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### **Flatness and defect of non-linear systems: introductory theory and examples**

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## Flatness and defect of non-linear systems: introductory theory and examples

MICHEL FLIESS<sup>†</sup>, JEAN LÉVINE<sup>‡</sup>, PHILIPPE MARTIN<sup>§</sup> and  
PIERRE ROUCHON<sup>||</sup>

We introduce flat systems, which are equivalent to linear ones via a special type of feedback called endogenous. Their physical properties are subsumed by a linearizing output and they might be regarded as providing another non-linear extension of Kalman's controllability. The distance to flatness is measured by a non-negative integer, the defect. We utilize differential algebra where flatness and defect are best defined without distinguishing between input, state, output and other variables. Many realistic classes of examples are flat. We treat two popular ones: the crane and the car with  $n$  trailers, the motion planning of which is obtained via elementary properties of plane curves. The three non-flat examples, the simple, double and variable length pendulums, are borrowed from non-linear physics. A high frequency control strategy is proposed such that the averaged systems become flat.

### 1. Introduction

We present here five case-studies: the control of a crane, of the simple, double and variable length pendulums and the motion planning of a car with  $n$ -trailers. They are all treated within the framework of dynamic feedback linearization which, contrary to static feedback linearization, has only been investigated by few authors (Charlet *et al.* 1989, 1991, Shadwick 1990). Our point of view will be probably best explained by the following calculations where all vector fields and functions are real-analytic.

Consider

$$\dot{x} = f(x, u) \quad (x \in \mathbb{R}^n, u \in \mathbb{R}^m) \quad (1)$$

where  $f(0, 0) = 0$  and

$$\text{rank} \frac{\partial f}{\partial u}(0, 0) = m$$

The dynamic feedback linearizability of (1) means, according to Charlet *et al.* (1989), the existence of

(a) a regular dynamic compensator

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$$\left. \begin{aligned} \dot{z} &= a(x, z, v) \\ u &= b(x, z, v) \end{aligned} \right\} \quad (z \in \mathbb{R}^q, v \in \mathbb{R}^m) \quad (2)$$

where  $a(0, 0, 0) = 0$ ,  $b(0, 0, 0) = 0$ . The regularity assumption implies the invertibility (see Li and Feng (1987) for a definition of this concept via the structure algorithm. See Di Benedetto *et al.* (1989), Delaleau and Fliess (1992) for a connection with the differential algebraic approach) of system (2) with input  $v$  and output  $u$ ; and

(b) a diffeomorphism

$$\xi = \Xi(x, z) \quad (\xi \in \mathbb{R}^{n+q}) \quad (3)$$

such that (1) and (2), whose  $(n + q)$ -dimensional dynamics is given by

$$\left. \begin{aligned} \dot{x} &= f(x, b(x, z, v)) \\ \dot{z} &= a(x, z, v) \end{aligned} \right\}$$

becomes, according to (3), a constant linear controllable system  $\dot{\xi} = F\xi + Gv$ .

Up to a static state feedback and a linear invertible change of coordinates, this linear system may be written in Brunovsky canonical form (see, e.g., Kailath 1980),

$$\left. \begin{aligned} y_1^{(v_1)} &= v_1 \\ &\vdots \\ y_m^{(v_m)} &= v_m \end{aligned} \right\}$$

where  $v_1, \dots, v_m$  are the controllability indices and  $(y_1, \dots, y_1^{(v_1-1)}, \dots, y_m, \dots, y_m^{(v_m-1)})$  is another basis of the vector space spanned by the components of  $\xi$ . Set  $Y = (y_1, \dots, y_1^{(v_1-1)}, \dots, y_m, \dots, y_m^{(v_m-1)})$ ; thus  $Y = T\xi$  where  $T$  is an invertible  $(n + q) \times (n + q)$  matrix. Otherwise stated,  $Y = T\Xi(x, z)$ . The invertibility of  $\Xi$  yields

$$\begin{bmatrix} x \\ z \end{bmatrix} = \Xi^{-1}(T^{-1}Y) \quad (4)$$

Thus from (2)  $u = b(\Xi^{-1}(T^{-1}Y), v)$ . From  $v_i = y_i^{(v_i)}$ ,  $i = 1, \dots, m$ ,  $u$  and  $x$  can be expressed as real-analytic functions of the components of  $y = (y_1, \dots, y_m)$  and of a finite number of their derivatives:

$$\left. \begin{aligned} x &= \mathcal{A}(y, \dot{y}, \dots, y^{(\alpha)}) \\ u &= \mathcal{B}(y, \dot{y}, \dots, y^{(\beta)}) \end{aligned} \right\} \quad (5)$$

The dynamic feedback (2) is said to be endogenous if, and only if, the converse holds, i.e., if, and only if, any component of  $y$  can be expressed as a real-analytic function of  $x$ ,  $u$  and a finite number of its derivatives:

$$y = \mathcal{C}(x, u, \dot{u}, \dots, u^{(\gamma)}) \quad (6)$$

Note that, according to (4), this amounts to expressing  $z$  as a function of  $(x, u, \dot{u}, \dots, u^{(\rho)})$  for some  $\rho$ . In other words, the dynamic extension does not contain exogenous variables, which are independent of the original system

variables and their derivatives. This justifies the word endogenous. Note that quasi-static feedbacks, introduced in the context of dynamic input-output decoupling (Delaleau and Fliess 1992), share the same property.

A dynamics (1) which is linearizable via such an endogenous feedback is said to be (differentially) flat;  $y$ , which might be regarded as a fictitious output, is called a linearizing or flat output. The terminology flat is due to the fact that  $y$  plays a somehow analogous role to the flat coordinates in the differential geometric approach to the Frobenius theorem (see, e.g., Isidori 1989, Nijmeijer and van der Schaft 1990). A considerable amount of realistic models are indeed flat. We treat here two case-studies, namely the crane (D'Andréa-Novel and Lévine 1990, Marttinen *et al.* 1990) and the car with  $n$  trailers (Murray and Sastry 1993, Rouchon *et al.* 1993 *a*). Notice that the use of a linearizing output was already known in the context of static state feedback (see Claude (1986) and Isidori (1989, p. 156)).

One major property of differential flatness is that, due to formulae (5) and (6), the state and input variables can be directly expressed, without integrating any differential equation, in terms of the flat output and a finite number of its derivatives. This general idea can be traced back to works by D. Hilbert (Hilbert 1912) and E. Cartan (Cartan 1915) on under-determined systems of differential equations, where the number of equations is strictly less than the number of unknowns. Let us emphasize on the fact that this property may be extremely useful when dealing with trajectories: from  $y$  trajectories,  $x$  and  $u$  trajectories are immediately deduced. We shall detail in the sequel various applications of this property from motion planning to stabilization of reference trajectories. The originality of our approach partly relies on the fact that the same formalism applies to study systems around equilibrium points as well as around arbitrary trajectories.

As demonstrated by the crane, flatness is best defined by not distinguishing between input, state, output and other variables. The equations moreover might be implicit. This standpoint, which matches with Willems's approach (Willems 1991), is here taken into account by utilizing differential algebra which has already helped clarifying several questions in control theory (see, e.g., Fliess 1989, 1990 *a*, Diop 1991, 1992, Fliess and Glad 1993).

Flatness might be seen as another nonlinear extension of Kalman's controllability. Such an assertion is surprising when having in mind the vast literature on this subject (see Isidori (1989), Nijmeijer and van der Schaft (1990) and the references therein). Remember, however, Willems' trajectory characterization (Willems 1991) of linear controllability which can be interpreted as the freeness of the module associated to a linear system (Fliess 1992). A linearizing output now is the nonlinear analogue of a basis of this free module.

We know from Charlet *et al.* (1989) that any single-input dynamics which is linearizable by a dynamic feedback is also linearizable by a static feedback. This implies the existence of non-flat systems which verify the strong accessibility property (Sussmann and Jurdjevic 1972). We introduce a non-negative integer, the defect, which measures the distance from flatness.

These new concepts and mathematical tools are providing the common formalism and the underlying structure of five physically motivated case studies. The first two, i.e., the control of a crane and the motion planning of a car with  $n$ -trailers, which are quite concrete, resort from flat systems. The three others,

i.e., the simple and double Kapitza pendulums and the variable-length pendulum exhibit a non-zero defect.

The characterization of the linearizing output in the crane is obvious when utilizing a non-classic representation, i.e., a mixture of differential and non-differential equations, where there are no distinctions between the system variables. It permits a straightforward tracking of a reference trajectory via an open-loop control. We not only take advantage of the equivalence to a linear system but also of the decentralized structure created by assuming that the engines are powerful with respect to the masses of the trolley and the load.

The motion planning of the car with  $n$ -trailer is perhaps the most popular example of path planning of nonholonomic systems (Laumond 1991, Monaco and Normand-Cyrot 1992, Martin and Rouchon 1993, Murray and Sastry 1993, Rouchon *et al.* 1993 *a*, 1993 *b*, Tilbury *et al.* 1993). It is a flat system where the linearizing output is the middle of the axle of the last trailer. Once the linearizing output is determined, the path planning problem becomes particularly easy: the reference trajectory as well as the corresponding open-loop control can be expressed in terms of the linearizing output and a finite number of its derivatives. Let us stress that no differential equations need to be integrated to obtain the open-loop control. The relative motions of the various components of the system are then obtained thanks to elementary geometric properties of plane curves. The resulting calculations, which are presented in the two-trailer case, are very fast and have been implemented on a standard personal microcomputer under MATLAB.

The control of the three non-flat systems is based on high frequency control and approximations by averaged and flat systems—for other approaches, see, e.g. Meerkov (1980), Bentsman (1987), Baillieul (1993). We exploit here an idea due to the Russian physicist Kapitza (Landau and Lifshitz 1982, Bogaevski and Povzner 1991) for stabilizing these three systems in the neighborhood of quite arbitrary positions and trajectories, and in particular positions which are not equilibrium points. This idea is closely related to a curiosity of classical mechanics that a double inverted pendulum (Stephenson 1908), and even the  $N$  linked pendulums which are inverted and balanced on top of one another (Acheson 1993), can be stabilized in the same way. Closed-loop stabilization around reference averaged trajectories becomes straightforward by utilizing the endogenous feedback equivalence to linear controllable systems.

The paper is organized as follows. After some differential algebraic preliminaries, we define equivalence by endogenous feedback, flatness and defect. Their implications for uncontrolled dynamics and linear systems are examined. We discuss the link between flatness and controllability. In order to verify that some systems are not linearizable by dynamic feedback, we demonstrate a necessary condition of flatness, which is of a geometric nature. The last two sections are devoted respectively to the flat and non-flat examples.

First drafts of various parts of this article have been presented in Fliess *et al.* (1991, 1992 *a*, *b*, 1993 *b*, *c*).

## 2. The algebraic framework

We consider variables related by algebraic differential equations. We start with a brief review of differential fields (see also Fliess 1990 *a*, Fliess and Glad

1993) and we refer to the books of Ritt (Ritt 1950) and Kolchin (Kolchin 1973) and Seidenberg's paper (Seidenberg 1952) for details. Basics on the customary (non-differential) field theory may be found in the textbooks by Jacobson (1985) and Winter (1974)—see also Fliess (1990 *a*) and Fliess and Glad (1993); they will not be repeated here.

### 2.1. Basics on differential fields

An (ordinary) differential ring  $R$  is a commutative ring equipped with a single derivation  $d/dt = \cdot$  such that

$$\begin{aligned}\forall a \in R, \quad \dot{a} &= \frac{da}{dt} \in R \\ \forall a, b \in R, \quad \frac{d}{dt}(a + b) &= \dot{a} + \dot{b} \\ \frac{d}{dt}(ab) &= \dot{a}b + a\dot{b}\end{aligned}$$

A constant  $c \in R$  is an element such that  $\dot{c} = 0$ . A ring of constants only contains constant elements. An (ordinary) differential field is an (ordinary) differential ring which is a field.

A differential field extension  $L/K$  is given by two differential fields,  $K$  and  $L$ , such that  $K \subseteq L$  and such that the restriction to  $K$  of the derivation of  $L$  coincides with the derivation of  $K$ .

An element  $\xi \in L$  is said to be differentially  $K$ -algebraic if, and only if, it satisfies an algebraic differential equation over  $K$ , i.e., if there exists a polynomial  $\pi \in K[x_0, x_1, \dots, x_v]$ ,  $\pi \neq 0$ , such that  $\pi(\xi, \dot{\xi}, \dots, \xi^{(v)}) = 0$ . The extension  $L/K$  is said to be differentially algebraic if, and only if, any element of  $L$  is differentially  $K$ -algebraic.

An element  $\xi \in L$  is said to be differentially  $K$ -transcendental if, and only if, it is not differentially  $K$ -algebraic. The extension  $L/K$  is said to be differentially transcendental if, and only if, there exists at least one element of  $L$  that is differentially  $K$ -transcendental.

A set  $\{\xi_i | i \in I\}$  of elements in  $L$  is said to be differentially  $K$ -algebraically independent if, and only if, the set of derivatives of any order,  $\{\xi_i^{(v)} | i \in I, v = 0, 1, 2, \dots\}$ , is  $K$ -algebraically independent. Such an independent set which is maximal with respect to inclusion is called a differential transcendence basis of  $L/K$ . Two such bases have the same cardinality, i.e., the same number of elements, which is called the differential transcendence degree of  $L/K$ : it is denoted by  $\text{diff tr } d^0 L/K$ . Notice that  $L/K$  is differentially algebraic if, and only if,  $\text{diff tr } d^0 L/K = 0$ .

