CTFT $x(t) \longleftrightarrow X(\omega)$

DTFT $x(n) \longleftrightarrow X(e^{j\omega})$

DFT $x[n] \longleftrightarrow X(k)$: DSP

Discrete-time Fourier Transform

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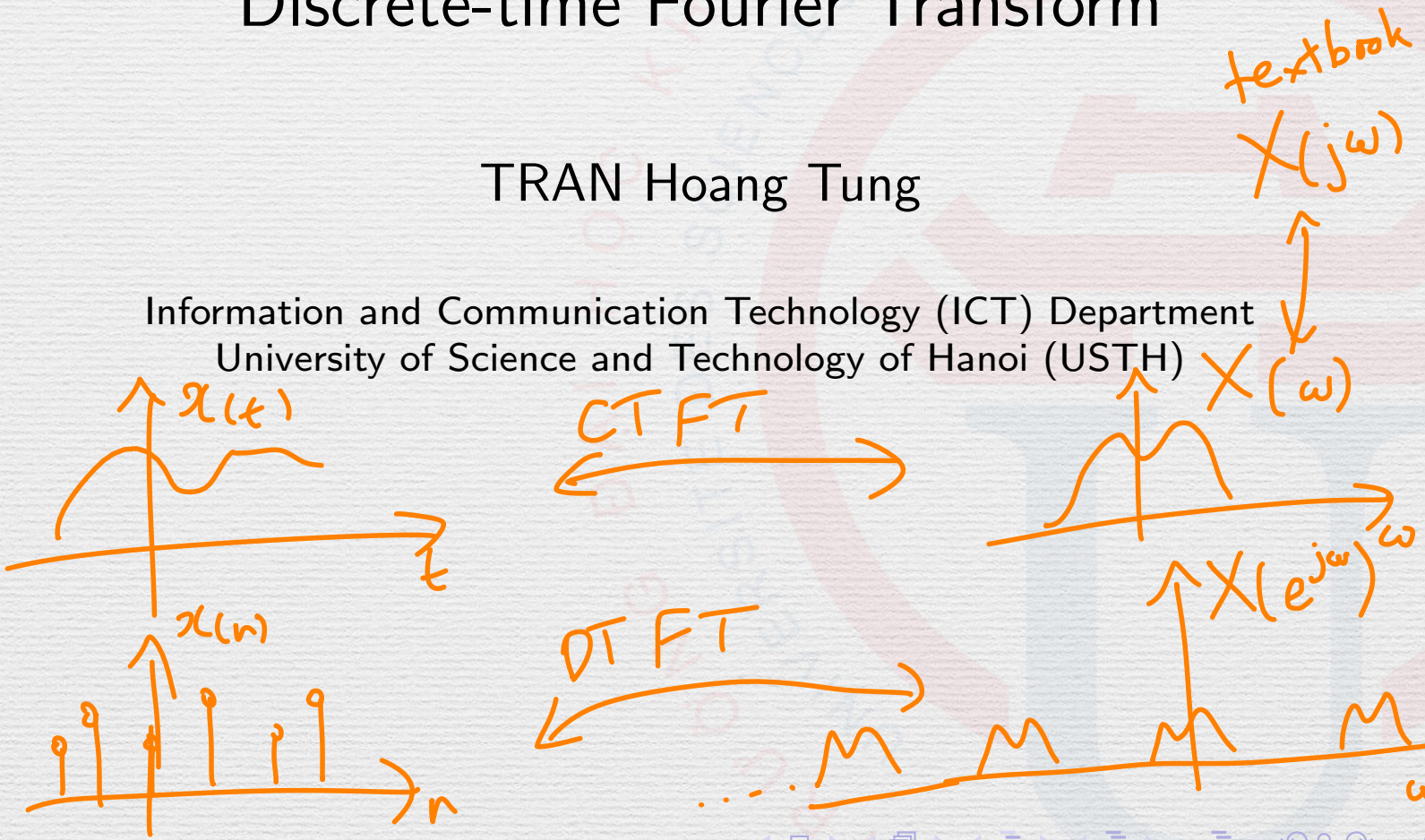

The Course So Far

What we have learnt?

- Signals and Systems
- LTI systems: convolution
- Fourier Series
- Fourier Transform in Continuous-time domain

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Today

Fourier Transform in Discrete-time domain

Lesson Objectives

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1. find Fourier Transform of any discrete-time signals
2. see the connection between CTFT and DTFT
3. understand properties of DTFT

Review: Discrete-time Fourier Series

Fourier Pair: $\omega_0 = 2\pi/N$

The synthesis equation

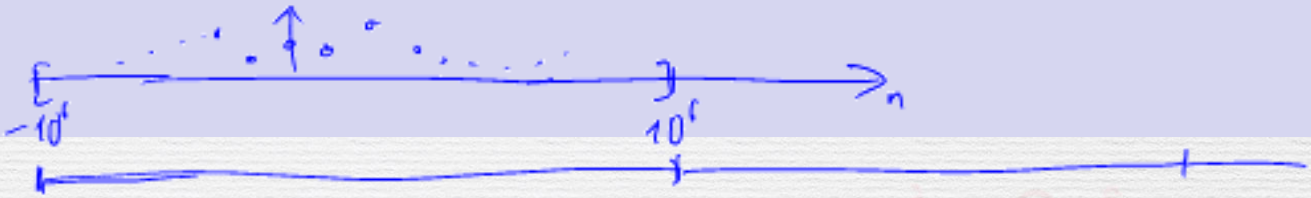
$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

The analysis equation

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

$0 \to N-1$

Definition



Consider a signal $x[n]$ that is of finite duration, and a periodic signal $\tilde{x}[n]$ for which $x[n]$ is one period.

$$\omega_0 = \frac{2\pi}{N} \rightarrow \frac{1}{N} = \frac{\omega_0}{2\pi}$$

$$Na_k \triangleq \sum_{<N>} \tilde{x}[n] e^{-jk\omega_0 n} = \sum_{-N/2}^{N/2} x[n] e^{-jk\omega_0 n} = \sum_{n=-\infty}^{+\infty} x[n] e^{-jk\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] \cdot e^{-jn\omega}$$

$$= X(e^{j\omega})\Big|_{\omega = k\omega_0}$$

$$\tilde{x}[n] = \sum_{<N>} a_k \cdot e^{jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{<N>} X(e^{jk\omega_0}) \cdot e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{<N>} X(e^{jk\omega_0}) \cdot e^{jk\omega_0 n} \cdot \omega_0$$

