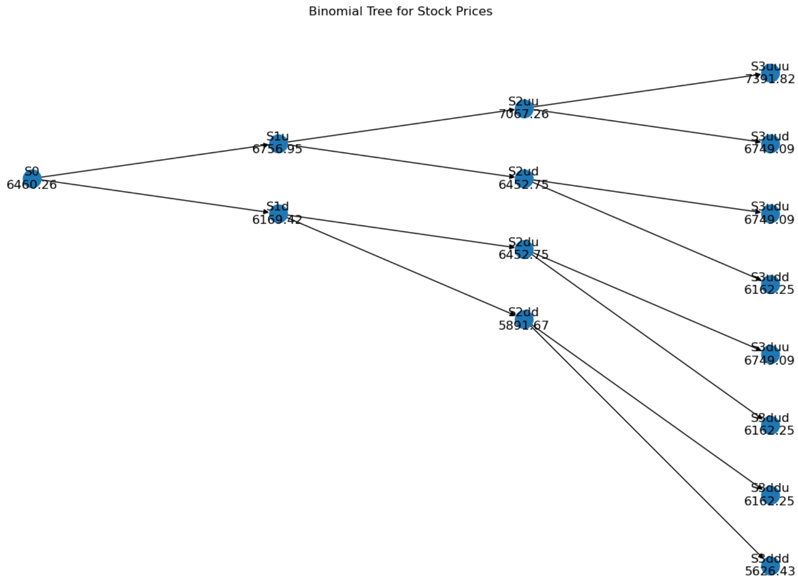
3.1 Question *(ab)*

3.2 Question *(c)*

The European call option price calculated using the Black-Scholes formula is approximately 205.89. In the Binomial tree method, we calculate the Call Option price using monthly compounding, while the Black Scholes pricing formula utilises continuous compounding. This can lead to divergence in the Call prices.

Now, we are going to increase the number of steps in the tree. Since we still want to price a 3-month option, this implies that the time between two consecutive steps decreases. However, the variance of the stock price after one month should remain the same (roughly). One way to accomplish this is to adjust the scale parameters u and d.

Figure 3.1: 3-step Binomial Tree for S&P-500 Index and European Put Option



3.3 Question *(d)*

Adjusting the up and down factors u and d when you increase the number of steps n (so that the option still has a 3-month maturity):

The idea is that the **variance of returns over the whole 3-month period must stay the same**, regardless of how finely you divide the interval. Since variance scales linearly with time in these discrete models, you adjust the distance between u and d by a factor of $\sqrt{\frac{3}{n}}$.

The rescaling formula is:

$$u_n \;=\; 1 + (u-1)\sqrt{\frac{3}{n}}, \quad d_n \;=\; 1 + (d-1)\sqrt{\frac{3}{n}}$$

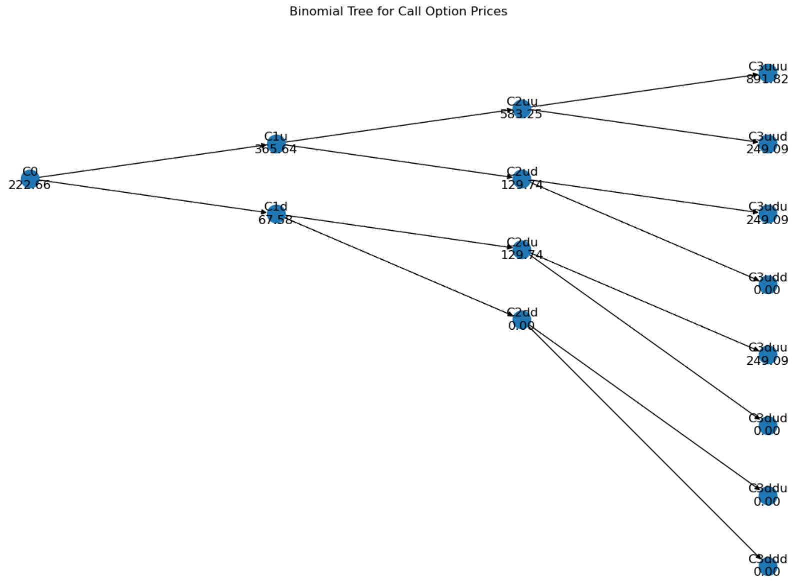
- If $n = 3$, then $\sqrt{3/3} = 1$, so you recover the original u and d.
- If $n = 6$ (half-month steps), the gap between u and d shrinks by $\sqrt{3/6} = \sqrt{1/2}$, keeping the total variance consistent.

This scaling ensures the binomial tree converges toward the continuous-time model (and eventually the Black-Scholes price) as $n \to \infty$.