**Theorem 1:** For a finitely generated differential extension  $L/K$ , the next two properties are equivalent:

- (i)  $L/K$  is differentially algebraic;
- (ii) the (non-differential) transcendence degree of  $L/K$  is finite, i.e.;  $\text{tr } d^0 L/K < \infty$ .

More details and some examples may be found in Fliess and Glad (1993).

## 2.2. Systems†

Let  $k$  be a given differential ground field. A system is a finitely generated differential extension  $\mathcal{D}/k$ ‡. Such a definition corresponds to a finite number of quantities which are related by a finite number of algebraic differential equations over  $k$ §. We do not distinguish in this setting between input, state, output and other types of variables. The differential order of the system  $\mathcal{D}/k$  is the differential transcendence degree of the extension  $\mathcal{D}/k$ .

**Example:** Set  $k = \mathbb{R}$ ;  $\mathcal{D}/k$  is the differential field generated by the four unknowns  $x_1, x_2, x_3, x_4$  related by the two algebraic differential equations:

$$\dot{x}_1 + \ddot{x}_3 \dot{x}_4 = 0, \quad \dot{x}_2 + (x_1 + \ddot{x}_3 x_4) x_4 = 0 \quad (7)$$

Clearly,  $\text{difftrd}^0 \mathcal{D}/k = 2$ : it is equal to the number of unknowns minus the number of equations.  $\square$

Denote by  $k\langle u \rangle$  the differential field generated by  $k$  and by a finite set  $u = (u_1, \dots, u_m)$  of differential  $k$ -indeterminates:  $u_1, \dots, u_m$  are differentially  $k$ -algebraically independent, i.e.  $\text{difftrd}^0 k\langle u \rangle/k = m$ . A dynamics with (independent) input  $u$  is a finitely generated differentially algebraic extension  $\mathcal{D}/k\langle u \rangle$ . Note that the number  $m$  of independent input channels is equal to the differential order of the corresponding system  $\mathcal{D}/k$ . An output  $y = (y_1, \dots, y_p)$  is a finite set of differential quantities in  $\mathcal{D}$ .

According to theorem 1, there exists a finite transcendence basis  $x = (x_1, \dots, x_n)$  of  $\mathcal{D}/k\langle u \rangle$ . Consequently, any component of  $\dot{x} = (\dot{x}_1, \dots, \dot{x}_n)$  and of  $y$  is  $k\langle u \rangle$ -algebraically dependent on  $x$ , which plays the role of a (generalized) state. This yields:

$$\left. \begin{aligned} A_1(\dot{x}_1, x, u, \dot{u}, \dots, u^{(\alpha_1)}) &= 0 \\ &\vdots \\ A_n(\dot{x}_n, x, u, \dot{u}, \dots, u^{(\alpha_n)}) &= 0 \\ B_1(y_1, x, u, \dot{u}, \dots, u^{(\beta_1)}) &= 0 \\ &\vdots \\ B_p(y_p, x, u, \dot{u}, \dots, u^{(\beta_p)}) &= 0 \end{aligned} \right\} \quad (8)$$

where the  $A_i$ s and  $B_j$ s are polynomial over  $k$ . The integer  $n$  is the dimension of the dynamics  $\mathcal{D}/k\langle u \rangle$ . We refer to Fliess and Hasler (1990) and Fliess *et al.* (1993 a) for a discussion of such generalized state-variable representations (8) and their relevance to practice.

† See also Fliess (1990 a), Fliess and Glad (1993).

‡ Two systems  $\mathcal{D}/k$  and  $\tilde{\mathcal{D}}/k$  are, of course, identified if, and only if, there exists a differential  $k$ -isomorphism between them (a differential  $k$ -isomorphism commutes with  $d/dt$  and preserves every element of  $k$ ).

§ It is a standard fact in classic commutative algebra and algebraic geometry (cf. Hartshorne 1977) that one needs prime ideals for interpreting 'concrete' equations in the language of field theory. In our differential setting, we of course need differential prime ideals—see Kolchin (1973) and also Fliess and Glad (1993) for an elementary exposition. The verification of the prime character of the differential ideals corresponding to all our examples is done in Appendix A.



**Example (continued):** Set  $u_1 = x_3$  and  $u_2 = \dot{x}_4$ . The extension  $\mathcal{D}/\mathbb{R}\langle u \rangle$  is differentially algebraic and yields the representation

$$\begin{cases} \dot{x}_1 = -\ddot{u}_1 u_2 \\ \dot{x}_2 = -(x_1 + \ddot{u}_1 x_4) x_4 \\ \dot{x}_4 = u_2. \end{cases} \quad (9)$$

The dimension of the dynamics is 3 and  $(x_1, x_2, x_4)$  is a generalized state. It would be 5 if we set  $u_1 = \ddot{x}_3$  and  $u_2 = \dot{x}_4$ , and the corresponding representation becomes causal in the classical sense.  $\square$

**Remark 1:** Take the dynamics  $\mathcal{D}/k\langle u \rangle$  and a finitely generated algebraic extension  $\tilde{\mathcal{D}}/\mathcal{D}$ . The two dynamics  $\mathcal{D}/k\langle u \rangle$  and  $\tilde{\mathcal{D}}/k\langle u \rangle$ , which are of course equivalent, have the same dimension and can be given the same state variable representation (11). In the sequel, a system  $\mathcal{D}/k\langle u \rangle$  will be defined up to a finitely generated algebraic extension of  $\mathcal{D}$ .  $\square$

### 2.3. Modules and linear systems (see also Fliess 1990 b)

Differential fields are too general for linear systems which are specified by linear differential equations. They are thus replaced by the following appropriate modules.

Let  $k$  be again a given differential ground field. Denote by  $k[d/dt]$  the ring of linear differential operators of the type

$$\sum_{\text{finite}} a_\alpha \frac{d^\alpha}{dt^\alpha} \quad (a_\alpha \in k)$$

This ring is commutative if, and only if,  $k$  is a field of constants. Nevertheless, in the general non-commutative case,  $k[d/dt]$  is still a principal ideal ring and the most important properties of left  $k[d/dt]$ -modules mimic those of modules over commutative principal ideal rings—see Cohn (1985).

Let  $M$  be a left  $k[d/dt]$ -module. An element  $m \in M$  is said to be torsion if, and only if, there exists  $\pi \in k[d/dt]$ ,  $\pi \neq 0$ , such that  $\pi \cdot m = 0$ . The set of all torsion elements of  $M$  is a submodule  $T$ , which is called the torsion submodule of  $M$ . The module  $M$  is said to be torsion if, and only if,  $M = T$ . The following result can be regarded as the linear counterpart of theorem 1.

**Proposition 1:** For a finitely generated left  $k[d/dt]$ -module  $M$ , the next two properties are equivalent:

- (i)  $M$  is torsion;
- (ii) the dimension of  $M$  as a  $k$ -vector space is finite.

A finitely generated module  $M$  is free if, and only if, its torsion submodule  $T$  is trivial, i.e.  $T = \{0\}$  (this is not the usual definition of free modules, but a characterization which holds for finitely generated modules over principal ideal rings, where any torsion-free module is free—see Cohn (1985)). Any finitely generated module  $M$  can be written  $M = T \oplus \Phi$  where  $T$  is the torsion submodule of  $M$  and  $\Phi$  is a free module. The rank of  $M$ , denoted by  $\text{rk } M$ , is the cardinality of any basis of  $\Phi$ . Thus,  $M$  is torsion if, and only if,  $\text{rk } M = 0$ .

A linear system is, by definition, a finitely generated left  $k[d/dt]$ -module  $\Lambda$ . We are thus dealing with a finite number of variables which are related by a finite number of linear homogeneous differential equations. The differential order of  $\Lambda$  is the rank of  $\Lambda$ .

A linear dynamics with input  $u = (u_1, \dots, u_m)$  is a linear system  $\Lambda$  which contains  $u$  such that the quotient module  $\Lambda/[u]$  is torsion, where  $[u]$  denotes the left  $k[d/dt]$ -module spanned by the components of  $u$ . The input is assumed to be independent, i.e., the module  $[u]$  is free. This implies that the differential order of  $\Lambda$  is equal to  $m$ . A classical Kalman state variable representation is always possible:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + B \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \quad (10)$$

where the dimension  $n$  of the state  $x = (x_1, \dots, x_n)$ , which is called the dimension of the dynamics, is equal to the dimension of the torsion module  $\Lambda/[u]$  as a  $k$ -vector space, and the matrices  $A$  and  $B$ , of appropriate sizes, have their entries in  $k$ .

An output  $y = (y_1, \dots, y_p)$  is a set of elements in  $\Lambda$ . It leads to the following output map:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = C \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{\text{finite}} D_v \frac{d^v}{dt^v} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

The controllability of (10) can be expressed in a module-theoretical language which is independent of any denomination of variables. Controllability is equivalent to the freeness of the module  $\Lambda$ . This is just an algebraic counterpart (Fliess 1992) of Willems's trajectory characterization (Willems 1991). When the system is uncontrollable, the torsion submodule corresponds to the Kalman uncontrollability subspace.

**Remark 2:** The relationship with the general differential field setting is obtained by producing a formal multiplication. The symmetric tensor product (Jacobson 1985) of a linear system  $\Lambda$ , where  $\Lambda$  is viewed as a  $k$ -vector space, is an integral differential ring. Its quotient field  $\mathcal{D}$ , which is a differential field, corresponds to the non-linear field theoretic description of linear systems.  $\square$

#### 2.4. Differentials and tangent linear systems

Differential calculus, which plays such a role in analysis and in differential geometry, admits a nice analogue in commutative algebra (Hartshorne 1977, Winter 1974), which has been extended to differential algebra by Johnson (Johnson 1969).

To a finitely generated extension  $L/K$ , associate a mapping  $d_{L/K}: L \rightarrow \Omega_{L/K}$ , called (Kähler) differential<sup>†</sup> and where  $\Omega_{L/K}$  is a finitely generated left

<sup>†</sup> For any  $a \in L$ ,  $d_{L/K}a$  should be intuitively understood, like in analysis and differential geometry, as a 'small' variation of  $a$ .

$L$   $[d/dt]$ -module, such that

$$\begin{aligned}\forall a \in L \quad d_{L/K} \left( \frac{da}{dt} \right) &= \frac{d}{dt} (d_{L/K} a) \\ \forall a, b \in L \quad d_{L/K} (a + b) &= d_{L/K} a + d_{L/K} b \\ d_{L/K} (ab) &= b d_{L/K} a + a d_{L/K} b \\ \forall c \in K \quad d_{L/K} c &= 0.\end{aligned}$$

Elements of  $K$  behave like constants with respect to  $d_{L/K}$ . Properties of the extension  $L/K$  can be translated into the linear module-theoretic framework of  $\Omega_{L/K}$ :

- (a) a set  $\xi = (\xi_1, \dots, \xi_m)$  is a differential transcendence basis of  $L/K$  if, and only if,  $d_{L/K} \xi = (d_{L/K} \xi_1, \dots, d_{L/K} \xi_m)$  is a maximal set of  $L[d/dt]$ -linearly independent elements in  $\Omega_{L/K}$ , thus,  $\text{diff tr } d^0 L/K = \text{rk } \Omega_{L/K}$ ;
- (b) the extension  $L/K$  is differentially algebraic if, and only if, the module  $\Omega_{L/K}$  is torsion. A set  $x = (x_1, \dots, x_n)$  is a transcendence basis of  $L/K$  if, and only if,  $d_{L/K} x = (d_{L/K} x_1, \dots, d_{L/K} x_n)$  is a basis of  $\Omega_{L/K}$  as  $L$ -vector space;
- (c) the extension  $L/K$  is algebraic if, and only if,  $\Omega_{L/K}$  is trivial, i.e.,  $\Omega_{L/K} = \{0\}$ .

The tangent (or variational) linear system associated to the system  $\mathcal{D}/k$  is the left  $\mathcal{D}[d/dt]$ -module  $\Omega_{\mathcal{D}/k}$ . To a dynamics  $\mathcal{D}/k\langle u \rangle$  is associated the tangent (or variational) dynamics  $\Omega_{\mathcal{D}/k}$  with the tangent (or variational) input  $d_{L/K} u = (d_{L/K} u_1, \dots, d_{L/K} u_m)$ . The tangent (or variational) output associated to  $y = (y_1, \dots, y_p)$  is  $d_{L/K} y = (d_{L/K} y_1, \dots, d_{L/K} y_p)$ .

### 3. Equivalence, flatness and defect

#### 3.1. Equivalence of systems and endogenous feedback

Two systems  $\mathcal{D}/k$  and  $\tilde{\mathcal{D}}/k$  are said to be equivalent or equivalent by endogenous feedback if, and only if, any element of  $\mathcal{D}$  (resp.  $\tilde{\mathcal{D}}$ ) is algebraic over  $\tilde{\mathcal{D}}$  (resp.  $\mathcal{D}$ )<sup>†</sup>. Two dynamics,  $\mathcal{D}/k\langle u \rangle$  and  $\tilde{\mathcal{D}}/k\langle \tilde{u} \rangle$ , are said to be equivalent if, and only if, the corresponding systems,  $\mathcal{D}/k$  and  $\tilde{\mathcal{D}}/k$ , are so.

