Exercises on Linear Programming

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LP formulations

Exercise 1. Transform the following LP into its standard formulation and its canonical formulation.

$$\begin{aligned} \min \quad & x_1 - x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_3 \leq 3 \\ & -x_1 + x_2 + x_3 \geq 2 \\ & x_1 - x_2 = 10 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \end{aligned}$$

Exercise 2. Transform the following LP into its standard formulation and its canonical formulation.

$$\begin{aligned} \max \quad & 2x_1 - 3x_2 + 4x_3 - 6x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 - x_4 = 7 \\ & -2x_2 + 8x_3 + 2x_4 \geq 5 \\ & 5x_1 + x_2 - 6x_3 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Exercise 3. (a) Transform the following LP into maximizing problem with unrestricted variables.

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 - 4x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 4 \\ & x_1 - x_3 \leq 5 \\ & x_1 \leq 0 \\ & x_2, x_3 \in \mathbb{R} \end{aligned}$$

(b) Transform the LP in part (a) into minimizing problem with equality constraints.

Structure of feasible sets of LPs

Exercise 4. Find an $\mathcal{H}$-representation of the polytope

$$P = \text{conv}\{(3,0),(6,0),(2,1),(3,2)\}.$$

Exercise 5. Find an $\mathcal{H}$-representation of the polyhedron

$$P = \text{conv}\{(3,0),(2,2)\} + \text{cone}((1,0),(1,1)).$$

Exercise 6. Find a $\mathcal{V}$-representation of the polytope

$$P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$$

in which

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ -1 & -2 \\ 2 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 12 \end{bmatrix}.$$

Exercise 7. (i) Is the point (-2, 5, 4) a convex combination of the points (1, 3, 2), (0, 1, 2), (-3, -2, 1)?

(ii) Is the point (3, 1, 0) an affine combination of the points (1, 3, 2) and (1, 0, -1)?

(iii) Is the point (10, 22, -2) in the conic hull of (2, 2, 4), (3, -3, -2), (1, -3, 3)?

Exercise 8. Let P be the polyhedron defined by the solution set of the following system

$$\begin{aligned} 3x_1 + x_2 + 2x_3 &= 5 \\ 2x_1 + x_2 + 3x_3 &= 5 \\ 2x_1 + x_3 + 2x_4 &= 9 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Is $\mathbf{x} = (1,0,1,3)$ an extreme point of P?

Level-set method for LPs

Exercise 9. Solve the following linear program with a graphical representation.

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \leq 12 \\ & 6x_1 + 5x_2 \leq 30 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Exercise 10. Solve the following linear program by 2D graphical solution method.

$$\begin{aligned} \min \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & -x_1 + x_2 \leq -1 \\ & x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Exercise 11. Solve the following linear program by 2D graphical solution method.

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & -x_1 + x_2 \leq -1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimality of LPs

Exercise 12. Without solving to optimality, show that the following LPs are unbounded. (i)

$$\begin{aligned} \min \quad & 3y_1 - 6y_2 - 6y_3 \\ \text{s.t.} \quad & y_1 - 2y_2 + y_3 \geq 4 \\ & y_1 + y_2 - 2y_3 \geq 7 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

(ii)

$$\begin{aligned} \max \quad & -3x_1 + 6x_2 + 6x_3 \\ \text{s.t.} \quad & -x_1 + 2x_2 - x_3 \leq 4 \\ & -x_1 - x_2 + 2x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Exercise 13. For which values of α the whole line segment AB (with $A = (2, 1)$ and $B = (3, 4)$) belongs to the feasible set of

$$\begin{aligned} 2x - y &\geq \alpha - 2 \\ x - 3y &\leq \alpha + 3 \\ x + y &\geq 2 - 3\alpha \end{aligned}$$

Exercise 14. Find a feasible solution of the following LP that has objective value no less than 70.

$$\begin{aligned} \max \quad & 3x_1 + 12x_2 + x_3 + 10x_4 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 + x_4 \leq 15 \\ & 2x_1 + x_2 + 3x_3 + 2x_4 = 21 \\ & x_1 + 2x_2 + x_3 + 2x_4 \geq 16 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Simplex method

Exercise 15. Solve the following LP by using simplex method.

