Please come back at 9AM

[with lots of questions]

$a_1, a_2, \dots, a_{100}$

Fourier Series Representation of Continuous-time Periodic Signals

aperiodic ?

F. Series

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F. Transform

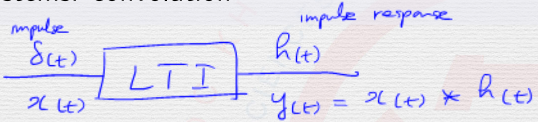
?

→ Friday

The Course So Far

Until Today

- Signals and Systems
- LTI systems: convolution



The Course So Far

Until Today

- Signals and Systems
- LTI systems: convolution

Today

Fourier Series in Continuous-time domain

Base

FS, FT

Lesson Objectives

At the end of this lesson, you should be able to

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1. be familiar with complex numbers

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1. be familiar with complex numbers
2. find Fourier coefficients of any continuous periodic signals

Lesson Objectives

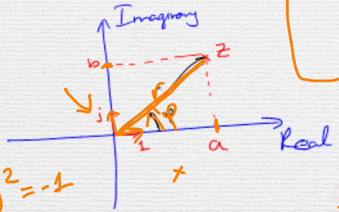
At the end of this lesson, you should be able to

1. be familiar with complex numbers
2. find Fourier coefficients of any continuous periodic signals
3. understand various properties of Fourier Series

Review: Complex Number, Conjugation

$(2,3)$

Complex plane with polar and Cartesian representation



$z = a + j.b \sim (a,b)$

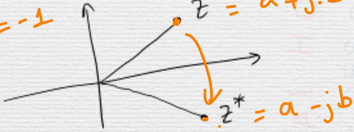
$z = (r, \varphi)$

$= r . e^{j\varphi}$

$(3, \frac{\pi}{4})$

$j^2 = -1$

$i^2 = -1$

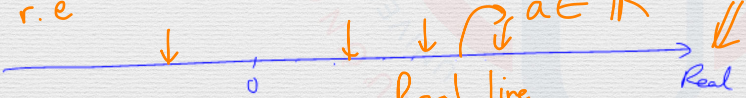


$\frac{r.e^{j\varphi}}{?}$ $\quad r.7^{j\varphi}$ ...

$r.e^{j\varphi}$

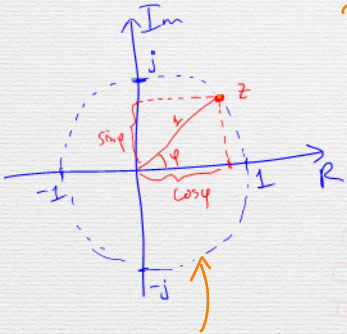
$r.e^{-j\varphi}$

$a \in \mathbb{R}$



Real line

Review: Euler Formula $e^{j\psi} = cos(\psi) + jsin(\psi)$



unit circle

root

$$z = \boxed{e^{j\varphi} = \cos\varphi + j \cdot \sin\varphi}$$

$$e^{-j\varphi} = \cos\varphi - j \cdot \sin\varphi$$

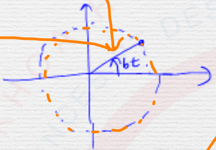
$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$$

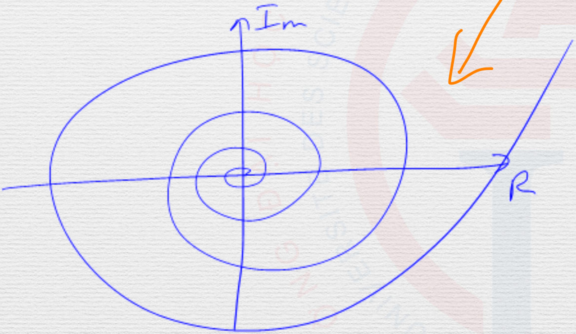
$$\sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$

Review: Complex Exponential Signal e^{st}

Real Exp $C.e^{at}$

$$x_{(t)} = e^{st} = e^{(a+jb).t} = e^{at} \cdot e^{jbt}$$





LTI Response to Complex Exponential

Signals & Systems

TRAN Hoang Tung

Convolution Integral

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\overbrace{x(t-\tau)}^{e^{s(t-\tau)}}d\tau$$

$s = \sigma + j\omega$

$x(t) = e^{st} \rightarrow \boxed{h(t)} \rightarrow y(t)$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\cdot e^{st}\, e^{-s\tau}\, d\tau = e^{st}\left[\int_{-\infty}^{\infty} h(\tau)\cdot e^{-s\tau} d\tau\right]$$

$$= x(t)\cdot |H(s)|$$

ALL LTI Systems

LTI Response to Complex Exponential

Convolution Integral

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.