**Proposition 2:** *Two equivalent systems (resp. dynamics) possess the same differential order, i.e., the same number of independent input channels.*

**Proof:** Denote by  $K$  the differential field generated by  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$ :  $K/\mathcal{D}$  and  $K/\tilde{\mathcal{D}}$  are algebraic extensions. Therefore,

$$\text{diff tr } d^0 \mathcal{D}/k = \text{diff tr } d^0 K/k = \text{diff tr } d^0 \tilde{\mathcal{D}}/k \quad \square$$

Consider two equivalent dynamics,  $\mathcal{D}/k\langle u \rangle$  and  $\tilde{\mathcal{D}}/k\langle \tilde{u} \rangle$ . Let  $n$  (resp.  $\tilde{n}$ ) be

<sup>†</sup> According to the second footnote to § 2.2, this definition of equivalence can also be read as follows: two systems  $\mathcal{D}/k$  and  $\tilde{\mathcal{D}}/k$  are equivalent if, and only if, there exist two differential extensions  $\mathcal{D}/\mathcal{D}$  and  $\tilde{\mathcal{D}}/\tilde{\mathcal{D}}$  which are algebraic (in the usual sense), and a differential  $k$ -automorphism  $\Phi$  between  $\mathcal{D}/k$  and  $\tilde{\mathcal{D}}/k$ .

the dimension of  $\mathcal{D}/k\langle u \rangle$  (resp.  $\tilde{\mathcal{D}}/k\langle \tilde{u} \rangle$ ). In general,  $n \neq \tilde{n}$ . Write

$$A_i(\dot{x}_i, x, u, \dot{u}, \dots, u^{(\alpha_i)}) = 0, \quad i = 1, \dots, n \quad (11)$$

and

$$\tilde{A}_i(\dot{\tilde{x}}_i, \tilde{x}, \tilde{u}, \dot{\tilde{u}}, \dots, \tilde{u}^{(\tilde{\alpha}_i)}) = 0, \quad i = 1, \dots, \tilde{n} \quad (12)$$

the generalized state variable representations of  $\mathcal{D}/k\langle u \rangle$  and  $\tilde{\mathcal{D}}/k\langle \tilde{u} \rangle$ , respectively. The algebraicity of any element of  $\mathcal{D}$  (respectively  $\tilde{\mathcal{D}}$ ) over  $\tilde{\mathcal{D}}$  (respectively  $\mathcal{D}$ ) yields the following relationships between (11) and (12):

$$\left. \begin{aligned} \varphi_i(u_i, \tilde{x}, \tilde{u}, \dot{\tilde{u}}, \dots, \tilde{u}^{(\tilde{\nu}_i)}) &= 0 & i = 1, \dots, m \\ \sigma_\alpha(x_\alpha, \tilde{x}, \tilde{u}, \dot{\tilde{u}}, \dots, \tilde{u}^{(\tilde{\mu}_\alpha)}) &= 0 & \alpha = 1, \dots, n \\ \tilde{\varphi}_i(\tilde{u}_i, x, u, \dot{u}, \dots, u^{(\nu_i)}) &= 0 & i = 1, \dots, m \\ \tilde{\sigma}_\alpha(\tilde{x}_\alpha, x, u, \dot{u}, \dots, u^{(\nu_\alpha)}) &= 0 & \alpha = 1, \dots, \tilde{n} \end{aligned} \right\} \quad (13)$$

where the  $\varphi_i$ s,  $\sigma_\alpha$ s,  $\tilde{\varphi}_i$ s and  $\tilde{\sigma}_\alpha$ s are polynomials over  $k$ .

The two dynamic feedbacks corresponding to (13) are called endogenous as they do not necessitate the introduction of any variable that is transcendental over  $\mathcal{D}$  and  $\tilde{\mathcal{D}}$ —see also Martin (1992). If we know  $\tilde{x}$  (resp.  $x$ ), we can calculate  $u$  (respectively  $\tilde{u}$ ) from  $\tilde{u}$  (respectively  $u$ ) without integrating any differential equation. The relationship with general dynamic feedbacks is given in Appendix B.

**Remark 3:** The tangent linear systems (see § 2.4) of two equivalent systems are strongly related and, in fact, are ‘almost identical’. Take two equivalent systems  $\mathcal{D}_1/k$  and  $\mathcal{D}_2/k$  and denote by  $\mathcal{D}$  the smallest algebraic extension of  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . It is straightforward to check that the three left  $\mathcal{D}[d/dt]$ -modules  $\Omega_{\mathcal{D}/k}$ ,  $\mathcal{D} \otimes_{\mathcal{D}_1} \Omega_{\mathcal{D}_1/k}$  and  $\mathcal{D} \otimes_{\mathcal{D}_2} \Omega_{\mathcal{D}_2/k}$  are isomorphic (see Hartshorne 1977, Jacobson 1985).  $\square$

### 3.2. Flatness and defect

As in the non-differential case, a differential extension  $L/K$  is said to be purely differentially transcendental if, and only if, there exists a differential transcendence basis  $\xi = \{\xi_i | i \in I\}$  of  $L/K$  such that  $L = K\langle \xi \rangle$ . A system  $\mathcal{D}/k$  is called purely differentially transcendental if, and only if, the extension  $\mathcal{D}/k$  is so.

A system  $\mathcal{D}/k$  is called (differentially) flat if, and only if, it is equivalent to a purely differentially transcendental system  $L/k$ . A differential transcendence basis  $y = (y_1, \dots, y_m)$  of  $L/k$  such that  $L = k\langle y \rangle$  is called a linearizing or flat output of the system  $\mathcal{D}/k$ .

**Example (continued):** Let us prove that  $y = (y_1, y_2)$  with

$$y_1 = x_2 + \frac{(x_1 + \ddot{x}_3 x_4)^2}{2x_3^{(3)}}, \quad y_2 = x_3$$

is a linearizing output for (7). Set  $\sigma = x_1 + \ddot{x}_3 x_4$ . Differentiating  $y_1 = x_2 + \sigma^2/2y_2^{(3)}$ , we have, using (7),  $\sigma^2 = -(2\dot{y}_1(y_2^{(3)})^2/y_2^{(4)})$ . Thus  $x_2 = y_1 - (\sigma^2/2y_2^{(3)})$  is an algebraic function of  $(y_1, \dot{y}_1, y_2^{(3)}, y_2^{(4)})$ . Since  $x_4 = -(\dot{x}_2/\sigma)$  and  $x_1 = \sigma - \ddot{y}_2 x_4$ ,  $x_4$  and  $x_1$  are algebraic functions of  $(y_1, \dot{y}_1, \ddot{y}_1, \ddot{y}_2, y_2^{(3)}, y_2^{(4)}, y_2^{(5)})$ .

Remark there exist many other linearizing outputs such as  $\tilde{y} = (\tilde{y}_1, \tilde{y}_2) = (2y_1y_2^{(3)}, y_2)$ , the inverse transformation being  $y = (\tilde{y}_1/2\tilde{y}_2^{(3)}, \tilde{y}_2)$ .  $\square$

Take an arbitrary system  $\mathcal{D}/k$  of differential order  $m$ . Among all the possible choices of sets  $z = (z_1, \dots, z_m)$  of  $m$  differential  $k$ -indeterminates which are algebraic over  $\mathcal{D}$ , take one such that  $\text{tr d}^0 \mathcal{D}\langle z \rangle / k\langle z \rangle$  is minimum, say  $\delta$ . This integer  $\delta$  is called the defect of the system  $\mathcal{D}/k$ . The next result is obvious.

**Proposition 3:** *A system  $\mathcal{D}/k$  is flat if, and only if, its defect is zero.*

**Example:** The defect of the system generated by  $x_1$  and  $x_2$  satisfying  $\dot{x}_1 = x_1 + (\dot{x}_2)^3$  is 1. Its general solution cannot be expressed without the integration of, at least, one differential equation.  $\square$

### 3.3. Basic examples

**3.3.1. Uncontrolled dynamical systems.** An uncontrolled dynamical system is, in our field-theoretic language (Fliess 1990 a), a finitely generated differentially algebraic extension  $\mathcal{D}/k$ :  $\text{diff tr d}^0 \mathcal{D}/k = 0$  implies the non-existence of any differential  $k$ -indeterminate algebraic over  $\mathcal{D}$ . Thus, the defect of  $\mathcal{D}/k$  is equal to  $\text{tr d}^0 \mathcal{D}/k$ , i.e., to the dimension of the dynamical system  $\mathcal{D}/k$ , which corresponds to the state variable representation  $A_i(\dot{x}_i, x) = 0$ , where  $x = (x_1, \dots, x_n)$  is a transcendence basis of  $\mathcal{D}/k$ . Flatness means that  $\mathcal{D}/k$  is algebraic in the (non-differential) sense: the dynamics  $\mathcal{D}/k$  is then said to be trivial.

**3.3.2. Linear systems.** The defect of  $\Lambda$  is, by definition, the defect of its associated differential field extension  $\mathcal{D}/k$  (see remark 2).

**Theorem 2:** *The defect of a linear system is equal to the dimension of its torsion submodule, i.e., to the dimension of its Kalman uncontrollable subspace. A linear system is flat if, and only if, it is controllable.*

**Proof:** Take the decomposition  $\Lambda = T \oplus \Phi$ , of § 2.3, where  $T$  is the torsion submodule and  $\Phi$  a free module. A basis  $b = (b_1, \dots, b_m)$  of  $\Phi$  plays the role of a linearizing output when  $\Lambda$  is free: the system then is flat. When  $T \neq \{0\}$ , the differential field extension  $\mathcal{T}/k$  generated by  $T$  is differentially algebraic and its (non-differential) transcendence degree is equal to the dimension of  $T$  as  $k$ -vector space. The conclusion follows at once.  $\square$

**Remark 4:** The above arguments can be made more concrete by considering a linear dynamics over  $\mathbb{R}$ . If it is controllable, we may write it, up to a static feedback, in its Brunovsky canonical form:

$$y_i^{(v_i)} = u_i \quad (i = 1, \dots, m)$$

where the  $v_i$ s are the controllability indices and  $y = (y_1, \dots, y_m)$  is a linearizing output. In the uncontrollable case, the defect  $d$  is the dimension of the uncontrollable subspace:

$$\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix} = M \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix}$$

where  $M$  is a  $d \times d$  matrix over  $\mathbb{R}$ .  $\square$

### 3.4. A necessary condition for flatness

Consider the system  $\mathcal{D}/k$  where  $\mathcal{D} = k\langle w \rangle$  is generated by a finite set  $w = (w_1, \dots, w_q)$ . The  $w_i$ s are related by a finite set,  $\Xi(w, \dot{w}, \dots, w^{(v)}) = 0$ , of algebraic differential equations. Define the algebraic variety  $S$  corresponding to  $\Xi(\xi^0, \dots, \xi^v) = 0$  in the  $(v+1)q$ -dimensional affine space with coordinates

$$\xi^j = (\xi_1^j, \dots, \xi_q^j), \quad j = 0, 1, \dots, v$$

**Theorem 3:** *If the system  $\mathcal{D}/k$  is flat, the affine algebraic variety  $S$  contains at each regular point a straight line parallel to the  $\xi^v$ -axes.*

**Proof:** The components of  $w, \dot{w}, \dots, w^{(v-1)}$  are algebraically dependent on the components of a linearizing output  $y = (y_1, \dots, y_m)$  and a finite number of their derivatives. Let  $\mu$  be the highest order of these derivatives. The components of  $w^{(v)}$  depend linearly on the components of  $y^{(\mu+1)}$ , which play the role of independent parameters for the coordinates  $\xi_1^v, \dots, \xi_q^v$ .  $\square$

The above condition is not sufficient. Consider the system  $\mathcal{D}/\mathbb{R}$  generated by  $(x_1, x_2, x_3)$  satisfying  $\dot{x}_1 = (\dot{x}_2)^2 + (\dot{x}_3)^3$ . This system does not satisfy the necessary condition: it is not flat. The same system  $\mathcal{D}$  can be defined via the quantities  $(x_1, x_2, x_3, x_4)$  related by  $\dot{x}_1 = (x_4)^2 + (\dot{x}_3)^3$  and  $x_4 = \dot{x}_2$ . Those new equations now satisfy our necessary criterion.

### 3.5. Flatness and controllability

Sussmann and Jurdjevic (1972) have introduced the concept of strong accessibility for dynamics of the form  $\dot{x} = f(x, u)$  in the differential geometric setting. Sontag (1988) showed that strong accessibility implies the existence of controls such that the linearized system around a trajectory passing through a point  $a$  of the state-space is controllable. Sontag (1992) and Coron (1994) demonstrated that, for any  $a$ , those controls are generic.

The above considerations with those of §§ 2.3 and 2.4 lead in our context to the following definition of controllability, which is independent of any distinction between variables: a system  $\mathcal{D}/k$  is said to be controllable (or strongly accessible) if, and only if, its tangent linear system is controllable, i.e., if, and only if, the module  $\Omega_{\mathcal{D}/k}$  is free.

Remark 3 shows that this definition is invariant under our equivalence via endogenous feedback.

