

Lecture 2

Linear Regression

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Outline



Linear Regression



Gradient Descent?

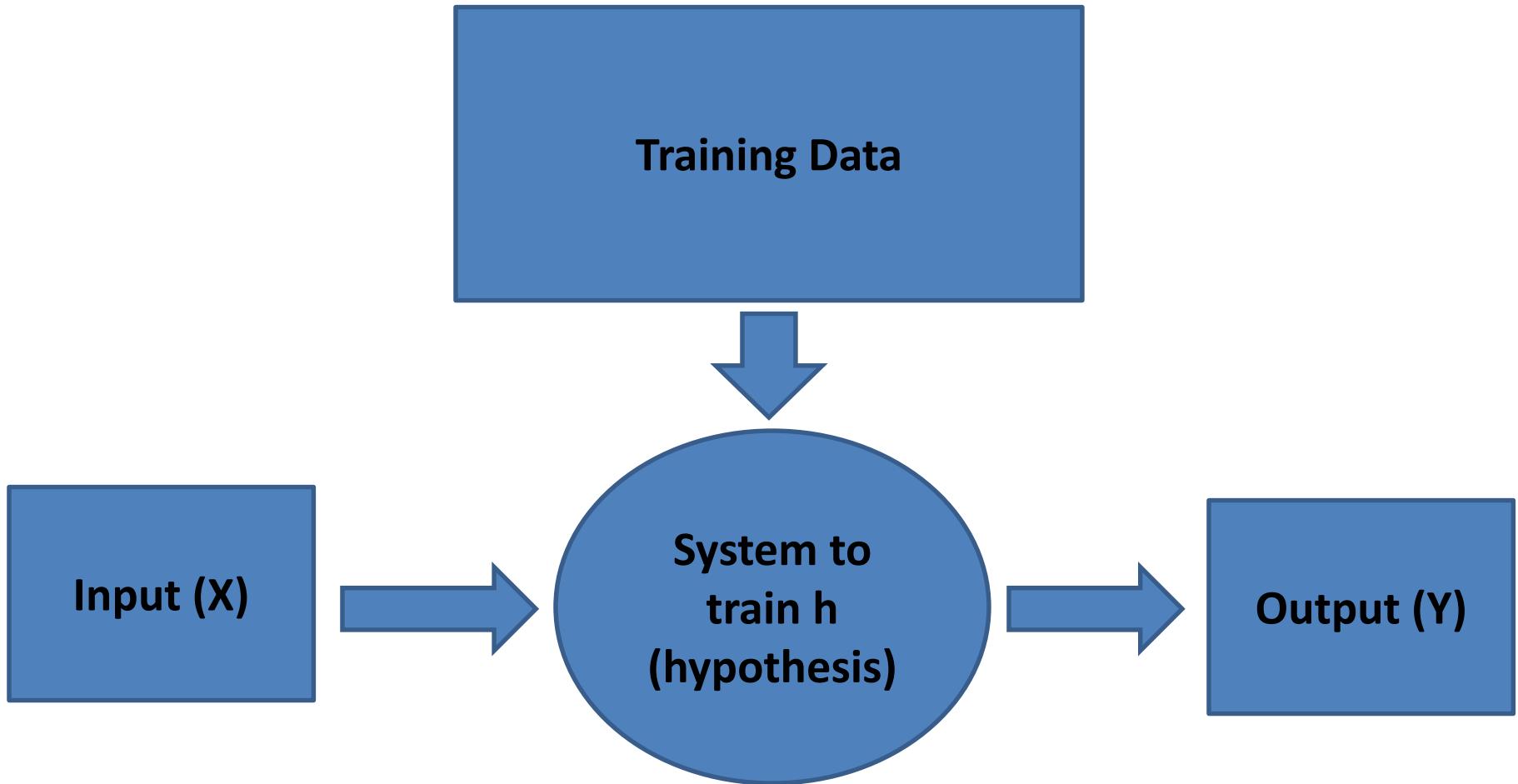


Multi Feature Representation



Questions and Answers

Review of Machine Learning



For each input x , output $y = h(x)$

Linear Regression

Size (m ²)	Price (billion VND)
30	2.5
43	3.4
25	1.8
51	4.5
40	3.2
20	1.6

$(x^{(2)}, y^{(2)})$ → (row 2)
← $(x^{(5)}, y^{(5)})$ (row 5)