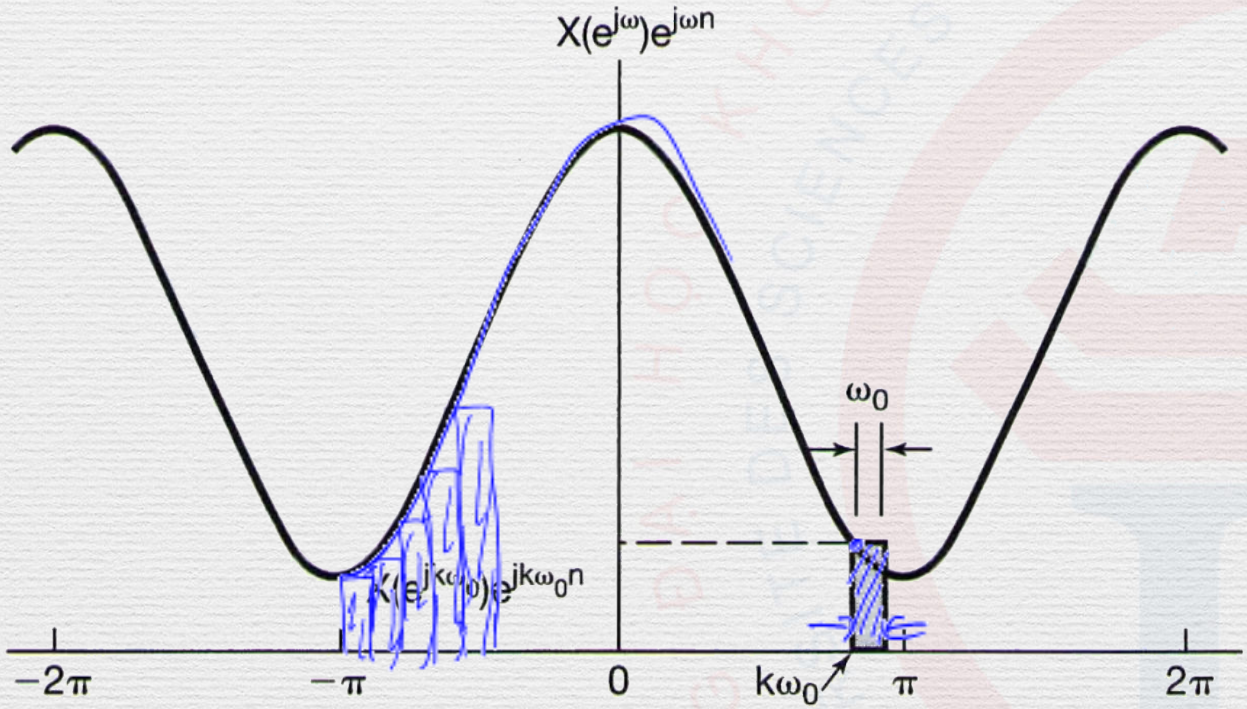
$$= X(e^{jk\omega_0})$$

$N \rightarrow \infty$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

Definition

$$\lim_{\omega_0 \to 0} \sum_{<N>} X(e^{jk\omega_0}) e^{-jk\omega_0 n} \omega_0 = \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$



Fourier Transform Pair

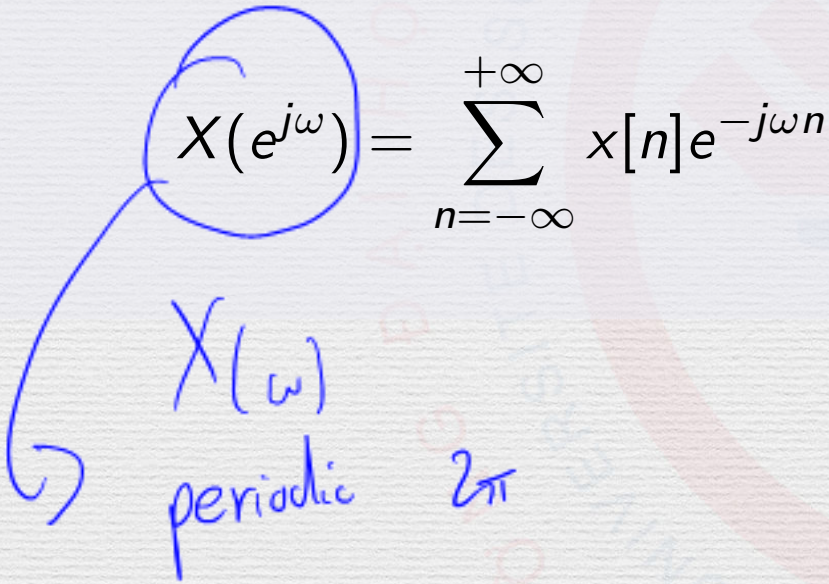
Definition

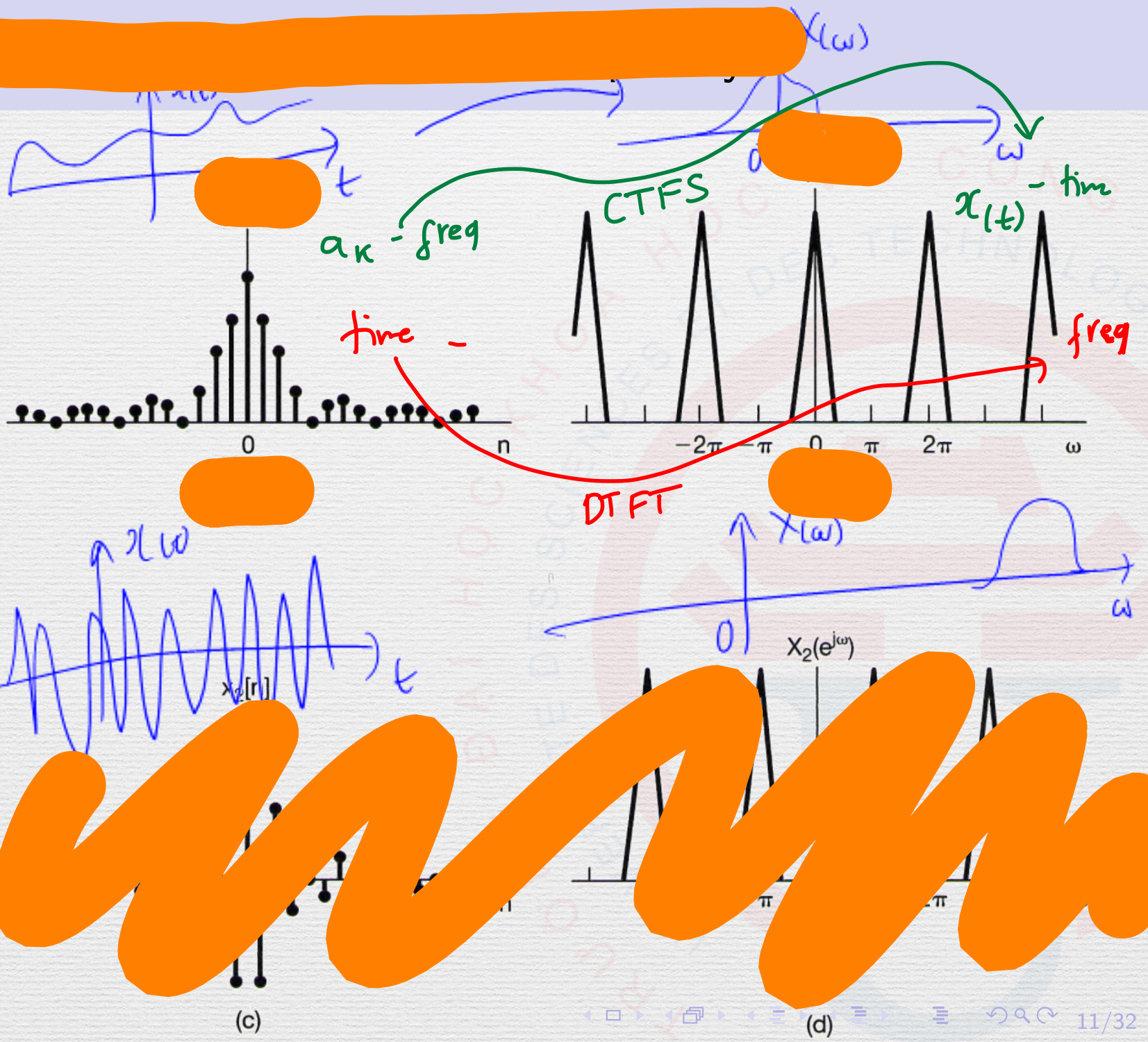
The synthesis equation (inverse Fourier transform)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

The analysis equation (Fourier transform)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$




CTFS
a_k - freq
$x_{(t)}$ - time
time -
freq
DT FT
0
n
-2π
$-\pi$
0
π
2π
ω
$X(\omega)$
$x_2[n]$
$X_2(e^{j\omega})$
(c)
(d)