3.4 Question *(e)*

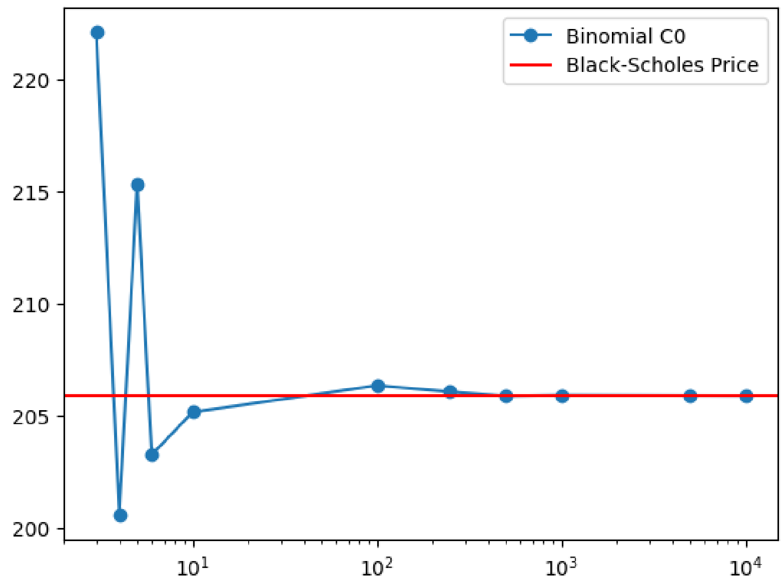
Essentially yes: under Q the expected simple 3-month return should equal the **risk-free** 3-month return. The assignment states "3% p.a. with **quarterly** compounding," which implies $1 + r_{3m} = 1.0075$ (i.e., **0.75%**). Your pipeline used annual→**monthly** compounding and then took the square root for half-months, giving 0.7418%. The tiny gap (~0.008%) is purely a compounding-convention mismatch.

Figure 3.2: 3-step Binomial Tree for S&P-500 Index and European Call Option



3.5 Question (f)

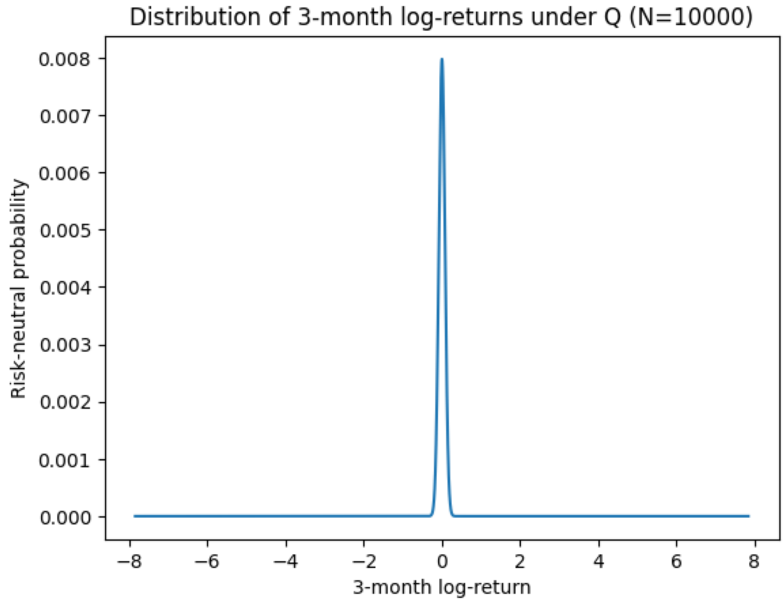
Figure 3.3: Convergence of Binomial Tree Method for European Call Option



As we increase the number of steps in the binomial tree, we see that the option value fluctuates around the Black–Scholes benchmark when the step size is very coarse (e.g. 3–10 steps). Once we move to a few hundred steps, the estimates stabilize and converge quickly toward the Black–Scholes price of about 205.89. The convergence is clear in the graph: the binomial values approach the continuous-time model as N grows, with only tiny differences (on the order of cents) beyond a few hundred steps. This confirms that the binomial method is consistent, and that with finer discretization we recover the Black–Scholes value for a European call.

3.6 Question (*g*)

Figure 3.4: Distribution of 3-Month Log-Returns Under Risk-Neutral Measure



With 10,000 steps the log-return distribution essentially collapses into a very narrow bell curve. On the plot it looks like a spike because most of the probability mass is concentrated around the mean, which is what we expect from the CLT as the number of steps grows.

We find that the expected 3-month log-return under the risk-neutral measure is about **0.43%**. This is lower than the risk-free simple return of about **0.74%**, because in log terms the expectation is shifted down by $\frac{1}{2}\sigma^2 T$. In other words, under Q the index grows on average at the risk-free rate in simple terms, but in log terms the mean is smaller due to volatility.

3.7 Question (*h*)

We extended our code to price European puts as well. For the put with strike 6,500 and maturity 3 months, the Black–Scholes value is **197.78**, while the corresponding call value is **205.89**.

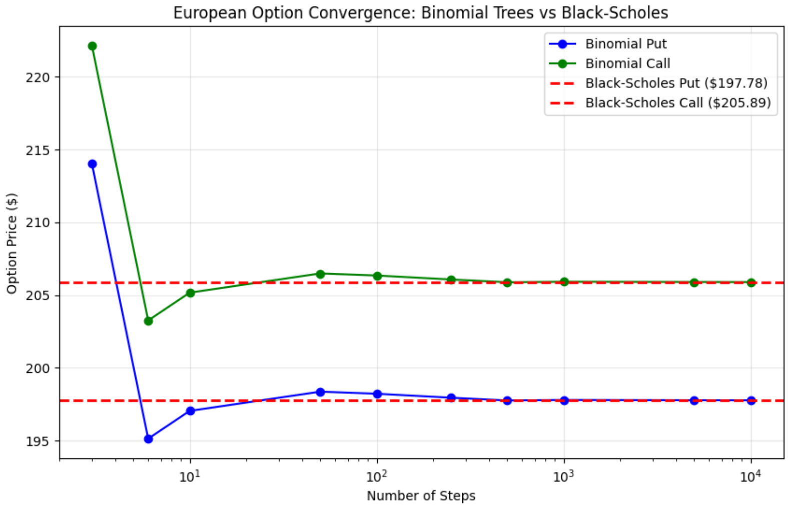
Looking at the convergence of the binomial tree, we see that with only a few steps the estimates are noisy (e.g. at $N = 3$, the put is 214.00 and the call 222.12). As we increase the number of steps, the values settle down: at $N = 100$ we already get a put of **198.22** and a call of **206.34**, and by $N = 10{,}000$ we converge to **197.77** for the put and **205.89** for the call—essentially identical to the Black–Scholes benchmarks.