Table 1: Training data of housing price in Hanoi

In Supervised Learning, each training data consists of 2 elements:
input x (features) and output y (response)

Notation

- (x, y) : One training data
 $(x^{(i)}, y^{(i)})$: the i^{th} training data

Linear Regression

Plotting of training data



Linear Regression: Assume that the output y is a linear function of input x

$$y = h(x) = a * x + b$$

Linear Regression

Objective:

Learning the function $y=h(x)=a*x+b$, such as it returns the minimize error- **cost function** for the training data (optimization problem)

Find the coefficient a,b to minimize the cost function

Linear Regression

Cost Function:

The error for training data:

$$e^{(1)} = \frac{1}{2} (h(x^{(1)}) - y^{(1)})^2 = \frac{1}{2} (30a + b - 2.5)^2$$

$$e^{(2)} = \frac{1}{2} (h(x^{(2)}) - y^{(2)})^2 = \frac{1}{2} (43a + b - 3.4)^2$$

.

.

$$e^{(m)} = \frac{1}{2} (h(x^{(m)}) - y^{(m)})^2 = \frac{1}{2} (x^{(m)}a + b - y^{(m)})^2$$

The cost function is define as:

$$E = \frac{1}{m} (e^{(1)} + e^{(2)} + \dots + e^{(m)}) = \frac{1}{m} \sum_{i=1}^m e^{(i)} = \frac{1}{2m} \sum_{i=1}^m (h(x^{(i)}) - y^{(i)})^2$$

$$E = \frac{1}{2m} \sum_{i=1}^m (ax^{(i)} + b - y^{(i)})^2$$

Size(m^2)	Price (b.VND)
30	2.5
43	3.4
25	1.8
51	4.5
40	3.2
20	1.6

Gradient Descent

Objective:

Use the gradient function to find a minimum of a function

$$E = \frac{1}{2m} \sum_{i=1}^m (ax^{(i)} + b - y^{(i)})^2$$

Note that E is a function of a and b, we have only 2 variables a and b.

Idea:

Choose random number for a and b, the algorithm is implemented in many steps. At each step, modify a and b such as the cost function is reduced

$$a := a - \alpha \frac{\partial}{\partial_a} E(a) \quad b := b - \alpha \frac{\partial}{\partial_b} E(b)$$

Gradient Descent

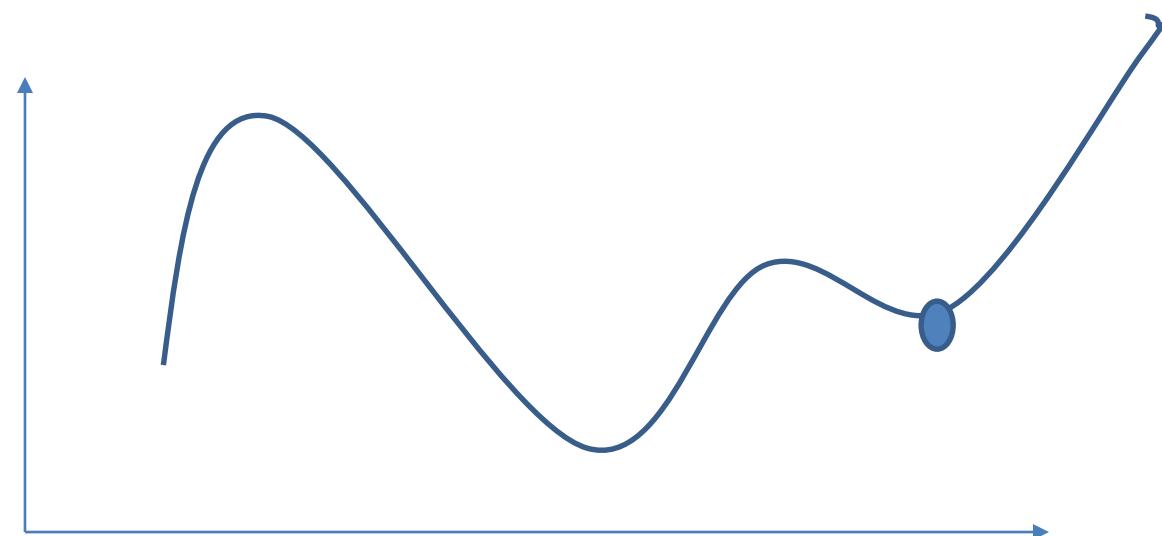
A demonstration and explanation of Gradient Descent algorithm can be found at the following website:

<http://www.onmyphd.com/?p=gradient.descent>

Gradient Descent

Suppose that (x_0, y_0) is a local minimum of the cost function, what will 1 iteration of gradient descent do?

1. Leave x_0 unchanged
2. Change x_0 in random direction
3. Move x_0 toward the global minimum
4. Decrease x_0



Gradient Descent

Calculate the derivations of the cost function: $\frac{\partial}{\partial_a} E(a) \quad \frac{\partial}{\partial_b} E(b)$

$$E = \frac{1}{2m} \sum_{i=1}^m (ax^{(i)} + b - y^{(i)})^2$$

Given the following formula:

$$\frac{\partial}{\partial_x} (x^2) = 2x \qquad \frac{\partial}{\partial_x} (f(x)^2) = 2f(x) \cdot \frac{\partial}{\partial_x} f(x)$$

Gradient Descent

$$\frac{\partial}{\partial_a} E(a) = \frac{1}{m} \sum_{i=1}^m (ax^{(i)} + b - y^{(i)}) \cdot x^{(i)}$$

$$\frac{\partial}{\partial_b} E(b) = \frac{1}{m} \sum_{i=1}^m (ax^{(i)} + b - y^{(i)})$$

Gradient Descent

Exercise:

Starting at $a=0$ and $b= 0$, $\alpha=0.01$, what is the cost function?

Calculate the value of a and b after first iteration (first step). Confirm if the cost function is reduced or not?

Size(m^2)	Price (b.VND)
30	2.5
43	3.4
25	1.8
51	4.5
40	3.2
20	1.6

$$a := a - \alpha \frac{\partial}{\partial_a} E(a) \quad b := b - \alpha \frac{\partial}{\partial_b} E(b)$$

$$\frac{\partial}{\partial_a} E(a) = \frac{1}{m} \sum_{i=1}^m (ax^{(i)} + b - y^{(i)}) \cdot x^{(i)}$$

$$\frac{\partial}{\partial_b} E(b) = \frac{1}{m} \sum_{i=1}^m (ax^{(i)} + b - y^{(i)})$$

Gradient Descent

Batch and Stochastic Gradient Descent

Batch gradient Descent:

Compute the Gradient Descent using the whole data set

Stochastic Gradient Descent:

Compute the Gradient Descent using 1 training example at a time

- Randomly reorder the training data
- Use $(x^{(1)}, y^{(1)})$ to calculate the gradient descent in order to update a, b
- Use $(x^{(2)}, y^{(2)})$ to update
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Gradient Descent

Mini-Batch Gradient Descent

Compute the Gradient Descent for t example at a time ($1 < t < m$)

Example:

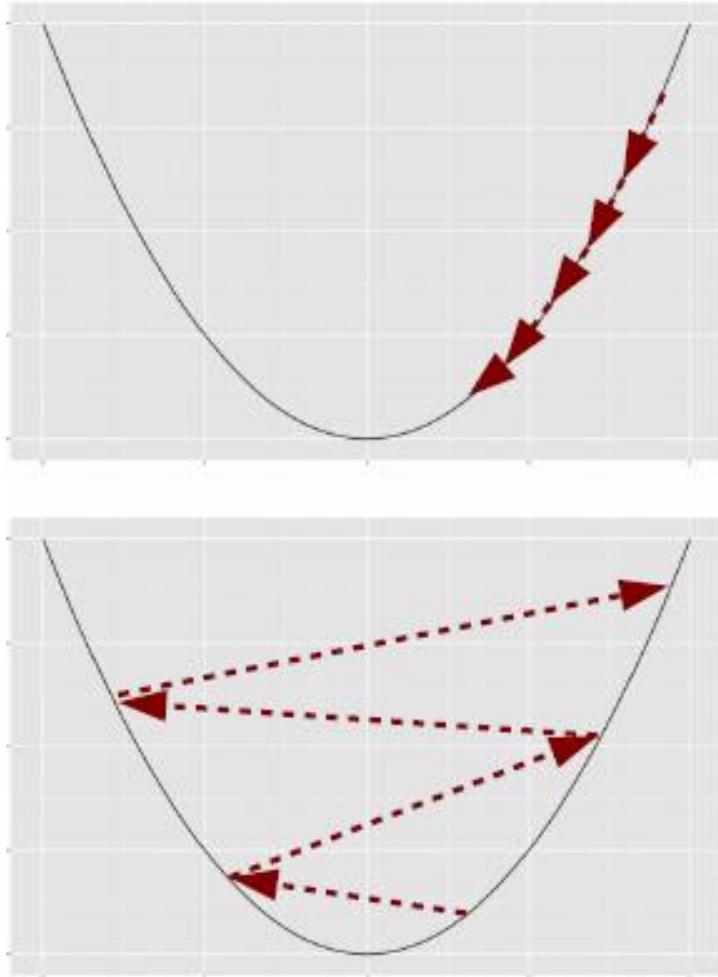
We have 1000 training data.

Step 1: Update the coefficient using 10 data 1-10

Step 2: Update the coefficient using 10 data 11-20

....

Gradient Descent



Big value of α may lead to the incensemement of cost function and not convergence

⇒ Gradient Descent works if the cost function decrease at each step

Gradient Descent

Convergence:

How to know if a function is converged or not?

- Cost function is smaller than a predefined threshold
- After a big enough number of step
- Cost function decreased less than a predefine threshold

Gradient Descent

Summarization:

1. Calculate the cost function
2. Select random value for coefficient a,b
3. Step by step modify a, b such as the cost function is decreased

While (not converged)

do

$$a := a - \frac{\partial}{\partial_a} E(a) \quad b := b - \frac{\partial}{\partial_b} E(b)$$

Multiple Input Representation

Example:

Consider the same example, but with more inputs

Size (m ²)	N ^o of floors	N ^o of rooms	Price (billion VND)
30	3	6	2.5
43	4	8	3.4
25	2	3	1.8
51	4	9	4.5
40	3	5	3.2
20	1	2	1.6

$x^{(i)}$: the input of i^{th} training data

$x_j^{(i)}$: the component j of i^{th} training data

Multiple Input Representation

Matrix representation:

$$y = h(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_n x_n$$

$$x = \begin{vmatrix} 1 \\ x_1 \\ \dots \\ x_n \end{vmatrix} \quad \theta = \begin{vmatrix} \theta_0 \\ \theta_1 \\ \dots \\ \theta_n \end{vmatrix}$$

$$h(x) = [\theta_0 \ \theta_1 \ \theta_2 \dots \theta_n] \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \theta^T x$$

Multiple Input Representation

Cost Function:

$$E(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)})^2 = \frac{1}{2m} \sum_{i=1}^m (\theta^T x^{(i)} - y^{(i)})^2$$

Multiple Input Representation

Gradient Descent

Start with random value of θ , step by step modify θ in order to decrease the cost function