Fundamental Period

The minimum positive, nonzero value of T so that

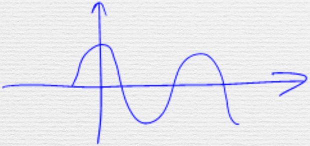
$$x(t) = x(t+T) = x(t+2T)$$

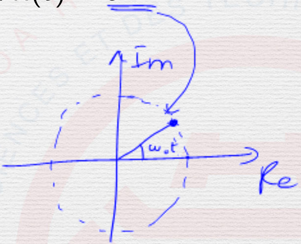
Fundamental frequency $\omega_0 = \frac{2\pi}{T}$

2 Basic Periodic Signals

the sinusoidal signal $x(t) = cos\omega_0 t$

the periodic complex exponential $x(t) = e^{j\omega_0 t}$





an Example

Consider a periodic signal $x(t)$ with fundamental frequency 2π

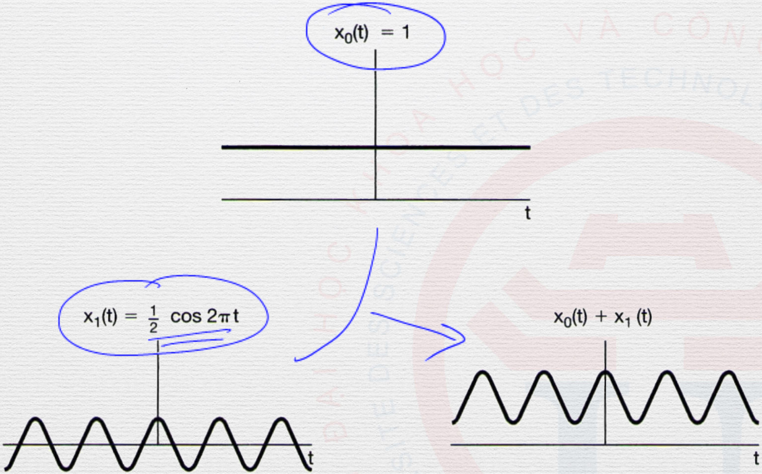
$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t} = \cos \ldots$$

where $a_0 = 1, a_1 = a_{-1} = \frac{1}{4}, a_2 = a_{-2} = \frac{1}{2}, a_3 = a_{-3} = \frac{1}{3}$

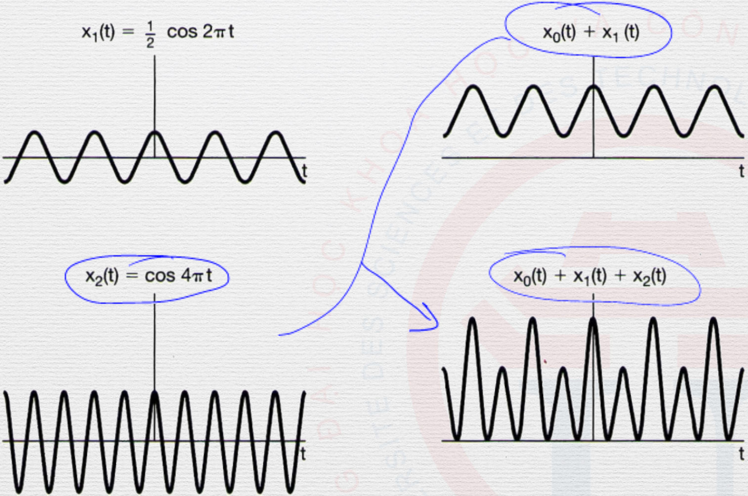
$$\begin{cases} e^{j\varphi} = \cos\varphi + j\sin\varphi \\ e^{-j\varphi} = \cos\varphi - j\sin\varphi \end{cases}$$

