Probability
Lecture 2: Conditional probability and Independence

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Content

Conditional probability: motivation

- Observe that *probability will change due to the additional information flow*.
- Consider a football game between Home team and Away team.

+ one day before the game, the winning probability of Home team is 0.6.
+ 2 hours before the game, a key player of the Home team got injury, the winning probability went down to 0.45.
+ after the first half, Home team got advantage 2-0, the winning probability was 0.8.
+ when the game ends with score 3-1, the winning probability was 0.1.

Conditional probability

Definition

Consider two events A and B with $\mathbb{P}(B) > 0$. The **conditional probability** of A given B is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

- Recall that in classical probability,

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}.$$

- Therefore,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{|A \cap B|}{|B|}.$$

$\Rightarrow$ Conditional probability: change the "sample space".

Examples

- Consider throwing a die. Let

$$A = \{\text{getting a } 3\},$$

$$B = \{\text{getting an odd number}\}.$$

Find $\mathbb{P}(A \mid B)$ and $\mathbb{P}(B \mid A)$.

We have

$$A = \{3\}, \quad B = \{1, 3, 5\}.$$

Hence

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/6}{3/6} = \frac{1}{3},$$

and

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{1/6}{1/6} = 1.$$

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Examples

Anne and Bob play the following game: they flip a coin three times, if the number of heads is bigger then Anne wins, otherwise Bob will win.
Is it a fair game?

YES. Since

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

and

$$A = \{HHH, HHT, HTH, THH\},$$
$$B = \{HTT, THT, TTH, TTT\}.$$

Now assume that we did already flipped the first time with head. This event can be expressed as

$$E = \{HHH, HHT, HTH, HTT\}.$$

How about the winning probability of each one now?

$$A \cap E = \{HHH, HHT, HTH\},$$
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Examples

- Consider a tetrahedral (4-sided) die. Let X be the first roll and Y be the second roll. Let B be the event that $\min(X, Y) = 2$ and M be the event that $\max(X, Y) = 3$. Find $\mathbb{P}[M|B]$.
- Calculate $\mathbb{P}[B] = 5/16$ since $B = \{(2,2);(2,3);(2,4);(3,2);(4,2)\}$.
- Calculate $\mathbb{P}[M \cap B]$. Since $M \cap B = \{(2,3);(3,2)\}$, $\mathbb{P}[M \cap B] = 2/16$.
- Hence $\mathbb{P}[M|B] = \dfrac{\mathbb{P}(M \cap B)}{\mathbb{P}(B)} = 2/5$.

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Product formula

Theorem

We have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A).$$

Theorem

Consider n events $A_1, \ldots, A_n$ such that $\mathbb{P}(A_1 \cap \ldots \cap A_{n-1}) > 0$. Then

$$\begin{aligned}&\mathbb{P}(A_1 \cap \ldots \cap A_n)\\ =&\mathbb{P}(A_n|A_1 \cap \ldots \cap A_{n-1})\mathbb{P}(A_{n-1}|A_1 \cap \ldots \cap A_{n-2}) \ldots \mathbb{P}(A_2|A_1)\mathbb{P}(A_1).\end{aligned}$$

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Example

- In a tier game with n tickets, among them, there is only one winning one. A group of n players will play by turn as follows: from the first person to the n-th one, he will take a ticket from the box without letting it back until getting the winning one. What is the wining probability of each one? It it a fair game?

- YES. It is a fair game.

$\mathbb{P}(A_1) = 1/n$.

$$\mathbb{P}(\overline{A_1} \cap A_2) = \mathbb{P}(A_2|\overline{A_1})\mathbb{P}(\overline{A_1}) = \frac{1}{n-1}\frac{n-1}{n} = 1/n.$$

$$\mathbb{P}(\overline{A_1} \cap \overline{A_2} \cap A_3) = \mathbb{P}(A_3|\overline{A_1} \cap \overline{A_2})\mathbb{P}(\overline{A_2}|\overline{A_1})\mathbb{P}(\overline{A_1}) = \frac{1}{n-2}\frac{n-2}{n-1}\frac{n-1}{n} = 1/n.$$

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Independence

Definition

Two events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

It is equivalent to

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A),$$

or

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B).$$

- Remark: **Disjoint $\neq$ Independent.**

Theorem

If two events A and B are independent, then the following couples are also independent: $(\overline{A}, B)$, $(A, \overline{B})$, *and* $(\overline{A}, \overline{B})$.

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Examples

- Throw a dice twice. Check the independence of the following couples of events.