Finally, we verify put–call parity. The theoretical difference is

$$C - P = S_0 - Ke^{-rT} \approx 8.12,$$

and both the Black–Scholes and binomial models (at $N = 10{,}000$) give **8.12** as well. The parity error is practically zero, which confirms that our implementation is consistent.

Figure 3.5: Put-Call Parity Verification



3.8 Question (*i*)

For the European benchmark, the Black–Scholes prices are **197.78** for the put and **205.89** for the call.

When we switch to American options and allow for early exercise, the results change only for the put. With $N = 2000$ steps, the American put is valued at **202.15**, which is about **4.37 higher** than the European put (a **2.2% premium**). The American call, on the other hand, converges to **205.90**, virtually identical to the European call.

This makes sense: early exercise is never optimal for a non-dividend-paying call, so its value is the same as the European call. For puts, early exercise can be advantageous when the option is sufficiently in-the-money, which explains why the American put carries a modest premium over its European counterpart.

References

[1] Björk, Tomas, Arbitrage Theory in Continuous Time, 2nd edn (Oxford, 2004; online edn, Oxford Academic, 1 Oct. 2005), https://doi-org.vu-nl.idm.oclc.org/10.1093/0199271267.001.0001

[2] Shreve, S.E. (2004) Stochastic Calculus for Finance II Continuous-Time Models. Springer, Berlin. https://doi.org/10.1007/978-1-4757-4296-1

[3] Mark-Jan Boes (2025). *Stochastic Processes - the Fundamentals - Lecture Slides.* Vrije Universiteit Amsterdam, School of Business and Economics.

Python Code

Listing 1: spf-assignment-1-script.py

```python
# Import libraries -----
import pandas as pd
import numpy as np

from scipy import stats
from scipy.stats import norm, binom

import matplotlib.pyplot as plt

import nbformat


# Import data -----

# Read the CSV file into a DataFrame
dfSP500 = pd.read_csv("SP500.csv")

# Rename columns for consistency
dfSP500 = dfSP500.rename(columns={"SP500": "index_value"})

# Convert the observation_date column to datetime format
dfSP500['observation_date'] =
    pd.to_datetime(dfSP500['observation_date'])

# Calculate the number of months to cover in our sample
iNumber_of_months = 10*(2 + 11) - 2*11

# Select the last iNumber_of_months observations and reset the index
dfSP500 =
    dfSP500[len(dfSP500)-iNumber_of_months:].reset_index(drop=True)

# Calculating simple net returns using the definition -----
for i in range(iNumber_of_months-1):
    dfSP500.loc[i, "simple_net_return"] = (dfSP500.loc[i+1,
        "index_value"] / dfSP500.loc[i, "index_value"]) - 1

# Define sample size
iSample_size = len(dfSP500) - 1

```

```python
# Solution to (a) part 1 -----

# Calculate sample mean and standard deviation
dSample_mean = np.mean(dfSP500["simple_net_return"])
dSample_std = np.std(dfSP500["simple_net_return"], ddof=1)

# Solution to (a) part 2 -----

# Define the significance level
dAlpha = 0.05

# Perform a two-sided t-test
dT_stat, dP_value = stats.ttest_1samp(dfSP500["simple_net_return"], popmean=0, nan_policy="omit", alternative="two-sided")

# Display the result of the t-test
print(f"t-statistic: {dT_stat:.4f}")
print(f"p-value: {dP_value:.4f}")

# Display the result of the hypothesis test
if dP_value < dAlpha:
    print("Reject H0: mean differs significantly from zero.")
else:
    print("Do not reject H0: no significant difference from zero.")

# Solution to (b) part 1 -----

# Define the population parameters based on the task and on the findings from part (a)
dPop_mean = 0.005
dPop_std = dSample_std

# Solution to (b) part 2 -----

# Calculate the required sample size to estimate the population mean within a margin of error of 0.005 with 95% confidence
dConfidence_interval = 0.95
dZ_975 = norm.ppf(1 - (1 - dConfidence_interval)/2)
dMargin_of_error = (0.0055 - 0.0045) / 2

# Calculate required months and convert to years
dMonths_required = ((dZ_975 * dPop_std) / dMargin_of_error) ** 2
dYears_required = dMonths_required / 12

# Display the results
print("Required months:", np.ceil(dMonths_required).astype(int))
print("Required years:", np.ceil(dYears_required).astype(int))

# Solution to (b) part 3 -----

# Set seed for reproducibility
np.random.seed(42)

# Generate random samples based on the defined population parameters
epsilon = np.random.normal(loc=0, scale=1, size=iSample_size)

# Calculate returns using the equation (2) provided in the task
```

```
dfSP500["return_by_equation_1"] = np.append((dPop_mean + dPop_std * epsilon), np.nan)

# Perform a two-sample t-test on the generated returns

dT_stat, dP_value = stats.ttest_ind(dfSP500["simple_net_return"], dfSP500["return_by_equation_1"], equal_var=True, nan_policy="omit", alternative="two-sided")

#
# Display the result of the t-test
print(f"t-statistic: {dT_stat:.4f}")
print(f"p-value: {dP_value:.4f}")

# Display the result of the hypothesis test
if dP_value < dAlpha:
    print("Reject H0: means differ significantly.")
else:
    print("Do not reject H0: no significant difference between means.")


# Solution to (c) -----

# Set seed for reproducibility
np.random.seed(42)

# Generate a large sample of returns to illustrate the distribution
dfSimulated_returns = np.random.normal(dPop_mean, dPop_std, int(np.ceil(dMonths_required)))

# Calculate the 97.5th percentile z-score (the margin of error for 95% confidence interval)
dMargin_of_Error = dZ_975 * (np.std(dfSimulated_returns) / np.sqrt(dMonths_required))

# Display the margin of error
print(f"Margin of Error for 95% confidence interval: {dMargin_of_Error:.8f}")

# Solution to (d) -----

# Calculating simple net returns according to the definition
for i in range(iNumber_of_months-1):
    dfSP500.loc[i, "log_normal_return"] = np.log(dfSP500.loc[i+1, "index_value"] / dfSP500.loc[i, "index_value"])

# Calculate mean and standard deviation of log-normal returns
dSample_std_tilda = np.std(dfSP500["log_normal_return"], ddof=1)
dSample_mean_tilda = (np.mean(dfSP500["log_normal_return"]) - 0.5 * dSample_std_tilda**2)

# Solution to (f) -----

# Define time horizon
iT = 60