**Proposition 4:** *A flat system is controllable.*

**Proof:** It suffices to prove it for a purely differentially transcendental extension  $k\langle y \rangle/k$ , where  $y = (y_1, \dots, y_m)$ . The module  $\Omega_{k\langle y \rangle/k}$ , which is spanned by  $d_{k\langle y \rangle/k} y_1, \dots, d_{k\langle y \rangle/k} y_m$ , is necessarily free.  $\square$

The converse is false as demonstrated by numerous examples of strongly accessible single-input dynamics  $\dot{x} = f(x, u)$  which are not linearizable by static feedback and therefore neither by dynamic ones (Charlet *et al.* 1989).

Flatness which is equivalent to the possibility of expressing any element of the system as a function of the linearizing output and a finite number of its derivatives, may be viewed as the non-linear extension of linear controllability, if the latter is characterized by free modules. Whereas the strong accessibility

property only is an 'infinitesimal' generalization of linear controllability, flatness should be viewed as a more 'global' and, perhaps, as a more tractable one. This will be enhanced in § 5 where controllable systems of non-zero defect are treated using high-frequency control that enables them to be approximated by flat systems for which the control design is straightforward.

#### 4. Examples and control of flat systems

The verification of the prime character of the differential ideals corresponding to all our examples is done in appendix A. This means that the equations defining all our examples can be rigorously interpreted in the language of differential field theory.

##### 4.1. The 2-D crane

Consider the crane displayed in Fig. 1 which is a classical object of control study (see, e.g., D'Andréa-Novel and Lévine 1990, Marttinen *et al.* 1990). The dynamics can be divided into two parts. The first part corresponds to the motor drives and industrial controllers for trolley travels and rolling up and down the rope. The second part is relative to the trolley load, the behaviour of which is very similar to the pendulum case. We concentrate here on the pendulum dynamics by assuming that

- (a) the traversing and hoisting are control variables;
- (b) the trolley load remains in a fixed vertical plane  $OXZ$ ;
- (c) the rope dynamics are negligible.

A dynamic model of the load can be derived by Lagrangian formalism. It can also be obtained, in a very simple way, by writing down all the differential (Newton law) and algebraic (geometric constraints) equations describing the pendulum behaviour:

$$\left. \begin{aligned} m\ddot{x} &= -T \sin \theta \\ m\ddot{z} &= -T \cos \theta + mg \\ x &= R \sin \theta + D \\ z &= R \cos \theta \end{aligned} \right\} \quad (14)$$

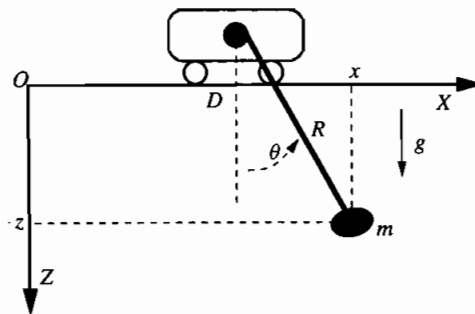


Figure 1. The two dimensional crane.

where

- (i)  $(x, z)$  (the coordinates of the load  $m$ ),  $T$  (the tension of the rope) and  $\theta$  (the angle between the rope and the vertical axis  $OZ$ ) are the unknown variables;
- (ii)  $D$  (the trolley position) and  $R$  (the rope length) are the input variables.

From (14), it is clear that  $\sin \theta$ ,  $T$ ,  $D$  and  $R$  are algebraic functions of  $(x, z)$  and their derivatives:

$$\sin \theta = \frac{x - D}{R}, \quad T = \frac{mR(g - \ddot{z})}{z}, \quad (\ddot{z} - g)(x - D) = \ddot{x}z$$

$$(x - D)^2 + z^2 = R^2$$

that is

$$\left. \begin{aligned} D &= x - \frac{\ddot{x}z}{\ddot{z} - g} \\ R^2 &= z^2 + \left( \frac{\ddot{x}z}{\ddot{z} - g} \right)^2 \end{aligned} \right\} \quad (15)$$

Thus, system (14) is flat with  $(x, z)$  as linearizing output.

**Remark 5:** Assume that the modelling equations (14) are completed with the following traversing and hoisting dynamics:

$$\left. \begin{aligned} M\ddot{D} &= \mathcal{F} - \lambda\dot{D} + T\sin\theta \\ \frac{J}{\rho^2}\ddot{R} &= \mathcal{C} - \frac{\mu}{\rho}\dot{R} - T\rho \end{aligned} \right\} \quad (16)$$

where the new variables  $\mathcal{F}$  and  $\mathcal{C}$  are, respectively the external force applied to the trolley and the hoisting torque. The other quantities  $(M, J, \rho, \lambda, \mu)$  are constant physical parameters. Then (14), (16) is also flat with the same linearizing output  $(x, z)$ . This explains without any additional computation why the system considered in D'Andréa-Novel and Lévine (1990) is linearizable via dynamic feedback.  $\square$

Let us now address the following question which is one of the basic control problems for a crane: how can one carry a load  $m$  from the steady-state  $R = R_1 > 0$  and  $D = D_1$  at time  $t_1$ , to the steady-state  $R = R_2 > 0$  and  $D = D_2$  at time  $t_2 > t_1$ ?

It is clear that any motion of the load induces oscillations that must be canceled at the end of the load transport. We propose here a very simple answer to this question when the crane can be described by (14). This answer just consists in using (15).

Consider a smooth curve  $[t_1, t_2] \ni t \rightarrow (\alpha(t), \gamma(t)) \in \mathbb{R} \times ]0, +\infty[$  such that

- (a) for  $i = 1, 2$ ,  $(\alpha(t_i), \gamma(t_i)) = (D_i, R_i)$ , and  $d^r/dt^r(\alpha, \gamma)(t_i) = 0$  with  $r = 1, 2, 3, 4$ .
- (b) for all  $t \in [t_1, t_2]$ ,  $\ddot{\gamma}(t) < g$ .

Then the solution of (14) starting at time  $t_1$  from the steady-state  $D_1$  and  $R_1$ ,



and with the control trajectory defined, for  $t \in [t_1, t_2]$ , by

$$\left. \begin{aligned} D(t) &= \alpha(t) - \frac{\ddot{\alpha}(t)\gamma(t)}{\ddot{\alpha}(t) - g} \\ R(t) &= \left[ \gamma^2(t) + \left( \frac{\ddot{\alpha}(t)\gamma(t)}{\ddot{\alpha}(t) - g} \right)^2 \right]^{1/2} \end{aligned} \right\} \quad (17)$$

and, for  $t > t_2$ , by  $(D(t), R(t)) = (D_2, R_2)$ , leads to a load trajectory  $t \rightarrow (x(t), z(t))$  such that  $(x(t), z(t)) = (\alpha(t), \gamma(t))$  for  $t \in [t_1, t_2]$  and  $(x(t), z(t)) = (D_2, R_2)$  for  $t \geq t_2$ . Notice that, since for all  $t \in [t_1, t_2]$ ,  $\ddot{z}(t) < g$ , the rope tension  $T = (mR(g - \ddot{z})/z)$  always remains positive and the description of the system by (14) remains reasonable.

The generalized state variable description of the system is the following (Fliess *et al.* 1991, 1993 *a*):

$$R\ddot{\theta} = -2\dot{R}\dot{\theta} - \ddot{D}\cos\theta - g\sin\theta$$

Since  $\alpha$  and  $\gamma$  are smooth,  $D$  and  $R$  are at least twice continuously differentiable. Thus, the classical existence and uniqueness theorem ensures that the above ordinary differential equation admits a unique smooth solution that is nothing but  $\theta(t) = \arctan(\alpha(t) - D(t))/\gamma(t)$ .

The approximation of the crane dynamics by (14) implies that the motor drives and industrial low-level controllers (trolley travels and rolling up and down the rope) produce fast and stable dynamics (see remark 5). Thus, if these dynamics are stable and fast enough, classical results of singular perturbation theory of ordinary differential equation (see, e.g., Tikhonov *et al.* 1980), imply that the control (17) leads to a final configuration close to the steady-state defined by  $D_2$  and  $R_2$ .

In the simulations displayed here below, we have verified that the addition of reasonable fast and stable regulator dynamics modifies only slightly the final position  $(R_2, D_2)$ . Classical proportional-integral controller for  $D$  and  $R$  are added to (14). The typical regulator time constants are equal to one tenth of the period of small oscillations ( $\frac{1}{10}2\pi\sqrt{R/g} \approx 0.3$  s) (see Fliess *et al.* 1991).

For the simulations presented in Fig. 2, the transport of the load  $m$  may be considered as a rather fast one: the horizontal motion of  $D$  is of 10 m in 3.5 s; the vertical motion of  $R$  is up to 5 m in 3.5 s. Compared with the low-level regulator time constants (0.1 and 0.3 s), such motions are not negligible. This explains the transient mismatch between the ideal and non-ideal cases. Nevertheless, the final control performances are not seriously altered: the residual oscillations of the load after 7 s admit less than 3 cm of horizontal amplitude. Such small residual oscillations can be canceled via a simple PID regulator with the vertical deviation  $\theta$  as input and the set-point of  $D$  as output.

The simulations illustrate the importance of the linearizing output  $(x, z)$ . When the regulations of  $R$  and  $D$  are suitably designed, it is possible to use the control given in (17) for fast transports of the load  $m$  from one point to another. The simplicity and the independence of (17) with respect to the system parameters (except  $g$ ) constitute its main practical interests.

**Remark 6:** Similar calculations can be performed when a second horizontal direction  $OX_2$ , orthogonal to  $OX_1 = OX$ , is considered. Denoting then by  $(x_1, x_2, z)$  the cartesian coordinates of the load,  $R$  the rope length and  $(D_1, D_2)$

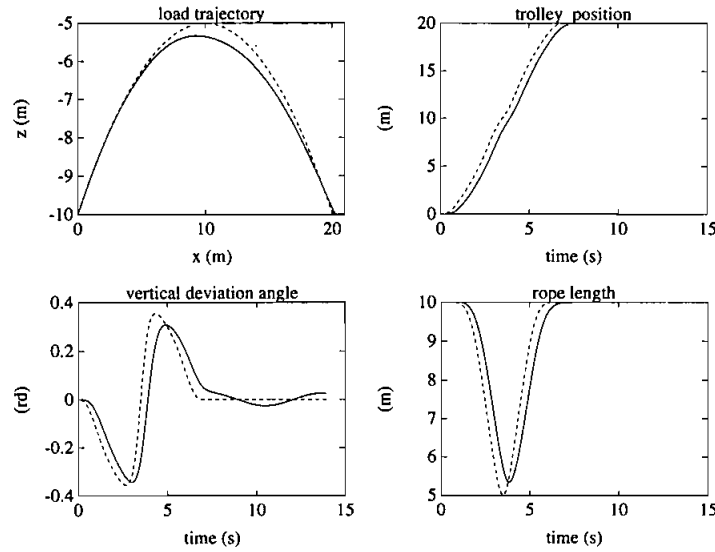


Figure 2. Simulation of the control defined by (17) without (solid lines) and with (dot lines) ideal low-level controllers for  $D$  and  $R$ .

the trolley horizontal position, the system is described by

$$\left. \begin{aligned} (\ddot{z} - g)(x_1 - D_1) &= \ddot{x}_1 z \\ (\ddot{z} - g)(x_2 - D_2) &= \ddot{x}_2 z \\ (x_1 - D_1)^2 + (x_2 - D_2)^2 + z^2 &= R^2 \end{aligned} \right\}$$

This system is clearly flat with the cartesian coordinates of the load,  $(x_1, x_2, z)$ , as flat output.  $\square$

**Remark 7:** In D'Andréa-Novel *et al.* (1992 b), the control of a body of mass  $m$  around a rotation axle  $\Delta$  of constant direction is investigated. This system is flat as a consequence of the following considerations. According to an old result due to Huygens (see, e.g. Whittaker 1937, p. 131–132), the equations describing the motion are equivalent to those of a pendulum of the same mass  $m$  and of length  $l = J/md$  where  $d \neq 0$  is the vertical distance between the mass center  $G$  and the axle  $\Delta$ ,  $J$  is the inertial moment around  $\Delta$ . Denoting by  $u$  and  $v$ , respectively, the vertical and horizontal positions of  $\Delta$ , the equations of motion are the following (compare to (15)):

$$\left. \begin{aligned} \frac{\ddot{u}}{u - x} &= \frac{\ddot{v} - g}{v - z} \\ (u - x)^2 + (v - z)^2 &= l^2 \end{aligned} \right\}$$

where  $(x, z)$  are the horizontal and vertical coordinates of the Huygens oscillation center. Clearly  $(x, z)$  is a linearizing output.  $\square$

**Remark 8:** The examples corresponding to the crane, Huygens' oscillation center (see remark 7) and the car with  $n$ -trailers below, illustrate the fact that linearizing outputs most admit a clear physical interpretation.  $\square$

#### 4.2. The car with $n$ -trailers

4.2.1. *Modeling equations.* Steering a car with  $n$  trailers is now the object of active researches (Laumond 1991, Monaco and Normand-Cyrot 1992, Murray and Sastry 1993, Rouchon *et al.* 1993 *a*, Tilbury *et al.* 1993). The flatness of a basic model (more realistic models where trailer  $i$  is not directly hitched to the center of the axle of trailer  $i - 1$  are considered by Martin and Rouchon 1993 and Rouchon *et al.* 1993 *b*) of this system combined with the use of Frénet formula lead to a complete and simple solution of the motion planning problem without obstacles. Notice that most of non-holonomic mobile robots are flat (D'Andréa-Novet *et al.* 1992 *a*, Campion *et al.* 1992).

