Introduction to Linear Programming

Lê Xuân Thanh

Contents

General formulation

- In words:
  Optimize a linear function
  subject to linear constraints on real-valued variables

- Explicit general formulation:

$$
\begin{array}{lrll}
\max \mid \min & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
\text{subject to} & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i & & (i = 1, \ldots, m_1) \\
 & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i & & (i = m_1 + 1, \ldots, m_2) \\
 & a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i & & (i = m_2 + 1, \ldots, m) \\
 & x_j \leq 0 & & (j = 1, \ldots, n_1) \\
 & x_j \geq 0 & & (j = n_1 + 1, \ldots, n_2) \\
 & x_j \in \mathbb{R} & & (j = n_2 + 1, \ldots, n)
\end{array}
$$

Standard formulation: case of maximization

- In words:
  Maximize a linear function subject to linear constraints of $\leq$-type on non-negative variables

- Explicit standard formulation:

$$
\begin{aligned}
\text{maximize} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

Note: standard formulation is starting point for simplex method

Standard formulation: case of minimization

- In words:
  Minimize a linear function subject to linear constraints of $\geq$-type on non-negative variables

- Explicit standard formulation:

$$
\begin{aligned}
\text{minimize} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\
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& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

Note: standard formulation is starting point for simplex method

Canonical formulation

- In words:
  Optimize a linear function subject to
  linear constraints of equality-type on non-negative variables
- Explicit standard formulation:

$$
\begin{aligned}
\text{maximize} \mid \text{minimize} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
& \ldots \\
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& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

Note: canonical formulation is starting point for simplex method

Transform general formulation to standard formulation

- $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$

  replace by $\sum_{j=1}^{n} -a_{ij}x_j \leq -b_i$

- $\sum_{j=1}^{n} a_{ij}x_j = b_i$

  replace by $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ and $\sum_{j=1}^{n} -a_{ij}x_j \leq -b_i$

- unrestricted variables $x_j \in \mathbb{R}$

  replace x_j by $x_j = x_j^+ - x_j^-$ with $x_j^+, x_j^- \geq 0$

- $\min \sum_{j=1}^{n} c_jx_j$

  replace by $\max \sum_{j=1}^{n} -c_jx_j$

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Example

Given

$$\begin{aligned} \min \quad & 2x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & 3x_1 + 2x_2 = 15 \\ & x_1 \geq 0 \end{aligned}$$

Transform to standard form

$$\begin{aligned} \max \quad & -2x_1 - 4x_2^+ + 4x_2^- \\ \text{s.t.} \quad & -x_1 - x_2^+ + x_2^- \leq -3 \\ & -3x_1 - 2x_2^+ + 2x_2^- \leq -15 \\ & 3x_1 + 2x_2^+ - 2x_2^- \leq 15 \\ & x_1, x_2^+, x_2^- \geq 0 \end{aligned}$$

Transform standard formulation to general formulation

- Change sign of $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ to get $\sum_{j=1}^{n} -a_{ij}x_j \geq -b_i$
- Add slack variable $s_i \geq 0$ to $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$

  to get $\sum_{j=1}^{n} a_{ij}x_j + s_i = b_i$
- Replace $x_j \geq 0$ by

$$
\begin{aligned}
x_j - y_j + z_j &= 0 \\
y_j - z_j &\geq 0 \\
x_j, y_j, z_j &\in \mathbb{R}
\end{aligned}
$$

  to get unrestricted variables $x_j, y_j, z_j \in \mathbb{R}$
- Change max $\sum_{j=1}^{n} c_j x_j$ to min $\sum_{j=1}^{n} -c_j x_j$

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$$

  to get unrestricted variables $x_j, y_j, z_j \in \mathbb{R}$
- Change max $\sum_{j=1}^{n} c_jx_j$ to min $\sum_{j=1}^{n} -c_jx_j$

Example

Given an LP in its standard form

$$\begin{aligned} \max \quad & 3x_1 - 2x_2 \\ \text{s.t.} \quad & -4x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Transform to minimizing problem s.t. equality constraints and non-negative variables

$$\begin{aligned} \min \quad & -3x_1 + 2x_2 \\ \text{s.t.} \quad & -4x_1 + x_2 + x_3 = 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Matrix formulation of a linear program (I)

Explicit formulation

$$
\begin{array}{rrcl}
\text{maximize} & c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq & b_2 \\
 & \ldots & & \\
 & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq & b_m \\
 & x_1, x_2, \ldots, x_n & \geq & 0
\end{array}
$$

Matrix formulation

$$
\begin{array}{rrcl}
\max & \mathbf{c}^t\mathbf{x} & & \\
\text{s.t.} & A\mathbf{x} & \leq & \mathbf{b} \\
 & \mathbf{x} & \geq & \mathbf{0}
\end{array}
$$

- $A = (a_{ij})_{m\times n}$: matrix of constraint coefficients
- $\mathbf{x} = (x_1, \ldots, x_n)^t$: vector of decision variables
- $\mathbf{c} = (c_1, \ldots, c_n)^t$: vector of coefficients of objective function
- $\mathbf{b} = (b_1, \ldots, b_m)^t$: vector of r.h.s. constants of constraints

Matrix formulation of a linear program (II)

Explicit formulation

$$
\begin{aligned}
\text{minimize} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}
$$

Matrix formulation

$$
\begin{aligned}
\text{min} \quad & \mathbf{c}^t\mathbf{x} \\
\text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

- $A = (a_{ij})_{m\times n}$: matrix of constraint coefficients
- $\mathbf{x} = (x_1, \ldots, x_n)^t$: vector of decision variables
- $\mathbf{c} = (c_1, \ldots, c_n)^t$: vector of coefficients of objective function
- $\mathbf{b} = (b_1, \ldots, b_m)^t$: vector of r.h.s. constants of constraints

Matrix formulation of a linear program (III)

Explicit formulation

$$\begin{aligned} \max \mid \min \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ & \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Matrix formulation

$$\begin{aligned} \max \mid \min \quad & \mathbf{c}^t\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- $A = (a_{ij})_{m\times n}$: matrix of constraint coefficients
- $\mathbf{x} = (x_1, \ldots, x_n)^t$: vector of decision variables
- $\mathbf{c} = (c_1, \ldots, c_n)^t$: vector of coefficients of objective function
- $\mathbf{b} = (b_1, \ldots, b_m)^t$: vector of r.h.s. constants of constraints

Contents

Geometrical structure of feasible set

- (Polyhedral) $\mathcal{V}$-cone
- (Convex) $\mathcal{V}$-polytope
- (Convex) $\mathcal{V}$-polyhedron

Cones in $\mathbb{R}^n$

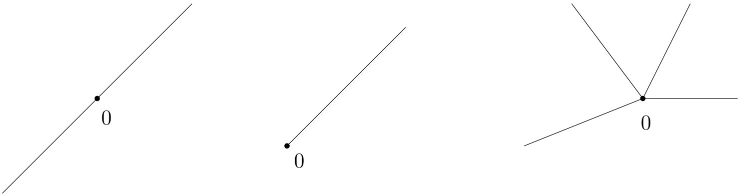
Definition

A set $K \subset \mathbb{R}^n$ is called a *cone* if

$$\mathbf{x} \in K \text{ and } \theta \geq 0 \quad \Rightarrow \quad \theta\mathbf{x} \in K$$

Examples:

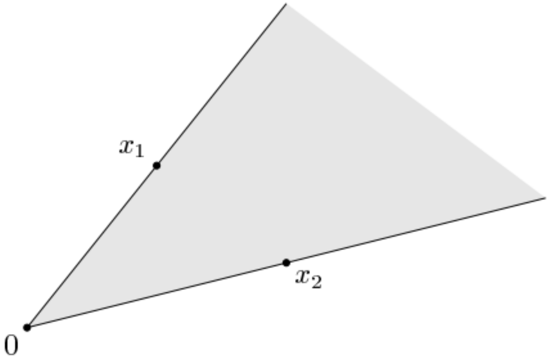
- A line passing through origin
- A ray based at origin (i.e., the set $\{\theta\mathbf{v} \mid \theta \geq 0\}$)
- The union of different rays based at origin



Convex cones in $\mathbb{R}^n$

Definition

A set $K \subset \mathbb{R}^n$ is a *convex cone* if it is convex and conic, i.e., for any $\mathbf{x}^1, \mathbf{x}^2 \in K$ and $\theta_1, \theta_2 \geq 0$ we have $\theta_1 \mathbf{x}^1 + \theta_2 \mathbf{x}^2 \in K$



Conic combination and conic hull

Definition

- A *conic combination* of the points $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^n$ is a point of the form
$$\theta_1 \mathbf{x}^1 + \ldots + \theta_k \mathbf{x}^k$$
where $\theta_1, \ldots, \theta_k \geq 0$
- The *conic hull* of a set $K \subset \mathbb{R}^n$ is the set of all conic combinations of points in K:
$$\operatorname{cone}(K) = \{\theta_1 \mathbf{x}^1 + \ldots + \theta_k \mathbf{x}^k \mid \mathbf{x}^i \in K, \theta_i \geq 0, i = 1, \ldots, k\}$$

- Examples
  - $\operatorname{cone}(\emptyset) = \{\mathbf{0}\}$
  - Conic hull of $\mathbf{x}$ is the ray based at origin and passing through $\mathbf{x}$

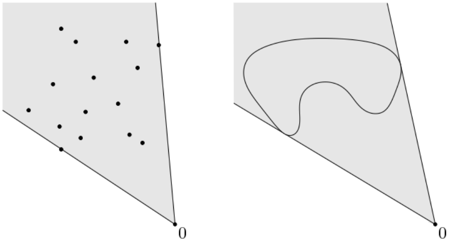
Conic hull

$$\mathrm{cone}(K) = \{\theta_1 \mathbf{x}^1 + \ldots + \theta_k \mathbf{x}^k \mid \mathbf{x}^i \in C, \theta_i \geq 0, i = 1, \ldots, k\}$$

Proposition

The conic hull $\mathrm{cone}(K)$ of a set $K \subset \mathbb{R}^n$ is the smallest convex cone containing K (in sense of set inclusion)

Corollary: $K \subset \mathbb{R}^n$ is a convex cone if and only if $K = \mathrm{cone}(K)$



(Polyhedral) $\mathcal{V}$-cone

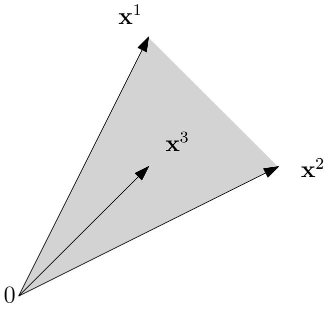
Definition

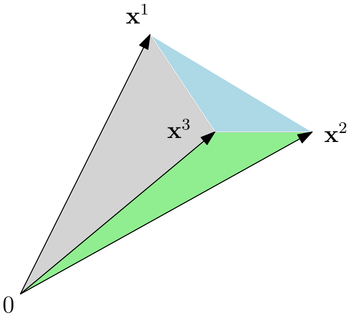
A $\mathcal{V}$*-cone* is the conic hull of a finite set of points in some $\mathbb{R}^n$

$$\mathrm{cone}(\mathbf{x}^1, \ldots, \mathbf{x}^k) = \{\theta_1 \mathbf{x}^1 + \ldots + \theta_k \mathbf{x}^k \mid \theta_1, \ldots, \theta_k \geq 0\}$$

Examples:

- $\mathrm{cone}(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ in $\mathbb{R}^2$ and $\mathbb{R}^3$





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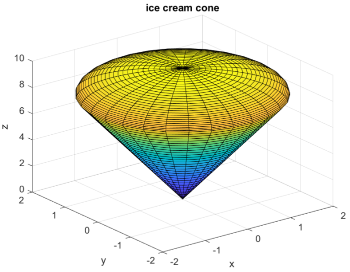
Examples:

- Lorentz cone (ice-cream cone)

$$\{(x_1, x_2, x_3) \mid \sqrt{x_1^2 + x_2^2} \leq x_3\}$$

is NOT a polyhedral cone



Geometrical structure of feasible set

- (Polyhedral) $\mathcal{V}$-cone
- (Convex) $\mathcal{V}$-polytope
- (Convex) $\mathcal{V}$-polyhedron

(Convex) $\mathcal{V}$-polytope

Definition

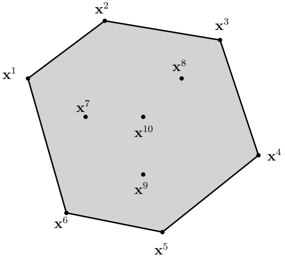
A *$\mathcal{V}$-polytope* is the convex hull of a finite set of points in some $\mathbb{R}^n$

$$\mathrm{conv}(\mathbf{x}^1, \ldots, \mathbf{x}^k) = \{\theta_1 \mathbf{x}^1 + \ldots + \theta_k \mathbf{x}^k \mid \theta_1, \ldots, \theta_k \geq 0, \sum_{i=1}^k \theta_i = 1\}$$

Examples:

- A polytope in $\mathbb{R}^2$

$$\mathrm{conv}(\{\mathbf{x}^1, \ldots, \mathbf{x}^{10}\}) = \mathrm{conv}(\{\mathbf{x}^1, \ldots, \mathbf{x}^6\})$$



(Convex) $\mathcal{V}$-polytope

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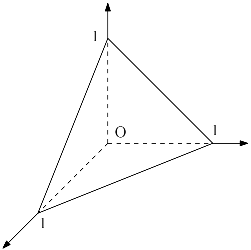
Examples:

- Standard d-simplex

$$\Delta_d := \mathrm{conv}(\{\mathbf{e}^1, \ldots, \mathbf{e}^{d+1}\})$$



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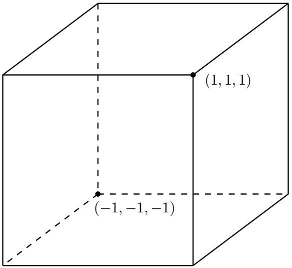
$$\mathrm{conv}(\mathbf{x}^1, \ldots, \mathbf{x}^k) = \{\theta_1 \mathbf{x}^1 + \ldots + \theta_k \mathbf{x}^k \mid \theta_1, \ldots, \theta_k \geq 0, \sum_{i=1}^k \theta_i = 1\}$$