Exercise 1

$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

Exercise

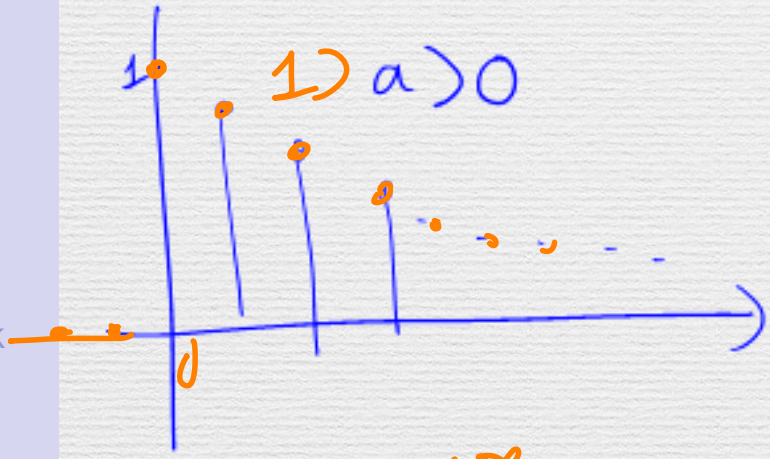
Determine the Fourier Transform of the signal

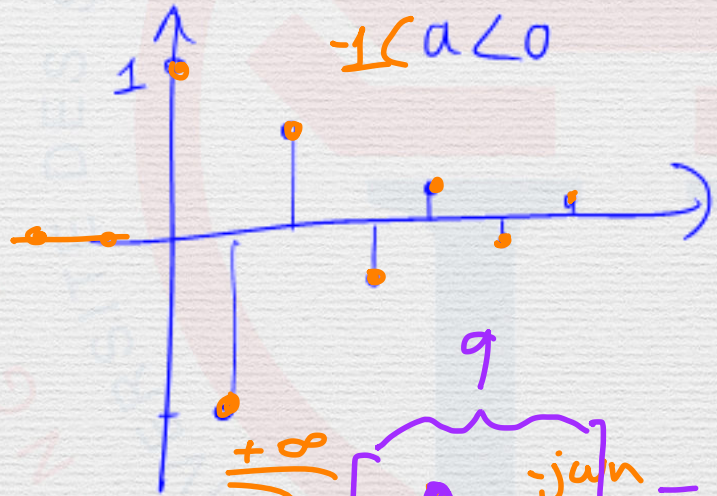
$$x(n) = a^n u[n], \text{ with } |a| < 1$$

$x(n) = a^n$, $n \geq 0$

$x(n) = 0$, $n < 0$

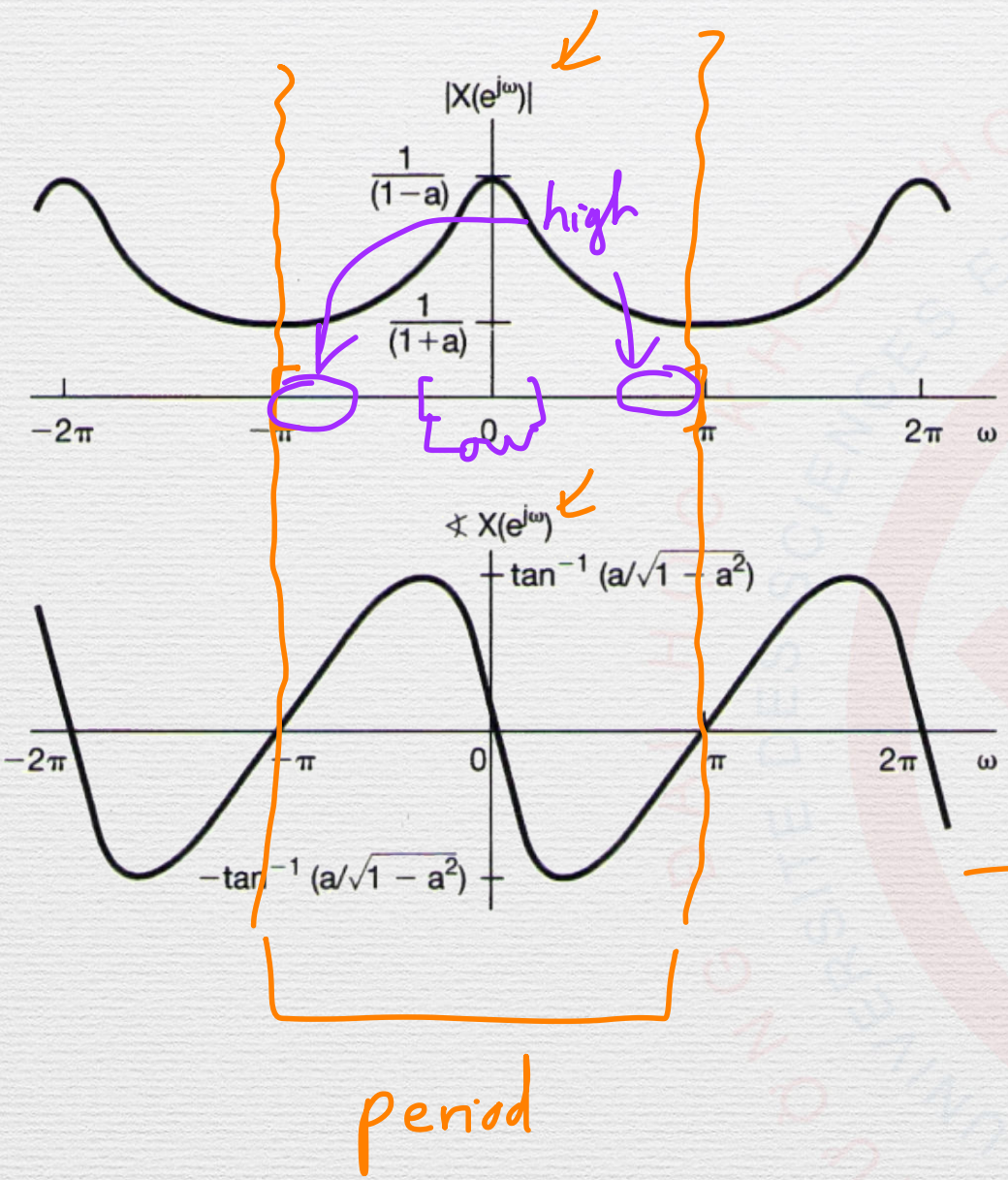
$\Rightarrow 1 + q + q^2 + q^3 + q^4 + q^5 + q^6 = \dfrac{1-q^7}{1-q}$





$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-j\omega n} = \sum_{n=0}^{+\infty} \left[a \cdot e^{-j\omega}\right]^n = \frac{1}{1 - a e^{-j\omega}}$$

