Fourier Series Representation of Discrete-time Periodic Signals

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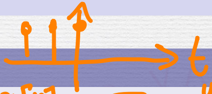
The Course So Far



Previous Lessons

$x[n] \to \boxed{S} \to y[n]$

Midterm (Thursday)

(Wednesday)

- Signals and Systems
- LTI systems: convolution
  - sum
  - integral
- Fourier Series in Continuous-time domain

Review:

- Complex exponential Signal: $\underline{e^{j\omega t}} = \underbrace{j\sin(\omega t)} + \underbrace{\cos(\omega t)}$

  $\delta(t)$

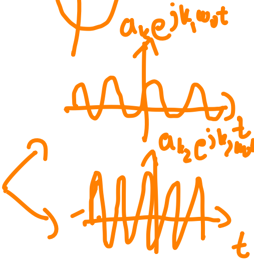
- Period Signal: $x(t) = x(t+T)$

  $\Downarrow$

  period

  Fundamental Frequency:

  $$\underline{\omega_0} = \frac{2\pi}{T}$$



- $$\underbrace{\underline{x(t)} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}}_{\text{Synthesis formula}}$$

- $\underbrace{\underline{a_k} = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt}_{\text{analysis formula}}$

- $e^{j\omega t} = j\sin(\omega t) + \cos(\omega t)$

$$\hookrightarrow \begin{cases} \sin(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j} \\[2ex] \cos(\omega t) = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2} \Rightarrow e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t) \end{cases}$$

The Course So Far

Previous Lessons

- Signals and Systems
- LTI systems: convolution
- Fourier Series in Continuous-time domain

Today

Fourier Series in Discrete-time domain

Lesson Objectives

At the end of this lesson, you should be able to

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1. be familiar with discrete complex signals

$e^{jk\omega_0 t} \rightarrow e^{jk\omega_0 n}$

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At the end of this lesson, you should be able to

1. be familiar with discrete complex signals
2. find Fourier coefficients of any discrete periodic signals

Lesson Objectives

At the end of this lesson, you should be able to

1. be familiar with discrete complex signals
2. find Fourier coefficients of any discrete periodic signals
3. understand various properties of Fourier Series

- Youtube link for lecture of FS in discrete time domain: 30m

(link: in classroom)

Fundamental Period

The minimum positive, nonzero value of N so that

Integer NB

$$x[n] = x[n+N]$$

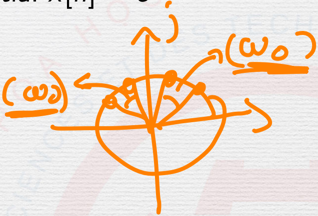
Fundamental frequency $\boxed{\omega_0 = \frac{2\pi}{N}}$

period (Interger)

2 Basic Periodic Signals

The sinusoidal signal $x[n] = cos(\frac{2\pi n}{10})$
The periodic complex exponential $x[n] = e^{j\omega_0 n}$

- $x[n] = e^{j\omega_0 n}$

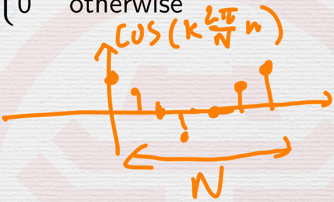


Exercises: Sum of $e^{j\omega_0 n}$

Given $\omega_0 = \frac{2\pi}{N}$, proof that

Integer

$$\sum_{n=\langle N\rangle} e^{jk\omega_0 n} = \sum_{n=\langle N\rangle} e^{k(2\pi/N)n} = \begin{cases} N & \text{if } k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$



$$\sum_{n=\langle N\rangle} e^{k\frac{2\pi}{N}n} = 0$$

$$\boxed{\sum_{n=\langle N\rangle} e^{2\pi r} = N \quad (\text{If } k=0, \pm N, \pm 2N, \ldots)}$$

$\Rightarrow$ exception

Definition

Assume that

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

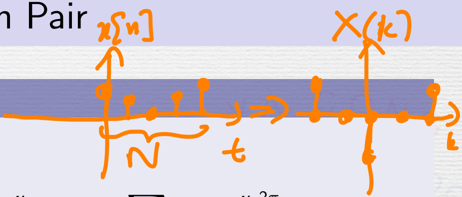
Then

$$a_k = ???$$

Hint: calculate $x[n]e^{-jr\omega_0 n}$, and its sum over N terms.

$$\sum_{n=N} x[n] e^{-jr\omega_0 n} = \sum_{n=N} \sum_{k=N} a_k e^{jk\omega_0 n} \cdot e^{jr\omega_0 n}$$

Fourier Series Equation Pair



Definition

The synthesis equation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

The analysis equation

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$ Synthesis in cont domain

Continuous

- $x(t) = e^{jk\omega_0 t}$, $t \rightarrow R$

- $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

(FS is aperiod Signal)

- $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$

Discrete

- $x[n] = e^{jk\omega_0 n}$, $n \rightarrow N$

- $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$

(FS Is period Signal)

- $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$

Examples

Number one

Consider the signal $x[n] = sin(\frac{2\pi}{5}n)$, determine and plot its Fourier Series coefficients.

