2.7 Problems

Exercise 1.
A space S and three of its subsets are given by $S = \{1, 3, 5, 7, 9, 11\}$, $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, and $C = \{1, 3, 9, 11\}$. Find $A \cap B \cap C$, $A^c \cap B$, $A \backslash C$, and $(A \backslash B) \cup B$.

Exercise 2.
Let $A = (-\infty, r]$ and $B = (-\infty, s]$ where $r \leq s$. Find an expression for $C = (r, s]$ in terms of A and B. Show that $B = A \cup C$, and $A \cap C = \emptyset$.

Exercise 3. (Video Solution)
Simplify the following sets.

(a) $[1, 4] \cap ([0, 2] \cup [3, 5])$

(b) $([0, 1] \cup [2, 3])^c$

(c) $\bigcap_{i=1}^{\infty} (-1/n, +1/n)$

(d) $\bigcup_{i=1}^{\infty} [5, 8 - (2n)^{-1}]$

Exercise 4.
We will sometimes deal with the relationship between two sets. We say that A implies B when A is a subset of B (why?). Show the following results.

(a) Show that if A implies B, and B implies C, then A implies C.

(b) Show that if A implies B, then B^c implies A^c.

Exercise 5.
Show that if $A \cup B = A$ and $A \cap B = A$, then $A = B$.

Exercise 6.
A space S is defined as $S = \{1, 3, 5, 7, 9, 22\}$, and three subsets as $A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, $C = \{1, 3, 9, 11\}$. Assume that each element has probability $1/6$. Find the following probabilities:

(a) $\mathbb{P}[A]$
(b) $\mathbb{P}[B]$
(c) $\mathbb{P}[C]$
(d) $\mathbb{P}[A \cup B]$
(e) $\mathbb{P}[A \cup C]$
(f) $\mathbb{P}[(A \backslash C) \cup B]$

Exercise 7. (VIDEO SOLUTION)
A collection of 26 letters, a-z, is mixed in a jar. Two letters are drawn at random, one after the other. What is the probability of drawing a vowel (a,e,i,o,u) and a consonant in either order? What is the sample space?

Exercise 8.
Consider an experiment consisting of rolling a die twice. The outcome of this experiment is an ordered pair whose first element is the first value rolled and whose second element is the second value rolled.

(a) Find the sample space.

(b) Find the set A representing the event that the value on the first roll is greater than or equal to the value on the second roll.

(c) Find the set B corresponding to the event that the first roll is a six.

(d) Let C correspond to the event that the first valued rolled and the second value rolled differ by two. Find $A \cap C$.

Note that A, B, and C should be subsets of the sample space specified in Part (a).

Exercise 9.
A pair of dice are rolled.

(a) Find the sample space Ω

(b) Find the probabilities of the events: (i) the sum is even, (ii) the first roll is equal to the second, (iii) the first roll is larger than the second.

Exercise 10.
Let A, B and C be events in an event space. Find expressions for the following:

(a) Exactly one of the three events occurs.

(b) Exactly two of the events occurs.

(c) Two or more of the events occur.

(d) None of the events occur.

Exercise 11.
A system is composed of five components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcomes of the experiment be given by all vectors $(x_1, x_2, x_3, x_4, x_5)$, where x_i is 1 if component i is working and 0 if component i is not working.

(a) How many outcomes are in the sample space of this experiment?

(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all of the outcomes in W.

(c) Let A be the event that components 4 and 5 have both failed. How many outcomes are in the event A?

(d) Write out all outcomes in the event $A \cap W$.

Exercise 12. (VIDEO SOLUTION)
A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x \,|\, x < 0\}$, $B = \{x \,|\, |x - 0.5| < 0.5\}$, $C = \{x \,|\, x > 0.75\}$. Find (a) $\mathbb{P}[A \,|\, B]$, (b) $\mathbb{P}[B \,|\, C]$, (c) $\mathbb{P}[A \,|\, C^c]$, (d) $\mathbb{P}[B \,|\, C^c]$.