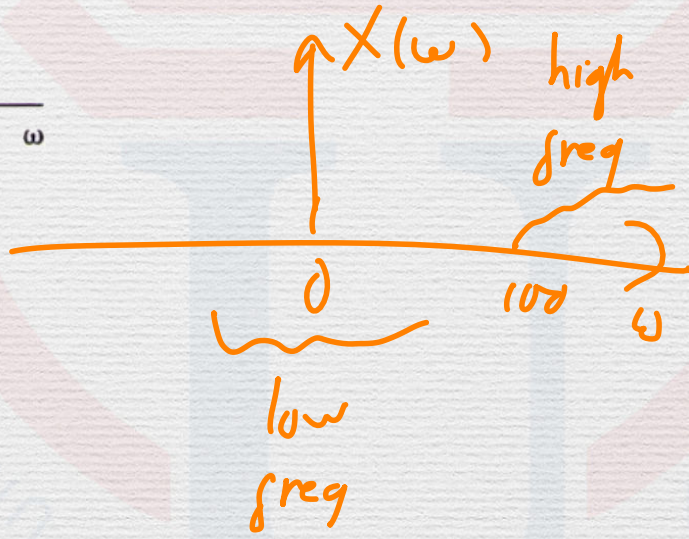
Exercise 1, $a > 0$



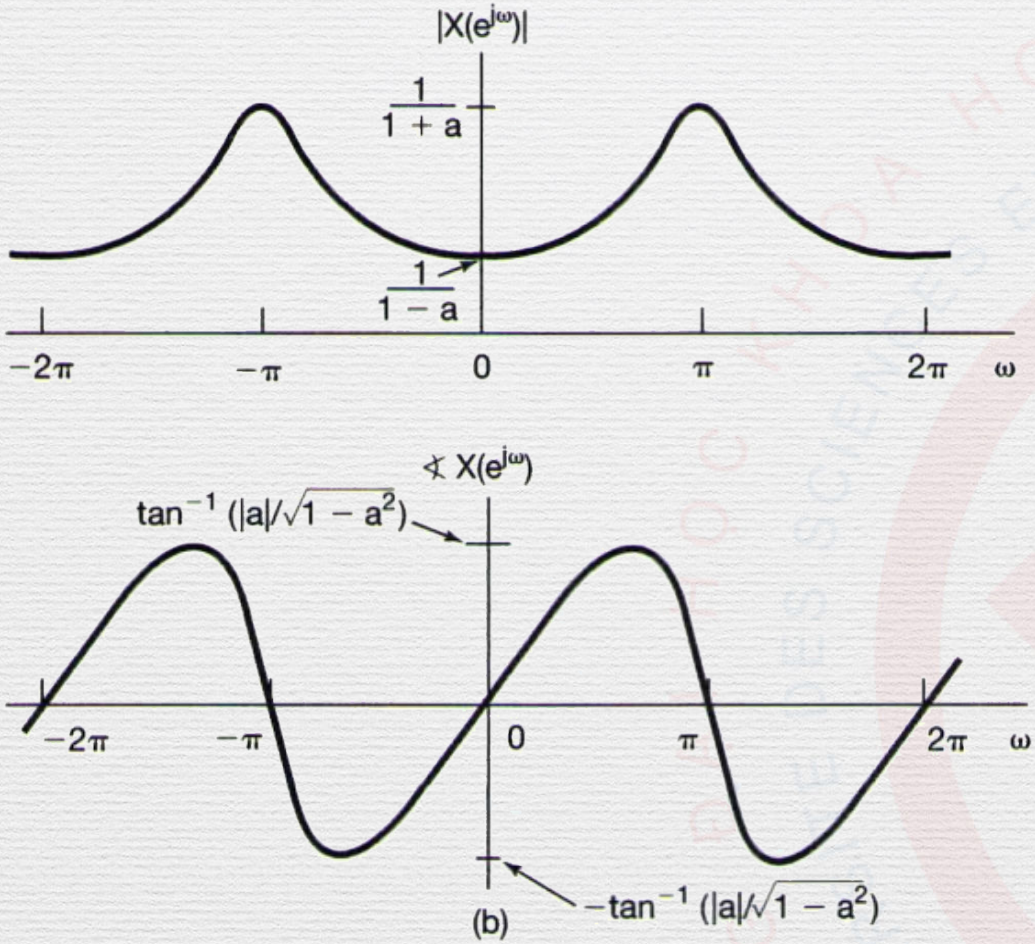
Complex

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$= |X(e^{j\omega})| \cdot e^{j\Delta X(\omega)}$



Exercise 1, $a < 0$



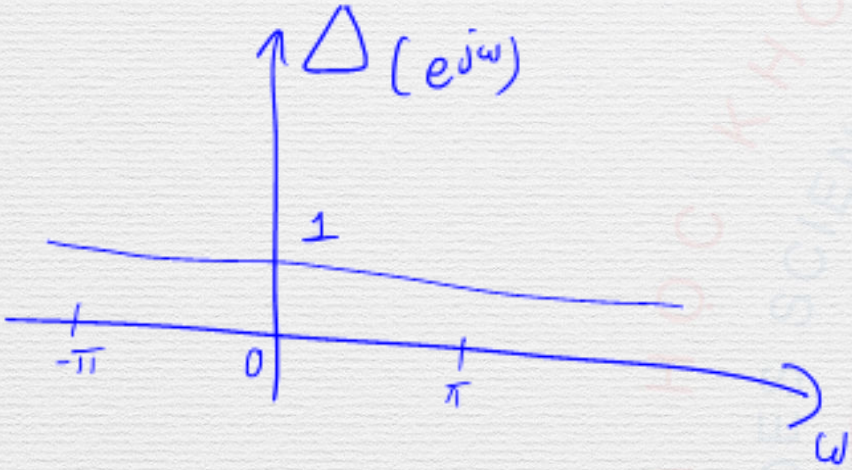
$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

Exercise 2

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$$

Exercise

Determine the Fourier Transform of the Delta signal.



$$\Delta(e^{j\omega}) = ?$$

Exercise 3

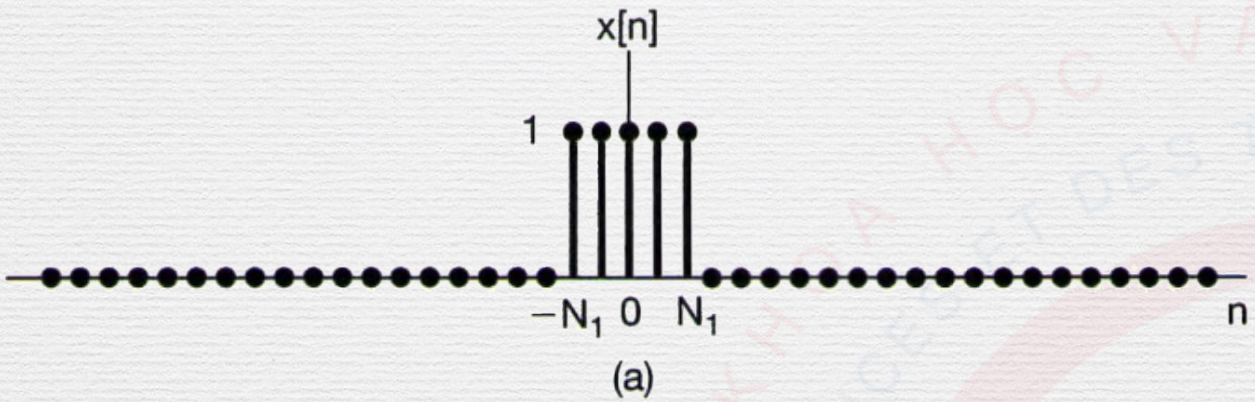
Exercise

Find the Fourier Transform of the rectangular pulse signal

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = 1 + 2\sum_{n=1}^{N_1} \cos(n\omega)$$

Exercise 3 for $N_1 = 2$



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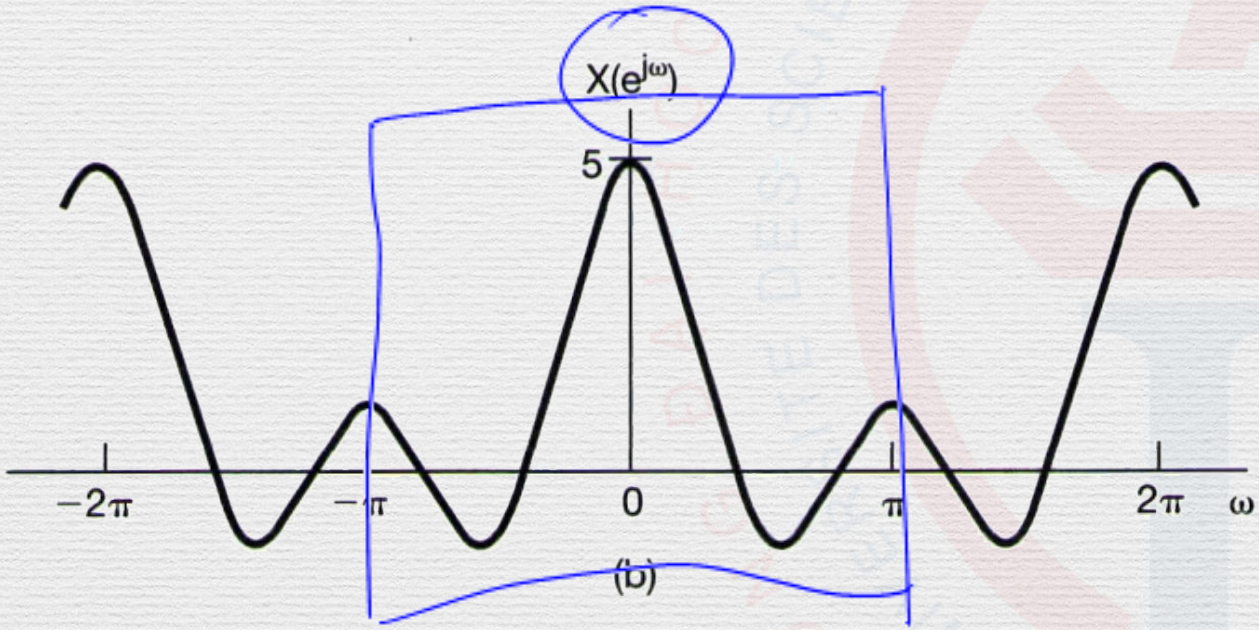