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} E(\theta)$$

$$\frac{\partial}{\partial \theta_j} E(\theta) = \frac{1}{2m} \sum_{i=1}^m \frac{\partial}{\partial \theta_j} (\theta^T x^{(i)} - y^{(i)})^2$$

$$= \frac{1}{m} \sum_{i=1}^m (\theta^T x^{(i)} - y^{(i)}) x_j^{(i)}$$

Multiple Input Representation

Gradient Descent

Start with random value of θ , step by step modify θ in order to decrease the cost function

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} E(\theta)$$

$$\frac{\partial}{\partial \theta_j} E(\theta) = \frac{1}{2m} \sum_{i=1}^m \frac{\partial}{\partial \theta_j} (\theta^T x^{(i)} - y^{(i)})^2$$

$$= \frac{1}{m} \sum_{i=1}^m (\theta^T x^{(i)} - y^{(i)}) x_j^{(i)}$$

Normal Equations

Linear Regression:

Minimize the value of the cost function:

$$E(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)})^2 = \frac{1}{2m} \sum_{i=1}^m (\theta^T x^{(i)} - y^{(i)})^2$$

Normal Equations:

Solve the following equation to find out the optimized value of θ

$$\frac{\partial}{\partial \theta} E(\theta) = 0$$

Normal Equations

Linear Regression:

Minimize the value of the cost function:

$$E(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)})^2 = \frac{1}{2m} \sum_{i=1}^m (\theta^T x^{(i)} - y^{(i)})^2$$

Normal Equations:

Solve the following equation to find out the optimized value of θ

$$\frac{\partial}{\partial \theta} E(\theta) = 0 \quad \forall j \in (0, 1, \dots, n), \frac{\partial}{\partial \theta_j} E(\theta) = 0$$

Normal Equations

Solution:

Given a training set of m training example, each contain n inputs, we have the matrix X ($m, n+1$) of inputs and vector of output Y

$$X = \begin{bmatrix} x_0^{(1)} & x_1^{(1)} & \dots & x_n^{(1)} \\ x_0^{(2)} & x_1^{(2)} & \dots & x_n^{(2)} \\ \dots & & & \\ x_0^{(m)} & x_1^{(m)} & \dots & x_n^{(m)} \end{bmatrix} = \begin{bmatrix} (x^{(1)})^T \\ (x^{(2)})^T \\ \dots \\ (x^{(m)})^T \end{bmatrix} \quad Y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \dots \\ y^{(m)} \end{bmatrix}$$

Solution of normal equations is:

$$\theta = (X^T X)^{-1} X^T Y$$

- Using normal equation to find the value of coefficients theta for this training data

x	y
3	0
2	1
1	2

$$\theta = (X^T X)^{-1} X^T Y$$

Homework

Size (m ²)	Nº of floors	Nº of rooms	Price (billion VND)
30	3	6	2.5
43	4	8	3.4
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$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} E(\theta)$$

$$\frac{\partial}{\partial \theta_j} E(\theta) = \frac{1}{2m} \sum_{i=1}^m \frac{\partial}{\partial \theta_j} (\theta^T x^{(i)} - y^{(i)})^2 = \frac{1}{m} \sum_{i=1}^m (\theta^T x^{(i)} - y^{(i)}) x_j^{(i)}$$

Polynomial Regression

Polynomial Regression

Output is an polynomial function of the input

For example

$$h(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \dots + \theta_n x^n$$

Assume $x_1 = x$

$$x_2 = x^2$$

$$h(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_n x_n$$

...

$$x_n = x^n$$

Linear Regression



References

<http://openclassroom.stanford.edu/MainFolder/VideoPage.php?course=MachineLearning&video=02.4-LinearRegression-GradientDescent&speed=100>

Feature Rescale

Objective: Scale all features to the same scale, in order to have easier computation

Popular scale : [0,1], [-0.5,0.5]

$$x = x / \max(x)$$

$$c = \text{mean}(x) \quad x = x - c / \max(x) \Rightarrow$$