```

```
# Calculate the expected index value after 60 months using the derived
    formula
dExpected_value = dfSP500.loc[iSample_size, "index_value"] * np.exp(iT
    * dSample_mean_tilda)

# Display the expected index value
print(f"The expected index value after {iT} months, using the derived
    formula is {dExpected_value:.2f}")

# Solution to (g) part 1 -----

# Set seed for reproducibility
np.random.seed(42)

# Number of simulations
iN = 10000

# Initial index value
iS0 = dfSP500.loc[iSample_size, "index_value"]

# Matrix to store simulated returns for 60 months
dfSimulated_returns = np.zeros((iN,60))

# Simulate the index value over 60 months for iN simulations
for i in range(60):
    # Simulate the index value using the GBM (Equation (2)
    vSt = iS0 * np.exp((dSample_mean_tilda - 0.5 * dSample_std_tilda
        ** 2) + dSample_std_tilda * np.random.normal(0,1,iN))
    # Store the simulated values
    dfSimulated_returns[:,i] = vSt
    # Update the initial index value for the next iteration
    iS0 = vSt

# Convert the simulated returns to a DataFrame for easier analysis
dfSimulated_returns = pd.DataFrame(dfSimulated_returns)

# Solution to (g) part 2 -----

# Create 60 monthly future dates (end of month)
dfDate_range =
    pd.date_range(start=pd.to_datetime(dfSP500["observation_date"].iloc[iSample_size])
    + pd.offsets.MonthEnd(1),
                            periods=iT,
                            freq="MS")

# Plotting the historical data and simulated paths
plt.figure(figsize=(12,6))

# Historical
plt.plot(dfSP500["observation_date"], dfSP500["index_value"],
         color='blue', lw=2, label='Historical S&P 500')

# Forecast mean
plt.plot(dfDate_range, dfSimulated_returns.mean(axis=0).to_numpy(),
         color='red', lw=2, label='Expected Path')

# Forecast interval (90% band)
```

```python
plt.fill_between(x=dfDate_range,
                 y1=dfSimulated_returns.quantile(0, axis=0),
                 y2=dfSimulated_returns.quantile(1, axis=0),
                 color='red', alpha=0.2, label='Expected Interval')

plt.xlabel("Date")
plt.ylabel("S&P 500 Index Level")
plt.title("S&P 500: Historical Data & 10,000 Simulated Paths (60
    Months Ahead)")

plt.legend()
plt.grid(alpha=0.3)
plt.tight_layout()
plt.show()


# Solution to (g) part 3 -----

# Test for normality of the index levels after 60 months using the
    Kolmogorov-Smirnov test
for i in range(iT):
    # Perform the Kolmogorov-Smirnov test
    dK_stat, dP_value = stats.kstest(dfSimulated_returns[i], 'norm',
        args=(dfSimulated_returns[i].mean(),
        dfSimulated_returns[i].std()))

    # Display the result of the Kolmogorov-Smirnov test
    print(f"Month {i+1}: D-statistic: {dK_stat:.4f}, p-value:
        {dP_value:.4f}")

# Solution to (h) -----

# Define the risk-free rate
dRf = 0.0

# Define the strike price
dStrike = 6300.0

# Digital put payoff: 1 if index < strike, else 0
dfPut_payoff = (dfSimulated_returns[iT-1] < dStrike).astype(int)

# Discount by risk-free rate (r=0%, so no effect)
dPut_prob = (dfPut_payoff * np.exp(-dRf * iT)).mean()

# Display the probability of the digital put option being in the money
print(dPut_prob)

# Solution to (i) -----

# Initial index value
dS0 = dfSP500.loc[iSample_size, "index_value"]

# Analytical digital put price under model (2)
dZ = (np.log(dStrike/dS0) - (0 - 0.5 * dSample_std_tilda**2) * iT) /
    (dSample_std_tilda * np.sqrt(iT))
dPut_analytical = np.exp(-0 * iT) * norm.cdf(dZ)