Examples:

- *d*-cube

$$C_d := \mathrm{conv}(\{+1, -1\}^d)$$



(Convex) $\mathcal{V}$-polytope

Definition

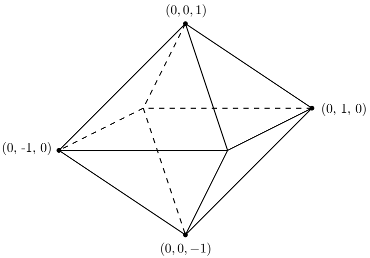
A *$\mathcal{V}$-polytope* is the convex hull of a finite set of points in some $\mathbb{R}^n$

$$\mathrm{conv}(\mathbf{x}^1, \ldots, \mathbf{x}^k) = \{\theta_1 \mathbf{x}^1 + \ldots + \theta_k \mathbf{x}^k \mid \theta_1, \ldots, \theta_k \geq 0, \sum_{i=1}^k \theta_i = 1\}$$

Examples:

- *d*-crosspolytope

$$C_d^{\Delta} := \mathrm{conv}(\{\mathbf{e}^1, -\mathbf{e}^1, \ldots, \mathbf{e}^d, -\mathbf{e}^d\})$$



Geometrical structure of feasible set

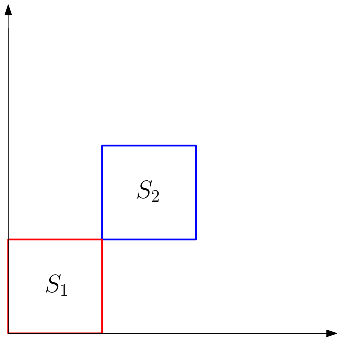
- (Polyhedral) $\mathcal{V}$-cone
- (Convex) $\mathcal{V}$-polytope
- (Convex) $\mathcal{V}$-polyhedron

Minkowski sum of sets

Definition

The *Minkowski sum* of two sets $P, Q \subset \mathbb{R}^n$ is defined by

$$P + Q := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in P, \mathbf{y} \in Q\}$$



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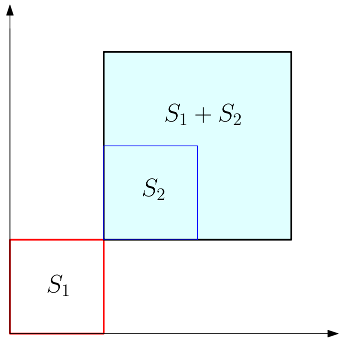

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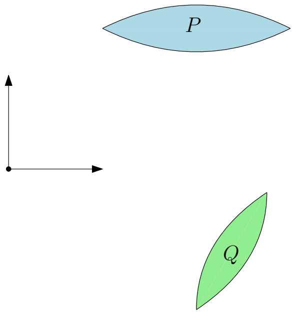

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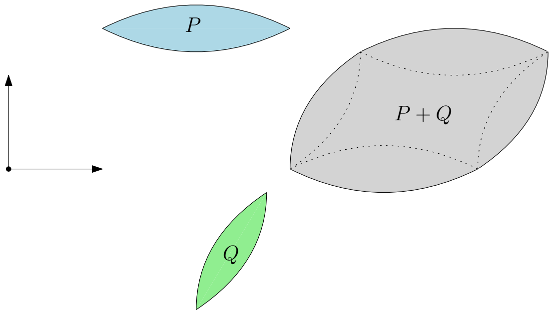

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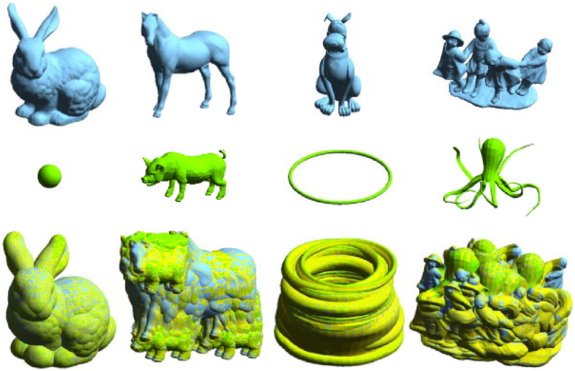


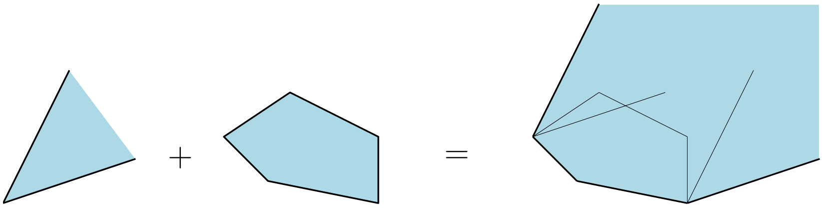
(Convex) $\mathcal{V}$-polyhedra

Definition

A *$\mathcal{V}$-polyhedron* is the Minkowski sum of a polytope and a polyhedral cone

$$P = \mathrm{conv}(\mathbf{x}^1, \dots, \mathbf{x}^k) + \mathrm{cone}(\mathbf{v}^1, \dots, \mathbf{v}^\ell)$$

for some $\mathbf{x}^1, \dots, \mathbf{x}^k, \mathbf{v}^1, \dots, \mathbf{v}^\ell \in \mathbb{R}^n$



Contents

Algebraic structure of feasible set

- (Polyhedral) $\mathcal{H}$-cone
- (Convex) $\mathcal{H}$-polyhedron

(Polyhedral) $\mathcal{H}$-cone

Definition

A *$\mathcal{H}$-cone* $K \subset \mathbb{R}^n$ is a set of form

$$K = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} \leq \mathbf{0}\}$$

for some matrix $A \subset \mathbb{R}^{m \times n}$

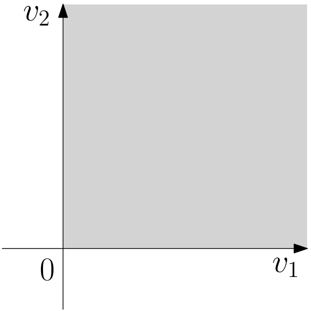
Examples:

- Non-negative orthant in $\mathbb{R}^2$:

$$\left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$



(Polyhedral) $\mathcal{H}$-cone

Definition

An $\mathcal{H}$*-cone* $K \subset \mathbb{R}^n$ is a set of form

$$K = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} \leq \mathbf{0}\}$$

for some matrix $A \subset \mathbb{R}^{m \times n}$

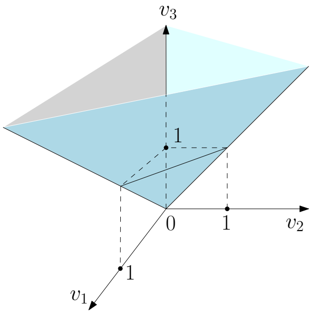
Examples:

- An $\mathcal{H}$-cone in 3D:

$$\{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} \leq \mathbf{0}\}$$

with

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$



Algebraic structure of feasible set

- (Polyhedral) $\mathcal{H}$-cone
- (Convex) $\mathcal{H}$-polyhedron

(Convex) $\mathcal{H}$-polyhedron

Definition

- An *$\mathcal{H}$-polyhedron* $P \subset \mathbb{R}^n$ is a set of form $P = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} \leq \mathbf{b}\}$ for some matrix $A \subset \mathbb{R}^{m \times n}$ and some $\mathbf{b} \in \mathbb{R}^m$
- An *$\mathcal{H}$-polytope* is an $\mathcal{H}$-polyhedron that is bounded

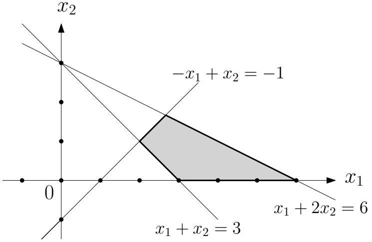
Examples:

- An $\mathcal{H}$-polytope in $\mathbb{R}^2$:

$$\{\mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} \leq \mathbf{b}\}$$

with

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ -1 & -1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -3 \\ 6 \end{pmatrix}$$



(Convex) $\mathcal{H}$-polyhedron

Definition

- An *$\mathcal{H}$-polyhedron* $P \subset \mathbb{R}^n$ is a set of form $P = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} \leq \mathbf{b}\}$ for some matrix $A \subset \mathbb{R}^{m \times n}$ and some $\mathbf{b} \in \mathbb{R}^m$
- An *$\mathcal{H}$-polytope* is an $\mathcal{H}$-polyhedron that is bounded

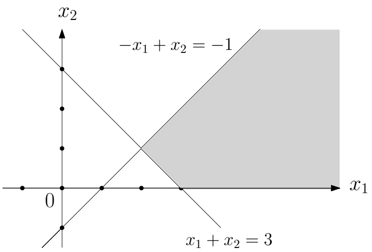
Examples:

- An unbounded $\mathcal{H}$-polyhedron in $\mathbb{R}^2$:

$$\{\mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} \leq \mathbf{b}\}$$

with

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$



Contents

Minkowski[1]-Weyl[2] theorem

Minkowski-Weyl theorem

A subset $P \subset \mathbb{R}^n$ is a $\mathcal{V}$-polyhedron

$$P = \mathrm{conv}(\mathbf{x}^1, \dots, \mathbf{x}^k) + \mathrm{cone}(\mathbf{v}^1, \dots, \mathbf{v}^\ell)$$

if and only if it is a finite intersection of closed halfspaces

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\} \quad \text{for some } A \in \mathbb{R}^{m \times n} \text{ and } \mathbf{b} \in \mathbb{R}^m$$

Remark:
The latter representation means that P is an $\mathcal{H}$-polyhedron

---

[1] Hermann Minkowski (22.06.1864-12.01.1909): a German mathematician

[2] Hermann Klaus Hugo Weyl (09.11.1885-08.12.1955): a German mathematician, theoretical physicist, and philosopher

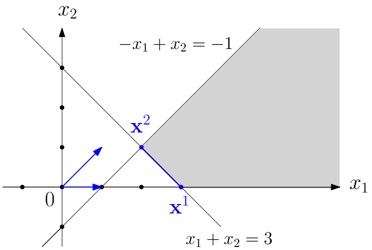
Minkowski-Weyl theorem: an illustrative example

- $\mathcal{H}$-representation

$$\{\mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} \leq \mathbf{b}\}$$

with

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$



- $\mathcal{V}$-representation

$$\mathrm{conv}(\mathbf{x}^1, \mathbf{x}^2) + \mathrm{cone}(\mathbf{v}^1, \mathbf{v}^2)$$

with

$$\mathbf{x}^1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \mathbf{x}^2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Minkowski-Weyl theorem: Corollary 1

Main theorem for polyhedral cones

A subset $C \subset \mathbb{R}^n$ is a $\mathcal{V}$-cone

$$C = \mathrm{cone}(\mathbf{x}^1, \ldots, \mathbf{x}^k)$$

if and only if it is a finite intersection of closed linear halfspaces

$$C = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{0}\} \quad \text{for some } A \in \mathbb{R}^{m \times n}$$

Remark:
The latter representation means that C is an $\mathcal{H}$-cone

Main theorem for polyhedral cones: Example

- $\mathcal{H}$-representation

$$\{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} \leq \mathbf{0}\}$$

with

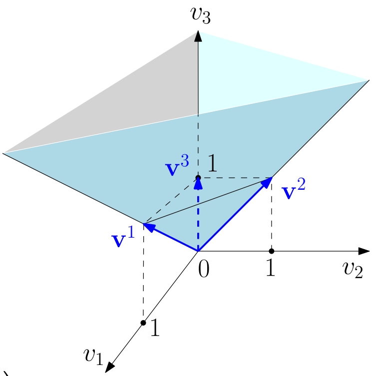
$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

- $\mathcal{V}$-representation

$$\mathrm{cone}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)$$

with

$$\mathbf{v}^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}^2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Minkowski-Weyl theorem: Corollary 2

Main theorem for convex polytopes

A subset $P \subset \mathbb{R}^n$ is a $\mathcal{V}$-polytope

$$P = \mathrm{conv}(\mathbf{x}^1, \dots, \mathbf{x}^k)$$

if and only if it is a bounded intersection of closed halfspaces

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\} \quad \text{for some } A \in \mathbb{R}^{m \times n} \text{ and } \mathbf{b} \in \mathbb{R}^m$$

Remark:
The latter representation means that P is an $\mathcal{H}$-polytope

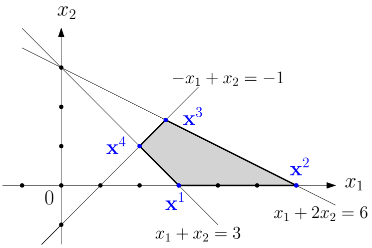
Main theorem for convex polytopes: Example 1

- $\mathcal{H}$-representation

$$\{\mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} \leq \mathbf{b}\}$$

with

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ -1 & -1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ -3 \\ 6 \end{pmatrix}$$



- $\mathcal{V}$-representation

$$\mathrm{conv}(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4)$$

with

$$\mathbf{x}^1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \mathbf{x}^2 = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \quad \mathbf{x}^3 = \begin{pmatrix} 8/3 \\ 5/3 \end{pmatrix}, \quad \mathbf{x}^4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Main theorem for convex polytopes: Example 2

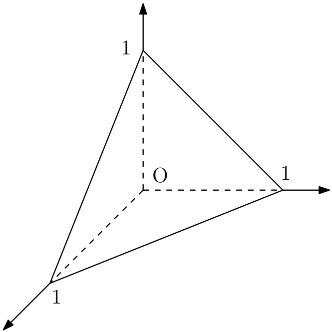
Standard d-simplex

- $\mathcal{V}$-representation

$$\Delta_d := \mathrm{conv}(\{\mathbf{e}^1, \ldots, \mathbf{e}^{d+1}\})$$

- $\mathcal{H}$-representation

$$\Delta_d := \{\mathbf{x} \in \mathbb{R}^{d+1} \mid \mathbf{1}^t\mathbf{x} = 1, \mathbf{x} \geq \mathbf{0}\}$$



