Exercises on Constrained Optimization

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Exercise 1. Solve the following optimization problem by using level-set method:

$$\begin{aligned} \min \quad & f(x_1, x_2) = x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 x_2 = 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Exercise 2. Solve the following optimization problem by using level-set method:

$$\begin{aligned} \min \quad & f(x_1, x_2) = x_1 - x_2^2 + 4x_2 \\ \text{s.t.} \quad & 4x_1 - x_2 \leq 12 \\ & -x_1 + x_2 \leq 3 \\ & x_1 + 2x_2 \leq 12 \\ & x_1 \geq 1 \\ & x_2 \geq 0 \end{aligned}$$

Exercise 3. Solve the following optimization problem by using level-set method:

$$\begin{aligned} \max \quad & f(x_1, x_2) = 3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 25 \\ & x_1 x_2 \geq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Exercise 4. Using level-set method, find the minimum and maximum of the following function

$$f(x_1, x_2) = (x_1 - 4)^2 + (x_2 - 3)^2$$

subjected to the following constraints

$$\begin{aligned} 2x_1 + 3x_2 &\geq 6 \\ 3x_1 - 2x_2 &\leq 18 \\ -x_1 + 2x_2 &\leq 8 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Exercise 5. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are farthest to $(3, 1, -1)$.

Exercise 6. Find global maximum and global minimum of the function:

$$\begin{aligned} f : [-2, 2] \times [-2, 2] &\to \mathbb{R} \\ (x, y) &\mapsto 4xy - 2x^2 - y^4. \end{aligned}$$

Exercise 7. Find the extrema (minimum and maximum) of the following functions f subject to the specified constraints.

1. $f(x, y) = x^2 + y^2$ subject to $xy = 1$.
2. $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 3z^2 = 6$.
3. $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x^4 + y^4 + z^4 = 1$.
4. $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$ subject to $x_1^2 + \ldots + x_n^2 = 1$.
5. $f(x, y, z) = yz + xy$ subject to $xy = 1, y^2 + z^2 = 1$.
6. $f(x, y) = xy$ subject to $x + y = 6$.
7. $f(x, y) = xy^2$ subject to $2x^2 + y^2 = 3$.
8. $f(x, y) = x^2 + y^2$ subject to $x^2 + xy + y^2 = 3$.
9. $f(x, y) = (x - 1)^2 + y^2$ subject to $y^2 - 8x = 0$.

Exercise 8. Find global maximum and global minimum of the following functions f subjected to the specified constraints.

1. $f(x, y) = x^2 + y^2 + y - 1$ subject to $x^2 + y^2 \leq 1$.
2. $f(x, y) = x^2 + 2y^2 - x$ subject to $x^2 + y^2 \leq 1$.
3. $f(x, y) = 3 + x^3 - x^2 - y^2$ subject to $x^2 + y^2 \leq 1$ and $x \geq 0$.
4. $f(x, y) = (x - 4)^2 + (y - 4)^2$ subject to $x + y \leq 4$ and $x + 3y \leq 9$.
5. $f(x, y) = x^2y^2$ subject to $2x + y \leq 2$ and $x, y \geq 0$.
6. $f(x, y) = x^2 + y^2 - 4x - 4y$ subject to $x^2 \leq y$, $x + y \geq 2$, and $x, y \geq 0$.
7. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ subject to $x^2 + y^2 \leq 16$.
8. $f(x, y) = e^{-xy}$ subject to $x^2 + 4y^2 \leq 1$.