```

```python
# Display both prices
print(f"Analytical digital put price: {dPut_analytical:.4f}")
print(f"Simulation price: {dPut_prob:.4f}")


# Solution to (j) -----

# Analytical digital call price under model (2)
dZ_call = (np.log(dStrike/dS0) - (0 - 0.5 * dSample_std_tilda**2) * iT) / (dSample_std_tilda * np.sqrt(iT))
dCall_analytical = np.exp(-0 * iT) * (1 - norm.cdf(dZ_call))

# Display both prices and their sum
print(f"Digital put price:  {dPut_analytical:.4f}")
print(f"Digital call price: {dCall_analytical:.4f}")
print(f"Sum: {dPut_analytical + dCall_analytical:.4f}")

# Solution to (a) and (b) -----

# Parameters for the binomial tree
net_return = 0.005
strike_call = 6500
risk_free_annual = 0.03
risk_free_quarter = 0.0075
Rm = (1+ risk_free_annual)**(1/12)

risk_free_monthly = Rm -1

p = 0.55
u = 1.045925
d = 0.95498
q = (1 + risk_free_monthly - d) / (u - d)

# Function to calculate the call option price C0 given S0, u, d, q, strike_call, and risk_free_rate
def S0_to_C0(S0,u,d,q,strike_call,risk_free_rate):
       S0 = dS0

       S1u = S0*u
       S1d = S0*d

       S2uu = S1u*u
       S2ud = S1u*d
       S2du = S1d*u
       S2dd = S1d*d

       S3uuu = S2uu*u
       S3uud = S2uu*d
       S3udu = S2ud*u
       S3udd = S2ud*d
       S3duu = S2du*u
       S3dud = S2du*d
       S3ddu = S2dd*u
       S3ddd = S2dd*d

       C3uuu = max(0, S3uuu - strike_call)
       C3uud = max(0, S3uud - strike_call)
```

```
        C3udu = max(0, S3udu - strike_call)
        C3udd = max(0, S3udd - strike_call)
        C3duu = max(0, S3duu - strike_call)
        C3dud = max(0, S3dud - strike_call)
        C3ddu = max(0, S3ddu - strike_call)
        C3ddd = max(0, S3ddd - strike_call)

        C2uu = (q * C3uuu + (1-q) * C3uud) / (1 + risk_free_rate)
        C2ud = (q * C3udu + (1-q) * C3udd) / (1 + risk_free_rate)
        C2du = (q * C3duu + (1-q) * C3dud) / (1 + risk_free_rate)
        C2dd = (q * C3ddu + (1-q) * C3ddd) / (1 + risk_free_rate)

        C1u = (q * C2uu + (1-q) * C2ud) / (1 + risk_free_rate)
        C1d = (q * C2du + (1-q) * C2dd) / (1 + risk_free_rate)

        C0 = (q * C1u + (1-q) * C1d) / (1 + risk_free_rate)

        binomial_tree_dict = {
                    'S1u': S1u, 'S1d': S1d,
                    'S2uu': S2uu, 'S2ud': S2ud, 'S2du': S2du, 'S2dd': S2dd,
                    'S3uuu': S3uuu, 'S3uud': S3uud, 'S3udu': S3udu, 'S3udd': S3udd,
                    'S3duu': S3duu, 'S3dud': S3dud, 'S3ddu': S3ddu, 'S3ddd': S3ddd,
                    'C1u': C1u, 'C1d': C1d,
                    'C2uu': C2uu, 'C2ud': C2ud, 'C2du': C2du, 'C2dd': C2dd,
                    'C3uuu': C3uuu, 'C3uud': C3uud, 'C3udu': C3udu, 'C3udd': C3udd,
                    'C3duu': C3duu, 'C3dud': C3dud, 'C3ddu': C3ddu, 'C3ddd': C3ddd
        }
        return C0 , binomial_tree_dict

binomial_tree = S0_to_C0(dS0,u,d,q,strike_call,risk_free_monthly)
C0 = binomial_tree[0]

S0 = dS0
S1u = binomial_tree[1]['S1u']
S1d = binomial_tree[1]['S1d']
S2uu = binomial_tree[1]['S2uu']
S2ud = binomial_tree[1]['S2ud']
S2du = binomial_tree[1]['S2du']
S2dd = binomial_tree[1]['S2dd']
S3uuu = binomial_tree[1]['S3uuu']
S3uud = binomial_tree[1]['S3uud']
S3udu = binomial_tree[1]['S3udu']
S3udd = binomial_tree[1]['S3udd']
S3duu = binomial_tree[1]['S3duu']
S3dud = binomial_tree[1]['S3dud']
S3ddu = binomial_tree[1]['S3ddu']
S3ddd = binomial_tree[1]['S3ddd']
C1u = binomial_tree[1]['C1u']
C1d = binomial_tree[1]['C1d']
C2uu = binomial_tree[1]['C2uu']
C2ud = binomial_tree[1]['C2ud']
```

```python
C2du = binomial_tree[1]['C2du']
C2dd = binomial_tree[1]['C2dd']
C3uuu = binomial_tree[1]['C3uuu']
C3uud = binomial_tree[1]['C3uud']
