

# Continuous-time Fourier Transform (Part 1)

TRAN Hoang Tung

Information and Communication Technology (ICT) Department  
University of Science and Technology of Hanoi (USTH)

Signals &  
Systems

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Objectives

Fourier  
Transform

Exercises

Periodic  
Signals

Homework

## Previous Lessons

- Signals and Systems
- Convolution
- Fourier Series

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- Signals and Systems
- Convolution
- Fourier Series

## Today

Fourier Transform in Continuous-time domain

*periodic*  
CT FS  
DT FS

*a periodic*  
CT FT  
DT FT

## 1 Lesson Objectives

## 2 Fourier Transform

## 3 Exercises

## 4 Fourier Transform for Periodic Signals

## 5 Homework

## Objectives

Fourier  
Transform

Exercises

Periodic  
Signals

Homework

# 1 Lesson Objectives

## 2 Fourier Transform

## 3 Exercises

## 4 Fourier Transform for Periodic Signals

## 5 Homework

At the end of this lesson, you should be able to

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Fourier  
Transform

Exercises

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At the end of this lesson, you should be able to

- 1 understand that Fourier Transform is mainly for aperiodic signals

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- 2 find Fourier Transform of any continuous-time signals

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- 1 understand that Fourier Transform is mainly for aperiodic signals
- 2 find Fourier Transform of any continuous-time signals
- 3 see the connection between Fourier Series and Fourier Transform

## 1 Lesson Objectives

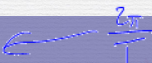
## 2 Fourier Transform

- Review: Continuous-time Fourier Series
- Definition

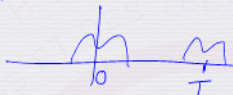
## 3 Exercises

## 4 Fourier Transform for Periodic Signals

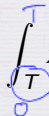
## 5 Homework

Fourier Pair:  $\omega_0 = 2\pi/T$  

The **synthesis** equation

$$\boxed{x(t)} = \sum_{k=-\infty}^{+\infty} \underline{a_k e^{jk\omega_0 t}}$$


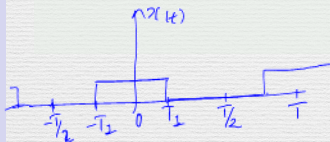
The **analysis** equation

$$\boxed{a_k} = \frac{1}{T} \int_{\boxed{T}} x(t) e^{-jk\omega_0 t} dt$$


## Example

Determine the Fourier series coefficients for the periodic square wave, defined over one period as

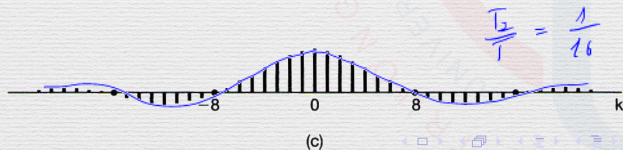
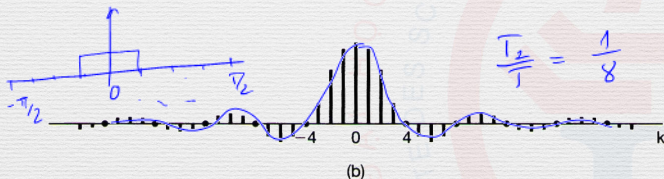
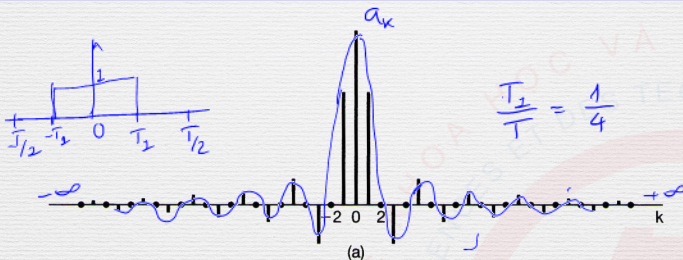
$$x(t) = \begin{cases} 1 & \text{if } |t| < T_1 \\ 0 & \text{if } T_1 < |t| < T/2 \end{cases}$$



$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \left. \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right|_{-T_1}^{T_1} = \frac{\sin(\pi k \frac{2T_1}{T})}{\pi k}$$