(b)

1 Lesson Objectives

2 Discrete-time Fourier Transform

3 Properties

- Periodicity
- Time Expansion
- Convolution
- Multiplication

4 Fourier Transform for Periodic Signals

5 Homework

Periodicity

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

$e^{j\omega} \cdot e^{j2\pi}$

$\downarrow$

1

Remind: Time Scaling in Continuous-time Domain

$$x(at) \xleftrightarrow{\mathcal{FT}} \frac{1}{|a|} X(\frac{\omega}{a})$$

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We define:

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & else \end{cases}$$

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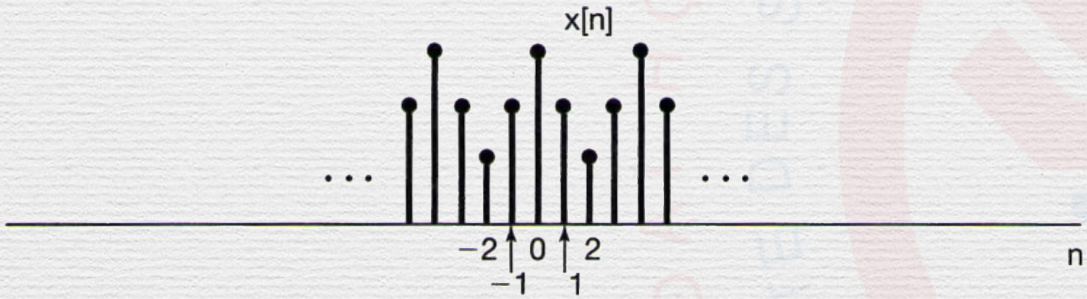
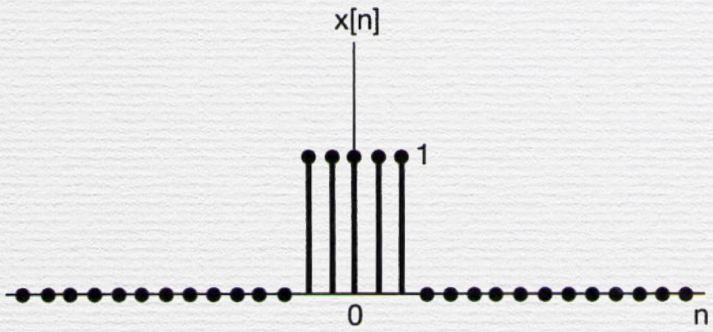
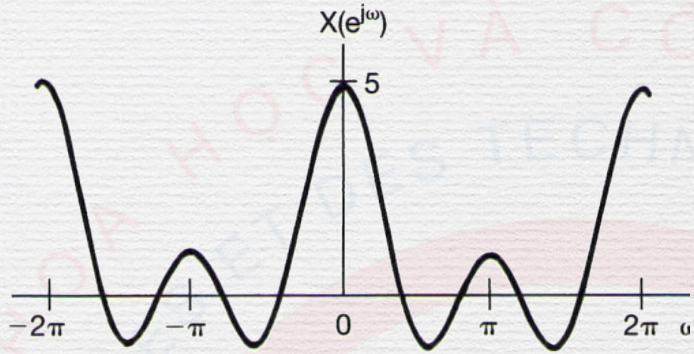
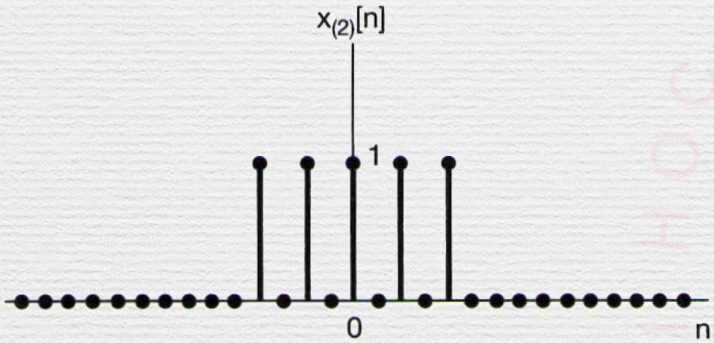
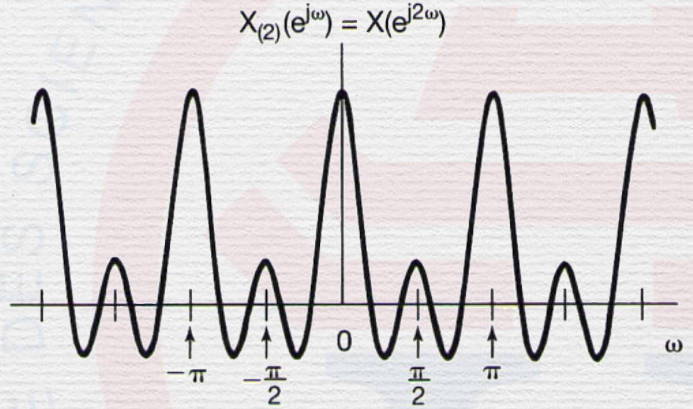


