4.11 Problems

Exercise 1. (Video Solution)
Let X be a Gaussian random variable with $\mu = 5$ and $\sigma^2 = 16$.

(a) Find $\mathbb{P}[X > 4]$ and $\mathbb{P}[2 \leq X \leq 7]$.

(b) If $\mathbb{P}[X < a] = 0.8869$, find a.

(c) If $\mathbb{P}[X > b] = 0.1131$, find b.

(d) If $\mathbb{P}[13 < X \leq c] = 0.0011$, find c.

Exercise 2. (Video Solution)
Compute $\mathbb{E}[Y]$ and $\mathbb{E}[Y^2]$ for the following random variables:

(a) $Y = A\cos(\omega t + \theta)$, where $A \sim \mathcal{N}(\mu, \sigma^2)$.

(b) $Y = a\cos(\omega t + \Theta)$, where $\Theta \sim \text{Uniform}(0, 2\pi)$.

(c) $Y = a\cos(\omega T + \theta)$, where $T \sim \text{Uniform}\left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right)$.

Exercise 3. (Video Solution)
Consider a CDF

$$F_X(x) = \begin{cases} 0, & \text{if } x < -1, \\ 0.5, & \text{if } -1 \leq x < 0, \\ (1+x)/2, & \text{if } 0 \leq x < 1, \\ 1, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{P}[X < -1]$, $\mathbb{P}[-0.5 < X < 0.5]$ and $\mathbb{P}[X > 0.5]$.

(b) Find $f_X(x)$.

Exercise 4. (Video Solution)
A random variable X has CDF:

$$F_X(x) = \begin{cases} 0, & \text{if } \quad x < 0, \\ 1 - \frac{1}{4}e^{-2x}, & \text{if } \quad x \geq 0. \end{cases}$$

(a) Find $\mathbb{P}[X \leq 2]$, $\mathbb{P}[X = 0]$, $\mathbb{P}[X < 0]$, $\mathbb{P}[2 < X < 6]$ and $\mathbb{P}[X > 10]$.

(b) Find $f_X(x)$.

Exercise 5. (VIDEO SOLUTION)
A random variable X has PDF

$$f_X(x) = \begin{cases} cx(1-x^2), & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find c, $F_X(x)$, and $\mathbb{E}[X]$.

Exercise 6. (VIDEO SOLUTION)
A continuous random variable X has a cumulative distribution

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.5 + c\sin^2(\pi x/2), & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

(a) What values can c assume?

(b) Find $f_X(x)$.

Exercise 7. (VIDEO SOLUTION)
A continuous random variable X is uniformly distributed in $[-2, 2]$.

(a) Let $Y = \sin(\pi X/8)$. Find $f_Y(y)$.

(b) Let $Z = -2X^2 + 3$. Find $f_Z(z)$.

Hint: Compute $F_Y(y)$ from $F_X(x)$, and use $\frac{d}{dy}\sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$.

Exercise 8.
Let $Y = e^X$.

(a) Find the CDF and PDF of Y in terms of the CDF and PDF of X.

(b) Find the PDF of Y when X is a Gaussian random variable. In this case, Y is said to be a lognormal random variable.

Exercise 9.
The random variable X has the PDF

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let Y be a new random variable

$$Y = \begin{cases} 0, & X < 0, \\ \sqrt{X}, & 0 \le X \le 1, \\ 1, & X > 1. \end{cases}$$

Find $F_Y(y)$ and $f_Y(y)$, for $-\infty < y < \infty$.

Exercise 10.
A random variable X has the PDF

$$f_X(x) = \begin{cases} 2xe^{-x^2}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Let

$$Y = g(X) = \begin{cases} 1 - e^{-X^2}, & X \geq 0, \\ 0, & X < 0. \end{cases}$$

Find the PDF of Y.

Exercise 11.
A random variable X has the PDF

$$f_X(x) = \frac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty.$$

Let $Y = g(X) = e^{-X}$. Find the PDF of Y.

Exercise 12.
A random variable X has the PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}, \qquad -\infty < x < \infty.$$

Find the PDF of Y where

$$Y = g(X) = \begin{cases} X, & |X| > K, \\ -X, & |X| < K. \end{cases}$$

Exercise 13.
A random variable X has the PDF

$$f_X(x) = \frac{1}{x^2\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.$$

Let $Y = g(X) = \frac{1}{X}$. Find the PDF of Y.

Exercise 14.
A random variable X has the CDF

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^\alpha, & 0 \leq x \leq 1, \\ 1, & x > 1, \end{cases}$$

with $\alpha > 0$. Find the CDF of Y if

$$Y = g(X) = -\log X.$$

Exercise 15.

Energy efficiency is an important aspect of designing electrical systems. In some modern buildings (e.g., airports), traditional escalators are being replaced by a new type of "smart" escalator which can automatically switch between a normal operating mode and a standby mode depending on the flow of pedestrians.

(a) The arrival of pedestrians can be modeled as a Poisson random variable. Let N be the number of arrivals, and let λ be the arrival rate (people per minute). For a period of t minutes, show that the probability that there are n arrivals is

$$\mathbb{P}(N = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

(b) Let T be a random variable denoting the interarrival time (i.e., the time between two consecutive arrivals). Show that

$$\mathbb{P}(T > t) = e^{-\lambda t}.$$

Also, determine $F_T(t)$ and $f_T(t)$. Sketch $f_T(t)$.

(Hint: Note that $\mathbb{P}(T > t) = \mathbb{P}(\text{no arrival in } t \text{ minutes})$.)

(c) Suppose that the escalator will go into standby mode if there are no pedestrians for $t_0 = 30$ seconds. Let Y be a random variable denoting the amount of time that the escalator is in standby mode. That is, let

$$Y = \begin{cases} 0, & \text{if } T \leq t_0, \\ T - t_0, & \text{if } T > t_0. \end{cases}$$

Find $\mathbb{E}[Y]$.