Main theorem for convex polytopes: Example 3

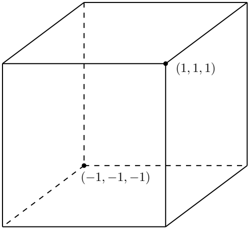
d-cube

- $\mathcal{V}$-representation

$$C_d := \mathrm{conv}(\{+1,-1\}^d)$$

- $\mathcal{H}$-representation

$$C_d := \{\mathbf{x} \in \mathbb{R}^d \mid -1 \le x_i \le 1 \; \forall i = 1,\dots,d\}$$



Contents

Motivation

Setting: Given

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \quad \text{for some } A \in \mathbb{R}^{m \times n} \text{ and } \mathbf{b} \in \mathbb{R}^m$$

- $\mathrm{rank}(A) = m$
- $\mathbf{a}^1, \ldots, \mathbf{a}^n$: columns of A

Theorem

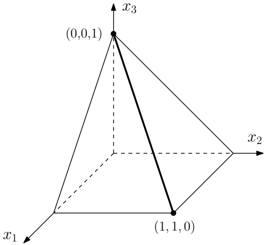
$\mathbf{x} = (x_1, \ldots, x_n)^t$ is an **extreme point** of P if and only if $\{\mathbf{a}^j \mid x_j > 0\}$ are **linearly independent**

Example: $P = \{\mathbf{x} \in \mathbb{R}^3 \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$

with $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\mathbf{x}^* = (1,1,0)^t$ we have

$$\{\mathbf{a}^j \mid x_j^* > 0\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$



Feasible basic solution: case of canonical form

Setting: Given

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \quad \text{for some } A \in \mathbb{R}^{m \times n} \text{ and } \mathbf{b} \in \mathbb{R}^m$$

- $\mathrm{rank}(A) = m$
- $A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad I_m \mathbf{x}_B + \overline{A}\mathbf{x}_N = \overline{b} \quad$ with $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$
- $\mathbf{x}^* = (x_1, \ldots, x_n)$ has $\mathbf{x}_N^* = \mathbf{0}$

Definition

- Such $\mathbf{x}^*$ is called a *basic solution* of P
- If $\mathbf{x}_B^* \geq 0$, then $\mathbf{x}^* = (\mathbf{x}_B^*, \mathbf{x}_N^*)$ is called a *feasible basic solution* of P
- If $\mathbf{x}_B^* > 0$, then $\mathbf{x}^* = (\mathbf{x}_B^*, \mathbf{x}_N^*)$ is called a *non-degenerate feasible basic solution* of P

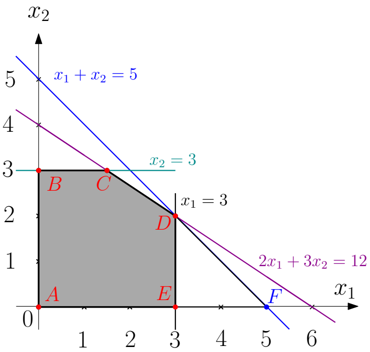
Example

Given $P = \{\mathbf{x} \in \mathbb{R}^2_+ \mid C\mathbf{x} \le \mathbf{d}\}$ with

$$C = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 5 \\ 12 \\ 3 \\ 3 \end{pmatrix}$$

Explicit form:

$$\begin{aligned} x_1 + x_2 &\le 5 \\ 2x_1 + 3x_2 &\le 12 \\ x_1 &\le 3 \\ x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$



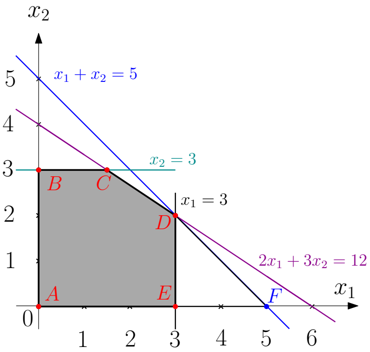
Example (cont.)

Given $P = \{\mathbf{x} \in \mathbb{R}^2_+ \mid C\mathbf{x} \leq \mathbf{d}\}$ with

$$C = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 5 \\ 12 \\ 3 \\ 3 \end{pmatrix}$$

Canonical form:

$$\begin{aligned}
x_1 + x_2 + x_3 \qquad\qquad &= 5 \\
2x_1 + 3x_2 \quad + x_4 \qquad &= 12 \\
x_1 \qquad\qquad + x_5 \quad &= 3 \\
x_2 \qquad\qquad\quad + x_6 &= 3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}$$



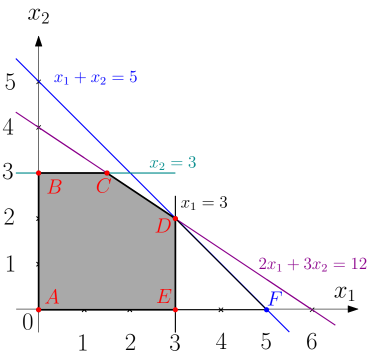
Example (cont.)

$\{\mathbf{x} \in \mathbb{R}^6 \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ with

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 3 \\ 3 \end{pmatrix}$$

Canonical form:

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 5 \\ 2x_1 + 3x_2 + x_4 & = & 12 \\ x_1 + x_5 & = & 3 \\ x_2 + x_6 & = & 3 \\ x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$

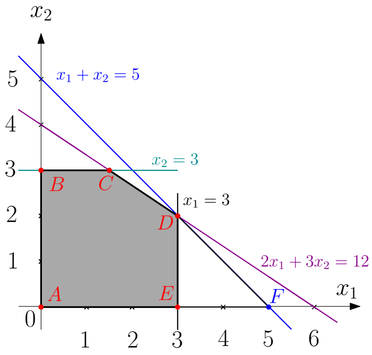


Example (cont.)

Canonical form:

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 5 \\ 2x_1 + 3x_2 \quad + x_4 \quad &= 12 \\ x_1 \quad + x_5 \quad &= 3 \\ x_2 \quad + x_6 &= 3 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

- Vertex A:
  $\mathbf{x}_N = (x_1, x_2) = (0, 0)$,
  $\mathbf{x}_B = (x_3, x_4, x_5, x_6) = (5, 12, 3, 3)$,
  $\mathbf{x} = (0, 0, 5, 12, 3, 3)$ is a FBS (non-degenerate)
- Vertex B:
  $\mathbf{x}_N = (x_1, x_6) = (0, 0)$,
  $\mathbf{x}_B = (x_2, x_3, x_4, x_5) = (3, 2, 5, 3)$,
  $\mathbf{x} = (0, 3, 2, 5, 3, 0)$ is a FBS (non-degenerate)



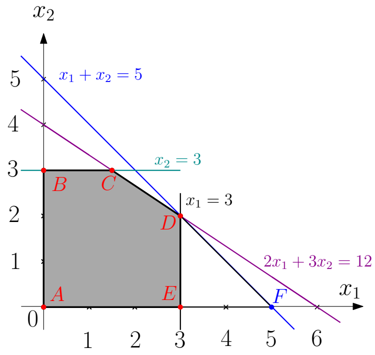
Example (cont.)

Canonical form:

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 5 \\
2x_1 + 3x_2 + x_4 &= 12 \\
x_1 + x_5 &= 3 \\
x_2 + x_6 &= 3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

- Vertex C:
  $\mathbf{x}_N = (x_4, x_6) = (0, 0)$,
  $\mathbf{x}_B = (x_1, x_2, x_3, x_5)$
  $= (1.5, 3, 0.5, 1.5)$,
  $\mathbf{x} = (1.5, 3, 0.5, 0, 1.5, 0)$ is a FBS (non-degenerate)

- Vertex E:
  $\mathbf{x}_N = (x_2, x_5) = (0, 0)$,
  $\mathbf{x}_B = (x_1, x_3, x_4, x_6) = (3, 2, 6, 2)$,
  $\mathbf{x} = (3, 0, 2, 6, 0, 2)$ is a FBS (non-degenerate)

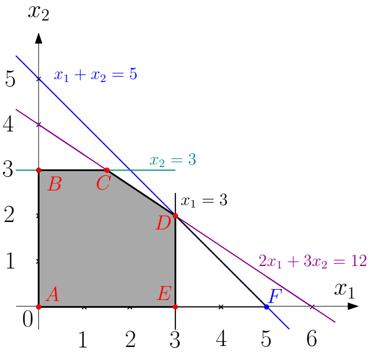


Example (cont.)

Canonical form:

$$
\begin{array}{rcrcrcrcrcrcl}
x_1 & + & x_2 & + & x_3 & & & & & & & = & 5 \\
2x_1 & + & 3x_2 & & & + & x_4 & & & & & = & 12 \\
x_1 & & & & & & & + & x_5 & & & = & 3 \\
 & & x_2 & & & & & & & + & x_6 & = & 3 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5, & x_6 & \geq & 0
\end{array}
$$

- Vertex D:
  Option 1:
  $\mathbf{x}_N = (x_3, x_4) = (0, 0)$,
  $\mathbf{x}_B = (x_1, x_2, x_5, x_6) = (3, 2, 0, 1)$
  Option 2:
  $\mathbf{x}_N = (x_4, x_5) = (0, 0)$,
  $\mathbf{x}_B = (x_1, x_2, x_3, x_6) = (3, 2, 0, 1)$
  Option 3:
  $\mathbf{x}_N = (x_3, x_5) = (0, 0)$,
  $\mathbf{x}_B = (x_1, x_2, x_4, x_6) = (3, 2, 0, 1)$
  Conclusion:
  $\mathbf{x} = (3, 2, 0, 0, 0, 1)$ is a FBS (degenerate)

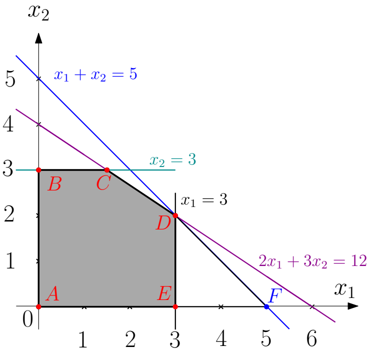


Example (cont.)

Canonical form:

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 5 \\ 2x_1 + 3x_2 + x_4 & = & 12 \\ x_1 + x_5 & = & 3 \\ x_2 + x_6 & = & 3 \\ x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$

- Vertex F:
  $\mathbf{x}_N = (x_2, x_3) = (0, 0)$,
  $\mathbf{x}_B = (x_1, x_4, x_5, x_6)$
  $= (5, 2, -2, -3)$,
  $\mathbf{x} = (5, 0, 0, 2, -2, -3)$ is a BS
  (non-feasible)



Feasible basic solution vs. extreme point

Setting: Given

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \quad \text{for some } A \in \mathbb{R}^{m\times n} \text{ and } \mathbf{b} \in \mathbb{R}^m$$

- $\mathrm{rank}(A) = m$

Theorem

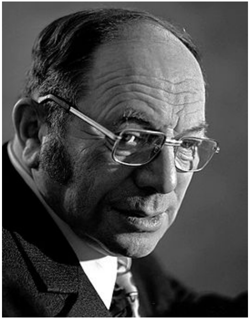
- Each feasible basic solution of P corresponds to an extreme point of P.
- Each extreme point of P corresponds to one or more basic feasible solutions of P.
- If $\mathbf{x}$ is a non-degenerate basic feasible solution of P, then the extreme point of P corresponding to $\mathbf{x}$ exists uniquely.

Contents

History

- 1826/1827 Fourier:
  rudimentory form of simplex method in 3 dimensions
- 1939 Kantorovich: foundations of linear programming

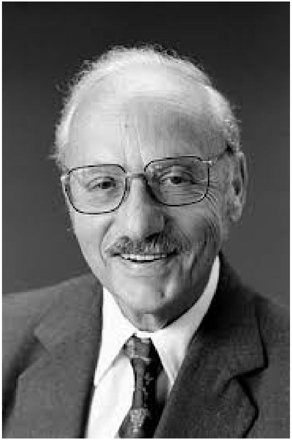
J. B. J. Fourier (1786-1830)



L. V. Kantorovich (1912-1986)
Nobel Memorial Prize
in Economic Sciences in 1975

History

- 1947 G. B. Dantzig: primal simplex algorithm
- 1954 C. E. Lemke: dual simplex algorithm



G. B. Dantzig (1914-2005)

History

- 1979 L. G. Khachiyan: ellipsoid method
- 1984 N. Karmarkar: interior point method

L. G. Khachiyan (1952-2005)

N. Karmarkar (1957)

List of algorithms for solving LPs

- Fourier-Motzkin elimination
- Primal simplex method
- Dual simplex method
- Ellipsoid method
- Interior point / barrier methods
- Lagrangian relaxation

Specialities of simplex method

- A **combinatorial method**
  to solve LPs (**continuous** problems)
- An exact method
  (i.e., find optimal solution exactly, not approximately)
- Very efficient in numerical practice
- Top 10 algorithms of 20th century

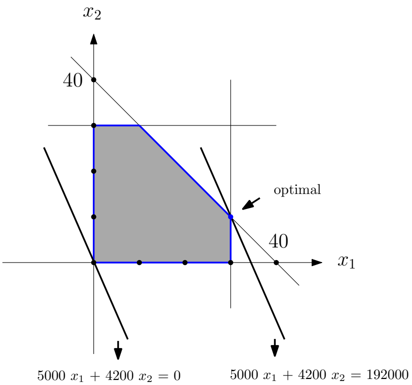
Contents

Example 1

$$
\begin{array}{rl}
\max & 5000x_1 + 4200x_2 \\
\text{s.t.} & x_1 + x_2 \leq 40 \\
& 200x_1 \leq 6000 \\
& 140x_2 \leq 4200 \\
& x_1, x_2 \geq 0
\end{array}
$$

For LPs with 2 variables:

- Draw (nonempty) feasible set
- Draw an objective level line
- Move parallelly level line
  - in direction w.r.t. objective of max or min
  - while crossing feasible set
- Stop if cannot move anymore

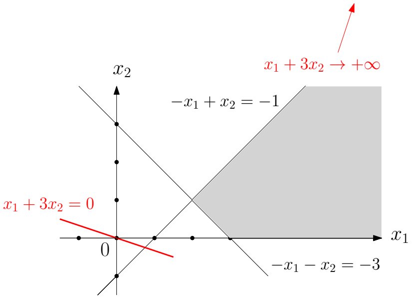


Example 2

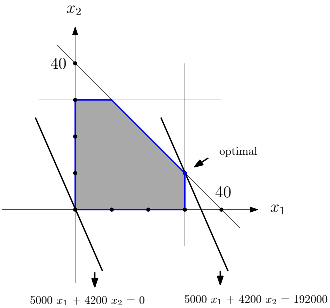
$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & -x_1 - x_2 \leq -3 \\ & -x_1 + x_2 \leq -1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

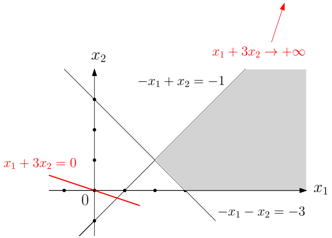
For LPs with 2 variables:

- Draw (nonempty) feasible set
- Draw an objective level line
- Move parallelly level line
  - in direction w.r.t. objective of max or min
  - while crossing feasible set
- Stop if cannot move anymore



Intuitions





- Feasible set (if nonempty) is convex polyhedral
- Feasible set (if nonempty) may be bounded or unbounded
- Optimal value may be finite or infinite
- Feasible set (if nonempty) has finite number of extreme points
- Optimal value (if finite) is achieved at extreme point of feasible set

Remark: These intuitions also hold for LPs in general $\mathbb{R}^n$

Feasibility

Proposition

A linear program is either *feasible* or *infeasible*, but not both

- **Feasible LP**: feasible set is *non-empty*

$$\begin{aligned} \max \quad & x_1 - x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \leq -1 \\ & -x_1 - 2x_2 \leq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- **Infeasible LP**: feasible set is *empty*

$$\begin{aligned} \max \quad & 3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & -2x_1 - 2x_2 \leq -10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

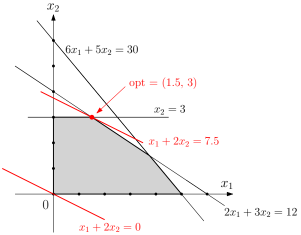
Optimality

Proposition

A feasible linear program is either *bounded* or *unbounded*, but not both

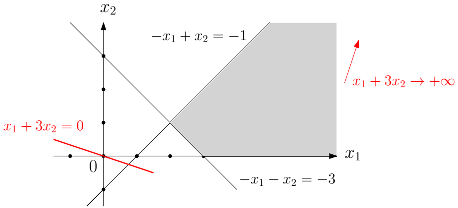
- **Bounded LP**:
  objective value is *finite*

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \le 12 \\ & 6x_1 + 5x_2 \le 30 \\ & x_2 \le 3,\ x_1, x_2 \ge 0 \end{aligned}$$



- **Unbounded LP**:
  objective value *can tend to* ∞

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & -x_1 - x_2 \le -3 \\ & -x_1 + x_2 \le -1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



Optimality

Theorem

The feasible set of a feasible linear program is convex polyhedral

Proof. Cf. Minkowski-Weyl theorem

Theorem

If the feasible set of a linear program is a convex polytope,
then the linear program is bounded

Proof. Follow from the facts:

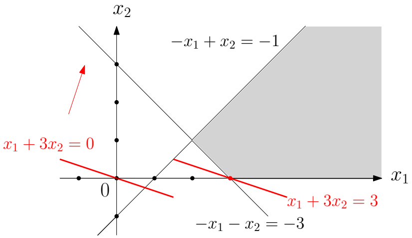
- Convex polytope is compact
- Objective function of linear program is continuous
- Continuous function on compact domain achieves finite optimum (Weierstrass theorem)

Optimality

Remark 1:
A linear program with unbounded feasible set may be bounded (i.e. having finite optimal value)

Example:

$$\begin{aligned} \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & -x_1 - x_2 \leq -3 \\ & -x_1 + x_2 \leq -1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

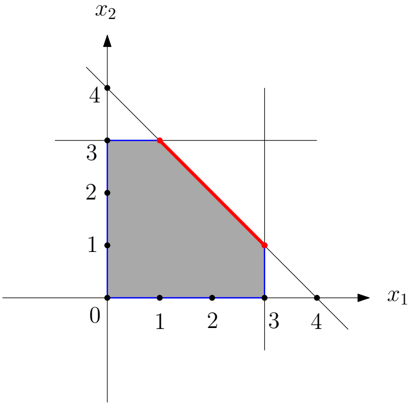


Optimality

Remark 2:
The optimal solution of a bounded linear program may be not unique

Example:

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & x_1 \leq 3 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Optimality

Theorem

The feasible set of a feasible linear program has finite number of extreme points

Theorem

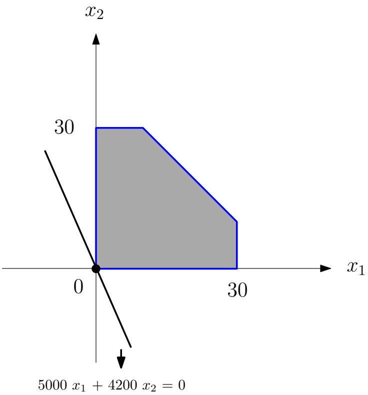
If a linear program is bounded, then it achieves the optimal value at some extreme point of its feasible set

Contents

Simplex algorithm: key ideas

- Principles:
  - Feasible set is convex polyhedral
  - Attain optimal value (if finite) at an extreme point of feasible set

- (Geometric) algorithm idea:
  - Start at some vertex
  - Iteratively move to an adjacent vertex of better objective value
  - Stop if
    - no better vertex found, or
    - a recession ray is visited
  - Finite number of vertices
    $\Rightarrow$ stop after finite iterations



A 2D illustrative example

Simplex algorithm: key ideas

- Principles:
  - Feasible set is convex polyhedral
  - Attain optimal value (if finite) at an extreme point of feasible set
- (Geometric) algorithm idea:
  - Start at some vertex
  - Iteratively move to an adjacent vertex of better objective value
  - Stop if
    - no better vertex found, or
    - a recession ray is visited
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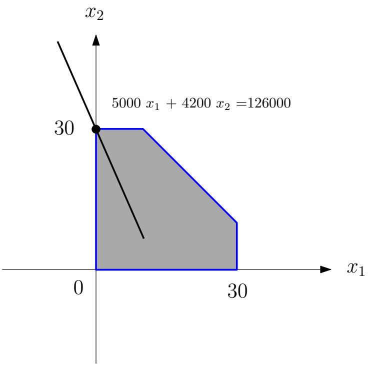
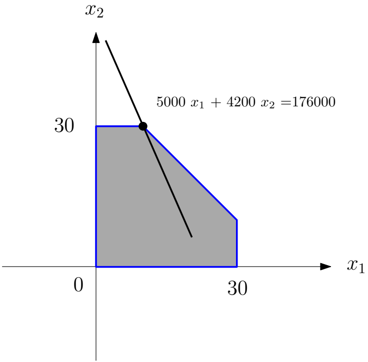
    $\Rightarrow$ stop after finite iterations



A 2D illustrative example

Simplex algorithm: key ideas

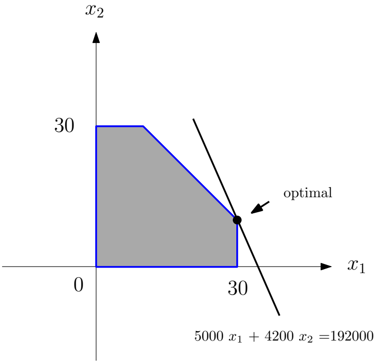
- Principles:
  - Feasible set is convex polyhedral
  - Attain optimal value (if finite) at an extreme point of feasible set
- (Geometric) algorithm idea:
  - Start at some vertex
  - Iteratively move to an adjacent vertex of better objective value
  - Stop if
    - no better vertex found, or
    - a recession ray is visited
  - Finite number of vertices
    $\Rightarrow$ stop after finite iterations



A 2D illustrative example

Simplex algorithm: key ideas

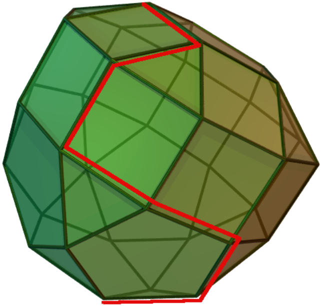
- Principles:
  - Feasible set is convex polyhedral
  - Attain optimal value (if finite) at an extreme point of feasible set
- (Geometric) algorithm idea:
  - Start at some vertex
  - Iteratively move to an adjacent vertex of better objective value
  - Stop if
    - no better vertex found, or
    - a recession ray is visited
  - Finite number of vertices
    $\Rightarrow$ stop after finite iterations



A 2D illustrative example

Simplex algorithm: key ideas

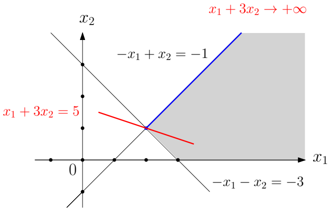
- Principles:
  - Feasible set is convex polyhedral
  - Attain optimal value (if finite) at a vertex
- (Geometric) algorithm idea:
  - Start at some vertex
  - Iteratively move to an adjacent vertex of better objective value
  - Stop if
    - no better vertex found, or
    - a recession ray is visited
  - Finite number of vertices
    $\Rightarrow$ stop after finite iterations



A 3D illustrative example

Simplex algorithm: key ideas

- Principles:
  - Feasible set is convex polyhedral
  - Attain optimal value (if finite) at a vertex
- (Geometric) algorithm idea:
  - Start at some vertex
  - Iteratively move to an adjacent vertex of better objective value
  - Stop if
    - no better vertex found, or
    - a recession ray is visited
  - Finite number of vertices
    $\Rightarrow$ stop after finite iterations



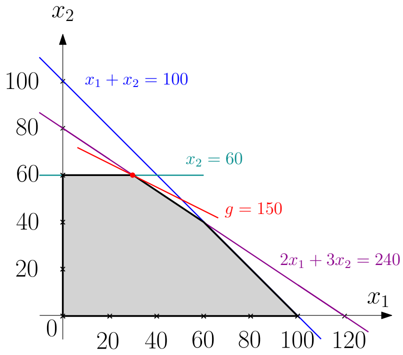
Example of recession ray

Contents

Simplex method via an example

- **Input:**
  an LP in standard form

$$
\begin{aligned}
\max \quad & g = x_1 + 2x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 100 \\
& 2x_1 + 3x_2 \leq 240 \\
& x_2 \leq 60 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



Simplex method via an example

- **Step 1:**
  Add non-negative slack variables to get equality constraints

$$
\begin{array}{rr}
\max & g = x_1 + 2x_2 \\
\text{s.t.} & x_1 + x_2 \le 100 \\
 & 2x_1 + 3x_2 \le 240 \\
 & x_2 \le 60 \\
 & x_1, x_2 \ge 0
\end{array}
\quad \Rightarrow \quad
\begin{array}{llr}
\max & x_1 + 2x_2 & \\
\text{s.t.} & x_1 + x_2 + x_3 & = 100 \\
 & 2x_1 + 3x_2 + x_4 & = 240 \\
 & x_2 + x_5 & = 60 \\
 & x_1, x_2, x_3, x_4, x_5 & \ge 0
\end{array}
$$

Simplex method via an example

- **Step 2:** Find an initial vertex of feasible set
  - **Step 2.1:** Express some variables and objective function in term of the other variables

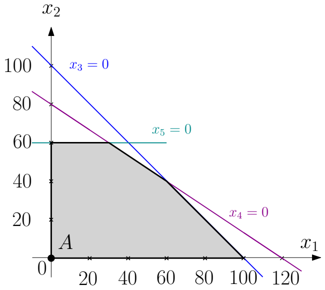
$$
\begin{array}{ll}
\max & g = x_1 + 2x_2 \\
\text{s.t.} & x_3 = 100 - x_1 - x_2 \\
& x_4 = 240 - 2x_1 - 3x_2 \\
& x_5 = 60 \qquad\quad - x_2 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
\quad \Leftarrow \quad
\begin{array}{llrrrrrl}
\max & x_1 + 2x_2 & & & & & & \\
\text{s.t.} & x_1 + & x_2 & + x_3 & & & = & 100 \\
& 2x_1 + & 3x_2 & & + x_4 & & = & 240 \\
& & x_2 & & & + x_5 & = & 60 \\
& & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

- **Related concepts:**
  - **Basic variables:** variables chosen to express (here x_3, x_4, x_5)
  - **Non-basic variables:** the remaining variables (here x_1, x_2)
  - **Dictionary:** formulation expressing basic variables and objective function in term of non-basic variables

Simplex method via an example

- **Step 2:** Find an initial vertex of feasible set
  - **Step 2.2:** Set all non-basic variables to 0, then compute basic variables

$$\begin{aligned} \max \quad & g = x_1 + 2x_2 \\ \text{s.t.} \quad & x_3 = 100 - x_1 - x_2 \\ & x_4 = 240 - 2x_1 - 3x_2 \\ & x_5 = 60 \qquad\quad - x_2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$



- **Related concepts:**
  - The obtained solution is called **basic solution** (here $x_1 = x_2 = 0 \Rightarrow (x_1, x_2, x_3, x_4, x_5) = (0, 0, 100, 240, 60) = A$)
  - **Feasible basic solution:** a basic solution with non-negative values of basic variables