Exercise 13. (VIDEO SOLUTION)
Let the events A and B have $\mathbb{P}[A] = x$, $\mathbb{P}[B] = y$ and $\mathbb{P}[A \cup B] = z$. Find the following probabilities: (a) $\mathbb{P}[A \cap B]$, (b) $\mathbb{P}[A^c \cap B^c]$, (c) $\mathbb{P}[A^c \cup B^c]$, (d) $\mathbb{P}[A \cap B^c]$, (e) $\mathbb{P}[A^c \cup B]$.

Exercise 14.

(a) By using the fact that $\mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$, show that $\mathbb{P}[A \cup B \cup C] \leq \mathbb{P}[A] + \mathbb{P}[B] + \mathbb{P}[C]$.

(b) By using the fact that $\mathbb{P}\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} \mathbb{P}[A_k]$, show that

$$\mathbb{P}\left[\bigcap_{k=1}^{n} A_k\right] \geq 1 - \sum_{k=1}^{n} \mathbb{P}[A_k^c].$$

Exercise 15.
Use the distributive property of set operations to prove the following generalized distributive law:

$$A \cup \left(\bigcap_{i=1}^{n} B_i\right) = \bigcap_{i=1}^{n} (A \cup B_i).$$

Hint: Use mathematical induction. That is, show that the above is true for $n = 2$ and that it is also true for $n = k + 1$ when it is true for $n = k$.

Exercise 16.
The following result is known as the Bonferroni's Inequality.

(a) Prove that for any two events A and B, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

(b) Generalize the above to the case of n events $A_1, A_2, \ldots, A_n$, by showing that

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n) - (n - 1).$$

Hint: You may use the generalized Union Bound $\mathbb{P}(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$.

Exercise 17. (VIDEO SOLUTION)
Let A, B, C be events with probabilities $\mathbb{P}[A] = 0.5$, $\mathbb{P}[B] = 0.2$, $\mathbb{P}[C] = 0.4$. Find

(a) $\mathbb{P}[A \cup B]$ if A and B are independent.

(b) $\mathbb{P}[A \cup B]$ if A and B are disjoint.

(c) $\mathbb{P}[A \cup B \cup C]$ if A, B and C are independent.

(d) $\mathbb{P}[A \cup B \cup C]$ if A, B and C are pairwise disjoint; can this happen?

Exercise 18. (Video Solution)
A block of information is transmitted repeated over a noisy channel until an error-free block is received. Let $M \geq 1$ be the number of blocks required for a transmission. Define the following sets.

(i) $A = \{M \text{ is even}\}$

(ii) $B = \{M \text{ is a multiple of } 3\}$

(iii) $C = \{M \text{ is less than or equal to } 6\}$

Assume that the probability of requiring one additional block is half of the probability without the additional block. That is:

$$\mathbb{P}[M = k] = \left(\frac{1}{2}\right)^k, \qquad k = 1, 2, \ldots.$$

Determine the following probabilities.

(a) $\mathbb{P}[A]$, $\mathbb{P}[B]$, $\mathbb{P}[C]$, $\mathbb{P}[C^c]$

(b) $\mathbb{P}[A \cap B]$, $\mathbb{P}[A \backslash B]$, $\mathbb{P}[A \cap B \cap C]$

(c) $\mathbb{P}[A \,|\, B]$, $\mathbb{P}[B \,|\, A]$

(d) $\mathbb{P}[A \,|\, B \cap C]$, $\mathbb{P}[A \cap B \,|\, C]$

Exercise 19. (Video Solution)
A binary communication system transmits a signal X that is either a $+2$-voltage signal or a -2-voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal. Possible values of Y are listed below.

	2 Heads	1 Head	No Head
$X = -2$	$Y = 0$	$Y = -1$	$Y = -2$
$X = +2$	$Y = 0$	$Y = +1$	$Y = +2$

Assume that the probability of having $X = +2$ and $X = -2$ is equal.