# Sinc Function: $\text{sinc}(\theta) = \sin(\pi\theta)/\pi\theta$

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Objectives

Fourier  
Transform

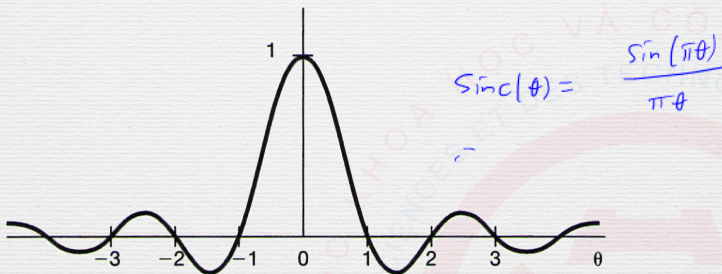
CTFS

Definition

Exercises

Periodic  
Signals

Homework



# The Envelope of $Ta_k$

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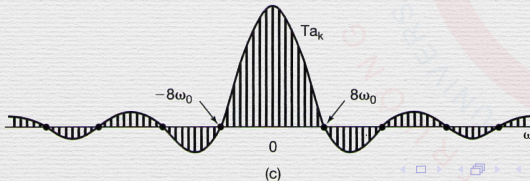
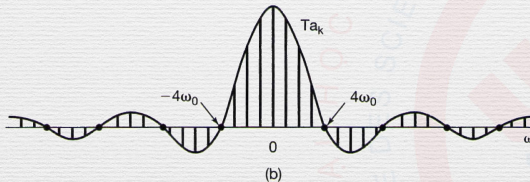
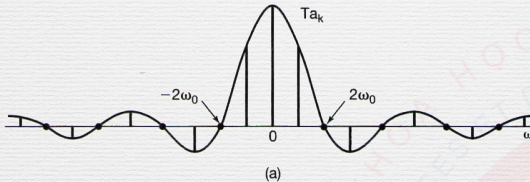
Fourier  
Transform

CTFS  
Definition

Exercises

Periodic  
Signals

Homework

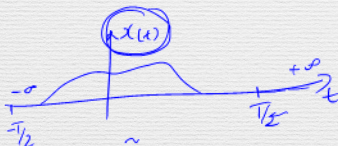


Consider a signal  $x(t)$  that is of finite duration, and a periodic signal  $\tilde{x}(t)$  for which  $x(t)$  is one period.

$$T a_k = \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt = \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$$= \int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt = X(\omega) \Big|_{\omega = k\omega_0}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

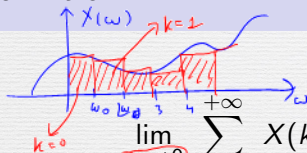


$$\Rightarrow a_k = \frac{X(k\omega_0)}{T}$$



# Definition

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$



$$\lim_{\omega_0 \rightarrow 0} \sum_{k=-\infty}^{+\infty} \underline{X(k\omega_0)} \underline{\omega_0} = \int_{-\infty}^{+\infty} \underline{X(\omega)} d\omega$$

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{+\infty} a_k \cdot e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{+\infty} \frac{X(k\omega_0)}{T} \cdot e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(k\omega_0) \cdot e^{jk\omega_0 t} \cdot \omega_0 \end{aligned}$$

$$\begin{aligned} \omega_0 \rightarrow 0 : \tilde{x}(t) &\rightarrow x(t) \\ \left\{ \begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \cdot e^{j\omega t} d\omega \\ X(\omega) &= \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt \end{aligned} \right. \end{aligned}$$