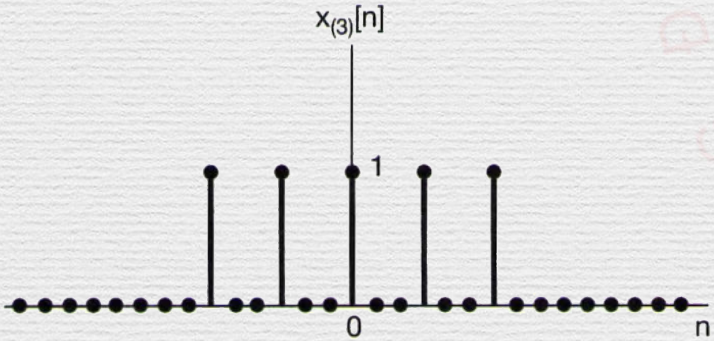
An example

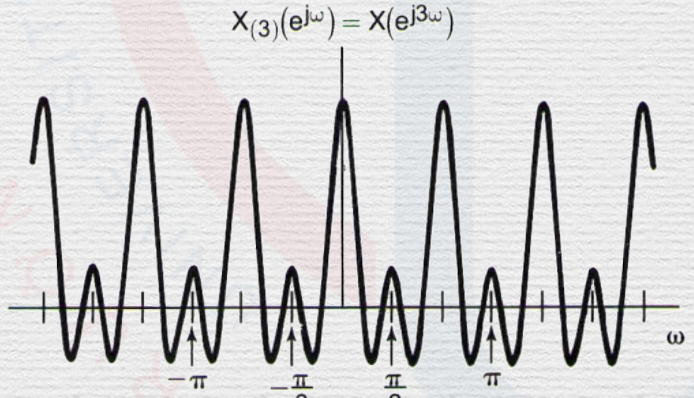












Convolution

Definition

Given

$$x[n] \xleftrightarrow{\mathcal{FT}} X(e^{j\omega})$$

$$h[n] \xleftrightarrow{\mathcal{FT}} H(e^{j\omega})$$

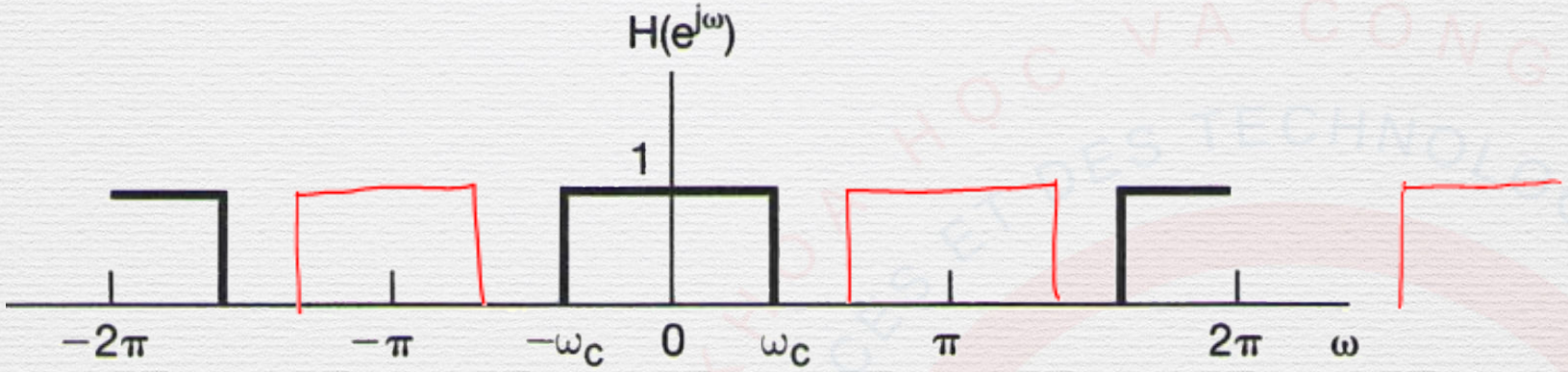
Then

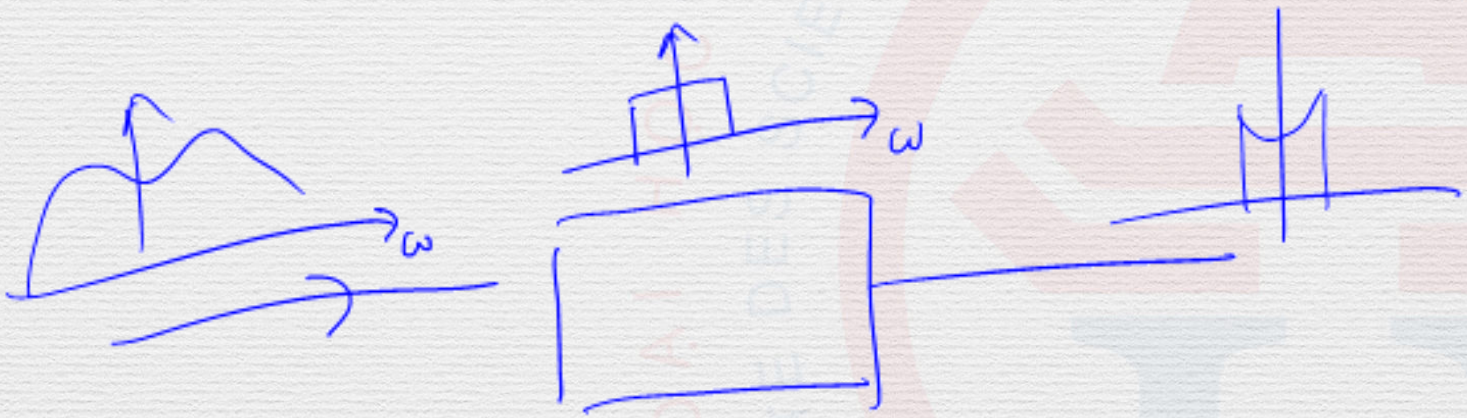
$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{FT}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Exercise 1

Consider a LTI system with impulse response $h[n] = \delta(n - n_0)$

Exercise 2 - Ideal Lowpass Filter





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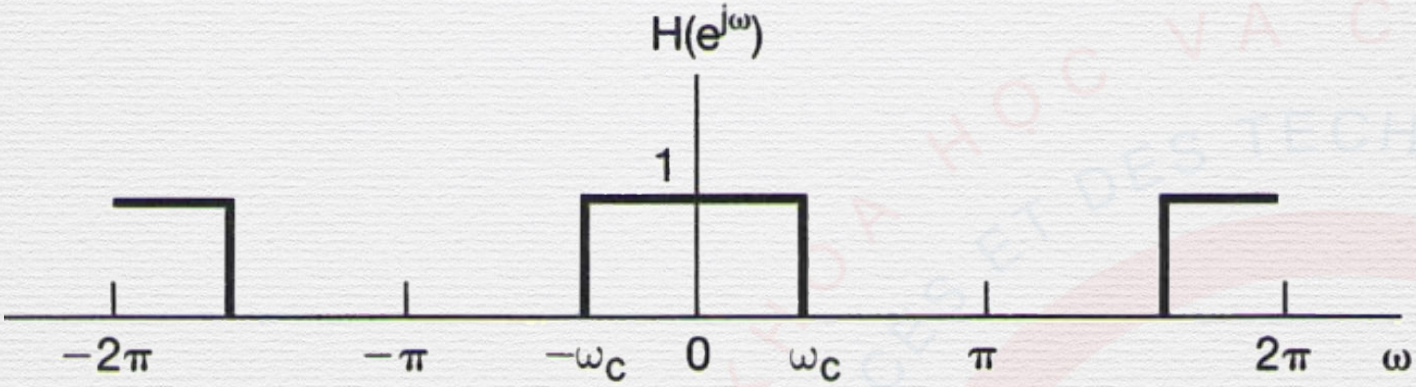
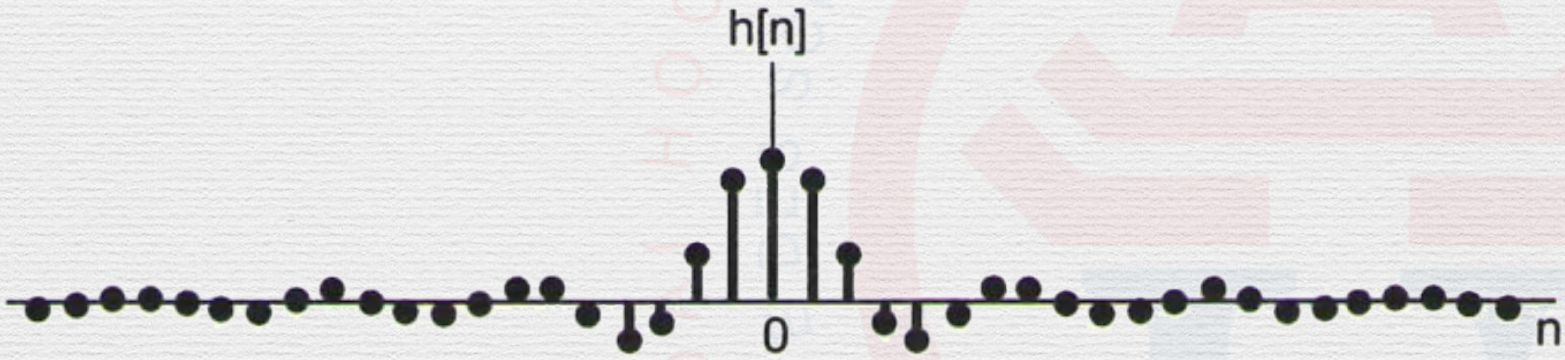


Here we go!