(a) Find the sample space of Y, and hence the probability of each value of Y.

(b) What are the probabilities $\mathbb{P}[X = +2 \,|\, Y = 1]$ and $\mathbb{P}[Y = 1 \,|\, X = -2]$?

Exercise 20. (Video Solution)
A block of 100 bits is transmitted over a binary communication channel with a probability of bit error $p = 10^{-2}$.

(a) If the block has 1 or fewer errors, then the receiver accepts the block. Find the probability that the block is accepted.

(b) If the block has more than 1 error, then the block is retransmitted. What is the probability that 4 blocks are transmitted?

Exercise 21. (Video Solution)
A machine makes errors in a certain operation with probability p. There are two types of errors. The fraction of errors that are type A is α and the fraction that are type B is $1-\alpha$.

(a) What is the probability of k errors in n operations?

(b) What is the probability of k_1 type A errors in n operations?

(c) What is the probability of k_2 type B errors in n operations?

(d) What is the joint probability of k_1 type A errors and k_2 type B errors in n operations? Hint: There are $\binom{n}{k_1}\binom{n-k_1}{k_2}$ possibilities of having k_1 type A errors and k_2 type B errors in n operations. (Why?)

Exercise 22. (Video Solution)
A computer manufacturer uses chips from three sources. Chips from sources A, B and C are defective with probabilities 0.005, 0.001 and 0.01, respectively. The proportions of chips from A, B and C are 0.5, 0.1 and 0.4 respectively. If a randomly selected chip is found to be defective, find

(a) the probability that the chips are from A.

(b) the probability that the chips are from B.

(c) the probability that the chips are from C.

Exercise 23. (Video Solution)
In a lot of 100 items, 50 items are defective. Suppose that m items are selected for testing. We say that the manufacturing process is malfunctioning if the probability that one or more items are tested to be defective. Call this failure probability p. What should be the minimum m such that $p \geq 0.99$?

Exercise 24. (Video Solution)
One of two coins is selected at random and tossed three times. The first coin comes up heads with probability $p_1 = 1/3$ and the second coin with probability $p_2 = 2/3$.

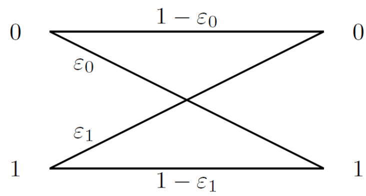
(a) What is the probability that the number of heads is $k = 3$?

(b) Repeat (a) for $k = 0, 1, 2$.

(c) Find the probability that coin 1 was tossed given that k heads were observed, for $k = 0, 1, 2, 3$.

(d) In part (c), which coin is more probably when 2 heads have been observed?

Exercise 25. (VIDEO SOLUTION)
Consider the following communication channel. A source transmits a string of binary symbols through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1 - p$, respectively, and is received incorrectly with probability ε_0 and ε_1. Errors in different symbols transmissions are independent.



Denote S as the source and R as the receiver.

(a) What is the probability that a symbol is correctly received? Hint: Find

$$\mathbb{P}[R = 1 \cap S = 1] \quad \text{and} \quad \mathbb{P}[R = 0 \cap S = 0].$$

(b) Find the probability of receiving 1011 conditioned on that 1011 was sent, i.e.,

$$\mathbb{P}[R = 1011 \,|\, S = 1011].$$

(c) To improve reliability, each symbol is transmitted three times, and the received string is decoded by the majority rule. In other words, a 0 (or 1) is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that the symbol is correctly decoded, given that we send a 0?

(d) Suppose that the scheme of part (c) is used. What is the probability that a 0 was sent if the string 101 was received?

(e) Suppose the scheme of part (c) is used and given that a 0 was sent. For what value of ε_0 is there an improvement in the probability of correct decoding? Assume that $\varepsilon_0 \neq 0$.