The hitch of trailer  $i$  is attached to the centre of the rear axle of trailer  $i - 1$ . The wheels are aligned with the body of the trailer. The two control inputs are the driving velocity (of the rear wheels of the car) and the steering velocity (of the front wheels of the car). The constraints are based on allowing the wheels to roll and spin without slipping. For the steering front wheels of the car, the derivation is simplified by assuming them as a single wheel at the midpoint of the axle. The resulting dynamics are described by the following equations (the notations are those of Murray and Sastry (1993) and are summarized on Fig. 3):

$$\left. \begin{aligned} \dot{x}_0 &= u_1 \cos \theta_0 \\ \dot{y}_0 &= u_1 \sin \theta_0 \\ \dot{\phi} &= u_2 \\ \dot{\theta}_0 &= \frac{u_1}{d_0} \tan \phi \\ \dot{\theta}_i &= \frac{u_1}{d_i} \left( \prod_{j=1}^{i-1} \cos(\theta_{j-1} - \theta_j) \right) \sin(\theta_{i-1} - \theta_i) \quad \text{for } i = 1, \dots, n \end{aligned} \right\} \quad (18)$$

where  $(x_0, y_0, \phi, \theta_0, \dots, \theta_n) \in \mathbb{R}^2 \times ]-\pi/2, +\pi/2[ \times (S^1)^{n+1}$  is the state,  $(u_1, u_2)$  is the control and  $d_0, d_1, \dots, d_n$  are positive parameters (lengths). As displayed in Fig. 3, we denote by  $P_i$ , the medium point of the wheel axle of trailer  $i$ , for  $i = 1, \dots, n$ . The medium point of the rear (resp. front) wheel axle of the car is denoted by  $P_0$  (resp.  $Q$ ).

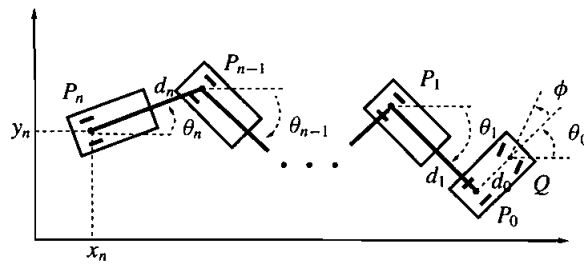


Figure 3. The kinematic car with  $n$  trailers.

4.2.2. *Cartesian coordinates of  $P_n$  as flat output.* Denote by  $(x_i, y_i)$  the cartesian coordinates of  $P_i$ ,  $i = 0, 1, \dots, n$ :

$$x_i = x_0 - \sum_{j=1}^i d_j \cos \theta_j$$

$$y_i = y_0 - \sum_{j=1}^i d_j \sin \theta_j.$$

A direct computation shows that  $\tan \theta_i = \dot{y}_i / \dot{x}_i$ . Since, for  $i = 0, \dots, n-1$ ,  $x_i = x_{i+1} + d_{i+1} \cos \theta_{i+1}$  and  $y_i = y_{i+1} + d_{i+1} \sin \theta_{i+1}$ , the variables  $\theta_n, x_{n-1}, y_{n-1}, \theta_{n-1}, \dots, \theta_1, x_0, y_0$  and  $\theta_0$  are functions of  $x_n$  and  $y_n$  and their derivatives up to the order  $n+1$ . But  $u_1 = \dot{x}_0 / \cos \theta_0$ ,  $\tan \phi = d_0 \dot{\theta}_0 / u_1$  and  $u_2 = \phi$ . Thus, the entire state and the control are functions of  $x_n$  and  $y_n$  and their derivatives up to order  $n+3$ .

This proves that the car with  $n$  trailers described by (18) is a flat system: the linearizing output corresponds to the cartesian coordinates of the point  $P_n$ , the medium point of the wheel axle of the last trailer.

Flatness implies that for generic values of the state, the strong accessibility rank associated to the control system (18) is maximum and equal to its state-space dimension: the system is thus controllable.

The singularity which might occur when dividing by  $\dot{x}_i = 0$  in  $\tan \theta_i = \dot{y}_i / \dot{x}_i$ , can be avoided by the following developments.

4.2.3. *Motion planning using flatness.* In Rouchon *et al.* (1993 *a, b*), the following result was sketched.

**Proposition 5:** Consider (18) and two different state-space configurations:  $\tilde{p} = (\tilde{x}_0, \tilde{y}_0, \tilde{\phi}, \tilde{\theta}_0, \dots, \tilde{\theta}_n)$  and  $\bar{p} = (\bar{x}_0, \bar{y}_0, \bar{\phi}, \bar{\theta}_0, \dots, \bar{\theta}_n)$ . Assume that the angles  $\tilde{\theta}_{i-1} - \tilde{\theta}_i$ ,  $i = 1, \dots, n$ ,  $\tilde{\phi}$ ,  $\bar{\theta}_{i-1} - \bar{\theta}_i$ ,  $i = 1, \dots, n$ , and  $\bar{\phi}$  belong to  $]-\pi/2, \pi/2[$ . Then, there exists a smooth open-loop control  $[0, T] \ni t \rightarrow (u_1(t), u_2(t))$  steering the system from  $\tilde{p}$  at time 0 to  $\bar{p}$  at time  $T > 0$ , such that the angles  $\theta_{i-1} - \theta_i$ ,  $i = 1, \dots, n$ , and  $\phi$  ( $i = 1, \dots, n$ ) always remain in  $]-\pi/2, \pi/2[$  and such that  $(u_1(t), u_2(t)) = 0$  for  $t = 0, T$ .

The conditions  $\theta_{i-1} - \theta_i \in ]-\pi/2, \pi/2[$  ( $i = 1, \dots, n$ ) and  $\phi \in ]-\pi/2, \pi/2[$  are meant to avoid some undesirable geometric configurations: trailer  $i$  should not be in front of trailer  $i-1$ .

The detailed proof is given in Appendix C and relies basically on the fact that the system is flat. It is constructive and gives explicitly  $(u_1(t), u_2(t))$ . The involved computations are greatly simplified by a simple geometric interpretation of the rolling without slipping conditions and the use of the Frénet formula. Here, we just recall this geometric construction and give the explicit formula for parking a car with two trailers. The Frénet formula are recalled in the appendix.

Denote by  $\mathcal{C}_i$  the curve followed by  $P_i$ ,  $i = 0, \dots, n$ . As displayed in Fig. 4, the point  $P_{i-1}$  belongs to the tangent to  $\mathcal{C}_i$  at  $P_i$  and at the fixed distance  $d_i$  from  $P_i$ :

$$P_{i-1} = P_i + d_i \tau_i$$

with  $\tau_i$  the unitary tangent vector to  $\mathcal{C}_i$ . Differentiating this relation with respect

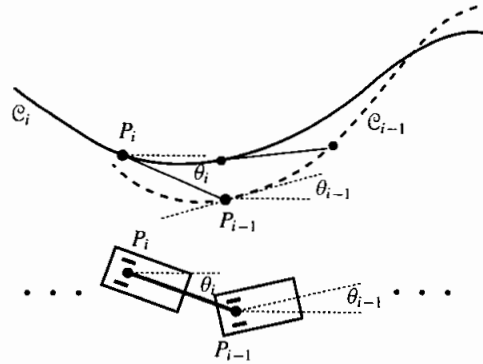


Figure 4. The geometric interpretation of the rolling without slipping conditions.

to  $s_i$ , the arc length of  $\mathcal{C}_i$ , leads to

$$\frac{d}{ds_i} P_{i-1} = \tau_i + d_i \kappa_i \nu_i$$

where  $\nu_i$  is the unitary vector orthogonal to  $\tau_i$  and  $\kappa_i$  is the curvature of  $\mathcal{C}_i$ . Since  $(d/ds_i)P_{i-1}$  gives the tangent direction to  $\mathcal{C}_{i-1}$ , we have

$$\tan(\theta_{i-1} - \theta_i) = d_i \kappa_i$$

**4.2.4. Parking simulations of the 2-trailer system.** We now restrict to the particular case  $n = 2$ . We show how the previous analysis can be employed to solve the parking problem. The simulations of Figs 5 and 6 have been written in MATLAB. They can be obtained upon request from the fourth author via electronic mail (rouchon@cas.ensmp.fr).

The car and its trailers are initially in  $A$  with angles  $\theta_2 = \theta_1 = \theta_0 = \pi/6$ ,  $\phi = 0$ . The objective is to steer the system to  $C$  with final angles  $(\theta_2, \theta_1, \theta_0,$

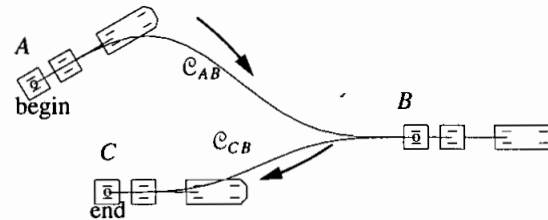


Figure 5. Parking the car with two trailers from A to B via C.

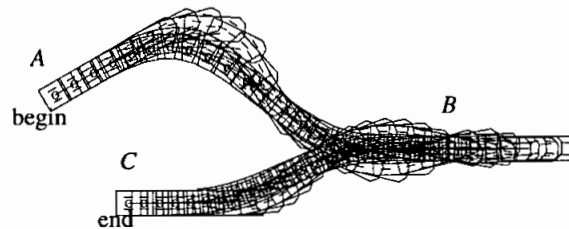


Figure 6. The successive motions of the car with two trailers.

$\phi) = 0$ . We consider the two smooth curves  $\mathcal{C}_{AB}$  and  $\mathcal{C}_{CB}$  of Fig. 5, defined by their natural parameterizations  $[0, L_{AB}] \ni s \rightarrow P_{AB}(s)$  and  $[0, L_{CB}] \ni s \rightarrow P_{CB}(s)$ , respectively ( $P_{AB}(0) = A$ ,  $P_{CB}(0) = C$ ,  $L_{AB}$  is the length of  $\mathcal{C}_{AB}$  and  $L_{CB}$  the length of  $\mathcal{C}_{CB}$ ). Their curvatures are denoted by  $\kappa_{AB}(s)$  and  $\kappa_{CB}(s)$ . These curves shall be followed by  $P_2$ . The initial and final system configuration in  $A$  and  $C$  impose  $\kappa_{AB}(0) = (d/ds)\kappa_{AB}(0) = (d^2/ds^2)\kappa_{AB}(0) = 0$  and  $\kappa_{CB}(0) = (d/ds)\kappa_{CB}(0) = (d^2/ds^2)\kappa_{CB}(0) = 0$ . We impose additionally that  $AB$  and  $CB$  are tangent at  $B$  and

$$\begin{aligned}\kappa_{AB}(L_{AB}) &= \frac{d}{ds}\kappa_{AB}(L_{AB}) = \frac{d^2}{ds^2}\kappa_{AB}(L_{AB}) = \kappa_{CB}(L_{CB}) = \frac{d}{ds}\kappa_{CB}(L_{CB}) \\ &= \frac{d^2}{ds^2}\kappa_{CB}(L_{CB}) = 0\end{aligned}$$

It is straightforward to find curves satisfying such conditions. For the simulation of Fig. 6, we take polynomial curves of degree 9.

Proposition 5 implies that, if  $P_2$  follows  $\mathcal{C}_{AB}$  and  $\mathcal{C}_{CB}$  as displayed on Fig. 5, then the initial and final states will be as desired. Take a smooth function  $[0, T] \ni t \rightarrow s(t) \in [0, L_{AB}]$  such that  $s(0) = 0$ ,  $s(T) = L_{AB}$  and  $\dot{s}(0) = \dot{s}(T) = 0$ . This leads to smooth control trajectories  $[0, T] \ni t \rightarrow u_1(t) \geq 0$  and  $[0, T] \ni t \rightarrow u_2(t)$  steering the system from  $A$  at time  $t = 0$  to  $B$  at time  $t = T$ . Similarly,  $[T, 2T] \ni t \rightarrow s(t) \in [0, L_{CB}]$  such that  $s(T) = L_{CB}$ ,  $s(2T) = 0$  and  $\dot{s}(T) = \dot{s}(2T) = 0$  leads to control trajectories  $[T, 2T] \ni t \rightarrow u_1(t) \leq 0$  and  $[T, 2T] \ni t \rightarrow u_2(t)$  steering the system from  $B$  to  $C$ . This gives the motions displayed in Fig. 6 with forward motion from  $A$  to  $B$ , backward motion from  $B$  to  $C$  and a stop in  $B$ .

