3.8 Problems

Exercise 1. (Video Solution)
Consider an information source that produces numbers k in the set $S_X = \{1, 2, 3, 4\}$. Find and plot the PMF in the following cases:

(a) $p_k = p_1/k$, for $k = 1, 2, 3, 4$. Hint: Find p_1.

(b) $p_{k+1} = p_k/2$ for $k = 1, 2, 3$.

(c) $p_{k+1} = p_k/2^k$ for $k = 1, 2, 3$.

(d) Can the random variables in parts (a)-(c) be extended to take on values in the set $\{1, 2, \ldots\}$? Why or why not? Hint: You may use the fact that the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges.

Exercise 2. (Video Solution)
Two dice are tossed. Let X be the absolute difference in the number of dots facing up.

(a) Find and plot the PMF of X.

(b) Find the probability that $X \leq 2$.

(c) Find $\mathbb{E}[X]$ and $\text{Var}[X]$.

Exercise 3. (Video Solution)
Let X be a random variable with PMF $p_k = c/2^k$ for $k = 1, 2, \ldots$.

(a) Determine the value of c.

(b) Find $\mathbb{P}(X > 4)$ and $\mathbb{P}(6 \leq X \leq 8)$.

(c) Find $\mathbb{E}[X]$ and $\text{Var}[X]$.

Exercise 4.
Let X be a random variable with PMF $p_k = c/2^k$ for $k = -1, 0, 1, 2, 3, 4, 5$.

(a) Determine the value of c.

(b) Find $\mathbb{P}(1 \leq X < 3)$ and $\mathbb{P}(1 < X \leq 5)$.

(c) Find $\mathbb{P}[X^3 < 5]$.

(d) Find the PMF and the CDF of X.

Exercise 5. (VIDEO SOLUTION)
A modem transmits a +2 voltage signal into a channel. The channel adds to this signal a noise term that is drawn from the set $\{0, -1, -2, -3\}$ with respective probabilities $\{4/10, 3/10, 2/10, 1/10\}$.

(a) Find the PMF of the output Y of the channel.

(b) What is the probability that the channel's output is equal to the input of the channel?

(c) What is the probability that the channel's output is positive?

(d) Find the expected value and variance of Y.

Exercise 6.
On a given day, your golf score takes values from numbers 1 through 10, with equal probability of getting each one. Assume that you play golf for three days, and assume that your three performances are independent. Let X_1, X_2, and X_3 be the scores that you get, and let X be the minimum of these three numbers.

(a) Show that for any discrete random variable X, $p_X(k) = \mathbb{P}(X > k-1) - \mathbb{P}(X > k)$.

(b) What is the probability $\mathbb{P}(X_1 > k)$ for $k = 1, \ldots, 10$?

(c) Use (a), determine the PMF $p_X(k)$, for $k = 1, \ldots, 10$.

(d) What is the average score improvement if you play just for one day compared with playing for three days and taking the minimum?

Exercise 7. (VIDEO SOLUTION)
Let

$$g(X) = \begin{cases} 1, & \text{if } X > 10 \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad h(X) = \begin{cases} X - 10, & \text{if } X - 10 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[g(X)]$ for X as in Problem 1(a) with $S_X = \{1, \ldots, 15\}$.

(b) Find $\mathbb{E}[h(X)]$ for X as in Problem 1(b) with $S_X = \{1, \ldots, 15\}$.

Exercise 8. (VIDEO SOLUTION)
A voltage X is uniformly distributed in the set $\{-3, \ldots, 3, 4\}$.

(a) Find the mean and variance of X.

(b) Find the mean and variance of $Y = -2X^2 + 3$.

(c) Find the mean and variance of $W = \cos(\pi X/8)$.

(d) Find the mean and variance of $Z = \cos^2(\pi X/8)$.

Exercise 9. (VIDEO SOLUTION)

(a) If X is Poisson(λ), compute $\mathbb{E}[1/(X+1)]$.

(b) If X is Bernoulli(p) and Y is Bernoulli(q), compute $\mathbb{E}[(X+Y)^3]$ if X and Y are independent.

(c) Let X be a random variable with mean μ and variance σ^2. Let $\Delta(\theta) = \mathbb{E}[(X-\theta)^2]$. Find θ that minimizes the error $\Delta(\theta)$.

(d) Suppose that $X_1, \ldots, X_n$ are independent uniform random variables in $\{0, 1, \ldots, 100\}$. Evaluate $\mathbb{P}[\min(X_1, \ldots, X_n) > \ell]$ for any $\ell \in \{0, 1, \ldots, 100\}$.

Exercise 10. (Video Solution)

(a) Consider the binomial probability mass function $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$. Show that the mean is $\mathbb{E}[X] = np$.

(b) Consider the geometric probability mass function $p_X(k) = p(1-p)^k$ for $k = 0, 1, \ldots$. Show that the mean is $\mathbb{E}[X] = (1-p)/p$.

(c) Consider the Poisson probability mass function $p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$. Show that the variance is $\text{Var}[X] = \lambda$.

(d) Consider the uniform probability mass function $p_X(k) = \frac{1}{L}$ for $k = 1, \ldots, L$. Show that the variance is $\text{Var}[X] = \frac{L^2-1}{12}$. Hint: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ and $1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.

Exercise 11. (Video Solution)
An audio player uses a low-quality hard drive. The probability that the hard drive fails after being used for one month is 1/12. If it fails, the manufacturer offers a free-of-charge repair for the customer. For the cost of each repair, however, the manufacturer has to pay \$20. The initial cost of building the player is \$50, and the manufacturer offers a 1-year warranty. Within one year, the customer can ask for a free repair up to 12 times.

(a) Let X be the number of months when the player fails. What is the PMF of X? Hint: $\mathbb{P}[X = 1]$ may not be very high because if the hard drive fails it will be fixed by the manufacturer. Once fixed, the drive can fail again in the remaining months. So saying $X = 1$ is equivalent to saying that there is only one failure in the entire 12-month period.

(b) What is the average cost per player?

Exercise 12. (Video Solution)
A binary communication channel has a probability of bit error of $p = 10^{-6}$. Suppose that transmission occurs in blocks of 10,000 bits. Let N be the number of errors introduced by the channel in a transmission block.

(a) What is the PMF of N?

(b) Find $\mathbb{P}[N = 0]$ and $\mathbb{P}[N \leq 3]$.

(c) For what value of p will the probability of 1 or more errors in a block be 99%?

Hint: Use the Poisson approximation to binomial random variables.

Exercise 13. (VIDEO SOLUTION)
The number of orders waiting to be processed is given by a Poisson random variable with parameter $\alpha = \lambda/n\mu$, where λ is the average number of orders that arrive in a day, μ is the number of orders that an employee can process per day, and n is the number of employees. Let $\lambda = 5$ and $\mu = 1$. Find the number of employees required so the probability that more than four orders are waiting is less than 10%.

Hint: You need to use trial and error for a few n's.

Exercise 14.
Let X be the number of photons counted by a receiver in an optical communication system. It is known that X is a Poisson random variable with a rate λ_1 when a signal is present and a Poisson random variable with the rate $\lambda_0 < \lambda_1$ when a signal is absent. The probability that the signal is present is p. Suppose that we observe $X = k$ photons. We want to determine a threshold T such that if $k \geq T$ we claim that the signal is present, and if $k < T$ we claim that the signal is absent. What is the value of T?