Multiplication

Definition

Given

$$x[n] \xleftrightarrow{\mathcal{FT}} X(e^{j\omega})$$

$$h[n] \xleftrightarrow{\mathcal{FT}} H(e^{j\omega})$$

Then

$$y[n] = x[n]h[n] \xleftrightarrow{\mathcal{FT}} Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$

is called the periodic convolution of $X(e^{j\omega})$ and $H(e^{j\omega})$

Example

Given $x_1[n] = \frac{sin(3\pi n/4)}{\pi n}$ and $x_2[n] = \frac{sin(\pi n/2)}{\pi n}$,
Find the Fourier Transform of $x[n] = x_1[n]x_2[n]$

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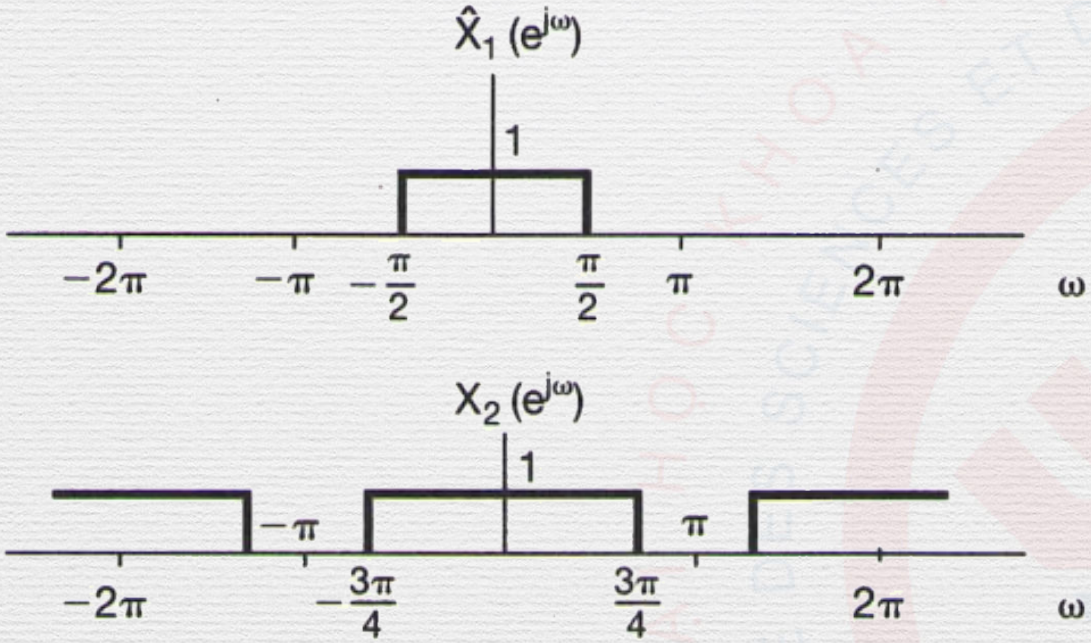

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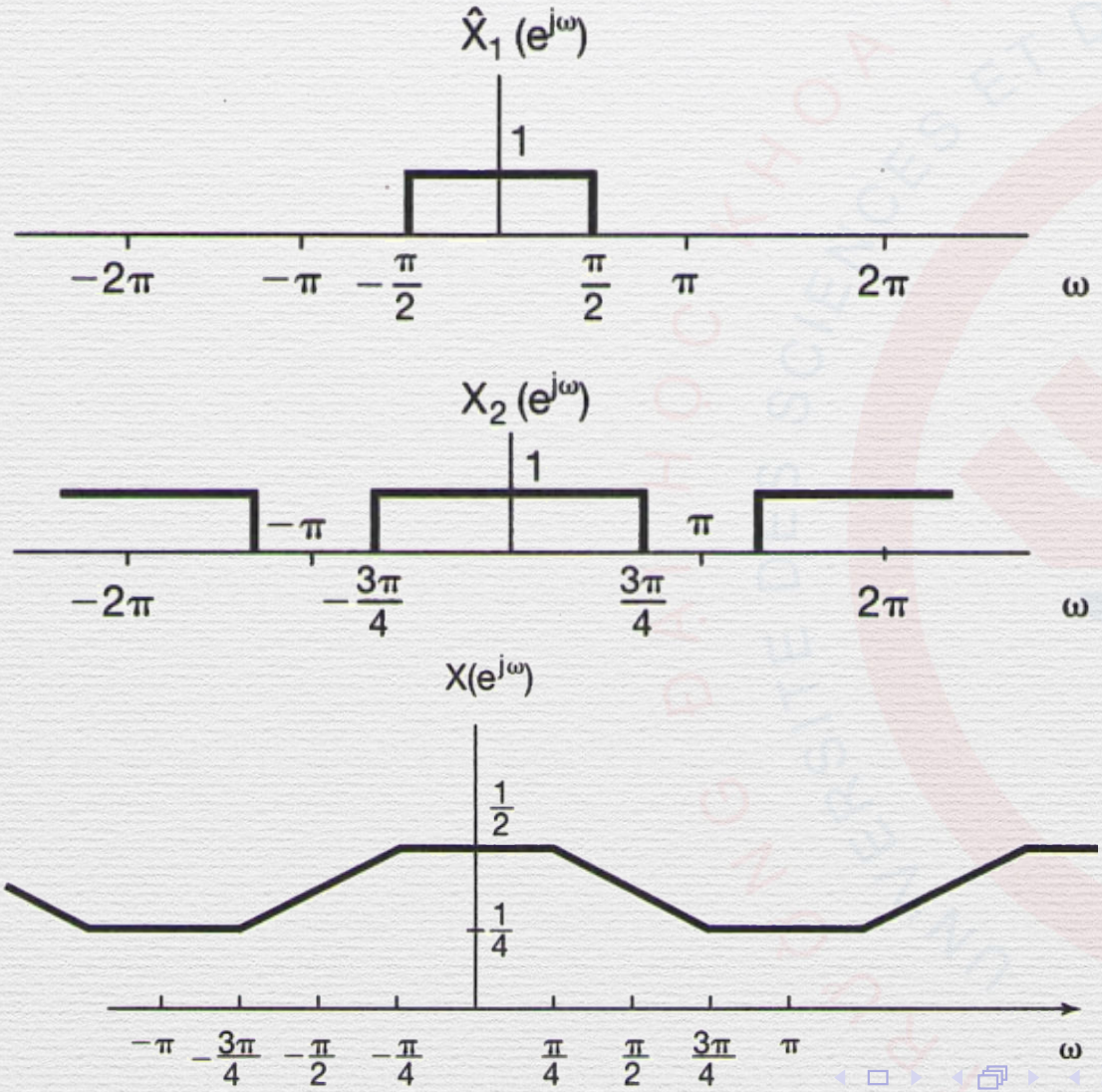
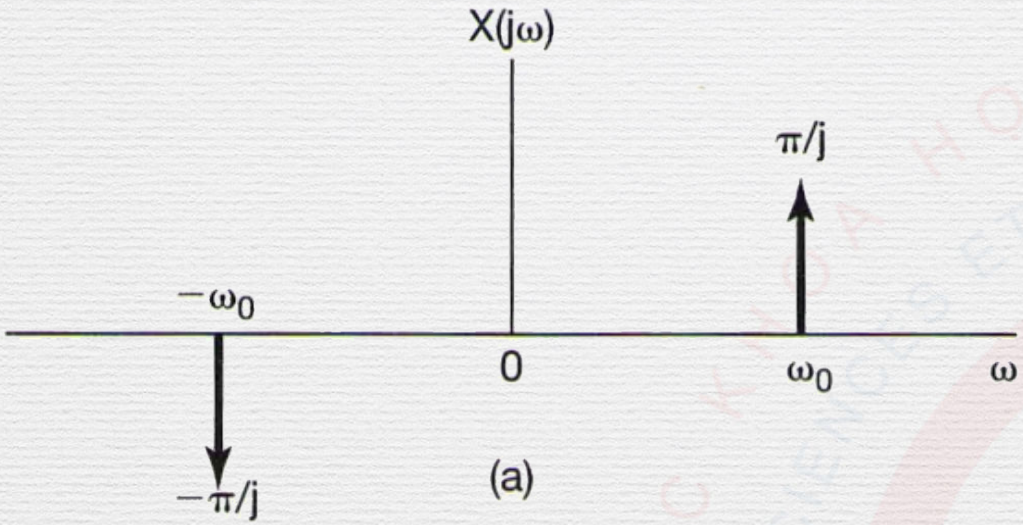