Let us detail the calculation of the control trajectories for the motion from  $A$  to  $B$ . Similar calculations can be done for the motion from  $B$  to  $C$ . The curve  $\mathcal{C}_{AB}$  corresponds to the curve  $\mathcal{C}_i$  of Fig. 4 with  $i = 2$ . Assume that  $\mathcal{C}_{AB}$  is given via the regular parameterization,  $y = f(x)$  ( $(x, y)$  are the cartesian coordinates and  $f$  is a polynomial of degree 9). Denote by  $s_i$  the arc length of curve  $\mathcal{C}_i$ ,  $i = 0, 1, 2$ . Then  $ds_2 = \sqrt{1 + (df/dx)^2} dx$  and the curvature of  $\mathcal{C}_2$  is given by

$$\kappa_2 = \frac{d^2 f/dx^2}{(1 + (df/dx)^2)^{3/2}}$$

We have

$$\kappa_1 = \frac{1}{(1 + d_2^2 \kappa_2^2)^{1/2}} \left( \kappa_2 + \frac{d_2}{1 + d_2^2 \kappa_2^2} \frac{d\kappa_2}{ds_2} \right)$$

and  $ds_1 = (1 + d_2^2 \kappa_2^2)^{1/2} ds_2$ . Similarly,

$$\kappa_0 = \frac{1}{(1 + d_1^2 \kappa_1^2)^{1/2}} \left( \kappa_1 + \frac{d_1}{1 + d_1^2 \kappa_1^2} \frac{d\kappa_1}{ds_1} \right)$$

and  $ds_0 = (1 + d_1^2 \kappa_1^2)^{1/2} ds_1$ . Thus  $u_1$  is given explicitly by

$$u_1 = \frac{ds_0}{dt} = (1 + d_1^2 \kappa_1^2)^{1/2} (1 + d_2^2 \kappa_2^2)^{1/2} (1 + (df/dx)^2)^{1/2} \dot{x}(t)$$

where  $[0, T] \ni t \rightarrow x(t)$  is any increasing smooth time function.  $(x(0), f(x(0)))$  (resp.  $(x(T), f(x(T)))$ ) are the coordinates of  $A$  (resp.  $B$ ) and  $\dot{x}(0) = \dot{x}(T) = 0$ .

Since  $\tan(\phi) = d_0 \kappa_0$ , we get

$$u_2 = \frac{d\phi}{dt} = \frac{d_0}{1 + d_0^2 \kappa_0^2} \frac{d\kappa_0}{ds_0} u_1$$

Here, we are not actually concerned with obstacles. The fact that the internal configuration depends only on the curvature results from the general following property: a plane curve is entirely defined (up to rotation and translation) by its curvature. For the  $n$ -trailer case, the angles  $\theta_n - \theta_{n-1}, \dots, \theta_1 - \theta_0$  and  $\phi$  describing the relative configuration of the system are only functions of  $\kappa_n$  and its first  $n$ -derivatives with respect to  $s_n$ .

Consequently, limitations due to obstacles can be expressed up to a translation (defined by  $P_n$ ) and a rotation (defined by the tangent direction ( $dP_n/ds_n$ )) via  $\kappa_n$  and its first  $n$ -derivatives with respect to  $s_n$ . Such considerations can be of some help in finding a curve avoiding collisions. More details on obstacle avoidance can be found in Laumond *et al.* (1994) where a car without trailer is considered.

The multi-steering trailer systems considered in Bushnell *et al.* (1993), Tilbury *et al.* (1993) and Tilbury and Chelouah (1993) are also flat: the flat output is then obtained by adding the angles of the trailers that are directly steered to the cartesian coordinates of the last trailer. This generalization is quite natural in view of the geometric construction of Fig. 4.

## 5. High-frequency control of non-flat systems

We address here a method for controlling non-flat systems via their approximations by averaged and flat ones. More precisely, we develop in three examples an idea due to the Russian physicist Kapitsa (Landau and Lifshitz 1982, Sagdeev *et al.* 1988, Bogaevski and Povzner 1991). He considers the motion of a particle in a highly oscillating field and proposes a method for deriving the equations of the averaged motion and potential. He shows that the inverted position of a single pendulum is 'stabilized' when the suspension point oscillates rapidly. Notice that some related calculations may be found in Baillicul (1993). For the use of high-frequency control in different contexts see also Stephenson (1908), Meerkov (1980), Bentsman (1987), Sussmann and Liu (1991), Acheson (1993).

### 5.1. The Kapitsa pendulum

The notation is summarized in Fig. 7. We assume that the vertical velocity  $\dot{z} = u$  of the suspension point is the control. The equations of motion are:

$$\left. \begin{aligned} \dot{\alpha} &= p + \frac{u}{l} \sin \alpha \\ \dot{p} &= \left( \frac{g}{l} - \frac{u^2}{l^2} \cos \alpha \right) \sin \alpha - \frac{u}{l} p \cos \alpha \\ \dot{z} &= u \end{aligned} \right\} \quad (19)$$

where  $p$  is proportional to the generalized impulsion;  $g$  and  $l$  are physical constants. This system is not flat since it admits only one control variable and is

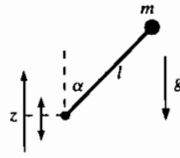


Figure 7. The Kapitza pendulum: the suspension point oscillates rapidly on a vertical axis.

not linearizable via static feedback (Charlet *et al.* 1989). However it is strongly accessible.

We state

$$u = u_1 + u_2 \cos(t/\varepsilon)$$

where  $u_1$  and  $u_2$  are auxiliary control and  $0 < \varepsilon \ll \sqrt{l/g}$ . It is then natural to consider the following averaged control system:

$$\left. \begin{aligned} \dot{\bar{\alpha}} &= \bar{p} + \frac{u_1}{l} \sin \bar{\alpha} \\ \dot{\bar{p}} &= \left( \frac{g}{l} - \frac{(u_1)^2}{l^2} \cos \bar{\alpha} - \frac{(u_2)^2}{2l^2} \cos \bar{\alpha} \right) \sin \bar{\alpha} - \frac{u_1}{l} \bar{p} \cos \bar{\alpha} \\ \dot{\bar{z}} &= u_1 \end{aligned} \right\} \quad (20)$$

It admits two control variables,  $u_1$  and  $u_2$ , whereas the original system (19) admits only one,  $u$ . Moreover (20) is flat with  $(\bar{\alpha}, \bar{z})$  as linearizing output.

The endogenous dynamic feedback

$$\left. \begin{aligned} \dot{\xi} &= v_1 \\ u_1 &= \xi \\ u_2 &= \left( \frac{2l}{\cos \bar{\alpha}} (g + v_1) - \frac{2l^2}{\cos \bar{\alpha} \sin \bar{\alpha}} v_2 \right)^{1/2} \end{aligned} \right\} \quad (21)$$

transforms (20) into

$$\left. \begin{aligned} \ddot{\bar{z}} &= v_1 \\ \ddot{\bar{\alpha}} &= v_2 \end{aligned} \right\} \quad (22)$$

Set

$$\left. \begin{aligned} v_1 &= -\left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \xi - \frac{1}{\tau_1 \tau_2} (\bar{z} - z^{sp}) \\ v_2 &= -\left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \left( \bar{p} + \frac{\xi}{l} \sin \bar{\alpha} \right) - \frac{1}{\tau_1 \tau_2} (\bar{\alpha} - \alpha^{sp}) \end{aligned} \right\} \quad (23)$$

where the parameters  $\tau_1, \tau_2 > 0$  and  $\alpha^{sp} \in ]-\pi/2, \pi/2[ \setminus \{0\}$ . Then, the closed-loop averaged system (20), (21), (23) admits an hyperbolic equilibrium point characterized by  $(z^{sp}, \alpha^{sp})$  that is asymptotically stable.

Now consider (19) and the high-frequency control  $u = u_1 + u_2 \sin(t/\varepsilon)$  with  $0 < \varepsilon \ll \sqrt{l/g}$  and  $(u_1, u_2)$  given by (21) and (23) where  $\bar{\alpha}, \bar{p}$  and  $\bar{z}$  are replaced by  $\alpha, p$  and  $z$ . Then, the corresponding averaged system is nothing but (22) with  $v_1$  and  $v_2$  given by (23). Since the averaged system admits a hyperbolic



asymptotically stable equilibrium, the perturbed system admits an hyperbolic asymptotically stable limit cycle around  $(\alpha, p, z) = (\alpha^{sp}, 0, z^{sp})$  (Guckenheimer and Holmes 1983, th. 4.1.1, p. 168): such control maintains  $(z, \alpha)$  near  $(z^{sp}, \alpha^{sp})$ . Moreover this control method is robust in the following sense: the existence and the stability of the limit cycle is not destroyed by small static errors in the parameters  $l$  and  $g$  and in the measurements of  $\alpha, p, z$  and  $u$ .

As illustrated by the simulations of Fig. 8, the generalization to trajectory tracking for  $\alpha$  and  $z$  is straightforward. These simulations give also a rough estimate of the errors that can be tolerated. The system parameter values are  $l = 0.10$  m and  $g = 9.81$  ms<sup>-2</sup>. The design control parameters are  $\varepsilon = 0.025/2\pi$  s and  $\tau_1 = \tau_2 = 0.10$  s. For the two upper graphics of Fig. 8, no error is introduced: control is computed with  $l = 0.10$  m and  $g = 9.81$  ms<sup>-2</sup>. For the two lower graphics of Fig. 8, parameter errors are introduced: control is computed with  $l = 0.11$  m and  $g = 9.00$  ms<sup>-2</sup>.

### 5.2. The variable-length pendulum

Let us consider the variable-length pendulum of (Bressan and Rampazzo 1993). The notations are summarized in Fig. 9. We assume as in Bressan and Rampazzo (1993) that the velocity  $\dot{u} = v$  is the control. The equations of motion are:

$$\left. \begin{aligned} \dot{q} &= p \\ \dot{p} &= -\cos u + qv^2 \\ \dot{u} &= v \end{aligned} \right\} \quad (24)$$

where mass and gravity are normalized to 1.

This system is not flat since it admits only one control variable and is not linearizable via static feedback (Charlet *et al.* 1989). It is, however, strongly accessible.

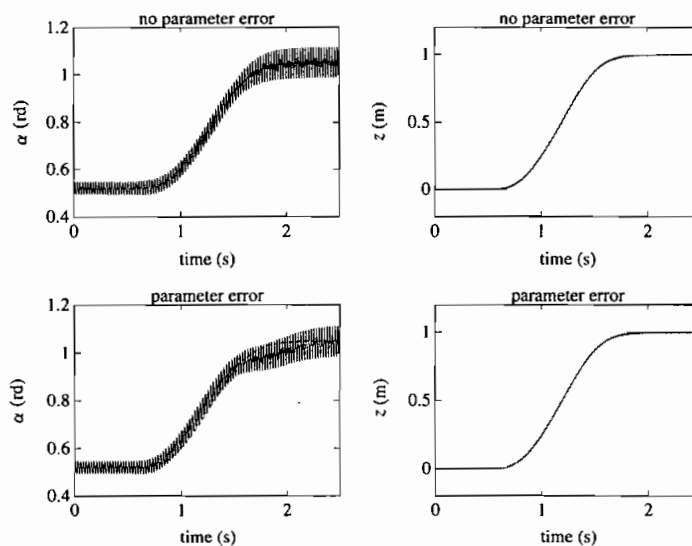


Figure 8. Robustness test of the high-frequency control for the inverted pendulum.

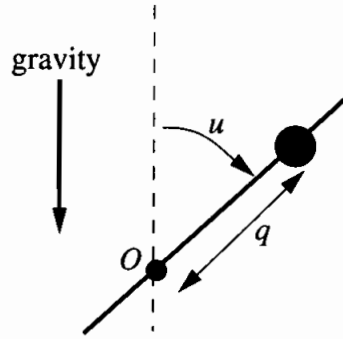


Figure 9. Pendulum with variable-length.

As for the Kapitza pendulum, we set

$$v = v_1 + v_2 \cos(t/\varepsilon)$$

where  $v_1$  and  $v_2$  are auxiliary controls,  $0 < \varepsilon \ll 1$ . We consider the averaged control system:

$$\left. \begin{aligned} \dot{\bar{q}} &= \bar{p} \\ \dot{\bar{p}} &= -\cos \bar{u} + q(v_1)^2 + q(v_2)^2/2 \\ \dot{\bar{u}} &= v_1 \end{aligned} \right\} \quad (25)$$

This system is obviously linearizable via static feedback with  $(\bar{q}, \bar{u})$  as linearizing output.

The static feedback

$$\left. \begin{aligned} v_1 &= w_1 \\ v_2 &= \left[ 2 \left( \frac{w_2 + \cos \bar{u}}{\bar{q}} - (w_1)^2 \right) \right]^{1/2} \end{aligned} \right\} \quad (26)$$

transforms (25) into

$$\left. \begin{aligned} \dot{\bar{u}} &= w_1 \\ \dot{\bar{q}} &= w_2 \end{aligned} \right\} \quad (27)$$

Set

$$\left. \begin{aligned} w_1 &= -\frac{\bar{u} - u^{sp}}{\tau_1} \\ w_2 &= -\left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \bar{p} - \frac{1}{\tau_1 \tau_2} (\bar{q} - q^{sp}) \end{aligned} \right\} \quad (28)$$

with  $\tau_1, \tau_2 > 0$ ,  $u^{sp} \in ]-\pi/2, \pi/2[$ ,  $q^{sp} > 0$ . The closed-loop averaged system (25), (26), (28) admits an hyperbolic equilibrium point  $(u^{sp}, q^{sp})$ , which is asymptotically stable.