$$1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cdot\cos(6\pi t)$$

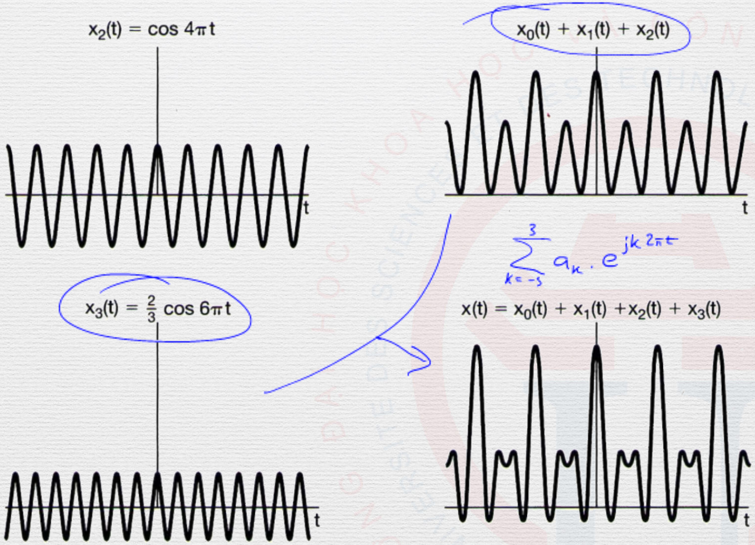
an Example



an Example



an Example


$x_2(t) = \cos 4\pi t$
$x_0(t) + x_1(t) + x_2(t)$
$\sum_{k=-3}^{3} a_k \cdot e^{jk2\pi t}$
$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t)$
$x_3(t) = \frac{2}{3} \cos 6\pi t$
t

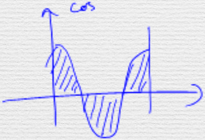
Review: Integral of $e^{j\omega_0 t}$

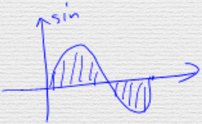
Given $\omega_0 = \frac{2\pi}{T}$, calculate

1

$$\int_0^T e^{j\omega_0 t} dt = 0$$

$$\int_0^T (\underbrace{\cos \omega_0 t}_{0} + j \underbrace{\sin \omega_0 t}_{0}) dt$$





Review: Integral of $e^{j\omega_0 t}$

Given $\omega_0 = \frac{2\pi}{T}$, calculate

1.
$$\int_0^T e^{j\omega_0 t} dt$$

2.
$$\int_0^T \underbrace{e^{jk\omega_0 t}}_{} dt = \begin{cases} T & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$k=0 \quad \xrightarrow{1} T$

Definition

Assume that

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}}$$

Then

$$a_k = \cancel{???}\ \frac{1}{T} \cdot \int_0^T x(t) \cdot e^{-jk\omega_0 t} dt$$

Hint: calculate $x(t)e^{-jn\omega_0 t}$, and its integral from 0 to T.

$$\int_0^T x(t) \cdot e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \underbrace{\int_0^T e^{j(k-n)\omega_0 t} \cdot dt}_{\begin{cases} T & \text{if } k=n \\ 0 & \text{else} \end{cases}} \doteq a_n \cdot T$$