C3udu = binomial_tree[1]['C3udu']
C3udd = binomial_tree[1]['C3udd']
C3duu = binomial_tree[1]['C3duu']
C3dud = binomial_tree[1]['C3dud']
C3ddu = binomial_tree[1]['C3ddu']
C3ddd = binomial_tree[1]['C3ddd']


# create the binomioal tree plot for both stock price and call option price
import matplotlib.pyplot as plt
import networkx as nx
import matplotlib.patches as mpatches
G = nx.DiGraph()

# Stock Price Tree
G.add_edges_from([(0, 1), (0, 2),
                  (1, 3), (1, 4),
                  (2, 5), (2, 6),
                  (3, 7), (3, 8),
                  (4, 9), (4, 10),
                  (5, 11), (5, 12),
                  (6, 13), (6, 14)])
pos = {0: (0, 0), 1: (1, 1), 2: (1, -1),
       3: (2, 2), 4: (2, 0), 5: (2, -2),
       6: (2, -4), 7: (3, 3), 8: (3, 1), 9: (3, -1), 10: (3, -3),
       11: (3, -5), 12: (3, -7), 13: (3, -9), 14: (3, -11)}
labels = {0: f"S0\n{S0:.2f}",
          1: f"S1u\n{S1u:.2f}", 2: f"S1d\n{S1d:.2f}",
          3: f"S2uu\n{S2uu:.2f}", 4: f"S2ud\n{S2ud:.2f}",
          5: f"S2du\n{S2du:.2f}", 6: f"S2dd\n{S2dd:.2f}",
          7: f"S3uuu\n{S3uuu:.2f}", 8: f"S3uud\n{S3uud:.2f}",
          9: f"S3udu\n{S3udu:.2f}", 10: f"S3udd\n{S3udd:.2f}",
          11: f"S3duu\n{S3duu:.2f}", 12: f"S3dud\n{S3dud:.2f}",
          13: f"S3ddu\n{S3ddu:.2f}", 14: f"S3ddd\n{S3ddd:.2f}"}
plt.figure(figsize=(12, 8))
nx.draw(G, pos, with_labels=False, arrows=True)
nx.draw_networkx_labels(G, pos, labels)
plt.title("Binomial Tree for Stock Prices")
plt.show()

# Call Option Price Tree
G_call = nx.DiGraph()
G_call.add_edges_from([(0, 1), (0, 2),
                       (1, 3), (1, 4),
                       (2, 5), (2, 6),
                       (3, 7), (3, 8),
                       (4, 9), (4, 10),
                       (5, 11), (5, 12),
                       (6, 13), (6, 14)])
pos_call = pos
labels_call = {0: f"C0\n{C0:.2f}",
               1: f"C1u\n{C1u:.2f}", 2: f"C1d\n{C1d:.2f}",
```

```
                3: f"C2uu\n{C2uu:.2f}", 4: f"C2ud\n{C2ud:.2f}",
                5: f"C2du\n{C2du:.2f}", 6: f"C2dd\n{C2dd:.2f}",
                7: f"C3uuu\n{C3uuu:.2f}", 8: f"C3uud\n{C3uud:.2f}",
                9: f"C3udu\n{C3udu:.2f}", 10: f"C3udd\n{C3udd:.2f}",
                11: f"C3duu\n{C3duu:.2f}", 12: f"C3dud\n{C3dud:.2f}",
                13: f"C3ddu\n{C3ddu:.2f}", 14: f"C3ddd\n{C3ddd:.2f}"}
plt.figure(figsize=(12, 8))
nx.draw(G_call, pos_call, with_labels=False, arrows=True)
nx.draw_networkx_labels(G_call, pos_call, labels_call)
plt.title("Binomial Tree for Call Option Prices")
plt.show()


# Direct calculation of C0 using all possible paths from t=0 to t=3
C0_direct = (
    q**3            * C3uuu +
    3*q**2*(1-q)    * C3uud +
    3*q*(1-q)**2    * C3udd +
    (1-q)**3        * C3ddd
) / (1+risk_free_monthly)**3


#compare C0 and C0_test
print(f"C0 from backward induction: {C0:.2f}")
print(f"C0 from direct calculation: {C0_direct:.2f}")

# Solution to (c) -----

S0 = dS0
K = 6500 # strike price

# Inputs
T = 0.25
rf_annual = 0.03
risk_free_rate = np.log(1 + rf_annual)  # convert to continuous
    compounding

monthly_std = dSample_std
sigma_annual = np.sqrt(12)*monthly_std

def black_scholes_call(S0, K, r, sigma, T):
    """Calculate Black-Scholes call option price"""
    d1 = (np.log(S0/K) + (r + 0.5 * sigma**2) * T) / (sigma *
        np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    call_price = S0 * norm.cdf(d1) - K * np.exp(-r*T) * norm.cdf(d2)
    return call_price

# Black-Scholes price
call_price_bs = black_scholes_call(S0, K,risk_free_rate,
    sigma_annual,T )

print(f"European Call Price by Black-Scholes: {call_price_bs:.2f}")


# Solution to (d) -----