Simplex method via an example

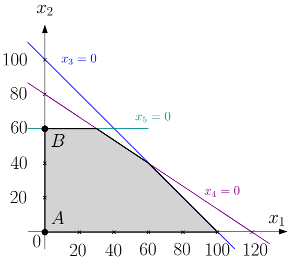
- **Step 3:** Find a better vertex of feasible set
  - **Step 3.1:** Increase a non-basic variable *as far as possible in its range* so that the objective value increases
  - **Step 3.2:** Re-compute values of variables

$$\begin{array}{rl} \max & g = x_1 + 2x_2 \\ \text{s.t.} & x_3 = 100 - x_1 - x_2 \\ & x_4 = 240 - 2x_1 - 3x_2 \\ & x_5 = 60 \qquad\quad - x_2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array} \Rightarrow \begin{array}{c} x_3 \geq 0 \Rightarrow x_2 \leq 100 \\ x_4 \geq 0 \Rightarrow x_2 \leq 80 \\ x_5 \geq 0 \Rightarrow x_2 \leq 60 \\ x_2 \text{ increases } \Rightarrow g \text{ increases} \\ \text{so we increase } x_2 \text{ from 0 to 60 and get} \\ (x_1, x_2, x_3, x_4, x_5) = (0, 60, 40, 60, 0) = B \end{array}$$

- **Related concepts:**
  - **Entering variable:** the non-basic variable chosen to increase (here x_2 is entering variable: it becomes basic in the next step)
  - **Leaving variable:** a new zero-value basic variable (here x_5)

Simplex method via an example

- **Step 3:** Find a better vertex of feasible set
  - **Step 3.1:** Increase a non-basic variable *as far as possible in its range* so that the objective value increases
  - **Step 3.2:** Re-compute values of variables



$\Leftarrow$

$$x_3 \geq 0 \Rightarrow x_2 \leq 100$$
$$x_4 \geq 0 \Rightarrow x_2 \leq 80$$
$$x_5 \geq 0 \Rightarrow x_2 \leq 60$$
$$x_2 \text{ increases } \Rightarrow g \text{ increases}$$

so we increase x_2 from 0 to 60 and get

$$(x_1, x_2, x_3, x_4, x_5) = (0, 60, 40, 60, 0) = B$$

- **Related concepts:**
  - **Entering variable:** the non-basic variable chosen to increase (here x_2 is entering variable: it becomes basic in the next step)
  - **Leaving variable:** a new zero-value basic variable (here x_5)

Simplex method via an example

- **Step 4:** Formulate a new dictionary
  - **Step 4.1:** Reset leaving variable as non-basic variable and reset entering variable as basic variable
  - **Step 4.2:** Express basic variables and objective function in term of (new set of) non-basic variables

Previous dictionary

$$
\begin{aligned}
\text{max} \quad & g = x_1 + 2x_2 \\
\text{s.t.} \quad & x_3 = 100 - x_1 - x_2 \\
& x_4 = 240 - 2x_1 - 3x_2 \\
& x_5 = 60 \qquad\quad - x_2 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

$$\Rightarrow$$

New dictionary

with non-basic variables x_1, x_5

$$
\begin{aligned}
\text{max} \quad & g = x_1 + 2(60 - x_5) \\
\text{s.t.} \quad & x_3 = 100 - x_1 - (60 - x_5) \\
& x_4 = 240 - 2x_1 - 3(60 - x_5) \\
& x_2 = 60 - x_5 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Simplex method via an example

- **Step 4:** Formulate a new dictionary
  - **Step 4.1:** Reset leaving variable as non-basic variable and reset entering variable as basic variable
  - **Step 4.2:** Express basic variables and objective function in term of (new set of) non-basic variables

Previous dictionary

$$\begin{aligned} \max \quad & g = x_1 + 2x_2 \\ \text{s.t.} \quad & x_3 = 100 - x_1 - x_2 \\ & x_4 = 240 - 2x_1 - 3x_2 \\ & x_5 = 60 \qquad\quad - x_2 \\ & \quad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

$$\Rightarrow$$

New dictionary

with non-basic variables x_1, x_5

$$\begin{aligned} \max \quad & g = 120 + x_1 - 2x_5 \\ \text{s.t.} \quad & x_3 = 40 - x_1 + x_5 \\ & x_4 = 60 - 2x_1 + 3x_5 \\ & x_2 = 60 - x_5 \\ & \quad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Simplex method via an example

- Repeat the procedures of Step 3 and Step 4 until
  - objective value cannot increase anymore
    (in this case the current feasible basic solution is optimal)
  - **or** objective value tends to ∞
    (in this case the LP is unbounded)

$$\begin{array}{ll} \max & g = 120 + x_1 - 2x_5 \\ \text{s.t.} & x_3 = 40 - x_1 + x_5 \\ & x_4 = 60 - 2x_1 + 3x_5 \\ & x_2 = 60 - x_5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array} \Rightarrow$$

The only way to increase g is to increase x_1

$$x_5 = 0, x_3 \geq 0 \Rightarrow x_1 \leq 40$$

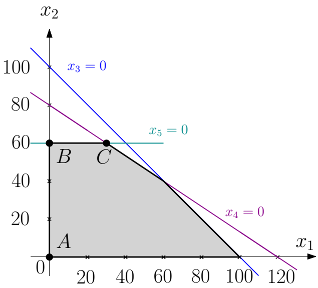
$$x_5 = 0, x_4 \geq 0 \Rightarrow x_1 \leq 30$$

so we increase x_1 from 0 to 30 and get

$$(x_1, x_2, x_3, x_4, x_5) = (30, 60, 10, 0, 0) = C$$

Simplex method via an example

- Repeat the procedures of Step 3 and Step 4 until
  - objective value cannot increase anymore
    (in this case the current feasible basic solution is optimal)
  - **or** objective value tends to ∞
    (in this case the LP is unbounded)



$\Leftarrow$

The only way to increase g is to increase x_1

$$x_5 = 0, x_3 \geq 0 \Rightarrow x_1 \leq 40$$
$$x_5 = 0, x_4 \geq 0 \Rightarrow x_1 \leq 30$$

so we increase x_1 from 0 to 30 and get

$$(x_1, x_2, x_3, x_4, x_5) = (30, 60, 10, 0, 0) = C$$

Simplex method via an example

- Repeat the procedures of Step 3 and Step 4 until
  - objective value cannot increase anymore
    (in this case the current feasible basic solution is optimal)
  - **or** objective value tends to ∞
    (in this case the LP is unbounded)

The only way to increase g is to increase x_1

Entering variable: x_1

Leaving variable: x_4

New set of non-basic variables: x_4, x_5

New set of basic variables: x_1, x_2, x_3

$\Leftarrow$

$$x_5 = 0, x_3 \geq 0 \Rightarrow x_1 \leq 30$$

$$x_5 = 0, x_4 \geq 0 \Rightarrow x_1 \leq 40$$

so we increase x_1 from 0 to 30 and get

$$(x_1, x_2, x_3, x_4, x_5) = (30, 60, 10, 0, 0) = C$$

Simplex method via an example

- Repeat the procedures of Step 3 and Step 4 until
  - objective value cannot increase anymore
    (in this case the current feasible basic solution is optimal)
  - **or** objective value tends to ∞
    (in this case the LP is unbounded)

Previous dictionary

$$
\begin{aligned}
\max \quad & g = 120 + x_1 - 2x_5 \\
\text{s.t.} \quad & x_3 = 40 - x_1 + x_5 \\
& x_4 = 60 - 2x_1 + 3x_5 \\
& x_2 = 60 - x_5 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

$\Rightarrow$

New dictionary
with non-basic variables x_4, x_5

$$
\begin{aligned}
\max \quad & g = 120 + (30 - x_4 + \frac{3}{2}x_5) - 2x_5 \\
\text{s.t.} \quad & x_2 = 60 - x_5 \\
& x_1 = 30 - x_4 + \frac{3}{2}x_5 \\
& x_3 = 40 - (30 - x_4 + \frac{3}{2}x_5) + x_5 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Simplex method via an example

- Repeat the procedures of Step 3 and Step 4 until
  - objective value cannot increase anymore
    (in this case the current feasible basic solution is optimal)
  - **or** objective value tends to ∞
    (in this case the LP is unbounded)

Previous dictionary

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\begin{aligned}
\max \quad & g = 120 + x_1 - 2x_5 \\
\text{s.t.} \quad & x_3 = 40 - x_1 + x_5 \\
& x_4 = 60 - 2x_1 + 3x_5 \\
& x_2 = 60 - x_5 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

$\Rightarrow$

New dictionary
with non-basic variables x_4, x_5

$$
\begin{aligned}
\max \quad & g = 150 - \frac{1}{2}x_4 - \frac{1}{2}x_5 \\
\text{s.t.} \quad & x_2 = 60 - x_5 \\
& x_1 = 30 - \frac{1}{2}x_4 + \frac{3}{2}x_5 \\
& x_3 = 10 + \frac{1}{2}x_4 - \frac{1}{2}x_5 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Simplex method via an example

- Repeat the procedures of Step 3 and Step 4 until
  - objective value cannot increase anymore
    (in this case the current feasible basic solution is optimal)
  - **or** objective value tends to ∞
    (in this case the LP is unbounded)

From current solution

$$(30, 60, 10, 0, 0)$$

we cannot improve g
so this is optimal solution

Conclusion:
Optimal solution:
$(x_1, x_2) = (30, 60)$
Optimal objective value:
$g = 150$

$\Leftarrow$

New dictionary
with non-basic variables x_4, x_5

$$\begin{aligned} \max \quad & g = 150 - \frac{1}{2}x_4 - \frac{1}{2}x_5 \\ \text{s.t.} \quad & x_2 = 60 - x_5 \\ & x_1 = 30 - \frac{1}{2}x_4 + \frac{3}{2}x_5 \\ & x_3 = 10 + \frac{1}{2}x_4 - \frac{1}{2}x_5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Simplex method via an example

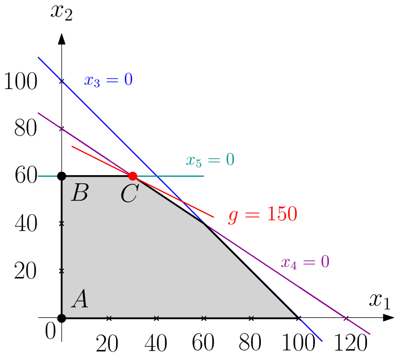
- Repeat the procedures of Step 3 and Step 4 until
  - objective value cannot increase anymore
    (in this case the current feasible basic solution is optimal)
  - **or** objective value tends to ∞
    (in this case the LP is unbounded)

From current solution

$$(30, 60, 10, 0, 0)$$

we cannot improve g
so this is optimal solution

Conclusion:
Optimal solution:
$(x_1, x_2) = (30, 60)$
Optimal objective value:
$g = 150$



Contents

Settings

Consider canonical LPs with minimization objective

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ & \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Shorten form:

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

where $\mathbf{a}^1, \ldots, \mathbf{a}^n$ are column vectors of $A = (a_{ij})_{m \times n}$

Assumption: $\text{rank}(A) = m$

Outline

- Simplex table w.r.t. a given extreme solution
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  - Unboundedness criterion on simplex table
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Basis w.r.t. extreme solution

$$
\begin{array}{ll}
\min & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{s.t.} & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} \\
 & x_1, x_2, \ldots, x_n \geq 0
\end{array}
$$

Recall

$\mathbf{x} = (x_1, \ldots, x_n)^t$ is an **extreme point** of feasible set of the canonical LP if and only if $\{\mathbf{a}^j \mid x_j > 0\}$ are linearly independent

Given: extreme solution $\mathbf{x}^0$

- $J = \{j \mid x_j^0 > 0\}$ is called the *basis* w.r.t. $\mathbf{x}^0$
- $\{\mathbf{a}^j \mid j \in J\}$ are linearly independent, called *basic vectors* w.r.t. $\mathbf{x}^0$

Representations via a basis

$$x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}$$
$$x_1, x_2, \ldots, x_n \geq 0$$

Given: extreme solution $\mathbf{x}^0$ with basis J, arbitrary feasible solution $\mathbf{x}$

Representations via basis J:

- Non-basic vector via basic vectors

$$\mathbf{a}^k = \sum_{j\in J} z_{jk}\mathbf{a}^j \qquad (\forall k \notin J) \tag{1}$$

- $\mathbf{b}$ via $\mathbf{x}^0$ and basic vectors (note that $x_j^0 = 0$ for all $j \notin J$)

$$\mathbf{b} = \sum_{j=1}^{n} x_j^0\mathbf{a}^j = \sum_{j\in J} x_j^0\mathbf{a}^j \tag{2}$$

- $\mathbf{b}$ via $\mathbf{x}$ and basic vectors

$$\mathbf{b} = \sum_{j=1}^{n} x_j\mathbf{a}^j = \sum_{j\in J} x_j\mathbf{a}^j + \sum_{k\notin J} x_k\mathbf{a}^k \stackrel{(1)}{=} \sum_{j\in J} x_j\mathbf{a}^j + \sum_{k\notin J} x_k \sum_{j\in J} z_{jk}\mathbf{a}^j = \sum_{j\in J}\left(x_j + \sum_{k\notin J} z_{jk}x_k\right)\mathbf{a}^j$$

Together with (2) and linear independence of basic vectors, we get

$$\mathbf{x}_j^0 = x_j + \sum_{k\notin J} z_{jk}x_k \quad \Leftrightarrow \quad \mathbf{x}_j = x_j^0 - \sum_{k\notin J} z_{jk}x_k \qquad (\forall j \in J) \tag{3}$$

Representations via a basis (cont.)

$$\begin{aligned} \min \quad & f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Given: extreme solution $\mathbf{x}^0$ with basis J, arbitrary feasible solution $\mathbf{x}$

Representations via basis J:

- Basic vector $\mathbf{a}^\ell$ ($\ell \in J$) via basic vectors

$$\mathbf{a}^\ell = \sum_{j\in J} z_{j\ell}\mathbf{a}^j \quad \text{with } z_{j\ell} = 1 \text{ if } j = \ell \text{ and } z_{j\ell} = 0 \text{ if } j \neq \ell \tag{4}$$