Fourier Series Equation Pair

periodic signals : T

$\omega_0 = \frac{2\pi}{T}$

Definition

The synthesis equation

k: ? ω_0

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T}t}$$

number

The analysis equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$$

Fourier coefficients

$|[\ldots]| = T$

$\int_0^T \quad \int_{-T/2}^{T/2} \quad \int_{6T}^{7T} \quad \int_{-5T}^{-4T}$

$x(t) \xrightarrow{\text{analy}} a_k$

$x(t) \xleftarrow{\text{syn}} a_k$

signal

Example 1

Example

Determine Fourier series coefficients of

$$x(t) = sin\omega_0 t$$

$$= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$a_1 = \frac{1}{2j}$$

$$a_0 = a_2 = a_{\pm 2} = \cdots = 0$$

$$a_{-1} = -\frac{1}{2j}$$

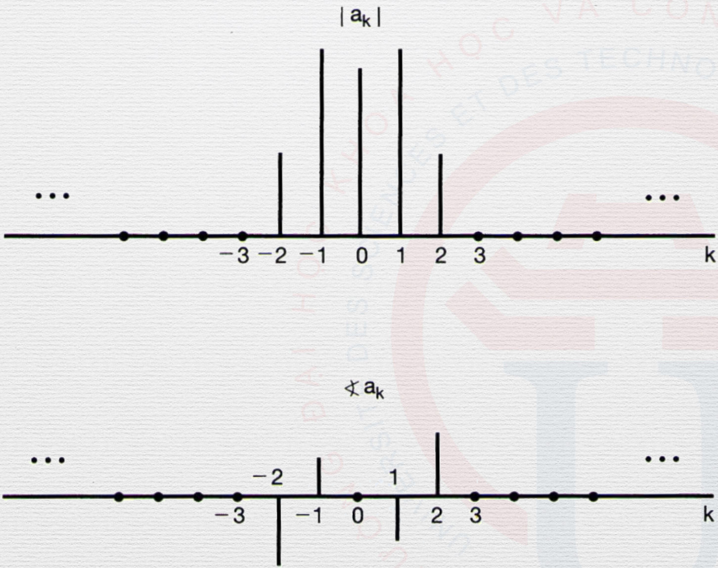
Example 2

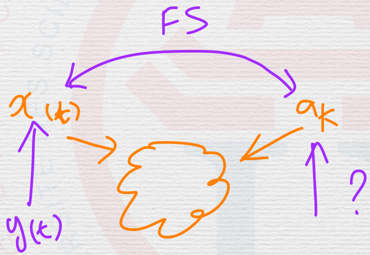
Example

Determine Fourier series coefficients of

$$x(t) = 1 + sin\omega_0 t + 2cos\omega_0 t + cos(2\omega_0 t + \frac{\pi}{4})$$

Example 2




FS
x(t)
a_k
y(t)
?

Notations

$x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Its Fourier series coefficients are denoted by a_k:

$$\boxed{x(t) \xleftrightarrow{\mathcal{FS}} a_k}$$

Fourier Pair

The synthesis equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T}t}$$

The analysis equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$$

Linearity

Definition

Given

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k$$

Then

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = \;?$$

Linearity

Definition

Given

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k$$

Then

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

Verify this property.

Time Shifting

Definition

Given

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

Then

$$x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$

Time Shifting

Definition

Given

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

Then

$$x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$$

Verify this property.

Time Reversal

Definition

Given

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

Then

$$x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow}$$

Time Reversal

Definition

Given

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

Then

$$x(-t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k}$$

!!!

Verify this property.

Time Scaling

Definition

Given

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

Then

$$x(\alpha t) \xleftrightarrow{\mathcal{FS}} ???$$

Time Scaling

Definition

Given

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

Then

$$x(\alpha t) \xleftrightarrow{\mathcal{FS}} \cancel{??}\ a_k$$

Attention:

1. the Fourier coefficients have not changed
2. the Fourier series representation has changed

$\omega.$

$\downarrow$

ω !!!

Multiplication

Definition

Given

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

Then

$$z(t) = x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow}$$

Multiplication

Definition

Given

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k$$

Then

$$z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$$

$a_k * b_k$

Verify this!

Conjugation

Definition

Given

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

Then

$$x^*(t) \stackrel{\mathcal{FS}}{\longleftrightarrow}$$

Conjugation

Definition

Given

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

Then

$$x^*(t) \xleftrightarrow{\mathcal{FS}} a^*_{-k}$$

Parseval's Relation

Definition

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 dt}_{} = \underbrace{\sum_{k=-\infty}^{+\infty} |a_k|^2}_{}$$

Power

Homework

Continuous Fourier Series Exercises

3.1, 3.4, 3.5, 3.17, 3.22, 3.46