$$x(t) = e^{-t} \cdot u(t+3)$$

$$\hookrightarrow X(\omega) = ?$$

## Definition

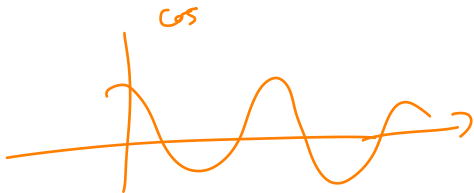
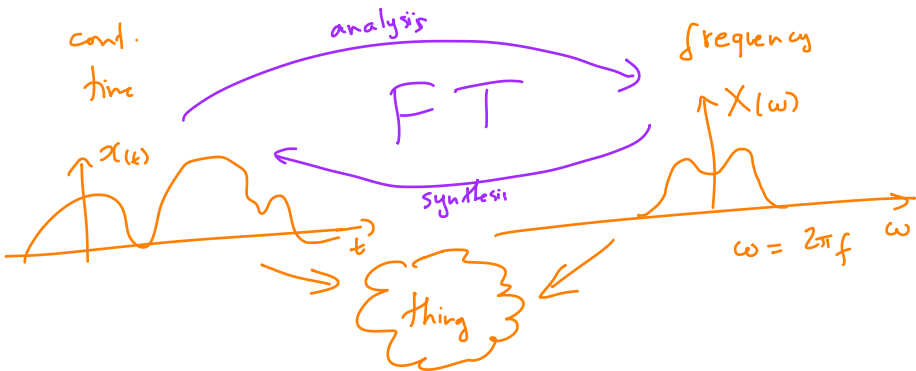
The **synthesis** equation (inverse Fourier transform)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

The **analysis** equation (Fourier transform)

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$





## Exercise

Determine the Fourier Transform of the signal

$$x(t) = \underline{e^{-at}} u(t), \text{ with } \underline{a} > 0 \rightarrow a = 1$$

$$x(t) = e^{-t} \cdot u(t)$$



$$X(\omega) = |X(\omega)| \cdot e^{j\angle X(\omega)}$$

# Exercise 1

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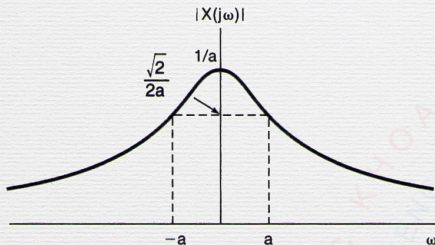
Objectives

Fourier  
Transform

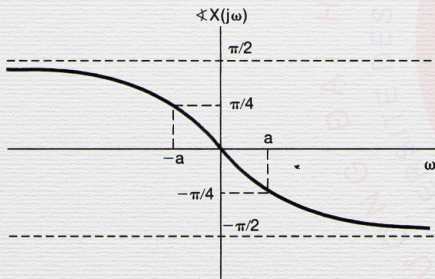
Exercises

Periodic  
Signals

Homework



(a)



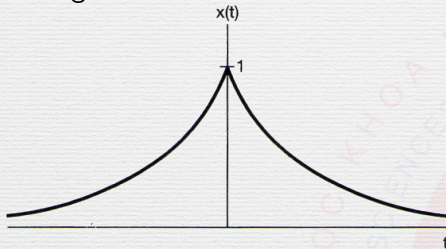
(b)

## Exercise

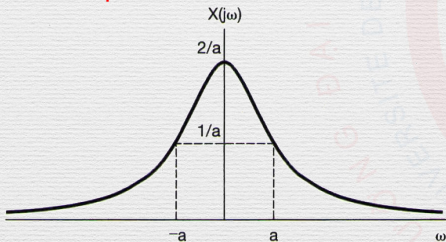
Determine the Fourier Transform of the signal

$$x(t) = e^{-a|t|}, \text{ with } a > 0$$

The signal



And its **spectrum**



# Exercise 3

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Objectives

Fourier  
Transform

Exercises

Periodic  
Signals

Homework

$$\delta(t) = \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{else} \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

## Exercise

Determine the Fourier Transform of the Delta signal.



$$\delta(t) \rightarrow \Delta(\omega) = ?$$

$$\Delta(\omega) = \int_{-\infty}^{+\infty} \delta(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \delta(t) \cdot dt = 1$$

if  $t \neq 0$ :

$$\delta(t) = 0 \rightarrow 0 \cdot e^{-j\omega t} = 0$$

$$\boxed{\Delta(\omega) = 1 \quad \forall \omega}$$

$$\begin{aligned} & \delta(t) \cdot e^{-j\omega t} = \delta(t) \cdot 1 \\ & \text{if } t = 0 \rightarrow e^{-j\omega t} = 1 \rightarrow \delta(t) \cdot e^{-j\omega t} = \delta(t) \cdot 1 = \delta(t) \end{aligned}$$