Similar to the Kapitza pendulum, the control law is as follows:  $v = v_1 + v_2 \sin(t/\varepsilon)$ ,  $0 < \varepsilon \ll 1$ ;  $(v_1, v_2)$  is given by (26) and (28) where  $\bar{q}$ ,  $\bar{p}$  and  $\bar{u}$  are replaced by  $q$ ,  $p$  and  $u$ . This control strategy leads to a small and attractive limit cycle. As illustrated by the simulations of Fig. 10, the size of these limit

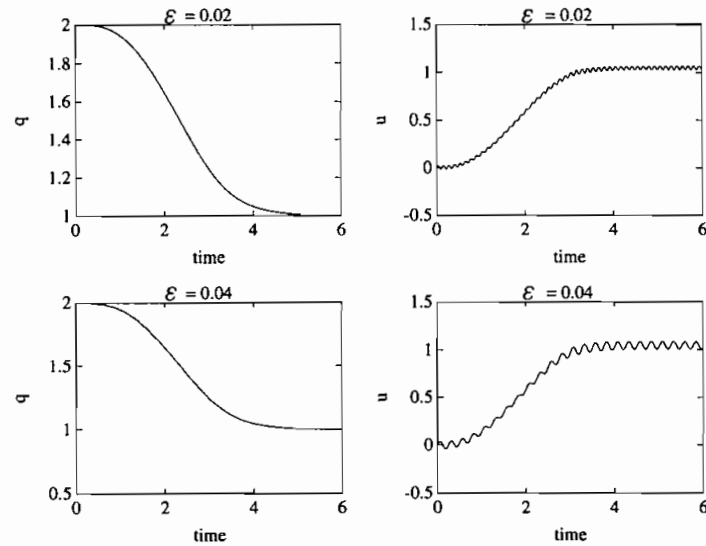
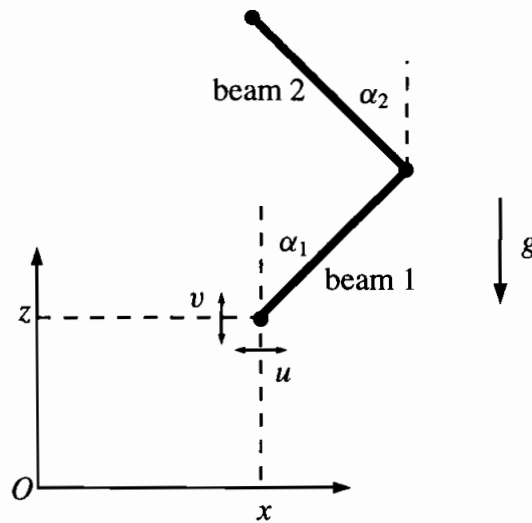


Figure 10. High-frequency control for the variable-length pendulum.

cycle is an increasing function of  $\varepsilon$  and tends to 0 as  $\varepsilon$  tends to  $0^+$ . The design control parameters are  $\tau_1 = 0.5$ ,  $\tau_2 = 0.4$ .

### 5.3. The inverted double pendulum

The double inverted pendulum of Fig. 11 moves in a vertical plane. Assume that  $u$  (resp.  $v$ ) the horizontal (resp. vertical) velocity of the suspension point

Figure 11. The inverted double pendulum: the horizontal velocity  $u$  and vertical velocity  $v$  of the suspension point are the two control variables.

$(x, z)$  is a control variable. The equations of motion are (implicit form):

$$\left. \begin{aligned} p_1 &= I_1 \dot{\alpha}_1 + I \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + n_1 \dot{x} \cos \alpha_1 - n_1 \dot{z} \sin \alpha_1 \\ p_2 &= I \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2) + I_2 \dot{\alpha}_2 + n_2 \dot{x} \cos \alpha_2 - n_2 \dot{z} \sin \alpha_2 \\ \dot{p}_1 &= n_1 g \sin \alpha_1 - n_1 \dot{\alpha}_1 \dot{x} \sin \alpha_1 - n_1 \dot{\alpha}_1 \dot{z} \cos \alpha_1 \\ \dot{p}_2 &= n_2 g \sin \alpha_2 - n_2 \dot{\alpha}_2 \dot{x} \sin \alpha_2 - n_2 \dot{\alpha}_2 \dot{z} \cos \alpha_2 \\ \dot{x} &= u \\ \dot{z} &= v \end{aligned} \right\} \quad (29)$$

where  $p_1$  and  $p_2$  are the generalized impulsions associated to the generalized coordinates  $\alpha_1$  and  $\alpha_2$ , respectively. The quantities  $g$ ,  $I$ ,  $I_1$ ,  $I_2$ ,  $n_1$  and  $n_2$  are constant physical parameters:

$$I_1 = \left( \frac{m_1}{3} + m_2 \right) (l_1)^2, \quad I_2 = \frac{m_2}{3} (l_2)^2, \quad I = \frac{m_2}{2} l_1 l_2$$

$$n_1 = \left( \frac{m_1}{2} + m_2 \right) l_1, \quad n_2 = \frac{m_2}{2} l_2$$

where  $m_1$  and  $m_2$  (resp.  $l_1$  and  $l_2$ ) are the masses (resp. lengths) of beams 1 and 2 which are assumed to be homogeneous.

**Proposition 6:** System (29) with the two control variables  $u$  and  $v$ , is not flat.

**Proof:** The proof is just an application of the necessary flatness condition of theorem 3. Since  $u = \dot{x}$  and  $v = \dot{z}$ , (29) is flat if, and only if, the reduced system,

$$\left. \begin{aligned} p_1 &= I_1 \dot{\alpha}_1 + I \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + n_1 \dot{x} \cos \alpha_1 - n_1 \dot{z} \sin \alpha_1 \\ p_2 &= I \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2) + I_2 \dot{\alpha}_2 + n_2 \dot{x} \cos \alpha_2 - n_2 \dot{z} \sin \alpha_2 \\ \dot{p}_1 &= n_1 g \sin \alpha_1 - n_1 \dot{\alpha}_1 \dot{x} \sin \alpha_1 - n_1 \dot{\alpha}_1 \dot{z} \cos \alpha_1 \\ \dot{p}_2 &= n_2 g \sin \alpha_2 - n_2 \dot{\alpha}_2 \dot{x} \sin \alpha_2 - n_2 \dot{\alpha}_2 \dot{z} \cos \alpha_2 \end{aligned} \right\} \quad (30)$$

is flat. Denote symbolically by  $F(\xi, \dot{\xi}) = 0$  the equations (30) where  $\xi = (\alpha_1, \alpha_2, x, z, p_1, p_2)$ . Consider  $(\xi, \zeta)$  such that  $F(\xi, \zeta) = 0$ . We are looking for a vector  $a = (a_{\alpha_1}, a_{\alpha_2}, a_x, a_z, a_{p_1}, a_{p_2})$  such that, for all  $\lambda \in \mathbb{R}$ ,  $F(\xi, \zeta + \lambda a) = 0$ . The second order conditions,

$$\left. \frac{d^2}{d\lambda^2} \right|_{\lambda=0} F(\xi, \zeta + \lambda a) = 0$$

lead to

$$a_{\alpha_1}(a_x \sin \alpha_1 + a_z \cos \alpha_1) = 0, \quad a_{\alpha_2}(a_x \sin \alpha_2 + a_z \cos \alpha_2) = 0$$

Two first order conditions,

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} F(\xi, \zeta + \lambda a) = 0$$

are

$$\left. \begin{aligned} -a_x \cos \alpha_1 + a_z \sin \alpha_1 &= \frac{I_1}{n_1} a_{\alpha_1} + \frac{I}{n_1} \cos(\alpha_1 - \alpha_2) a_{\alpha_2} \\ -a_x \cos \alpha_2 + a_z \sin \alpha_2 &= \frac{I}{n_2} \cos(\alpha_1 - \alpha_2) a_{\alpha_1} + \frac{I_2}{n_2} a_{\alpha_2} \end{aligned} \right\}$$

Simple computations show that, if  $(I/n_1) \neq (I_2/n_2)$  and  $(I_1/n_1) \neq (I/n_2)$  (these conditions are always satisfied for homogeneous identical beams), then  $(a_{\alpha_1}, a_{\alpha_2}, a_x, a_z) = 0$ . The two remaining first order conditions imply that  $(a_{p_1}, a_{p_2}) = 0$ . Thus  $a = 0$  and the inverted double pendulum is not flat.  $\square$

The same control method as the one explained in details for the Kapitza pendulum (19) can be also used for the double pendulum. The only difference relies on the calculations that are here more tedious. We just sketch some simulations (Fliess *et al.* 1993 b).

To approximate the non-flat system (29) by a flat one, we set  $u = u_1 + u_2 \cos(t/\varepsilon)$  and  $v = v_1 + v_2 \cos(t/\varepsilon)$  where  $0 < \varepsilon \ll \min((I_1/n_1 g)^{1/2}, (I_2/n_2 g)^{1/2})$  and  $u_1, u_2, v_1, v_2$  are new control variables. This leads to a flat averaged system with  $(\alpha_1, \alpha_2, x, z)$  as the linearizing output. The endogenous dynamic feedback that linearized the averaged system provides then  $(u_1, u_2, v_1, v_2)$ . For the simulations of Fig. 12, the angles  $\alpha_1$  and  $\alpha_2$  follow approximately prescribed trajectories whereas, simultaneously, the suspension point  $(x, z)$  is maintained approximately constant.

## 6. Conclusion

Our five examples, as well as other ones in preparation in various domains of engineering, indicate that flatness and defect ought to be considered as physical and/or geometric properties. This explains why flat systems are so often

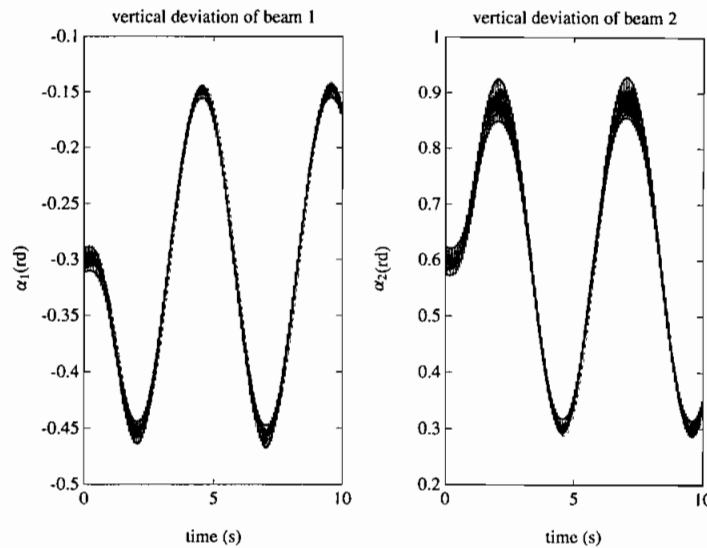


Figure 12. Simulation of the inverted double pendulum via high-frequency control.

encountered in spite of the non-genericity of dynamic feedback linearizability in some customary mathematical topologies (Tchoń 1994, Rouchon 1994).

We hope to have convinced the reader that flatness and defect bring a new theoretical and practical insight in control. We briefly list some important open problems.

(a) Ritt's work (Ritt 1950) shows the differential algebra provides powerful algorithmic means (see Diop (1991, 1992) for a survey and connections with control). Can flatness and defect be determined by this kind of procedures?

(b) Great progress have recently been made in non-linear time-varying feedback stabilization (see, e.g., Coron 1992, 1994). Most of the examples which were considered happen to be flat (see, e.g., Coron and D'Andréa-Novel 1992). The utilization of this property is related to the understanding of the notion of singularity (see, e.g., Martin (1993) for a first step in this direction and the references therein).

(c) The two averaged systems associated to high-frequency control are flat. Can this result be generalized to a large class of devices?

(d) Differential algebra is not the only possible language for investigating flatness and defect. The extension of the differential algebraic formalism to smooth and analytic functions (Jakubczyk 1992) and the differential geometric approach (Martin 1992, Fliess *et al.* 1993 *d*, 1993 *e*, Pomet 1993) should also be examined in this context.

### Appendix A: Prime differential ideals

We know from Diop (1992, lem. 5.2, p. 158) (see also Moog *et al.* 1989) that, for  $x = (x_1, \dots, x_n)$  ( $n \geq 0$ ) and  $u = (u_1, \dots, u_m)$  ( $m \geq 0$ ), the differential ideal corresponding to

$$\dot{x}_i = \frac{a_i(x, u, \dot{u}, \dots, u^{(\alpha_i)})}{b_i(x, u, \dot{u}, \dots, u^{(\beta_i)})}, \quad i = 1, \dots, n$$

where the  $a_i$ s and  $b_i$ s are polynomials over  $k$ , is prime. It is then immediate that the differential ideal corresponding to the tutorial example (7) is prime: set  $x = (x_1, x_2)$  and  $u = (x_3, x_4)$ . Let us now list our five case-studies.

#### Kapitsa pendulum (19)

Let us replace  $\alpha$  by  $\sigma = \tan(\alpha/2)$ . Then, using

$$\dot{\sigma} = \frac{1 + \sigma^2}{2} \dot{\alpha}, \quad \cos \alpha = \frac{1 - \sigma^2}{1 + \sigma^2}, \quad \sin \alpha = \frac{2\sigma}{1 + \sigma^2}$$

the equations (19) become explicit and rational

$$\left. \begin{aligned} \dot{\sigma} &= \frac{1 + \sigma^2}{2} \left( p + \frac{2\sigma u}{l(1 + \sigma^2)} \right) \\ \dot{p} &= \left( \frac{g}{l} - \frac{u^2(1 - \sigma^2)}{l^2(1 + \sigma^2)} \right) \frac{2\sigma}{1 + \sigma^2} - \frac{pu(1 - \sigma^2)}{l(1 + \sigma^2)} \\ \dot{z} &= u. \end{aligned} \right\}$$

The associated differential ideal is thus prime and leads to a finitely generated differential field extension over  $\mathbb{R}$ .