- Correlation between $f(\mathbf{x})$ and $f(\mathbf{x}^0)$

$$\begin{aligned} f(\mathbf{x}) &= \sum_{j=1}^{n} c_jx_j = \sum_{j\in J} c_jx_j + \sum_{k\notin J} c_kx_k \overset{(3)}{=} \sum_{j\in J}\left(x_j^0 - \sum_{k\notin J} z_{jk}x_k\right)c_j + \sum_{k\notin J} c_kx_k \\ &= \sum_{j\in J} c_jx_j^0 - \sum_{k\notin J}\left(\sum_{j\in J} z_{jk}c_j - c_k\right)x_k = f(\mathbf{x}^0) - \sum_{k\notin J}\Delta_kx_k \end{aligned}$$

in which

$$\Delta_k = \sum_{j\in J} z_{jk}c_j - c_k \quad \forall k \notin J \tag{5}$$

Simplex table w.r.t. extreme solution

$$\begin{array}{lrl} \min & f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ \text{s.t.} & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} & \\ & x_1, x_2, \ldots, x_n \geq 0 & \end{array}$$

Given: extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$

Simplex table w.r.t. $\mathbf{x}^0$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_k	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 k}$	$\ldots$	$z_{j_1 n}$
j_2	c_{j_2}	$x_{j_2}^0$	$z_{j_2 1}$	$z_{j_2 2}$	$\ldots$	$z_{j_2 k}$	$\ldots$	$z_{j_2 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m k}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_k	$\ldots$	Δ_n

- For $j \in J$ and $k \notin J$, z_{jk} is determined by (1): $\mathbf{a}^k = \sum_{j\in J} z_{jk}\mathbf{a}^j$
- For $j \in J$ and $\ell \in J$, $z_{j\ell}$ is determined by (4):

$$z_{j\ell} = 1 \text{ if } j = \ell \quad \text{and} \quad z_{j\ell} = 0 \text{ if } j \neq \ell$$

- $f(\mathbf{x}^0)$ is computed by $f(\mathbf{x}^0) = J^t\mathbf{x}_J^0 = c_{j_1}x_{j_1}^0 + c_{j_2}x_{j_2}^0 + \ldots + c_{j_m}x_{j_m}^0$
- For $j \in J$: $\Delta_j = 0$
- For $k \notin J$, Δ_k is determined by (5): $\Delta_k = \sum_{j\in J} z_{jk}c_j - c_k$

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Optimality criterion

$$\begin{aligned} \min \quad & f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Given: extreme solution $\mathbf{x}^0$ with basis J

For any feasible solution $\mathbf{x} \geq \mathbf{0}$:

- $f(\mathbf{x}) = f(\mathbf{x}^0) - \sum_{k \notin J} \Delta_k x_k$ with $\Delta_k = \sum_{j \in J} z_{jk} c_j - c_k$
- $\Delta_k \leq 0 \ \forall k \notin J \implies f(\mathbf{x}) \geq f(\mathbf{x}^0)$

Optimality criterion

If $\Delta_k \leq 0$ for all $k \notin J$, then $\mathbf{x}^0$ is an optimal solution

Remark:

- If $\mathbf{x}^0$ is not an optimal solution, then $\exists k \notin J$ such that $\Delta_k > 0$

Optimality criterion on simplex table

$$\begin{aligned} \min \quad & f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Given: extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$

Simplex table w.r.t. $\mathbf{x}^0$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_k	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 k}$	$\ldots$	$z_{j_1 n}$
j_2	c_{j_2}	$x_{j_2}^0$	$z_{j_2 1}$	$z_{j_2 2}$	$\ldots$	$z_{j_2 k}$	$\ldots$	$z_{j_2 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m k}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_k	$\ldots$	Δ_n

Optimality criterion: If all $\Delta_1, \Delta_2, \ldots, \Delta_n \leq 0$, then $\mathbf{x}^0$ is optimal

Outline

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Change of basis

$$\begin{aligned} \min \quad & f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Given: extreme solution $\mathbf{x}^0$ that is not optimal

Goal: construct extreme solution $\mathbf{x}^1$ of better objective value

Construction idea: replace one element in basis of $\mathbf{x}^0$

Construction steps:

- Let J be basis of $\mathbf{x}^0$
- Since $\mathbf{x}^0$ is extreme but not optimal, there exists $k \notin J$ such that $\Delta_k > 0$
- Choose such a non-basic index to enter basis
  - Often take $\Delta_s = \max\{\Delta_k \mid k \notin J, \Delta_k > 0\}$
- Choose some basic index $r \in J$ to leave basis (determine later)
- Basis of $\mathbf{x}^1$ will be $J^1 = (J \backslash \{r\}) \cup \{s\}$
- Components of $\mathbf{x}^1$:
  - Non-basic components: $x_k^1 = 0$ for all $k \notin J^1$
  - New basic component: $x_s^1 = \theta$ with $\theta > 0$ determined later
  - Old basic components satisfy (3):
    $\mathbf{x}_j^1 = x_j^0 - \sum_{k \notin J} z_{jk} x_k^1 = x_j^0 - \theta z_{js} \quad \forall j \in J \backslash \{r\}$

Change of basis (cont.)

- $\mathbf{x}^1$ has form

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin J, j \neq s \\ \theta & \text{if } j = s \\ x_j^0 - \theta z_{js} & \text{if } j \in J \end{cases} \quad \text{with} \quad \begin{cases} s \notin J : \Delta_s > 0 \\ x_j^0 \geq 0 \ (j \in J) \text{ are given} \\ \theta > 0 \text{ to be determined} \end{cases}$$

- $\mathbf{x}^1$ needs to be feasible (i.e., $A\mathbf{x}^1 = \mathbf{b}$ and $\mathbf{x}^1 \geq \mathbf{0}$)
  - $\sum_{j=1}^n x_j^1 \mathbf{a}^j = \sum_{j \in J} x_j^1 \mathbf{a}^j + \sum_{j \notin J} x_j^1 \mathbf{a}^j = \sum_{j \in J} \left(x_j^0 - \theta z_{js} \right) \mathbf{a}^j + \theta \mathbf{a}^s$
    $= \sum_{j \in J} x_j^0 - \theta \sum_{j \in J} z_{js} \mathbf{a}^j + \theta \mathbf{a}^s = \mathbf{b} - \theta \mathbf{a}^s + \theta \mathbf{a}^s = \mathbf{b}$
    hence $A\mathbf{x}^1 = \mathbf{b}$ holds for any choice of θ
  - In case $z_{js} \leq 0$ for all $j \in J$:
    - $x_j^1 = x_j^0 - \theta z_{js} \geq 0 \ \forall j \in J$, so $\mathbf{x}^1 \geq \mathbf{0}$ and hence $\mathbf{x}^1$ is feasible
    - $f(\mathbf{x}^1) = f(\mathbf{x}^0) - \sum_{k \notin J} \Delta_k x_k^1 = f(\mathbf{x}^0) - \Delta_s \theta \to -\infty$ as $\theta \to +\infty$, hence the LP is unbounded
  - In case $\exists\, j \in J$ such that $z_{js} > 0$:
    - $x_j^1 = x_j^0 - \theta z_{js} \geq 0 \ \forall j \in J \quad \Longrightarrow \quad \theta \leq \min \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\}$
    - Take $r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\}$ and $\theta = \frac{x_r^0}{z_{rs}}$
    - $f(\mathbf{x}^1) = f(\mathbf{x}^0) - \Delta_s \frac{x_r^0}{z_{rs}}$

Change of basis (cont.)

- Formula of $\mathbf{x}^1$:

$$(*) \quad x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \quad \text{with} \quad \begin{cases} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{cases}$$

- **Claim:** $\mathbf{x}^1$ is an extreme solution with basis $J^1 = (J\backslash\{r\}) \cup \{s\}$
  - $\{\mathbf{a}^j \mid j \in J^1\}$ are linearly independent:
    - Consider expression

$$\mathbf{0} = \sum_{j \in J^1} \alpha_j \mathbf{a}^j = \alpha_s \mathbf{a}^s + \sum_{j \in J, j \neq r} \alpha_j \mathbf{a}^j \overset{(1)}{=} \alpha_s \sum_{j \in J} z_{js} \mathbf{a}^j + \sum_{j \in J, j \neq r} \alpha_j \mathbf{a}^j$$

$$= \alpha_s z_{rs} \mathbf{a}^r + \sum_{j \in J, j \neq r} (\alpha_j + \alpha_s z_{js}) \, \mathbf{a}^j$$

    - By linear independence of $\{\mathbf{a}^j \mid j \in J\}$ (basic vectors of $\mathbf{x}^0$):

$$\begin{cases} \alpha_s z_{rs} & = 0 \\ \alpha_j + \alpha_s z_{js} & = 0 \quad \forall j \in J\backslash\{r\} \end{cases} \overset{z_{rs} > 0}{\Longrightarrow} \begin{cases} \alpha_s & = 0 \\ \alpha_j & = 0 \quad \forall j \in J\backslash\{r\} \end{cases} \Leftrightarrow \quad \alpha_j = 0 \; \forall j \in J^1$$

  - By (*): $\mathbf{x}^1 \geq \mathbf{0}$ and $x_j^1 = 0 \; \forall j \notin J^1$, hence $\{j \mid x_j^1 > 0\} \subset J^1$
    so $\{\mathbf{a}^j \mid x_j^1 > 0\} \subset \{\mathbf{a}^j \mid j \in J^1\}$ are linearly independent

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Unboundedness criterion on simplex table

$$\begin{aligned} \min \quad & f(\mathbf{x}) := c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\ \text{s.t.} \quad & x_1 \mathbf{a}^1 + \ldots + x_n \mathbf{a}^n = \mathbf{b} \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Given: extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$

Unboundedness criterion on simplex table w.r.t. $\mathbf{x}^0$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 k} \leq 0$	$\ldots$	$z_{j_1 n}$
j_2	c_{j_2}	$x_{j_2}^0$	$z_{j_2 1}$	$z_{j_2 2}$	$\ldots$	$z_{j_2 s} \leq 0$	$\ldots$	$z_{j_2 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s} \leq 0$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	$\Delta_s > 0$	$\ldots$	Δ_n

If $\exists s \notin J$ such that $\Delta_s > 0$ and $z_{js} \leq 0 \ \forall j \in J$, then the LP is unbounded

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Recap

$$\begin{aligned} \min \quad & f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b} \\ & x_1, x_2, \ldots, x_n \geq 0 \end{aligned}$$

Given:

- Extreme (but not optimal) solution $\mathbf{x}^0$ with basis J
- Representations via basis J:
  - For $j \in J$ and $k \notin J$: $\mathbf{a}^k = \sum_{j\in J} z_{jk}\mathbf{a}^j$
  - For $j \in J$ and $\ell \in J$: $z_{j\ell} = 1$ if $j = \ell$ and $z_{j\ell} = 0$ if $j \neq \ell$
  - $f(\mathbf{x}^0) = J^t\mathbf{x}^0_J = c_{j_1}x^0_{j_1} + c_{j_2}x^0_{j_2} + \ldots + c_{j_m}x^0_{j_m}$
  - $\Delta_j = 0 \quad \forall j \in J$ and $\Delta_k = \sum_{j\in J} z_{jk}c_j - c_k \quad \forall k \notin J$

Computed:

- new extreme solution $\mathbf{x}^1$ with basis $J^1 = (J\setminus\{r\}) \cup \{s\}$
  - $s \in \{k \notin J \mid \Delta_k > 0\}$ (often take $\Delta_s = \max\{\Delta_k \mid k \notin J, \Delta_k > 0\}$)
  - $r \in \operatorname{argmin}\left\{\frac{x^0_j}{z_{js}} \mid j \in J : z_{js} > 0\right\}$
  - $x^1_s = \frac{x^0_r}{z_{rs}}, \quad x^1_j = x^0_j - \frac{x^0_r}{z_{rs}}z_{js} \quad \forall j \in J\setminus\{r\}, \quad x^1_j = 0 \quad \forall j \notin J^1$

Aim: Representations via new basis J^1

Representations via new basis

- For z-coefficients:

$$\mathbf{a}^k = \sum_{j \in J} z_{jk} \mathbf{a}^j = \sum_{j \in J^1} z^1_{jk} \mathbf{a}^j \qquad (6)$$

  - First observation:

$$\mathbf{a}^s = \sum_{j \in J} z_{js} \mathbf{a}^s = z_{rs} \mathbf{a}^r + \sum_{j \in J \setminus \{r\}} z_{js} \mathbf{a}^j \quad \overset{z_{rs} > 0}{\Longrightarrow} \quad \mathbf{a}^r = \frac{1}{z_{rs}} \left(\mathbf{a}^s - \sum_{j \in J \setminus \{r\}} z_{js} \mathbf{a}^j \right) \qquad (7)$$

  - Second observation:

$$\begin{aligned} \sum_{j \in J} z_{jk} \mathbf{a}^j &= z_{rk} \mathbf{a}^r + \sum_{j \in J \setminus \{r\}} z_{jk} \mathbf{a}^j \overset{(7)}{=} \frac{z_{rk}}{z_{rs}} \left(\mathbf{a}^s - \sum_{j \in J \setminus \{r\}} z_{js} \mathbf{a}^j \right) + \sum_{j \in J \setminus \{r\}} z_{jk} \mathbf{a}^j \\ &= \frac{z_{rk}}{z_{rs}} \mathbf{a}^s + \sum_{j \in J \setminus \{r\}} \left(z_{jk} - \frac{z_{rk}}{z_{rs}} z_{js} \right) \mathbf{a}^j \qquad (8) \end{aligned}$$

  - (6) & (7) & (8) & linear independence of basic vectors $\{\mathbf{a}^j \mid j \in J^1\}$:

$$z^1_{sk} = \frac{z_{rk}}{z_{rs}} \quad \text{and} \quad z^1_{jk} = z_{jk} - \frac{z_{rk}}{z_{rs}} z_{js} \ \forall j \in J \setminus \{r\} \qquad (9)$$

Representations via new basis (cont.)