Remind: FT of $sin(\omega_0 t)$ and $cos(\omega_0 t)$



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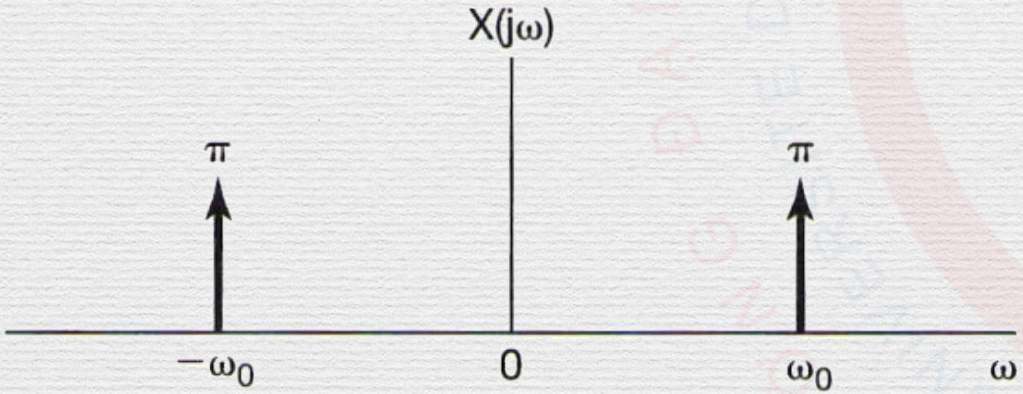

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Exercise 1

Determine Fourier transform of $cos(\omega_0 n)$

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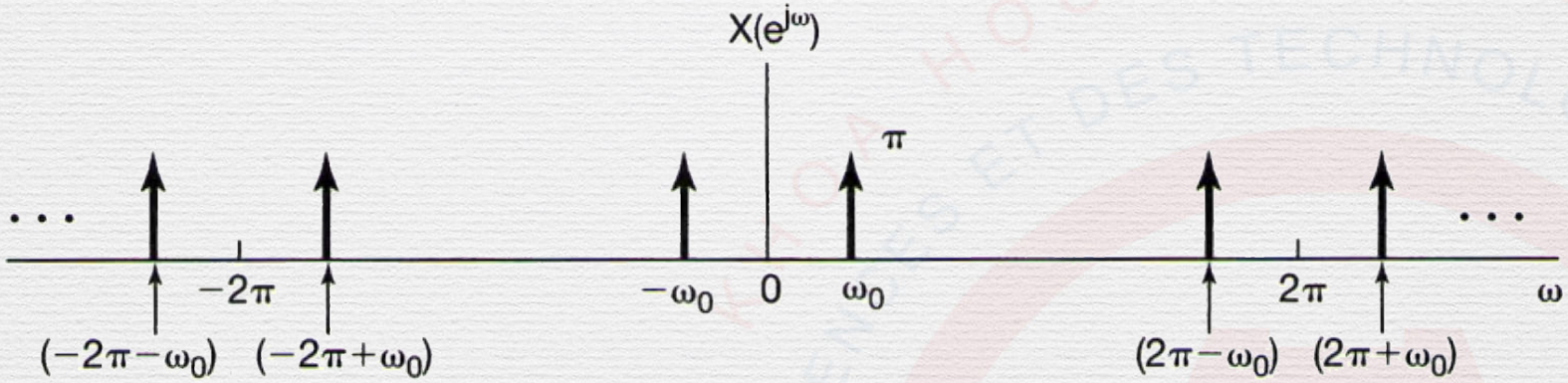

$$X(e^{j\omega}) = \pi \cdot \delta(\omega - \omega_0) + \pi \cdot \delta(\omega + \omega_0)$$

$[-\pi, \pi)$

$x(n) = ?$

Exercise 2

Determine Fourier transform of $x[n] = \sum_{-\infty}^{+\infty} \delta[n - kN]$

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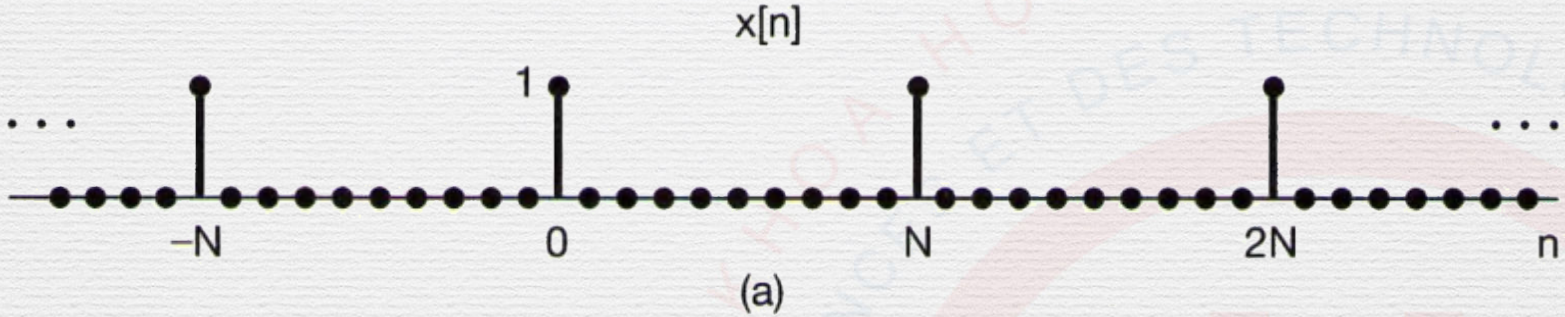

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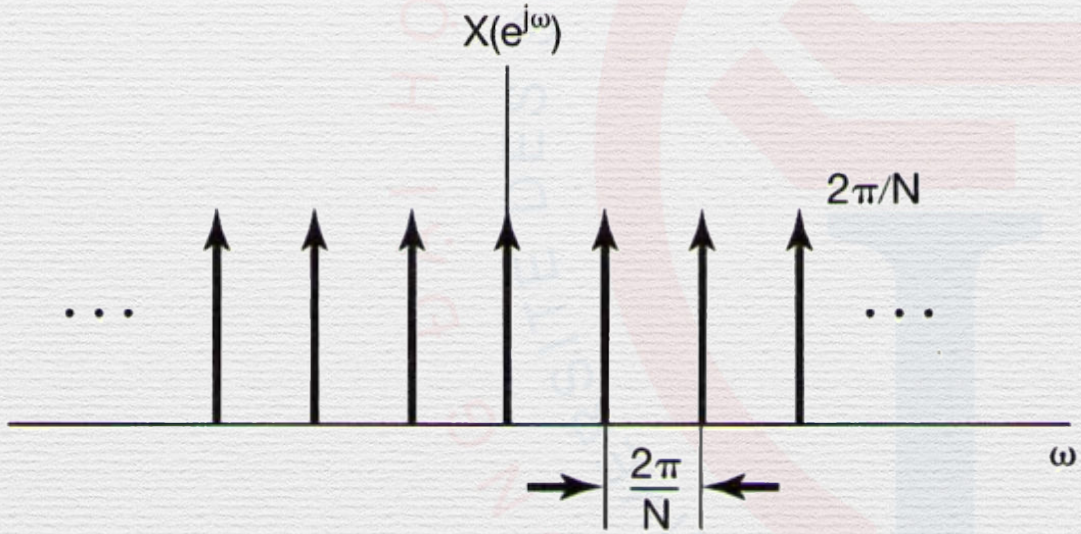

(b)

Homework

Discrete-time Fourier Transform Exercises

5.2, 5.4, 5.6, 5.20, 5.25