Exercises on Convex Optimization

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Convex set

Exercise 1. Prove that the following sets are convex.

1. An affine set in $\mathbb{R}^n$.
2. A half space in $\mathbb{R}^n$.
3. The solution set of the system of linear inequalities $A\mathbf{x} \leq \mathbf{b}$ where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

Exercise 2. Determine if each set below is convex.

1. $\{(x, y) \in \mathbb{R}^2_+ \mid x/y \leq 1, x, y > 0\}$.
2. $\{(x, y) \in \mathbb{R}^2_+ \mid x/y \geq 1, x, y > 0\}$.
3. $\{(x, y) \in \mathbb{R}^2_+ \mid xy \leq 1, x, y > 0\}$.
4. $\{(x, y) \in \mathbb{R}^2_+ \mid xy \geq 1, x, y > 0\}$.

Exercise 3. Which of the following sets are convex?

1. A slab, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \alpha \leq \mathbf{a}^T\mathbf{x} \leq \beta\}$.
2. A rectangle, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$.
3. A wedge, i.e., $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_1^T\mathbf{x} \leq b_1, \mathbf{a}_2^T\mathbf{x} \leq b_2\}$.
4. The set of points closer to a given point than a given set, i.e.,

$$\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S\}$$

   where $S \subseteq \mathbb{R}^{\mathbf{n}}$.

5. The set of points whose distance to $\mathbf{a}$ does not exceed a fixed fraction θ of the distance to $\mathbf{b}$, i.e., the set $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta\|\mathbf{x} - \mathbf{b}\|_2\}$. You can assume $\mathbf{a} \neq \mathbf{b}$ and $0 \leq \theta \leq 1$.

Exercise 4. Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality

$$C = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c \leq 0\}$$

with $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that C is convex if A is positive semidefinite.

Exercise 5. Let $C_1, C_2 \subset \mathbb{R}^n$ be two convex sets. Prove that $C_1 \cap C_2$ is also convex. Give an example that $C_1 \cup C_2$ may not be convex.

Exercise 6. Let $C_1, C_2 \subset \mathbb{R}^n$ be two convex sets. Prove that $C_1 + C_2$ is also convex. Here, $C_1 + C_2$ is the Minkowski sum of C_1 and C_2, i.e.,

$$C_1 + C_2 = \{\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2 \mid \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}.$$

Convex function

Exercise 7. Determine if each function below is convex or concave:

1. $f(x, y) = 2x - y - x^2 + 2xy - y^2$
2. $f(x, y, z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$
3. $f(x, y) = x + y - e^x - e^{x+y}$
4. $f(x, y) = 12x^{1/3}y^{1/2}$, $x, y \geq 0$

Exercise 8. For each of the following functions determine whether it is convex, concave.

1. $f(x) = e^x - 1$ on $\mathbb{R}$
2. $f(x, y) = xy$ on $\mathbb{R}^2_{++}$
3. $f(x, y) = 1/(xy)$ on $\mathbb{R}^2_{++}$
4. $f(x, y) = x/y$ on $\mathbb{R}^2_{++}$

Exercise 9. Let $f(x, y) = x^3 + 2x^2 + 2xy + \frac{1}{2}y^2 - 8x - 2y - 8$. Find the range of values (x, y) for which the function is convex.

Exercise 10. Let $f(x, y, z) = 2x^2 + 2xz + 2ayz + 2z^2$. Determine the values of a for which the function is concave and for which it is convex.

Exercise 11. Consider the quadratic function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T P\mathbf{x} + q^T\mathbf{x} + r$, where $P \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $\mathbf{q} \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Prove that f is convex if and only if P is positive semidefinite.

Convex Optimization

Exercise 12. Solve the following optimization problems.

1. $\min x^2 + 1 \quad \text{s.t.} \quad (x-2)(x-4) \leq 0.$
2. $\min x + 3y \quad \text{s.t.} \quad x^2 + y^2 \leq 1.$
3. $\max xy \quad \text{s.t.} \quad x + y^2 \leq 2, x \geq 0, y \geq 0.$