- $A = \{\text{1st die is 3}\}$ and $B = \{\text{2nd die is 4}\}$.
- $A = \{\text{1st die is 3}\}$ and $B = \{\text{sum is 7}\}$.
- $A = \{\text{max is 2}\}$ and $B = \{\text{min is 2}\}$.

Examples

- Throw a dice twice. Check the independence of the following couples of events.

- $A = \{\text{1st die is } 3\}$ and $B = \{\text{2nd die is } 4\}$ are independent.

$$\mathbb{P}(A) = 1/6, \quad \mathbb{P}(B) = 1/6, \quad \mathbb{P}(A \cap B) = 1/36.$$

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$$\mathbb{P}(A) = 1/6, \quad \mathbb{P}(B) = 1/6, \quad \mathbb{P}(A \cap B) = 1/36.$$

  - Let $B' = \{\text{sum is } 8\}$. Are A and B' independent?
  NO. Since

$$\mathbb{P}(B') = 5/36, \quad \mathbb{P}(A \cap B') = 1/36.$$

- $A = \{\max \text{ is } 2\}$ and $B = \{\min \text{ is } 2\}$ are not independent (or dependent).

$$\mathbb{P}(A) = \frac{3}{36} = \frac{1}{12}, \quad \mathbb{P}(B) = \frac{9}{36} = \frac{1}{4}, \quad \mathbb{P}(A \cap B) = 1/36.$$

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Independence: more events

Definition

A collection of events $\{A_i\}_{i\in I}$ is called an independent collection if for any finite subset $\{i_1, \ldots, i_n\}$ of I,

$$\mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_n}) = \mathbb{P}(A_{i_1}) \ldots \mathbb{P}(A_{i_n}).$$

- In particular, any two events in an independent collection are independent.
- Examples: throw a die many times and A_i be the obtained number in the i-th throw.

Independence: Counterexample

- **A pairwisely independent collection may not be independent!!!**
- Throw a die three times. Consider the events $A_{i,j}$ such that in i-th and j-th turn, we obtain the same number. Then

$$\mathbb{P}(A_{1,2}) = \mathbb{P}(A_{2,3}) = \mathbb{P}(A_{1,3}) = 1/6.$$

$+$ The intersection event $A_{1,2} \cap A_{2,3}$ means that we obtain the same number for all three turns, hence

$$\mathbb{P}(A_{1,2} \cap A_{2,3}) = \frac{6}{6^3} = 1/36 = \mathbb{P}(A_{1,2}).\mathbb{P}(A_{2,3}),$$

It implies that two events $A_{1,2}$ are $A_{2,3}$ independent. And similarly for other couples.

$+$ But the collection of three events is not independent, since

$$\mathbb{P}(A_{1,2} \cap A_{2,3} \cap A_{1,3}) = \frac{6}{6^3} = 1/36 \neq \mathbb{P}(A_{1,2}).\mathbb{P}(A_{2,3})\mathbb{P}(A_{1,3}).$$

Example

- Consider a match of two players, the winner is the one will get 6 winning games first. However, by some reason, the match has to stop when the first player got 5 winning games, and the second one got 3 ones. What is the winning probability of each one if the match continues? Assume that in each game, the winning probability of each one is equal.

- Consider the second player, he will win the match only if for the next three consecutive games, he wins all. Then

$$\mathbb{P}(A_2) = \mathbb{P}(G_1 \cap G_2 \cap G_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 1/8.$$

- Hence $\mathbb{P}(A_1) = 1 - \mathbb{P}(A_2) = 7/8$.

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Bayes formula

Definition

A collection $\{A_1, \ldots, A_n\}$ is called a partition of Ω if

- $A_1, \ldots, A_n$ are disjoint.
- $\Omega = A_1 \cup \ldots \cup A_n$.

Theorem (Bayes theorem)

For a partition $\{A_1, \ldots, A_n\}$ *and an event* B *,*

- *Law of Total Probability,* $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$.
- *If* $\mathbb{P}(B) > 0$*,*

$$\mathbb{P}(A_k|B) = \frac{\mathbb{P}(B|A_k)\mathbb{P}(A_k)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_k)\mathbb{P}(A_k)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}.$$

Bayes theorem

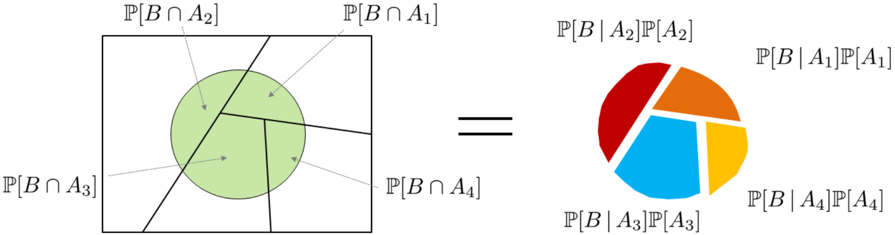

Figure: Proof of Bayes theorem.