```

```python
# Function (formula) to rescale u and d for a different number of steps
def rescale_factors_u_d(u,d,steps):
    u_rescale = (u-1) * np.sqrt(3/ steps) + 1
    d_rescale = (d-1) * np.sqrt(3/ steps) + 1
    return u_rescale, d_rescale

# Solution to (e) -----

u_rescaled, d_rescaled = rescale_factors_u_d(u,d,6)

# half a month risk free rate
R_hm = (1 + risk_free_monthly)**0.5
r = R_hm - 1

q_rescaled = (1 + r - d_rescaled) / (u_rescaled - d_rescaled)

print(f"Rescaled u: {u_rescaled:.6f}, Rescaled d: {d_rescaled:.6f},
    Rescaled q: {q_rescaled:.6f}")

# Solution to (f) -----

# Convert discrete parameters to continuous
def binomial_tree_european_call(S0, K, r, sigma, T, N):
    dt = T / N
    u = np.exp(sigma * np.sqrt(dt))
    d = np.exp(-sigma * np.sqrt(dt))
    disc = np.exp(-r * dt)
    q = (np.exp(r * dt) - d) / (u - d)

    # terminal payoffs with Cox-Ross-Rubinstein (# of steps agnostic)
    ST = np.array([S0 * (u ** j) * (d ** (N - j)) for j in range(N+1)])
    C = np.maximum(ST - K, 0)

    # backward induction
    for _ in range(N):
        C = disc * (q * C[1:] + (1 - q) * C[:-1])
    return C[0]

steps_all = [3, 4, 5, 6, 10, 100, 250, 500, 1000, 5000, 10000]
C0_values = []
S0 = dS0
K = 6500
T = 0.25
r = np.log(1+risk_free_annual)   # continuous, annualized
sigma = dSample_std*np.sqrt(12)     # fill with your annualized value

for steps in steps_all:
    C0 = binomial_tree_european_call(S0, K, r, sigma, T, steps)
    C0_values.append(C0)
    print(f"Steps: {steps}, C0: {C0:.6f}")


# Reference Black-Scholes price
plt.plot(steps_all, C0_values, 'o-', label='Binomial C0')
plt.axhline(y=call_price_bs, color='r', label='Black-Scholes Price')
plt.xscale('log')
plt.legend()
```

```
plt.show()


# Solution to (g) -----

N = 10000
dt = T / N
u = np.exp(sigma * np.sqrt(dt))
d = np.exp(-sigma * np.sqrt(dt))
disc = np.exp(-r * dt)
q = (np.exp(r * dt) - d) / (u - d)

# All possible numbers of up-moves
k_vals = np.arange(N + 1)

# Log-returns at each terminal node
log_returns = k_vals * np.log(u/d) + N * np.log(d)

# Binomial probabilities under risk-neutral measure
probabilities = binom.pmf(k_vals, N, q)

# Plot distribution
plt.plot(log_returns, probabilities)
plt.xlabel('3-month log-return')
plt.ylabel('Risk-neutral probability')
plt.title(f'Distribution of 3-month log-returns under Q (N={N})')
plt.show()

expected_log_return = np.sum(probabilities * log_returns)
print(f"Expected 3-month log-return (risk-neutral): {expected_log_return:.6f}")


# Solution to (h) -----

S0 = dS0
K = 6500
T = 0.25
r_continuous = np.log(1+risk_free_annual)   # continuous, annualized
sigma_annual = dSample_std*np.sqrt(12)

def black_scholes_put(S0, K, r, sigma, T):
    d1 = (np.log(S0/K) + (r + 0.5 * sigma**2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    put_price = K * np.exp(-r*T) * norm.cdf(-d2) - S0 * norm.cdf(-d1)
    return put_price

def binomial_tree_european_put(S0, K, r, sigma, T, N): #same with call function but P = max(ST-K)
    dt = T / N
    u = np.exp(sigma * np.sqrt(dt))
    d = np.exp(-sigma * np.sqrt(dt))
    q = (np.exp(r * dt) - d) / (u - d)
    disc = np.exp(-r * dt)

    ST = np.array([S0 * (u ** j) * (d ** (N - j)) for j in range(N+1)])
```

```
    P = np.maximum(K - ST, 0)

    # Backward induction
    for _ in range(N):
        P = disc * (q * P[1:] + (1 - q) * P[:-1])
    return P[0]

bs_put_price = black_scholes_put(S0, K, r_continuous, sigma_annual, T)
bs_call_price = black_scholes_call(S0, K, r_continuous, sigma_annual, T)
print(f"Black-Scholes Put Price: ${bs_put_price:.2f}")
print(f"Black-Scholes Call Price: ${bs_call_price:.2f}")

# Convergence
steps_all = [3, 6, 10, 50, 100, 250, 500, 1000, 5000, 10000]
put_values = []
call_values = []

for N in steps_all:
    put_price = binomial_tree_european_put(S0, K, r_continuous, sigma_annual, T, N)
    call_price = binomial_tree_european_call(S0, K, r_continuous, sigma_annual, T, N)

    put_values.append(put_price)
    call_values.append(call_price)

    print(f"{N:5d}  | ${put_price:8.2f} | ${call_price:9.2f}")

# Check put call parity
PV_strike = K * np.exp(-r_continuous * T)
theoretical_diff = S0 - PV_strike

print(f"\nPut-Call Parity Verification:")
print(f"Formula: C - P = S0 - K*e^(-rT)")
print(f"Theoretical Difference: ${theoretical_diff:.2f}")

# Using final binomial values (N=10000)
final_put = put_values[-1]
final_call = call_values[-1]
binomial_diff = final_call - final_put

print(f"\nBlack-Scholes: C - P = ${bs_call_price:.2f} - ${bs_put_price:.2f} = ${bs_call_price - bs_put_price:.2f}")
print(f"Binomial (N=10000): C - P = ${final_call:.2f} - ${final_put:.2f} = ${binomial_diff:.2f}")
print(f"Parity Error (Binomial): ${abs(binomial_diff - theoretical_diff):.4f}")


# Convergence Plot
plt.figure(figsize=(10, 6))
plt.plot(steps_all, put_values, 'bo-', label='Binomial Put', markersize=6)
plt.plot(steps_all, call_values, 'go-', label='Binomial Call', markersize=6)
plt.axhline(y=bs_put_price, color='r', linestyle='--',
```

```
             label=f'Black-Scholes Put (${bs_put_price:.2f})', linewidth=2)
plt.axhline(y=bs_call_price, color='r', linestyle='--',
             label=f'Black-Scholes Call (${bs_call_price:.2f})', linewidth=2)

plt.xscale('log')
plt.xlabel('Number of Steps')
plt.ylabel('Option Price ($)')
plt.title('European Option Convergence: Binomial Trees vs Black-Scholes')
plt.legend()
plt.grid(True, alpha=0.3)
plt.show()

# Solution to (i) -----

def binomial_tree_american_put(S0, K, r, sigma, T, N):
    dt = T / N
    u = np.exp(sigma * np.sqrt(dt))
    d = np.exp(-sigma * np.sqrt(dt))
    q = (np.exp(r * dt) - d) / (u - d)
    disc = np.exp(-r * dt)

    stock_tree = np.zeros((N+1, N+1))
    for i in range(N+1):
        for j in range(i+1):
            stock_tree[j, i] = S0 * (u ** j) * (d ** (i - j))

    # Initialize options tree
    option_tree = np.zeros((N+1, N+1))
    for j in range(N+1):
        option_tree[j, N] = max(0, K - stock_tree[j, N])  # put payoff

    # Backward induction
    for i in range(N-1, -1, -1):
        for j in range(i+1):
            continuation_value = disc * (q * option_tree[j+1, i+1] + (1-q) * option_tree[j, i+1])
            exercise_value = max(0, K - stock_tree[j, i])
            option_tree[j, i] = max(continuation_value, exercise_value)

    return option_tree[0, 0]

def binomial_tree_american_call(S0, K, r, sigma, T, N):
    dt = T / N
    u = np.exp(sigma * np.sqrt(dt))
    d = np.exp(-sigma * np.sqrt(dt))
    q = (np.exp(r * dt) - d) / (u - d)
    disc = np.exp(-r * dt)

    # Initialize stock tree
    stock_tree = np.zeros((N+1, N+1))
    for i in range(N+1):
        for j in range(i+1):
            stock_tree[j, i] = S0 * (u ** j) * (d ** (i - j))

