Classify Optimization Problems

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Contents

General form of an optimization problem

- **In words:**
  Make *the best decision* with respect to *some criteria* while satisfying *some constraints*

- **General formulation:**

$$\text{maximize / minimize} \quad f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{x} \in C$$

in which $C \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$

  - $\mathbf{x} = (x_1, \ldots, x_n)$: *decision variables*
  - $f(\mathbf{x})$: *objective function*
  - C: *feasible set*

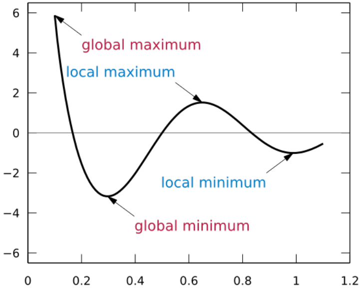
Particular case:

$$C = \{\mathbf{x} \in X \subseteq \mathbb{R}^n \mid g_i(\mathbf{x}) \leq 0, i = 1, \ldots, m; h_j(\mathbf{x}) = 0, j = 1, \ldots, \ell\}$$

  - X: *domain of decision variables*
  - $g_i(x) \leq 0$ $(i = 1, \ldots, m)$, $h_j(\mathbf{x}) = 0$ $(j = 1, \ldots, \ell)$: *constraints*

Solutions of optimization problems: illustration

$$\begin{array}{rl} \text{optimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in C \subset \mathbb{R}^n \end{array}$$



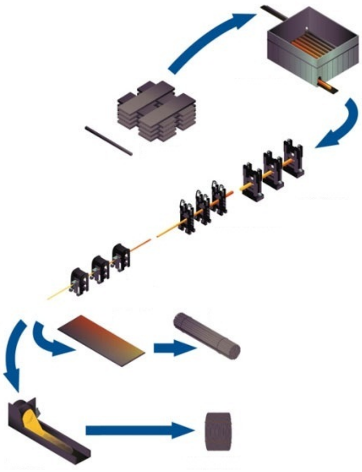
- *Feasible solution:* any $x \in C$
- *(Global) optimal solution:* best solution in feasible set
- *Optimal objective value:* value of f at optimal solution
- *Local optimal solution:* best solution in comparison to feasible solutions near by
- *Local optimum:* best objective value in comparison with objective values near by

Contents

List of examples

- Rolling mill
- Transportation problem
- Production problem
- Knapsack problem
- Shortest path problem
- Traveling salesman problem
- Diet problem of Stigler
- Isoperimetric problem

Rolling mill



http://smwires.com/Process.aspx

- **Input:** steel slabs
- **Output:** steel bands & coils

	Bands	Coils
Producing rate	200 tons/h	140 tons/h
Profit	25$/ton	30$/ton
Capacity	6000 tons	4200 tons

- **Constraints:**
  - $\leq$ 40 hours/week
  - either bands or coils at a time, but not both
- **Objective:** producing plan of maximum profit

Rolling mill

x_1 = hours of producing bands
x_2 = hours of producing coils

$$\max \quad 5000x_1 + 4200x_2$$

subject to

$$x_1 + x_2 \leq 40$$
$$200x_1 \leq 6000$$
$$140x_2 \leq 4200$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

- **Input:** steel slabs
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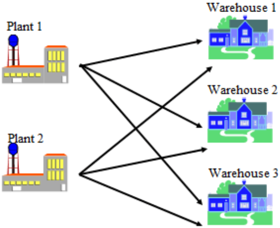
Transportation problem

- **Goal:** Minimize cost of transporting products from plants to warehouses regarding plants' supply and warehouses' demand

- **Input:**
  - m plants, n warehouses
  - s_i = supply of plant i
  - d_j = demand of warehouse j
  - c_{ij} = transportation cost from plant i to warehouse j



- **Variables:** x_{ij} = amount of products to send $i \to j$

Transportation problem

- **Formulation**

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} = d_j & (j = 1, \dots, n) \\
& \sum_{j=1}^{n} x_{ij} = s_i & (i = 1, \dots, m) \\
& x_{ij} \geq 0 & (i = 1, \dots, m; j = 1, \dots, n)
\end{aligned}
$$

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Production problem

- **Goal:** Maximize revenue of goods made from raw materials
- **Input:**
  - m different raw materials, n different goods
  - b_i = available amount of material i
  - a_{ij} = amount of material i to produce good j
  - c_j = revenue of one unit of good j
- **Variables:** x_j = amount of good j that should be produced
- **Formulation**

$$
\begin{array}{lrl}
\max & c_1x_1 + \ldots + c_nx_n & \\
\text{s.t.} & a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i & (i = 1, \ldots, m) \\
& x_j \geq 0 & (j = 1, \ldots, n)
\end{array}
$$