Example

- Suppose there are three types of players in a tennis tournament: A, B, and C. Fifty percent of the contestants in the tournament are A players, 25% are B players, and 25% are C players. Your chance of beating the contestants depends on the class of the player, as follows:

0.3 against an A player
0.4 against a B player
0.5 against a C player

If you play a match in this tournament, what is the probability of your winning the match? Supposing that you have won a match, what is the probability that you played against an A player?

- Summary $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.25$ and $\mathbb{P}(C) = 0.25$.
$\mathbb{P}(W|A) = 0.3$, $\mathbb{P}(W|B) = 0.3$ and $\mathbb{P}(W|C) = 0.5$.
$+\mathbb{P}(W) =? \Rightarrow \mathbb{P}(W|A)\mathbb{P}(A) + \mathbb{P}(W|B)\mathbb{P}(B) + \mathbb{P}(W|C)\mathbb{P}(C) = 0.375.$

and $\mathbb{P}(A|W) =? \Rightarrow \mathbb{P}(A|W) = \dfrac{\mathbb{P}(W|A)\mathbb{P}(A)}{\mathbb{P}(W)} = 0.4.$

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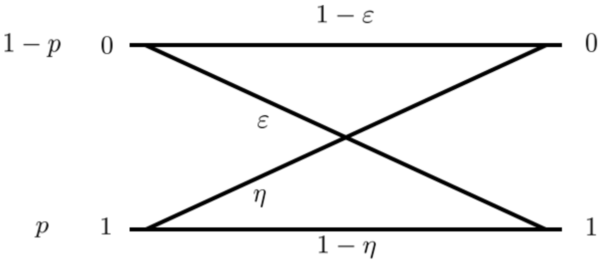
Example

- Consider the communication channel shown below. The probability of sending a 1 is p and the probability of sending a 0 is $1 - p$. Given that 1 is sent, the probability of receiving 1 is $1 - \eta$. Given that 0 is sent, the probability of receiving 0 is $1 - \epsilon$. Find the probability that the signal 1 has been received. Find the probability that the signal has been right received.



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- Define the events

$S_0 = 0$ is sent, and $R_0 = 0$ is received.
$S_1 = 1$ is sent, and $R_1 = 1$ is received.

- $\mathbb{P}(R_1|S_1) = 1-\eta$, $\mathbb{P}(R_0|S_0) = 1-\epsilon$.
- $\mathbb{P}(R_1) = \mathbb{P}(R_1|S_1)\mathbb{P}(S_1) + \mathbb{P}(R_1|S_0)\mathbb{P}(S_0) = p(1-\eta) + (1-p)\epsilon$. - $\mathbb{P}(S_1R_1 \cup S_0R_0) = p(1-\eta) + (1-p)(1-\epsilon)$.

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- Hint: consider the result of next match.
- If player 1 wins the next match, then we have the situation 5-3. So

$$\mathbb{P}(W_1|W) = 7/8.$$

$$\mathbb{P}(W_1) = \mathbb{P}(W_1|W)\mathbb{P}(W) + \mathbb{P}(W_1|L)\mathbb{P}(L) = p(5,3)/2 + p(4,4)/2$$

$$= \frac{7}{16} + \frac{1}{4} = 11/16.$$

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Example

Because a new medical procedure has been shown to be effective in the early detection of an illness, a medical screening of the population is proposed. The probability that the test correctly identifies someone with the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test, and the result is positive. What is the probability that you have the illness?

Example

Dubrovsky sits down to a night of gambling with his fellow officers. Each time he stakes u dollars there is a probability $1/2$ that he will win and receive back $2u$ dollars (including his stake). At the beginning of the night he has 8000 dollars. If ever he has 256000 dollars he will marry the beautiful Natasha and retire to his estate in the country. Otherwise, he will commit suicide. He decides to follow one of two courses of action:
(i) to stake 1000 dollars each time until the issue is decided;
(ii) to stake ALL IN each time until the issue is decided.

Homework

+ Stanley Chan: Chapter2. 17-25.
+ Ngo Hoang Long: Chapter1. 33, 39, 59, 61, 64, 68.

Thank you!