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Exercise 16. Solve the following LP by using simplex method.

$$\begin{array}{rl} \max & x_1 + 2x_2 \\ \text{s.t.} & x_1 + 3x_2 + x_3 = 9 \\ & 2x_1 + 3x_2 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Exercise 17. Solve the following LP by using simplex method.

$$\begin{array}{rl} \min & -x_1 - 3x_2 \\ \text{s.t.} & x_1 + 3x_2 \geq 3 \\ & -x_1 + x_2 \leq -1 \\ & x_1, x_2 \geq 0 \end{array}$$

Exercise 18. Solve the following LP by using simplex table.

$$\begin{array}{rl} \min & x_2 - 3x_3 + 2x_5 \\ \text{s.t.} & x_1 + \frac{1}{4}x_2 - x_3 + 2x_5 = 7 \\ & -2x_2 + 4x_3 + x_4 = 12 \\ & -4x_2 + 3x_3 + 8x_5 + x_6 = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Exercise 19. Solve the following LP by using simplex table.

$$\begin{array}{rl} \max & -x_1 + 3x_2 - 3x_3 \\ \text{s.t.} & 3x_1 - x_2 - 2x_3 \leq 7 \\ & -2x_1 - 4x_2 + 4x_3 \leq 3 \\ & x_1 - 2x_3 \leq 4 \\ & -2x_1 + 2x_2 + x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Exercise 20. Solve the following LP by using simplex table.

$$\begin{array}{rl} \min & -3x_2 + 2x_4 + 2x_6 \\ \text{s.t.} & x_1 + 5x_4 + x_5 + 2x_6 = 10 \\ & x_2 - x_4 + x_5 = 3 \\ & x_3 - 5x_4 - 3x_5 + 8x_6 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Exercise 21. Solve the following LP by using simplex table.

$$\begin{array}{rl} \min & x_1 + x_2 + x_3 \\ \text{s.t.} & x_1 - x_4 - 2x_6 = 5 \\ & x_2 + 2x_4 - 3x_5 + x_6 = 3 \\ & x_3 + 2x_4 - 5x_5 + 6x_6 = 5 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Exercise 22. Solve the following LP when $\alpha \geq -\frac{5}{3}$.

$$\begin{array}{lrcl} \min & x_1 + \alpha x_2 + x_3 & & \\ \text{s.t.} & x_1 - x_4 - 2x_6 & = & 5 \\ & x_2 + 2x_4 - 3x_5 + x_6 & = & 3 \\ & x_3 + 2x_4 - 5x_5 + 6x_6 & = & 5 \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$

Exercise 23. Solve the following LP with respect to α.

$$\begin{array}{lrcl} \min & -x_1 + 2x_2 - \alpha x_3 & & \\ \text{s.t.} & x_1 + x_2 - x_3 & \leq & 8 \\ & 2x_1 + 3x_3 & \leq & 4 \\ & x_1 - x_2 + x_3 & \leq & 6 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Exercise 24. Solve the following LP with respect to $\alpha \in [-3, 2]$.

$$\begin{array}{lrcl} \min & (1-\alpha)x_1 + (2-\alpha)x_2 & & \\ \text{s.t.} & x_1 - 2x_2 + x_3 & = & 0 \\ & 2x_1 + 3x_2 + x_4 & = & 5 \\ & -x_1 + 2x_2 + x_5 & = & 5 \\ & x_1, \ldots, x_5 & \geq & 0 \end{array}$$

Exercise 25. For which values of α the following LP is feasible? Solve the LP for such values of α.

$$\begin{array}{lrcl} \min & -3x_1 - 2x_2 - 5x_3 & & \\ \text{s.t.} & x_1 + 2x_2 + x_3 + x_4 & = & 40 - \alpha \\ & 3x_1 + 2x_3 + x_5 & = & 60 + 2\alpha \\ & x_1 + 4x_2 + x_6 & = & 30 - 5\alpha \\ & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$