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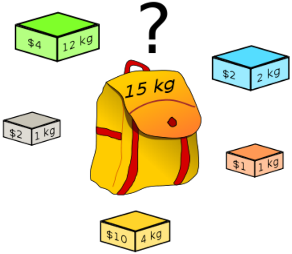
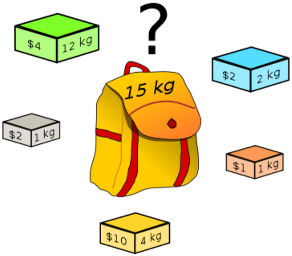
Classical knapsack problem

- Given a set of items $\{1, \ldots, n\}$
- Item i has weight w_i and value c_i
- Given a knapsack that can hold a weight at most W
- Which items should be put into the knapsack
  so that the total weight $\leq W$
  and the total value is as large as possible



Classical knapsack problem



Variables: $x_i = 1$ if item i is chosen, $= 0$ otherwise

Formulation:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} c_i x_i \\
\text{subject to} \quad & \sum_{i=1}^{n} w_i x_i \leq W \\
& x_i \in \{0, 1\} \qquad \forall i = 1, \ldots, n
\end{aligned}
$$

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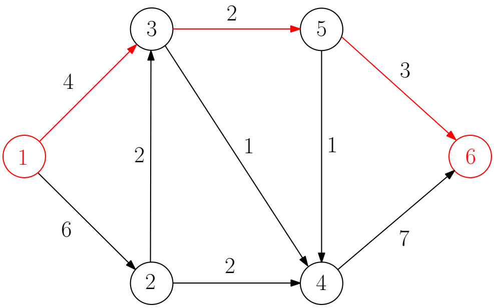
Shortest path problem

Input:

- a directed network $G = (N, A)$
- arc length c_{ij} associated with each $(i, j) \in A$
- two distinct nodes $s, t \in N$

Definition: Length of a path = sum of lengths of arcs in the path

Objective: Find the shortest path from s to t



Shortest path problem

Variables:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is in the shortest path from } s \text{ to } t, \\ 0 & \text{otherwise.} \end{cases}$$

Formulation:

$$\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} & \\
\text{s.t.} \quad & \sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0 & \forall i \in N \backslash \{s,t\} \\
& \sum_{(s,j)\in A} x_{sj} - \sum_{(j,s)\in A} x_{js} = 1 & \\
& \sum_{(t,j)\in A} x_{tj} - \sum_{(j,t)\in A} x_{jt} = -1 & \\
& x_{ij} \in \{0,1\} & \forall (i,j) \in A
\end{aligned}$$

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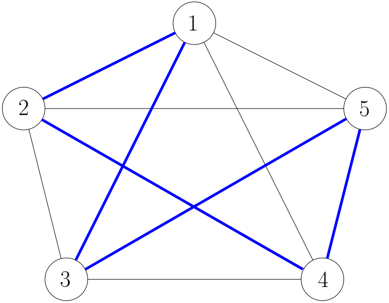
Traveling salesman problem (TSP)

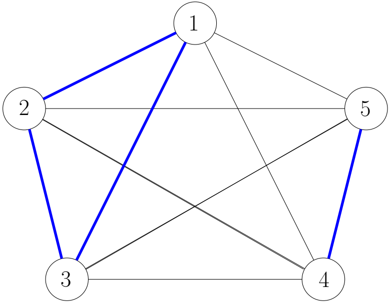
Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?

49-city USA tour. *Newsweek*, July 26, 1954

Observations

- Each node is arrived at from exactly one other node
- From each node there is a departure to exactly one other node
- Sub-tours must be excluded





Representation

- Set of nodes: $N = \{1, \ldots, n\}$
- Set of edges: $E = \{\{i,j\} \mid i,j \in N, i < j\}$
- **Variables:**

$$x_{ij} = \begin{cases} 1 & \text{if edge } \{i,j\} \text{ is in the cycle} \\ 0 & \text{otherwise} \end{cases}$$

- **Formulation:**

$$\begin{aligned}
\min \quad & \sum_{\{i,j\} \in E} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{\{i,j\} \in E} x_{ij} + \sum_{\{j,i\} \in E} x_{ji} = 2 && \forall i \in N \\
& \sum_{i,j \in Q : \{i,j\} \in E} x_{ij} \leq |Q| - 1 && \forall\ Q : \emptyset \neq Q \subsetneq N \\
& x_{ij} \in \{0, 1\} && \forall i,j \in N
\end{aligned}$$

Remark: the last constraints exclude sub-tours

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Diet problem of Stigler[2]

Posed by Stigler in 1945[1]:

> *For a moderately active man (economist) weighing 154 pounds, how much of each of 77 foods should be eaten on a daily basis so that the man's intake of nine nutrients (including calories) will be at least equal to the recommended dietary allowances (RDAs) suggested by the National Research Council in 1943, with the cost of the diet being minimal?*

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[1] G. J. Stigler. The cost of subsistence. *Journal of Farm Economics*, 27(2):303–314, 1945

[2] George Joseph Stigler (17.01.1911-01.12.1991): an American economist, Nobel Memorial Prize in Economic Sciences 1982

Diet problem of Stigler (cont.)