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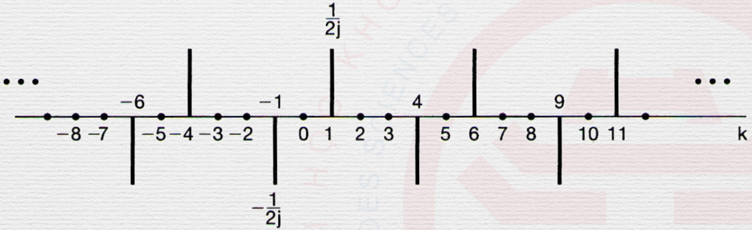

Examples

Number two

Consider the signal $x[n] = sin3(2\pi/5)n$, determine and plot its Fourier Series coefficients.

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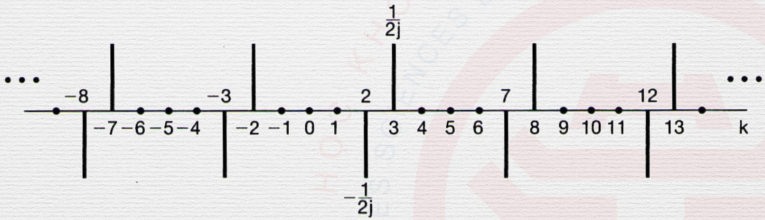
Consider the signal $x[n] = sin3(2\pi/5)n$, determine and plot its Fourier Series coefficients.



Multiplication

Definition

Given

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

Then

$$z[n] = x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow}$$

Multiplication

Definition

Given

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$y[n] \xleftrightarrow{\mathcal{FS}} b_k$$

Then

$$\underline{z[n] = x[n]y[n]} \xleftrightarrow{\mathcal{FS}} c_k = \underline{\sum_{l=<N>} a_l b_{k-l}}$$

Multiplication in time Repr

CS in FS Repr

Multiplication

Definition

Given

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$y[n] \xleftrightarrow{\mathcal{FS}} b_k$$

Then

$$z[n] = x[n]y[n] \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=<N>} a_l b_{k-l}$$

The right-hand side is a periodic convolution between the two periodic sequences of Fourier coefficients.

Parseval's Relation

Definition

$$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 \cancel{dt} = \sum_{k=<N>} |a_k|^2$$

power Signal In time domain

power Signal In FS domain

Parseval's Relation

Definition

$$\frac{1}{N}\sum_{n=<N>}|x[n]|^2 dt = \sum_{k=<N>}|a_k|^2$$

The average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Homework

Homework

3.2, 3.9, 3.11, 3.28

3.1)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$= a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_3 e^{j3\omega_0 t}$$

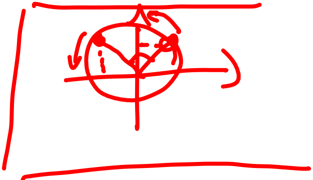
$$+ a_{-3} e^{-j3\omega_0 t}$$

$$= 2\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + 4j\left(e^{j3\omega_0 t}\right.$$

$$\left. - e^{-j3\omega_0 t}\right)$$

$$= 4\cos(\omega_0 t) - 8\sin(3\omega_0 t)$$

$$\left(\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}\right)$$



$$= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{3\pi}{4}t\right)$$

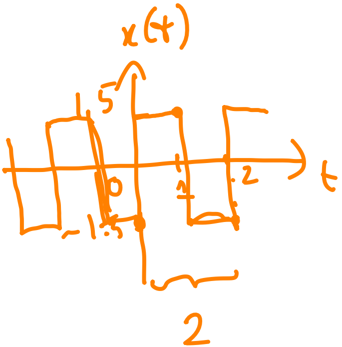
$$= 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right)$$

$$\boxed{x(t) = 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4} + \frac{\pi}{2}\right)}$$