## Exercise 4

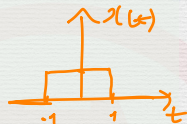
Euler

$$e^{j\varphi} = \cos \varphi + j \cdot \sin \varphi$$

### Exercise

Find the Fourier Transform of the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



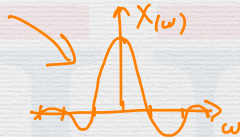
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt$$

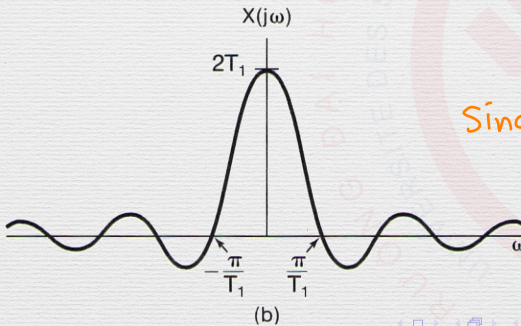
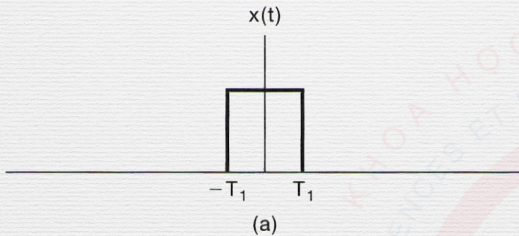
$$= \int_{-1}^{1} e^{-j\omega t} dt = \dots$$

remember:

$$\sin \varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$

$$X(\omega) = ?$$



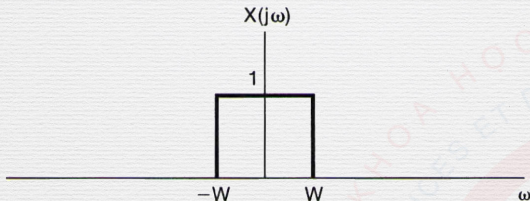


$$\text{Sinc} \cdot = \frac{\text{Sin} \cdot}{\dots}$$

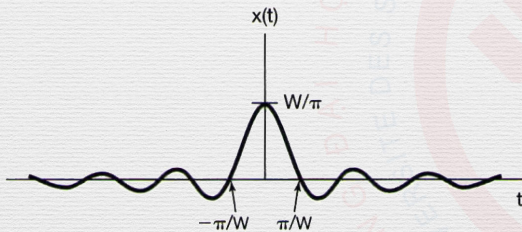
## Exercise

Determine the signal  $x(t)$  whose Fourier transform is

$$X(\omega) = \begin{cases} 1, & |\omega| < \cancel{4} \quad 1 \\ 0, & |\omega| > \cancel{4} \quad 1 \end{cases}$$



(a)



(b)

1 Lesson Objectives

2 Fourier Transform

3 Exercises

4 Fourier Transform for Periodic Signals

5 Homework

Consider a signal with Fourier transform  $X(\omega) = 2\pi\delta(\omega - \omega_0)$

what is  $x(t)$ ?  $\rightarrow e^{j\omega_0 t}$

$$X(\omega) = 2\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\hookrightarrow x(t) = ?$$

Determine Fourier transform of  $\sin(\omega_0 t)$  and  $\cos(\omega_0 t)$

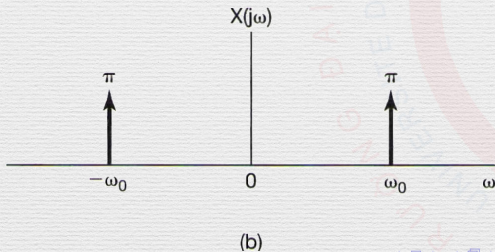
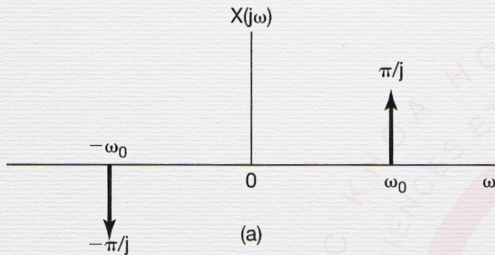
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Periodic  
Signals

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## Continuous-time Fourier Transform Exercises

4.1, 4.3, 4.4, 4.13, 4.22

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Periodic  
Signals

Homework