#### Variable-length pendulum (24)

Similar computations with  $\sigma = \tan(u/2)$  prove that the associated differential ideal is prime.

#### Double pendulum (29)

Similar computations with  $\sigma_1 = \tan(\alpha_1/2)$  and  $\sigma_2 = \tan(\alpha_2/2)$  prove that the associated differential ideal is prime.

#### Car with $n$ -trailers (18)

Similar computations with  $\sigma = \tan(\varphi/2)$  and  $\sigma_i = \tan(\theta_i/2)$  prove that the associated differential ideal is prime.

#### Crane (17)

Analogous calculations on the generalized state variable equation  $R\ddot{\theta} = -2\dot{R}\dot{\theta} - \dot{D}\cos\theta - g\sin\theta$  given in Fliess *et al.* (1991, 1993 *a*) lead to a prime differential ideal.

Another more direct way for obtaining the differential field corresponding to the crane is the following. Take (17) and consider the differential field  $\mathbb{R}\langle x, z \rangle$  generated by the two differential indeterminates  $x$  and  $z$ . The variable  $D$  belongs to  $\mathbb{R}\langle x, z \rangle$  and the variable  $R$  belongs to an obvious algebraic extension  $\mathcal{D}$  of  $\mathbb{R}\langle x, z \rangle$ , which defines the system.

### Appendix B: Dynamic feedbacks versus endogenous feedbacks

A dynamic feedback between two systems  $\mathcal{D}/k$  and  $\tilde{\mathcal{D}}/k$  consists in a finitely differential extension  $\mathcal{E}/k$  such that  $\mathcal{D} \subset \mathcal{E}$  and  $\tilde{\mathcal{D}} \subset \mathcal{E}$ . Assume moreover that the extension  $\mathcal{E}/\tilde{\mathcal{D}}$  is differentially algebraic. According to theorem 1, the (non-differential) transcendence degree of  $\mathcal{E}/\tilde{\mathcal{D}}$  is finite, say  $v$ . Choose a transcendence basis  $z = (z_1, \dots, z_v)$  of  $\mathcal{E}/\tilde{\mathcal{D}}$ . It yields like (8):

$$\left. \begin{aligned} A_\alpha(\dot{z}_\alpha, z) &= 0 \quad \alpha = 1, \dots, v \\ B(\xi, z) &= 0 \end{aligned} \right\}$$

where  $\xi$  is any element of  $\mathcal{E}$  and the  $A_\alpha$ s and  $B$  are polynomials over  $\tilde{\mathcal{D}}$ .

The above formulas are the counterpart in the field theoretic language of the usual ones for defining general dynamic feedbacks (see, e.g., Isidori 1989, Nijmeijer and van der Schaft 1990). The dynamic feedback is said to be regular if, and only if,  $\mathcal{E}/\mathcal{D}$  and  $\mathcal{E}/\tilde{\mathcal{D}}$  are both differentially algebraic. The following generalization of proposition 2 is immediate: the systems  $\mathcal{D}/k$  and  $\tilde{\mathcal{D}}/k$  possess the same differential order, i.e., the same number of independent input channels.

The situation of endogenous feedbacks is recovered when  $\mathcal{E}/\mathcal{D}$  and  $\mathcal{E}/\tilde{\mathcal{D}}$  are both algebraic, i.e.,  $v = 0$ .

### Appendix C: Proof of Proposition 5

#### The Frénet formula

Let us recall some terminology and relations relative to planar smooth curves that are displayed on Fig. 13 (see, e.g., Dubrovin *et al.* 1984). A curve parameterization  $\mathbb{R} \ni s \rightarrow P(s) \in \mathbb{R}^2$  is called regular if, and only if, for all  $s$ ,  $(dP/ds) \neq 0$ . A curve is called smooth if, and only if, it admits a regular parameterization. A parameterization is called natural if, and only if, for all  $s$ ,  $\|(dP/ds)\| = 1$  where  $\|\cdot\|$  denotes the Euclidian norm. For smooth curves with a natural parameterization  $s \rightarrow P(s)$ , its signed curvature  $\kappa$  is defined by  $(d\tau/ds) = \kappa v$ , where  $\tau = (dP/ds)$  is the unitary tangent vector and  $v$  is the oriented normal vector ( $(\tau, v)$  is a direct orthonormal frame of the oriented Euclidian plane  $\mathbb{R}^2$ ). Notice that  $(dv/ds) = -\kappa\tau$ . Every smooth curve admits a natural parameterization: every regular parameterization  $t \rightarrow P(t)$  leads to a natural parameterization  $s \rightarrow P(s)$  via the differential relation  $ds = \|(dP/dt)\| dt$ .

**Lemma** Consider a trajectory of (18) such that the curve  $\mathcal{C}_n$  followed by  $P_n$  is smooth with the natural parameterization  $[0, L_n] \ni s_n \rightarrow P_n(s_n)$ :  $s_n = 0$  (resp.  $s_n = L_n$ ) corresponds to the starting point (resp. end point);  $L_n$  is the length of  $\mathcal{C}_n$ . Assume also that for  $s_n = 0$ ,  $\theta_{i-1} - \theta_i$  ( $i = 1, \dots, n$ ) and  $\phi$  belong to  $] -\pi/2, \pi/2[$ . Then,

- (i) for all  $s_n \in [0, L_n]$ ,  $\theta_{i-1} - \theta_i$  ( $i = 1, \dots, n$ ) and  $\phi$  belong to  $] -\pi/2, \pi/2[$ .
- (ii) the curves  $\mathcal{C}_i$  and  $\mathcal{C}$  followed by  $P_i$  and  $Q$  are smooth ( $i = 0, 1, \dots, n$ ).
- (iii)  $\tan(\theta_{i-1} - \theta_i) = d_i \kappa_i$  ( $i = 1, \dots, n$ ) and  $\tan \phi = d_0 \kappa_0$ , where  $\kappa_i$  and  $\kappa_0$  are the curvatures of  $\mathcal{C}_i$  and  $\mathcal{C}_0$ , respectively;
- (iv) the curvature  $\kappa_i$  can be expressed as a smooth function of  $\kappa_n$  and of its first  $n - i$  derivatives with respect to  $s_n$ ; moreover the mapping (which is independent of  $s_n$ )

$$\begin{pmatrix} \kappa_n \\ \frac{d\kappa_n}{ds_n} \\ \vdots \\ \frac{d^n \kappa_n}{ds_n^n} \end{pmatrix} \rightarrow \begin{pmatrix} \kappa_n \\ \kappa_{n-1} \\ \vdots \\ \kappa_0 \end{pmatrix}$$

is a global diffeomorphism from  $\mathbb{R}^{n+1}$  to  $\mathbb{R}^{n+1}$ .

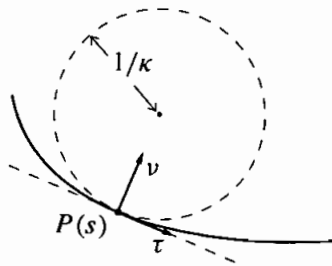


Figure 13. Frénet frame  $(\tau, v)$  and curvature  $\kappa$  of a smooth planar curve.



**Proof of the lemma:** As displayed in Fig. 4, the point  $P_{i-1}$  belongs to the tangent to  $\mathcal{C}_i$  at  $P_i$  and at the fixed distance  $d_i$  from  $P_i$ . By assumption  $\tau_n = (dP_n/ds_n)$  admits the good orientation:  $P_{n-1} = P_n + d_n \tau_n$  (we do not have  $P_{n-1} = P_n - d_n \tau_n$ ). Thus  $\mathcal{C}_{n-1}$  is given by the parameterization  $s_n \rightarrow P_n + d_n \tau_n$  which is regular since  $\|(dP_{n-1}/ds_n)\| = \sqrt{1 + d_n^2 \kappa_n^2}$ . A natural parameterization  $s_{n-1} \rightarrow P_{n-1}$  is given by

$$ds_{n-1} = (1 + d_n^2 \kappa_n^2)^{1/2} ds_n \quad (\text{C } 1)$$

The unitary tangent vector,  $\tau_{n-1}$ , is given by

$$(1 + d_n^2 \kappa_n^2)^{1/2} \tau_{n-1} = \tau_n + d_n \kappa_n v_n$$

where  $v_n$  is the oriented normal to  $\mathcal{C}_n$ . The angle  $\theta_{n-1} - \theta_n$  is the angle between  $\tau_n$  and  $\tau_{n-1}$ . Thus  $\tan(\theta_{n-1} - \theta_n) = d_n \kappa_n$ . Since  $\kappa_n$  is always finite and  $\theta_{n-1} - \theta_n$  belongs  $]-\pi/2, \pi/2[$  for  $s_n = 0$ ,  $\theta_{n-1} - \theta_n$  cannot escape from  $]-\pi/2, \pi/2[$  for any  $s_n \in [0, L_n]$ . The oriented normal to  $\mathcal{C}_{n-1}$ ,  $v_{n-1}$ , is given by

$$(1 + d_n^2 \kappa_n^2)^{1/2} v_{n-1} = -d_n \kappa_n \tau_n + v_n$$

and the signed curvature  $\kappa_{n-1}$  of  $\mathcal{C}_{n-1}$  is, after some calculations,

$$\kappa_{n-1} = \frac{1}{(1 + d_n^2 \kappa_n^2)^{1/2}} \left( \kappa_n + \frac{d_n}{1 + d_n^2 \kappa_n^2} \frac{d\kappa_n}{ds_n} \right) \quad (\text{C } 2)$$

Since  $\theta_{n-1} - \theta_n$  remains in  $]-\pi/2, \pi/2[$ , the unitary tangent vector  $\tau_{n-1}$  has the good direction, i.e.,  $P_{n-2} = P_{n-1} + d_{n-1} \tau_{n-1}$ . The analysis can be continued for  $P_{n-2}, \dots, P_0$  and  $Q$ . This proves (i), (ii) and (iii).

Assertion (iv) comes from the following formula derived from (C 2) and (C 1) ( $i = 1, \dots, n$ ):

$$\kappa_{i-1} = \frac{1}{(1 + d_i^2 \kappa_i^2)^{1/2}} \left( \kappa_i + \frac{d_i}{1 + d_i^2 \kappa_i^2} \frac{d\kappa_i}{ds_i} \right) \quad (\text{C } 3)$$

where  $s_{i-1}$  is the natural parameterization of  $\mathcal{C}_{i-1}$  defined by

$$ds_{i-1} = (1 + d_i^2 \kappa_i^2)^{1/2} ds_i \quad (\text{C } 4)$$

Consequently,  $\kappa_i$  is an algebraic function of  $\kappa_n$  and its first  $n - i$  derivatives with respect to  $s_n$ . Moreover, the dependence with respect to  $(d^{n-i} \kappa_n / ds_n^{n-i})$  is linear via the term

$$\frac{d_{i+1}}{(1 + d_{i+1}^2 \kappa_{i+1}^2)^{3/2}} \frac{d_{i+2}}{(1 + d_{i+2}^2 \kappa_{i+2}^2)^{3/2}} \cdots \frac{d_n}{(1 + d_n^2 \kappa_n^2)^{3/2}} \frac{d^{n-i} \kappa_n}{ds_n^{n-i}}$$

The map of assertion (iv) has a triangular structure with a diagonal dependence that is linear and always invertible: it is a global diffeomorphism.  $\square$

**Proof of Proposition 5:** Denote by  $(\tilde{x}_n, \tilde{y}_n)$  and  $(\bar{x}_n, \bar{y}_n)$  the cartesian coordinates of  $\tilde{P}_n$  and  $\bar{P}_n$ , the initial and final positions of  $P_n$ . There always exists a smooth planar curve  $\mathcal{C}_n$  with a natural parameterization  $s_n \rightarrow P_n(s_n)$  satisfying the following constraints:

- (a)  $P_n(0) = \tilde{P}_n$  and  $P_n(L_n) = \bar{P}_n$  for some  $L_n > 0$
- (b) the direction of tangent at  $\tilde{P}_n$  (respectively  $\bar{P}_n$ ) is given by the angle  $\tilde{\theta}_n$  (respectively  $\bar{\theta}_n$ );

- (c) the first  $n$  derivatives of the signed curvature  $\kappa_n$  at points  $\tilde{P}_n$  and  $\bar{P}_n$  have prescribed values.

According to (iii) and (iv) of the above lemma, the initial and final values of the angles ( $i = 1, \dots, n$ )  $\theta_{i-1} - \theta_i$  and  $\phi$  define entirely the initial and final first  $n$  derivatives of  $\kappa_n$ . It suffices now to choose a smooth function  $[0, T] \ni t \rightarrow s_n(t) \in [0, L_n]$  such that  $s_n(0) = 0$ ,  $s_n(T) = L_n$  and  $\dot{s}_n(0) = \dot{s}_n(L_n) = 0$ , to obtain the desired control trajectory via the relations (the notations are those of the above lemma):

$$\left. \begin{aligned} \dot{s}_0 &= u_1 = \left( \prod_{i=1}^n (1 + d_i^2 \kappa_i^2)^{1/2} \right) \dot{s}_n \\ u_2 &= \left( \prod_{i=1}^n (1 + d_i^2 \kappa_i^2)^{1/2} \right) \frac{d_0}{1 + d_0^2 \kappa_0^2} \frac{d\kappa_0}{ds_0} \dot{s}_n \end{aligned} \right\} \quad \square$$

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