3.4)

$$a_k = \frac{1}{T}\int_T x(t)\, e^{jk\omega_0 t}\, dt$$

$$a_k = \int_0^2 x(t)\, e^{jk\omega_0 t}\, dt$$



$$a_0 = \int_0^2 x(t)\, dt \quad , k=0$$

$$= \int_0^1 x(t)\, dt + \int_1^2 x(t)\, dt$$

T=2

$$= \int_0^1 1.5\, dt - \int_1^2 1.5\, dt = 0$$

- $a_k = \int_0^2 x(t)\,dt \quad , k \neq 0$

$$= \int_0^1 1.5 e^{jk\omega_0 t}\,dt - \int_1^2 1.5 e^{jk\omega_0 t}\,dt$$

$$= 1.5 \left(\int_0^1 e^{jk\omega_0 t}\,dt - \int_1^2 e^{jk\omega_0 t}\,dt \right)$$

$$\left(\int e^{at}\,dt = \frac{e^{at}}{a} \right)$$

$$= 1.5 \left(\frac{e^{jk\omega_0 t}}{jk\omega_0} \Big|_0^1 - \frac{e^{jk\omega_0 t}}{jk\omega_0} \Big|_1^2 \right)$$

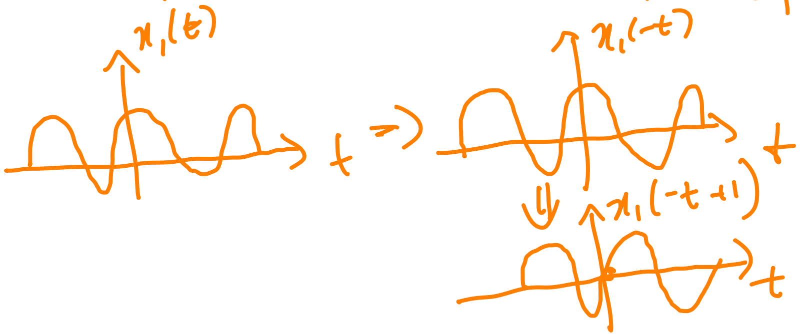
$$= 1.5\left(\frac{e^{jk\omega_0} - 1 - e^{jk\omega_0 2} + e^{jk\omega_0}}{jk\omega_0}\right)$$

$$= 1.5 * \frac{2e^{jk\omega_0} - e^{j2k\omega_0} - 1}{jk\omega_0}$$

3.5) $x_1(t) = x_1(t+T)$, $T = \frac{2\pi}{\omega_1}$

$x_1(t-1) = x_1(t-1+T)$, $T = \frac{2\pi}{\omega_1}$

$x_1(-t+1) = x_1(-t+1+T)$, $T = \frac{2\pi}{\omega_1}$



$$x_2(t) = \underbrace{x_1(t-1)}_{\omega_1} + \underbrace{x_1(-t+1)}_{\omega_1}$$

$$= x_1(t-1+T) + x_1(-t+1+T)$$

$$= x_2(t+T)$$

$$\left(T = \frac{2\pi}{\omega_1}\right)$$

$$\Rightarrow \omega_2 = \omega_1$$

$x_1(t) \xleftrightarrow{FS} a_k$

$x_2(t) \xleftrightarrow{FS} b_k = ?\, a_k$

$$\boxed{\begin{array}{l} \text{Time Reverse: } \left(x(t) \xleftrightarrow{FS} a_k \right) \\ \quad x(-t) \xleftrightarrow{FS} a_{-k} \\ \text{Time Shift: } \\ \quad x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k \end{array}}$$

$x(t) \xleftrightarrow{FS} a_k$

$$\boxed{x(t-1) \xleftrightarrow{FS} e^{-jk\omega_0} a_k}$$

$y(t) \xleftrightarrow{FS} c_k$

$y(-t) \xleftrightarrow{FS} c_{-k}$

$$\boxed{x(-t+1) \xleftrightarrow{FS} e^{-jk\omega_0} a_{-k}}$$

$$x_2(t) = x_1(t-1) + x_1(-t+1)$$

$$b_k = \frac{1}{T}\int_T x_2(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \boxed{\frac{1}{T}\int_T x_1(t-1)\, e^{-jk\omega_0 t}\, dt} \;\; \text{FS}\,(x(t-1))$$

$$+ \boxed{\frac{1}{T}\int_T x_1(-t+1)\, e^{-jk\omega_0 t}\, dt} \;\; \frac{\text{FS}}{(x(-t+1))}$$

$$= e^{-jk\omega_0 t_0}\, a_k + e^{-jk\omega_0 t_0}\, a_{-k}$$

3.2)

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$a_0 = 1$

$a_2 = e^{j\pi/4}$

$a_{-2}^{*} = e^{j\pi/4} = j\sin(\pi/4) + \underline{\cos}(\pi/4)$