- For Δ-parameters:

$$\begin{aligned}
\Delta_k^1 &= \sum_{j \in J^1} z_{jk}^1 c_j - c_k \\
&\overset{(9)}{=} \sum_{j \in J \setminus \{r\}} \left(z_{jk} - \frac{z_{rk}}{z_{rs}} z_{js} \right) c_j + \frac{z_{rk}}{z_{rs}} c_s - c_k \\
&= \sum_{j \in J} \left(z_{jk} - \frac{z_{rk}}{z_{rs}} z_{js} \right) c_j + \frac{z_{rk}}{z_{rs}} c_s - c_k \qquad \left(\text{since } \left(z_{jk} - \frac{z_{rk}}{z_{rs}} z_{js} \right) \Big|_{j=r} = 0\right) \\
&= \sum_{j \in J} z_{jk} c_j - c_k - \frac{z_{rk}}{z_{rs}} \left(\sum_{j \in J} z_{js} c_j - c_s \right) \\
&= \Delta_k - \frac{z_{rk}}{z_{rs}} \Delta_s
\end{aligned}$$

- For objective value:

$$f(\mathbf{x}^1) = f(\mathbf{x}^0) - \Delta_s \frac{x_r^0}{z_{rs}}$$

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \quad \text{s.t.} \quad x_1 \mathbf{a}^1 + \ldots + x_n \mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \setminus \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J \setminus \{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J \setminus \{r\} \end{cases} \quad \text{with} \quad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	$\ldots$	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$\ldots$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$\ldots$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	$\ldots$	$\ldots$	Δ_n

Step 0: Start with simplex table w.r.t. extreme solution $\mathbf{x}^0$

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \quad \text{with} \quad \begin{cases} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin}\left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{cases}$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n

Step 1: Look for some $\Delta_s > 0$ (often take $\Delta_s = \max\{\Delta_k \mid k \neq J, \Delta_k > 0\}$)

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \quad \text{with} \quad \begin{cases} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin}\left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{cases}$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	c_j	x_j^0	z_{j1}	z_{j2}	$\ldots$	$z_{js} > 0$	$\ldots$	z_{jn}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n

Step 2: On so-called *pivot column* of Δ_s, focus on all positive elements

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \quad \text{with} \quad \begin{cases} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin}\left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{cases}$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	c_j	x_j^0	z_{j1}	z_{j2}	$\ldots$	$z_{js} > 0$	$\ldots$	z_{jn}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n

Step 3: Compute quotients $\frac{x_j^0}{z_{js}}$ corresponding to such positive elements

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \setminus \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\setminus\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\setminus\{r\} \end{cases} \quad \text{with} \quad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin}\left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	c_r	x_r^0	z_{r1}	z_{r2}	$\ldots$	z_{rs}	$\ldots$	z_{rn}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n

Step 4: The minimum of computed quotients attains at row $r \in J$ (called *pivot row*)

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \setminus \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\setminus\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\setminus\{r\} \end{cases} \quad \text{with} \quad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J	J	$\mathbf{x}_J^0$	c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	c_r	x_r^0	z_{r1}	z_{r2}	$\ldots$	z_{rs}	$\ldots$	z_{rn}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n

Step 5: The element z_{rs} is called *pivot element*

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \setminus \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\setminus\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\setminus\{r\} \end{cases} \quad \text{with} \quad \begin{cases} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{cases}$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J	J		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	c_r	x_r^0/z_{rs}	z_{r1}/z_{rs}	z_{r2}/z_{rs}	$\ldots$	z_{rs}/z_{rs}	$\ldots$	z_{rn}/z_{rs}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n

Step 6(i): Divide elements on pivot row by pivot element and get *normalized row*

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \quad \text{with} \quad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J^1	J^1		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	c_s	x_s^1	z_{s1}^1	z_{s2}^1	$\ldots$	1	$\ldots$	z_{sn}^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n

Step 6(ii): Replace index r by s, the obtained quotients are $x_s^1 = \frac{x_r^0}{z_{rs}}$ and $z_{sk}^1 = \frac{z_{rk}}{z_{rs}}$

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

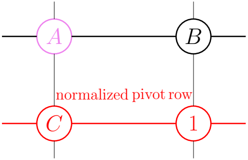
Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \setminus \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\setminus\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\setminus\{r\} \end{cases} \quad \text{with} \quad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin}\left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J^1	J^1		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^0$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	c_s	x_s^1	z_{s1}^1	z_{s2}^1	$\ldots$	1	$\ldots$	z_{sn}^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^0$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n



Replace A by $A - BC$

Step 7: Update elements outside pivot row by pivot formula

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

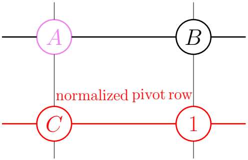
Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \qquad \text{with} \qquad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J^1	J^1		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^1$	$z_{j_1 1}$	$z_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	c_s	x_s^1	z_{s1}^1	z_{s2}^1	$\ldots$	1	$\ldots$	z_{sn}^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^1$	$z_{j_m 1}$	$z_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n



Replace A by $A - BC$

Step 7(i): $x_j^1 = x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} = x_j^0 - x_s^1 z_{js} \quad \forall j \in J\backslash\{r\}$

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

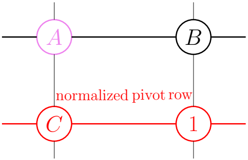
Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x^1_j = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x^0_r}{z_{rs}} & \text{if } j = s \\ x^0_j - \frac{x^0_r}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \qquad \text{with} \qquad \begin{cases} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x^0_j}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{cases}$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J^1	J^1		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x^1_{j_1}$	$z^1_{j_1 1}$	$z^1_{j_1 2}$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z^1_{j_1 n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	c_s	x^1_s	z^1_{s1}	z^1_{s2}	$\ldots$	1	$\ldots$	z^1_{sn}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x^1_{j_m}$	$z^1_{j_m 1}$	$z^1_{j_m 2}$	$\ldots$	$z_{j_m s}$	$\ldots$	$z^1_{j_m n}$
		$f(\mathbf{x}^0)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n



Replace A by $A - BC$

Step 7(ii): $z^1_{jk} = z_{jk} - \frac{z_{rk}}{z_{rs}} z_{js} = z_{jk} - z^1_{sk} z_{js} \quad \forall j \in J\backslash\{r\}, k \neq s$

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

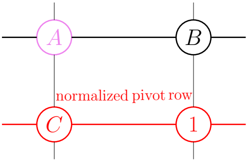
Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \quad \text{with} \quad \begin{cases} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin}\left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{cases}$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J^1	J^1		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^1$	$z_{j_1 1}^1$	$z_{j_1 2}^1$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	c_s	x_s^1	z_{s1}^1	z_{s2}^1	$\ldots$	1	$\ldots$	z_{sn}^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^1$	$z_{j_m 1}^1$	$z_{j_m 2}^1$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}^1$
		$f(\mathbf{x}^1)$	Δ_1	Δ_2	$\ldots$	Δ_s	$\ldots$	Δ_n



Replace A by $A - BC$

Step 7(iii): $f(\mathbf{x}^1) = f(\mathbf{x}^0) - \frac{x_r^0}{z_{rs}}\Delta_s = f(\mathbf{x}^0) - x_s^1\Delta_s$

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

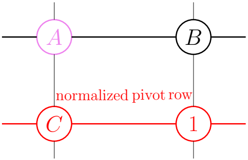
Given:

- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \backslash \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\backslash\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\backslash\{r\} \end{cases} \quad \text{with} \quad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin} \left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J^1	J^1		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^1$	$z_{j_1 1}^1$	$z_{j_1 2}^1$	$\ldots$	$z_{j_1 s}$	$\ldots$	$z_{j_1 n}^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	c_s	x_s^1	z_{s1}^1	z_{s2}^1	$\ldots$	1	$\ldots$	z_{sn}^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^1$	$z_{j_m 1}^1$	$z_{j_m 2}^1$	$\ldots$	$z_{j_m s}$	$\ldots$	$z_{j_m n}^1$
		$f(\mathbf{x}^1)$	Δ_1^1	Δ_2^1	$\ldots$	Δ_s	$\ldots$	Δ_n^1



Replace A by $A - BC$

Step 7(iv): $\Delta_k^1 = \Delta_k - \frac{z_{rk}}{z_{rs}} \Delta_s = \Delta_k - z_{sk}^1 \Delta_s$

Change of simplex table w.r.t. new extreme solution

$$\min \quad f(\mathbf{x}) := c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \text{s.t.} \quad x_1\mathbf{a}^1 + \ldots + x_n\mathbf{a}^n = \mathbf{b}, \quad x_1, x_2, \ldots, x_n \geq 0$$

Given:

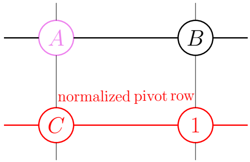
- Simplex table w.r.t. extreme solution $\mathbf{x}^0$ with basis $J = \{j_1, \ldots, j_m\}$
- New extreme solution $\mathbf{x}^1$ with basis $J^1 = (J \setminus \{r\}) \cup \{s\}$, in which

$$x_j^1 = \begin{cases} 0 & \text{if } j \notin (J\setminus\{r\}) \cup \{s\} \\ \frac{x_r^0}{z_{rs}} & \text{if } j = s \\ x_j^0 - \frac{x_r^0}{z_{rs}} z_{js} & \text{if } j \in J\setminus\{r\} \end{cases} \quad \text{with} \quad \left\{ \begin{array}{l} s \notin J : \Delta_s > 0 \\ r \in \operatorname{argmin}\left\{ \frac{x_j^0}{z_{js}} \mid j \in J : z_{js} > 0 \right\} \end{array} \right.$$

Aim: compute simplex table w.r.t. extreme solution $\mathbf{x}^1$

J^1	J^1		c_1	c_2	$\ldots$	c_s	$\ldots$	c_n
j_1	c_{j_1}	$x_{j_1}^1$	$z_{j_1 1}^1$	$z_{j_1 2}^1$	$\ldots$	0	$\ldots$	$z_{j_1 n}^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s	c_s	x_s^1	z_{s1}^1	z_{s2}^1	$\ldots$	1	$\ldots$	z_{sn}^1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j_m	c_{j_m}	$x_{j_m}^1$	$z_{j_m 1}^1$	$z_{j_m 2}^1$	$\ldots$	0	$\ldots$	$z_{j_m n}^1$
		$f(\mathbf{x}^1)$	Δ_1^1	Δ_2^1	$\ldots$	0	$\ldots$	Δ_n^1

Step 7(v): $z_{js} = 0 \; \forall j \in J^1 \setminus \{s\}$



Replace A by $A - BC$

Outline

- Simplex table w.r.t. a given extreme solution
  - Optimality criterion on simplex table
- From a given extreme solution to a new one
  - Unboundedness criterion on simplex table
- Updating simplex table w.r.t. a new extreme solution
- Simplex algorithm with simplex tables

Contents

Infeasible dictionary

Example:

$$
\begin{aligned}
\min \quad & 2x_1 + x_2 \\
\text{s.t.} \quad & -x_1 + x_2 + x_3 = -1 \\
& -x_1 - 2x_2 + x_4 = -2 \\
& x_2 + x_5 = 1 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

As usual:

- x_3, x_4, x_5: basic variables
- x_1, x_2: non-basic variables
- Set non-basic variables to 0 to obtain **initial basic solution**

$$(x_1, x_2, x_3, x_4, x_5) = (0, 0, -1, -2, 1).$$

This is an **infeasible** basic solution!

Principles of two-phase method

Original program:

$$\begin{aligned} \min \quad & t\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Auxiliary program:

$$\begin{aligned} \min \quad & \mathbf{1}^t\mathbf{u} \\ \text{s.t.} \quad & A\mathbf{x} + \mathbf{u} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{u} \geq \mathbf{0} \end{aligned}$$

- Setting: $\mathbf{b} \geq \mathbf{0}$ (multiply both sides of constraint with -1 if needed)
- Observation: Auxiliary program always has $(\mathbf{x}^0, \mathbf{u}^0) = (\mathbf{0}, \mathbf{b})$ as FBS
- Result: *Original program* has a feasible solution $\mathbf{x}^*$ if and only if *auxiliary program* has an optimal solution $(\mathbf{x}^*, \mathbf{0})$
  - *Proof?*

Steps in two-phase method

Original program:

$$\begin{aligned} \min \quad & tx \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Auxiliary program:

$$\begin{aligned} \min \quad & \mathbf{1}^t\mathbf{u} \\ \text{s.t.} \quad & A\mathbf{x} + \mathbf{u} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{u} \geq \mathbf{0} \end{aligned}$$

Two-phase method:

- **Phase I:** solve auxiliary program to optimality.
- Let $(\mathbf{x}^*, \mathbf{u}^*)$ be optimal solution of auxiliary program.
- If $\mathbf{u}^* \neq \mathbf{0}$, then original program is infeasible.
- If $\mathbf{u}^* = \mathbf{0}$, then go to Phase II.
- **Phase II:**
  - If $\mathbf{u}$ is non-basic, then $\mathbf{x}^*$ is initial FBS for original program.
  - Otherwise, eliminate the columns corresponding to basic u_i's, and repeat Phase II.

