The Man Who Flew Into Space

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1 Abstract

1. **Analysis**

   (a) We aim to analyse the behaviour of the differential equation

$$\frac{d^2\theta}{d\tau^2} + \nu\theta + \epsilon\cos(2\tau)\theta = 0 \tag{1}$$

for $\nu \approx 1$ and determine if this θ will grow exponentially or linearly. To do this we use an expansion of $\nu = \nu_0 + \epsilon\nu_1 + \epsilon^2\nu_2 \ldots$ where in this case $\nu_0 = 1$, as well as the method of multiple time scales, to find the leading order solution to Equation 1. The amplitude of this solution will show us the behaviour of this system.

Firstly, we define an operator

$$\begin{aligned} \frac{d}{d\tau} &= (D_0 + \epsilon D_1 + \epsilon^2 D_2 + \ldots) \\ \implies \frac{d^2}{d\tau^2} &= (D_0 + \epsilon D_1 + \epsilon^2 D_2 + \ldots) \\ &= D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0 D_2) \end{aligned}$$

where $D_n = \frac{\partial}{\partial T_n}$ and $T_0 = \tau, T_1 = \epsilon\tau, T_2 = \epsilon^2\tau$ etc. We can also expand $\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \ldots$ and then we can rewrite Equation 1 as

$$\begin{gathered} (D_0^2 + 2\epsilon D_0 D_1 + \ldots)(\theta_0 + \epsilon\theta_1 + \ldots) + (\nu_0 + \epsilon\nu_1 + \ldots)(\theta_0 + \epsilon\theta_1 + \ldots) \\ +\epsilon\cos(2\tau)(\theta_0 + \epsilon\theta_1 + \ldots) = 0 \end{gathered} \tag{2}$$

We can then multiply these brackets out and set coefficients of powers of ϵ to 0, starting with ϵ^0, which gives us

$$D_0^2\theta_0 + \theta_0 \overset{1}{\cancel{\nu_1}} = 0$$

which has the solution $\theta_0 = Ae^{iT_0} + A^*e^{-iT_0}$ where A^* is the complex conjugate of A. Now the coefficients of ϵ^1:

$$\begin{aligned}
D_0^2\theta_1 + \theta_1 =& -2D_0D_1\theta_0 - \cos(2T_0)\theta_0 - \nu_1\theta_0 \\
=& -2D_0D_1(Ae^{iT_0} + A^*e^{-iT_0}) - \cos(2T_0)(Ae^{iT_0} + A^*e^{-iT_0}) \\
& -\nu_1(Ae^{iT_0} + A^*e^{-iT_0}) \\
=& -2D_0D_1(Ae^{iT_0} + A^*e^{-iT_0}) - \left(\frac{e^{2iT_0} + e^{-2iT_0}}{2}\right)(Ae^{iT_0} + A^*e^{-iT_0}) \\
& -\nu_1(Ae^{iT_0} + A^*e^{-iT_0}) \\
=& -2D_1(iAe^{iT_0} - iA^*e^{-iT_0}) - \nu_1(Ae^{iT_0} + A^*e^{-iT_0}) \\
& -\frac{1}{2}(Ae^{3iT_0} + Ae^{-iT_0} + A^*e^{iT_0} + A^*e^{-3iT_0})
\end{aligned}$$

At this point, we kill the secular terms by setting each of them to 0:

$$\begin{aligned}
-2iD_1A - \nu_1A - \frac{1}{2}A^* = 0 \\
2iD_1A^* - \nu_1A^* - \frac{1}{2}A = 0
\end{aligned} \tag{3}$$

We can solve this with an Ansatz. If we guess $A = a + ib$; $A^* = a - ib$ and substitute in, we end up with