```

```python
    # Initialize option tree
    option_tree = np.zeros((N+1, N+1))
    for j in range(N+1):
        option_tree[j, N] = max(0, stock_tree[j, N] - K)  # Call payoff

    # Backward induction
    for i in range(N-1, -1, -1):
        for j in range(i+1):
            continuation_value = disc * (q * option_tree[j+1, i+1] + (1-q) * option_tree[j, i+1])
            exercise_value = max(0, stock_tree[j, i] - K)
            option_tree[j, i] = max(continuation_value, exercise_value)
    return option_tree[0, 0]

european_put = black_scholes_put(S0, K, r_continuous, sigma_annual, T)
european_call = black_scholes_call(S0, K, r_continuous, sigma_annual, T)

print(f"Black-Scholes Put: ${european_put:.2f}")
print(f"Black-Scholes Call: ${european_call:.2f}")

print("\nCalculating American options with different steps:")
steps = [100, 250, 500, 1000, 2000]

for N in steps:
    american_put = binomial_tree_american_put(S0, K, r_continuous, sigma_annual, T, N)
    american_call = binomial_tree_american_call(S0, K, r_continuous, sigma_annual, T, N)

    put_diff = american_put - european_put
    call_diff = american_call - european_call

    print(f"N={N}:")
    print(f"Put: ${american_put:.2f} (diff: ${put_diff:.2f})")
    print(f"Call: ${american_call:.2f} (diff: ${call_diff:.2f})")

    if N == steps[-1]:
        final_put = american_put
        final_call = american_call

print("\nFinal comparison:")
put_diff = final_put - european_put
call_diff = final_call - european_call

print(f"Put options:")
print(f"European: ${european_put:.2f}")
print(f"American: ${final_put:.2f}")
print(f"Difference: ${put_diff:.2f} ({(put_diff/european_put*100):.1f}%)\n")

print(f"Call options:")
print(f"European: ${european_call:.2f}")
print(f"American: ${final_call:.2f}")
print(f"Difference: ${call_diff:.2f} ({(call_diff/european_call*100):.1f}%)")
```

```
# Export python code / markdown cells separately


# Load notebook
nb = nbformat.read("/Users/milanpeter/Documents/University/Vrije
    Universiteit Amsterdam/Stochastic Processes - the
    Fundamentals/Computer Assignment/spf-assignment-1-script.ipynb",
    as_version=4)

# Write code cells into one python script
with open("spf-a1-script.py", "w", encoding="utf-8") as f:
    for cell in nb.cells:
        if cell.cell_type == "code":
            f.write(cell.source.replace("-", "-") + "\n\n")

# Collect markdown cells
md_cells = [cell['source'] for cell in nb.cells if cell['cell_type']
    == 'markdown']

# Write them into one markdown file
with open("spf-a1-markdown.md", "w") as f:
    f.write("\n\n".join(md_cells))

# Convert markdown to LaTeX using pandoc
# run in bash
# pandoc spf-a1-markdown.md -o spf-a1-markdown.tex
```