TABLE 1. DAILY ALLOWANCES OF NUTRIENTS FOR A MODERATELY ACTIVE MAN (weighing 154 pounds)*

Nutrient	Allowance
Calories	3,000 calories
Protein	70 grams
Calcium	.8 grams
Iron	12 milligrams
Vitamin A	5,000 International Units
Thiamine (B_1)	1.8 milligrams
Riboflavin (B_2 or G)	2.7 milligrams
Niacin (Nicotinic Acid)	18 milligrams
Ascorbic Acid (C)	75 milligrams

* National Research Council, *Recommended Dietary Allowances*, Reprint and Circular Series No. 115, January, 1943.

Diet problem of Stigler (cont.)

Nutrients and prices of 77 foods (in USD, year 1945)

https://github.com/lxthanh86/StiglerDiet

Solution of Stigler

TABLE 2. MINIMUM COST ANNUAL DIETS, AUGUST 1939 AND 1944

Commodity	August 1939		August 1944	
	Quantity	Cost	Quantity	Cost
Wheat Flour	370 lb.	$13.33	535 lb.	$34.53
Evaporated Milk	57 cans	3.84	—	—
Cabbage	111 lb.	4.11	107 lb.	5.23
Spinach	23 lb.	1.85	13 lb.	1.56
Dried Navy Beans	285 lb.	16.80	—	—
Pancake Flour	—	—	134 lb.	13.08
Pork Liver	—	—	25 lb.	5.48
Total Cost		$39.93		$59.88

Diet problem of Stigler: Modeling

Data:

- F: set of 77 foods
- N: set of 9 nutrients
- a_n: RDA of nutrient $n \in N$ per day
- d_{fn}: amount of nutrient $n \in N$ in each unit of food $f \in F$

Variables:
x_f = payment (in USD) to buy food $f \in F$ in the diet

Model:

$$\begin{aligned} \min \quad & \sum_{f \in F} x_f & \\ \text{s.t.} \quad & \sum_{f \in F} d_{fn} x_f \geq a_n & \forall n \in N \\ & x_f \geq 0 & \forall f \in F \end{aligned}$$

Fact: Need 9 accountants working in 120 days to solve by hand

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- Isoperimetric problem: see video

Contents

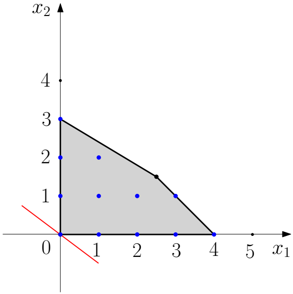
Types of optimization problems

- Continuous optimization
- Discrete (combinatorial) optimization
- Constrained optimization
- Unconstrained optimization
- Convex continuous optimization
- Non-convex continuous optimization
- Linear programming
- Nonlinear programming

Combinatorial vs. Continuous

Combinatorial Optimization:
feasible set is discrete and finite

$$
\begin{aligned}
\max \quad & 3x_1 + 4x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 4 \\
& 3x_1 + 5x_2 \leq 15 \\
& x_1, x_2 \in \mathbb{Z}_+
\end{aligned}
$$



Continuous Optimization:
feasible set has continuous structure

$$
\begin{aligned}
\max \quad & 5000x_1 + 4200x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 40 \\
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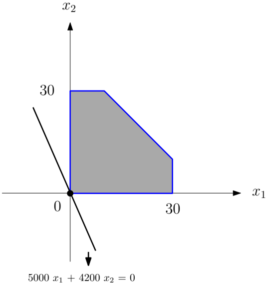

Unconstrained vs. Constrained

Unconstrained Optimization:

feasible set is the whole domain

$$\begin{aligned} \min \quad & x^2 - 3x + 2 \\ \text{s.t.} \quad & x \in \mathbb{R} \end{aligned}$$

Constrained Optimization:

feasible set is a subset of domain

$$\begin{aligned} \min \quad & x^2 - 3x + 2 \\ \text{s.t.} \quad & x \in [-2, 1] \end{aligned}$$

Convex vs. Nonconvex

Convex Optimization:

feasible set is convex, and
objective function is convex

Nonconvex Optimization:

either feasible set is nonconvex
or objective function is nonconvex

Linear vs. Nonlinear

Convex Optimization:

feasible set is polyhedral, and objective function is linear

Nonlinear Optimization:

either feasible set is not polyhedral or objective function is nonlinear

Thank you for your attention!