5.11 Problems

Exercise 1. (VIDEO SOLUTION)
Alex and Bob each flips a fair coin twice. Use "1" to denote heads and "0" to denote tails. Let X be the maximum of the two numbers Alex gets, and let Y be the minimum of the two numbers Bob gets.

(a) Find and sketch the joint PMF $p_{X,Y}(x,y)$.

(b) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.

(c) Find the conditional PMF $P_{X|Y}(x\,|\,y)$. Does $P_{X|Y}(x\,|\,y) = P_X(x)$? Why or why not?

Exercise 2.
Two fair dice are rolled. Find the joint PMF of X and Y when

(a) X is the larger value rolled, and Y is the sum of the two values.

(b) X is the smaller, and Y is the larger value rolled.

Exercise 3.
The amplitudes of two signals X and Y have joint PDF

$$f_{XY}(x,y) = e^{-x/2}ye^{-y^2}$$

for $x > 0, y > 0$.

(a) Find the joint CDF.

(b) Find $\mathbb{P}(X^{1/2} > Y)$.

(c) Find the marginal PDFs.

Exercise 4. (VIDEO SOLUTION)
Find the marginal CDFs $F_X(x)$ and $F_Y(y)$ and determine whether or not X and Y are independent, if

$$F_{XY}(x,y) = \begin{cases} x - 1 - \frac{e^{-y}-e^{-xy}}{y}, & \text{if } 1 \le x \le 2, y \ge 0 \\ 1 - \frac{e^{-y}-e^{-2y}}{y}, & \text{if } x > 2, y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 5. (VIDEO SOLUTION)

(a) Find the marginal PDF $f_X(x)$ if

$$f_{XY}(x,y) = \frac{\exp\{-|y-x| - x^2/2\}}{2\sqrt{2\pi}}.$$

(b) Find the marginal PDF $f_Y(y)$ if

$$f_{XY}(x,y) = \frac{4e^{-(x-y)^2/2}}{y^2\sqrt{2\pi}}.$$

Exercise 6. (Video Solution)
Let X, Y be two random variables with joint CDF

$$F_{X,Y}(x,y) = \frac{y + e^{-x(y+1)}}{y+1}.$$

Show that

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{X,Y}(x,y).$$

What is the implication of this result?

Exercise 7. (Video Solution)
Let X and Y be two random variables with joint PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}.$$

(a) Find the PDF of $Z = \max(X, Y)$.

(b) Find the PDF of $Z = \min(X, Y)$.

You may leave your answers in terms of the $\Phi(\cdot)$ function.

Exercise 8.
The random vector (X, Y) has a joint PDF

$$f_{XY}(x,y) = 2e^{-x}e^{-2y}$$

for $x > 0,\ y > 0$. Find the probability of the following events:

(a) $\{X + Y \le 8\}$.

(b) $\{X - Y \le 10\}$.

(c) $\{X^2 < Y\}$.

Exercise 9.
Let X and Y be zero-mean, unit-variance independent Gaussian random variables. Find the value of r for which the probability that (X, Y) falls inside a circle of radius r is $1/2$.

Exercise 10.
The input X to a communication channel is $+1$ or -1 with probabilities p and $1-p$, respectively. The received signal Y is the sum of X and noise N, which has a Gaussian distribution with zero mean and variance $\sigma^2 = 0.25$.

(a) Find the joint probability $\mathbb{P}(X = j,\ Y \leq y)$.

(b) Find the marginal PMF of X and the marginal PDF of Y.

(c) Suppose we are given that $Y > 0$. Which is more likely, $X = 1$ or $X = -1$?

Exercise 11. (VIDEO SOLUTION)
Let

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y}, & \text{if} \quad 0 \leq y \leq x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find c.

(b) Find $f_X(x)$ and $f_Y(y)$.

(c) Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$, $\text{Var}[X]$ and $\text{Var}[Y]$.

(d) Find $\mathbb{E}[XY]$, $\text{Cov}(X,Y)$ and ρ.

Exercise 12. (VIDEO SOLUTION)
In class, we have used the Cauchy-Schwarz inequality to show that $-1 \leq \rho \leq 1$. This exercise asks you to prove the Cauchy-Schwarz inequality:

$$(\mathbb{E}[XY])^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2].$$

Hint: Consider the expectation $\mathbb{E}[(tX+Y)^2]$. Note that this is a quadratic equation in t and $\mathbb{E}[(tX+Y)^2] \geq 0$ for all t. Consider the discriminant of this quadratic equation.

Exercise 13. (VIDEO SOLUTION)
Let $\Theta \sim \text{Uniform}[0, 2\pi]$.

(a) If $X = \cos\Theta$, $Y = \sin\Theta$. Are X and Y uncorrelated?

(b) If $X = \cos(\Theta/4)$, $Y = \sin(\Theta/4)$. Are X and Y uncorrelated?

Exercise 14. (VIDEO SOLUTION)
Let X and Y have a joint PDF

$$f_{X,Y}(x,y) = c(x+y),$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

(a) Find c, $f_X(x)$, $f_Y(y)$, and $\mathbb{E}[Y]$.

(b) Find $f_{Y|X}(y|x)$.

(c) Find $\mathbb{P}[Y > X \mid X > 1/2]$.

(d) Find $\mathbb{E}[Y|X = x]$.

(e) Find $\mathbb{E}[\mathbb{E}[Y|X]]$, and compare with the $\mathbb{E}[Y]$ computed in (a).

Exercise 15. (Video Solution)
Use the law of total expectation to compute the following:

1. $\mathbb{E}[\sin(X+Y)]$, where $X \sim \mathcal{N}(0,1)$, and $Y \mid X \sim \text{Uniform}[x-\pi, x+\pi]$
2. $\mathbb{P}[Y < y]$, where $X \sim \text{Uniform}[0,1]$, and $Y \mid X \sim \text{Exponential}(x)$
3. $\mathbb{E}[Xe^Y]$, where $X \sim \text{Uniform}[-1,1]$, and $Y \mid X \sim \mathcal{N}(0, x^2)$

Exercise 16.
Let $Y = X + N$, where X is the input, N is the noise, and Y is the output of a system. Assume that X and N are independent random variables. It is given that $\mathbb{E}[X] = 0$, $\text{Var}[X] = \sigma_X^2$, $\mathbb{E}[N] = 0$, and $\text{Var}[N] = \sigma_N^2$.

(a) Find the correlation coefficient ρ between the input X and the output Y.

(b) Suppose we estimate the input X by a linear function $g(Y) = aY$. Find the value of a that minimizes the mean squared error $\mathbb{E}[(X - aY)^2]$.

(c) Express the resulting mean squared error in terms of $\eta = \sigma_X^2/\sigma_N^2$.

Exercise 17. (Video Solution)
Two independent random variables X and Y have PDFs

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad f_Y(y) = \begin{cases} 0, & y > 0, \\ e^{y}, & y \leq 0. \end{cases}$$

Find the PDF of $Z = X - Y$.

Exercise 18.
Let X and Y be two independent random variables with densities

$$f_X(x) = \begin{cases} xe^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} ye^{-y}, & y \geq 0, \\ 0, & y < 0. \end{cases}$$

Find the PDF of $Z = X + Y$.

Exercise 19.
The random variables X and Y have the joint PDF

$$f_{XY}(x,y) = e^{-(x+y)}$$

for $0 < y < x < 1$. Find the PDF of $Z = X + Y$.

Exercise 20.
The joint density function of X and Y is given by

$$f_{XY}(x,y) = e^{-(x+y)}$$

for $x > 0, y > 0$. Find the PDF of the random variable $Z = X/Y$.