Two-phase method: Example 1

Solve the LP

$$\min \quad -x_1 + 3x_3 - x_4 \quad \text{s.t.}$$

$$\begin{array}{rcrcrcrcl}
x_1 & & & & x_3 & + & 2x_4 & = & 1 \\
-2x_1 & - & x_2 & - & 4x_3 & + & 2x_4 & = & -2 \\
3x_1 & + & x_2 & + & 3x_3 & & & = & 3 \\
 & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

Step 0: Make right hand side parameters non-negative

$$\min \quad -x_1 + 3x_3 - x_4 \quad \text{s.t.}$$

$$\begin{array}{rcrcrcrcl}
x_1 & & & & x_3 & + & 2x_4 & = & 1 \\
2x_1 & + & x_2 & + & 4x_3 & - & 2x_4 & = & 2 \\
3x_1 & + & x_2 & + & 3x_3 & & & = & 3 \\
 & & & & & & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

Step 1: Formulate auxiliary program

$$\min \quad x_5 + x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rcrcrcrcrcrcrcl}
x_1 & & & & x_3 & + & 2x_4 & + & x_5 & & & & & = & 1 \\
2x_1 & + & x_2 & + & 4x_3 & - & 2x_4 & & & + & x_6 & & & = & 2 \\
3x_1 & + & x_2 & + & 3x_3 & & & & & & & + & x_7 & = & 3 \\
 & & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}$$

Two-phase method: Example 1 - Phase I

$$\min \quad x_5 + x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rcrcrcrcrcrcrcl}
x_1 & & & & x_3 & + & 2x_4 & + & x_5 & & & & & = & 1 \\
2x_1 & + & x_2 & + & 4x_3 & - & 2x_4 & & & + & x_6 & & & = & 2 \\
3x_1 & + & x_2 & + & 3x_3 & & & & & & & + & x_7 & = & 3 \\
 & & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}$$

- Initital feasible basic solution $(0, 0, 0, 0, 1, 2, 3)$ with basis $J = \{5, 6, 7\}$

J	c_J	x_J	0	0	0	0	1	1	1
5	1	1	1	0	-1	2	1	0	0
6	1	2	2	1	4	-2	0	1	0
7	1	3	3	1	3	0	0	0	1
		6	6	2	6	0	0	0	0
1	0	1	1	0	-1	2	1	0	0
6	1	0	0	1	6	-6	-2	1	0
7	1	0	0	1	6	-6	-3	0	1
		0	0	2	12	-12	-5	0	0
1	0	1	1	1/6	0	1	2/3	1/6	0
3	0	0	0	1/6	1	-1	-1/3	1/6	0
7	1	0	0	0	0	0	-1	-1	1
		0	0	0	0	0	-2	-2	0

- Optimal solution (1, 0, 0, 0, 0, 0, 0) with basis $\{1, 3, 7\}$, 7 can be out of basis

Two-phase method: Example 1 - Phase I

$$\min \quad x_5 + x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rcrcrcrcrcrcrcl}
x_1 & & & & x_3 & + & 2x_4 & + & x_5 & & & & & = & 1 \\
2x_1 & + & x_2 & + & 4x_3 & - & 2x_4 & & & + & x_6 & & & = & 2 \\
3x_1 & + & x_2 & + & 3x_3 & & & & & & & + & x_7 & = & 3 \\
 & & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}$$

- Initital feasible basic solution $(0, 0, 0, 0, 1, 2, 3)$ with basis $J = \{5, 6, 7\}$

J	c_J	x_J	0	0	0	0	1	1	1
5	1	1	1	0	-1	2	1	0	0
6	1	2	2	1	4	-2	0	1	0
7	1	3	3	1	3	0	0	0	1
		6	6	2	6	0	0	0	0
1	0	1	1	0	-1	2	1	0	0
6	1	0	0	1	6	-6	-2	1	0
7	1	0	0	1	6	-6	-3	0	1
		0	0	2	12	-12	-5	0	0
1	0	1	1	1/6	0	1	2/3	1/6	0
3	0	0	0	1/6	1	-1	-1/3	1/6	0
7	1	0	0	0	0	0	-1	-1	1
		0	0	0	0	0	-2	-2	0

- Optimal solution (1, 0, 0, 0, 0, 0, 0) with basis $\{1, 3, 7\}$, 7 can be out of basis

Two-phase method: Example 1 - Phase I

$$\min \quad x_5 + x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rcrcrcrcrcrcrcl}
x_1 & & & & x_3 & + & 2x_4 & + & x_5 & & & & & = & 1 \\
2x_1 & + & x_2 & + & 4x_3 & - & 2x_4 & & & + & x_6 & & & = & 2 \\
3x_1 & + & x_2 & + & 3x_3 & & & & & & & + & x_7 & = & 3 \\
& & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}$$

- Initital feasible basic solution $(0, 0, 0, 0, 1, 2, 3)$ with basis $J = \{5, 6, 7\}$

J	c_J	x_J	0	0	0	0	1	1	1
5	1	1	1	0	-1	2	1	0	0
6	1	2	2	1	4	-2	0	1	0
7	1	3	3	1	3	0	0	0	1
		6	6	2	6	0	0	0	0
1	0	1	1	0	-1	2	1	0	0
6	1	0	0	1	6	-6	-2	1	0
7	1	0	0	1	6	-6	-3	0	1
		0	0	2	12	-12	-5	0	0
1	0	1	1	1/6	0	1	2/3	1/6	0
3	0	0	0	1/6	1	-1	-1/3	1/6	0
7	1	0	0	0	0	0	-1	-1	1
		0	0	0	0	0	-2	-2	0

- Optimal solution (1, 0, 0, 0, 0, 0, 0) with basis $\{1, 3, 7\}$, 7 can be out of basis

Two-phase method: Example 1 - Phase II

- Last simplex table of auxiliary program

J	c_J	x_J	0	0	0	0	1	1	1
1	0	1	1	1/6	0	1	2/3	1/6	0
3	0	0	0	1/6	1	-1	-1/3	1/6	0
7	1	0	0	0	0	0	-1	-1	1
		0	0	0	0	0	-2	-2	0

- Remove rows corresponding to redundant elements in basis
- Remove all columns corresponding to auxiliary variables
- Recompute w.r.t. original program

J	c_J	x_J	-1	0	3	-1
1	-1	1	1	1/6	0	1
3	3	0	0	1/6	1	-1
		-1	0	1/3	0	-3
1	-1	1	1	0	-1	2
2	3	0	0	1	6	-6
		-1	0	0	-2	-1

Conclusion: Optimal solution (1, 0, 0, 0), optimal objective value = -1

Experiences

Original program:

$$\begin{aligned} \min \quad & t\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Reduce computational efforts:

- If $A \in \mathbb{R}^{m \times n}$ has k identity columns, then just add $m - k$ auxiliary variables
- If an auxiliary variable is out of basis, then remove the column corresponding to that variable in the next steps

Two-phase method: Example 2

Solve the LP

$$\min \quad 7x_1 + x_2 - 4x_3 \quad \text{s.t.}$$

$$\begin{array}{rcrcrcrcrcl}
6x_1 & - & 4x_2 & - & 5x_3 & + & x_4 & & & = & 20 \\
x_1 & + & x_2 & + & x_3 & & & & & = & 8 \\
3x_1 & - & 2x_2 & - & x_3 & & & + & x_5 & = & -8 \\
 & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0
\end{array}$$

Step 0: Make right hand side parameters non-negative

$$\min \quad 7x_1 + x_2 - 4x_3 \quad \text{s.t.}$$

$$\begin{array}{rrcrcrcrcrcl}
 & 6x_1 & - & 4x_2 & - & 5x_3 & + & x_4 & & & = & 20 \\
 & x_1 & + & x_2 & + & x_3 & & & & & = & 8 \\
- & 3x_1 & + & 2x_2 & + & x_3 & & & - & x_5 & = & 8 \\
 & & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0
\end{array}$$

Step 1: Formulate auxiliary program

$$\min \quad x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rrcrcrcrcrcrcrcl}
 & 6x_1 & - & 4x_2 & - & 5x_3 & + & x_4 & & & & & & & = & 20 \\
 & x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & & & = & 8 \\
- & 3x_1 & + & 2x_2 & + & x_3 & & & - & x_5 & & & + & x_7 & = & 8 \\
 & & & & & & & & \multicolumn{6}{r}{x_1, x_2, x_3, x_4, x_5, x_6, x_7} & \geq & 0
\end{array}$$

Two-phase method: Example 2 - Phase I

$$\min \quad x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rrrrrrrrrrrrrrrr}
 & 6x_1 & - & 4x_2 & - & 5x_3 & + & x_4 & & & & & & & = & 20 \\
 & x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & & & = & 8 \\
- & 3x_1 & + & 2x_2 & + & x_3 & & & - & x_5 & & & + & x_7 & = & 8 \\
 & & & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}$$

- Initital feasible basic solution $(0, 0, 0, 20, 0, 8, 8)$ with basis $J = \{4, 6, 7\}$

J	c_J	x_J	0	0	0	0	0	1	1
4	0	20	6	-4	-5	1	0	0	0
6	1	8	1	2	1	0	0	1	0
7	1	8	-3	2	1	0	-1	0	1
		16	-2	4	2	0	-1	0	0
4	0	36	8	0	-3	1	0		0
2	0	4	1/2	1	1/2	0	0		0
7	1	0	-4	0	0	0	-1		1
		0	-4	0	0	0	-1		0

- Auxiliary variable x_7 is still in basis $\implies$ pivot with x_1

Two-phase method: Example 2 - Phase I

$$\min \quad x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rrrrrrrrrrrrrrrr}
 & 6x_1 & - & 4x_2 & - & 5x_3 & + & x_4 & & & & & & & = & 20 \\
 & x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & & & = & 8 \\
- & 3x_1 & + & 2x_2 & + & x_3 & & & - & x_5 & & & + & x_7 & = & 8 \\
 & & & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}$$

- Initital feasible basic solution $(0, 0, 0, 20, 0, 8, 8)$ with basis $J = \{4, 6, 7\}$

J	c_J	x_J	0	0	0	0	0	1	1
4	0	20	6	-4	-5	1	0	0	0
6	1	8	1	2	1	0	0	1	0
7	1	8	-3	2	1	0	-1	0	1
		16	-2	4	2	0	-1	0	0
4	0	36	8	0	-3	1	0		0
2	0	4	1/2	1	1/2	0	0		0
7	1	0	-4	0	0	0	-1		1
		0	-4	0	0	0	-1		0

- Auxiliary variable x_7 is still in basis $\implies$ pivot with x_1

Two-phase method: Example 2 - Phase I

$$\min \quad x_6 + x_7 \quad \text{s.t.}$$

$$\begin{array}{rrrrrrrrrrrrrrrr}
 & 6x_1 & - & 4x_2 & - & 5x_3 & + & x_4 & & & & & & & = & 20 \\
 & x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & & & = & 8 \\
- & 3x_1 & + & 2x_2 & + & x_3 & & & - & x_5 & & & + & x_7 & = & 8 \\
 & & & & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0
\end{array}$$

- Initital feasible basic solution $(0, 0, 0, 20, 0, 8, 8)$ with basis $J = \{4, 6, 7\}$

J	c_J	x_J	0	0	0	0	0	1	1
4	0	20	6	-4	-5	1	0	0	0
6	1	8	1	2	1	0	0	1	0
7	1	8	-3	2	1	0	-1	0	1
		16	-2	4	2	0	-1	0	0
4	0	36	8	0	-3	1	0		0
2	0	4	1/2	1	1/2	0	0		0
7	1	0	-4	0	0	0	-1		1
		0	-4	0	0	0	-1		0

- Auxiliary variable x_7 is still in basis $\implies$ pivot with x_1

Two-phase method: Example 2 - Phase I

- Pivot x_7 with x_1

J	c_J	x_J	0	0	0	0	0	1	1
4	0	36	8	0	-3	1	0		0
2	0	4	1/2	1	1/2	0	0		0
7	1	0	-4	0	0	0	-1		1
		0	-4	0	0	0	-1		0
4	0	36	0	0	-3	1	-2		
2	0	4	0	1	1/2	0	-1/8		
1	0	0	1	0	0	0	1/4		
		0	0	0	0	0	0		

- Optimal solution (0,4,0,36,0,0,0) with basis $\{1, 2, 4\}$

Two-phase method: Example 2 - Phase I

- Pivot x_7 with x_1

J	c_J	x_J	0	0	0	0	0	1	1
4	0	36	8	0	-3	1	0		0
2	0	4	1/2	1	1/2	0	0		0
7	1	0	-4	0	0	0	-1		1
		0	-4	0	0	0	-1		0
4	0	36	0	0	-3	1	-2		
2	0	4	0	1	1/2	0	-1/8		
1	0	0	1	0	0	0	1/4		
		0	0	0	0	0	0		

- Optimal solution (0,4,0,36,0,0,0) with basis $\{1, 2, 4\}$

Two-phase method: Example 2 - Phase II

- Last simplex table of auxiliary program

J	c_J	x_J	0	0	0	0	0	1	1
4	0	36	0	0	-3	1	-2		
2	0	4	0	1	1/2	0	-1/8		
1	0	0	1	0	0	0	1/4		
		0	0	0	0	0	0		

- Remove all columns corresponding to auxiliary variables
- Recompute w.r.t. original program

J	c_J	x_J	7	1	-4	0	0
4	0	36	0	0	-3	1	-2
2	1	4	0	1	1/2	0	-1/8
1	7	0	1	0	0	0	1/4
		4	0	0	9/2	0	13/8
4	0	60	0	6	0	1	-2
3	-4	8	0	2	1	0	-11/4
1	7	0	1	0	0	0	1/4
			0	-9	0	0	11/4
4	0	60	11	6	0	1	0
3	-4	8	1	2	1	0	0
5	0	0	4	0	0	0	1
		-32	-11	-9	0	0	0

Conclusion: Optimal solution (0, 0, 8, 60, 0), optimal objective value = -32

Two-phase method: Example 2 - Phase II

- Last simplex table of auxiliary program

J	c_J	x_J	0	0	0	0	0	1	1
4	0	36	0	0	-3	1	-2		
2	0	4	0	1	1/2	0	-1/8		
1	0	0	1	0	0	0	1/4		
		0	0	0	0	0	0		

- Remove all columns corresponding to auxiliary variables
- Recompute w.r.t. original program

J	c_J	x_J	7	1	-4	0	0
4	0	36	0	0	-3	1	-2
2	1	4	0	1	1/2	0	-1/8
1	7	0	1	0	0	0	1/4
		4	0	0	9/2	0	13/8
4	0	60	0	6	0	1	-2
3	-4	8	0	2	1	0	-11/4
1	7	0	1	0	0	0	1/4
			0	-9	0	0	11/4
4	0	60	11	6	0	1	0
3	-4	8	1	2	1	0	0
5	0	0	4	0	0	0	1
		-32	-11	-9	0	0	0

Conclusion: Optimal solution (0, 0, 8, 60, 0), optimal objective value = -32

Thank you for your attention!