Chaotic Ingredients

KDSMIL001 MAM2046W 2ND

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1 Introduction

The purpose of this project is to investigate one of the simplest system in which chaotic behaviour can be observed. We know from the Poincaré-Bendixon theorem that we must be looking at a 3-dimensional system in order to have chaos, however to illustrate our point we begin in 2-D.

2 A 2-dimensional Prologue

We consider the following system:

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &- x + ay \end{aligned}$$

where we have $a = 0.1$. This system is clearly represented by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}}_{Jacobian} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.1}$$

The system clearly has a fixed point at the origin and we can analyse the stability of that fixed point by looking at the Jacobian. The trace is $\tau = a > 0$ and the determinant is $\Delta = 1 > 0$, meaning our fixed point is an unstable spiral. Figure 2.1 is a plot of a trajectory starting at $(0.3, 0.3)$.

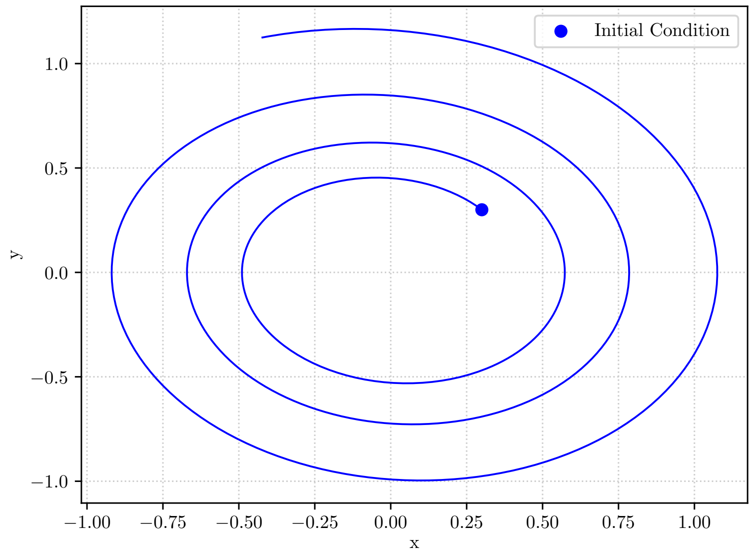


Figure 2.1: A trajectory of the system in Equation 2.1 starting from $(0.3, 0.3)$

3 A 3-dimensional Linear System

As stated before, we require 3 dimensions to even be able to observe chaotic behaviour, so let us add on to our original system the following:

$$\dot{z} = a - bz$$

where b is some positive parameter that we can vary. We can notice that z is decoupled from the other two coordinates, and that it's stable, so it will return to its fixed point as time goes on, that fixed point being easily found to be $(0, 0, \frac{a}{b})$. To confirm this stability in z, we look at the Jacobian:

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & a & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{3.1}$$

The eigenvalues of this Jacobian are

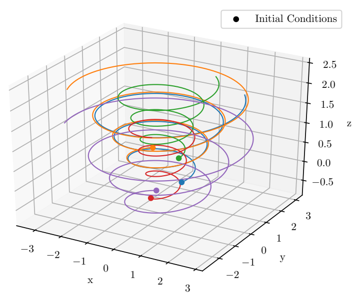
$$\lambda_{1,2} = \frac{1}{2}(0.1 \pm \sqrt{0.1^2 - 4}), \quad \lambda_3 = -b$$

Since b is a positive parameter, we can see that $\lambda_3 < 0$, so the eigenvector associated with it is stable, while the other two eigenvectors are unstable, as their real part is

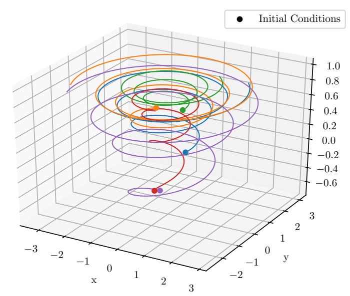
> 0. The eigenvector associated with λ_3 is

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

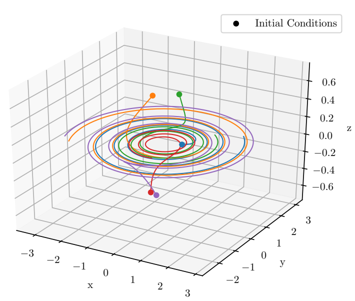
so z will always return to its fixed point, regardless of the behaviour of x or y. To visualise this, we have plotted the trajectories from random initial conditions near to the fixed point for various values of b, and it's clear to see that z always returns to its fixed point $\frac{a}{b}$.



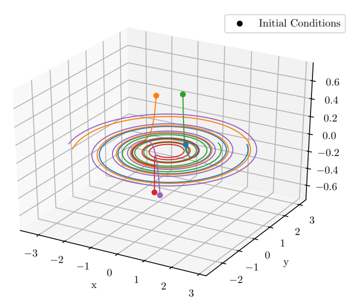
(a) $b = 0.01$



(b) $b = 0.1$



(c) $b = 1$



(d) $b = 10$

Figure 3.1: Trajectories starting from the same random initial conditions near to the origin, for the system in Equation 3.1, for various values of b.

As can be see from the plots in Figure 3.1, any initial condition near to the origin will lead to an unstable spiral around the origin of the xy-plane, with the z coordinate returning to $\frac{a}{b}$. For smaller b, the fixed point of z is further from the origin, so it takes longer to get there.

4 A 3-dimensional Nonlinear System

Clearly just adding a third dimension doesn't automatically give us chaotic behaviour, it's a little bit more complicated than that. We actually need to have all three equations coupled, and feeding back into each other, so we change the system to

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= a + z(x - b)\end{aligned}$$

This means that as x grows, so will z, leading to some lovely feedback. As always we want to examine this system to understand its fixed points and their stability. We can find the fixed points to be

$$\begin{aligned}x &= \frac{b \pm \sqrt{b^2 - 0.04}}{2} \\ y &= -\frac{b \pm \sqrt{b^2 - 0.04}}{0.2} \\ z &= \frac{b \pm \sqrt{b^2 - 0.04}}{0.2}\end{aligned}$$

which we will call $\vec{x}_+$ and $\vec{x}_-$. The Jacobian of the system is

$$J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - b \end{pmatrix} \tag{4.1}$$

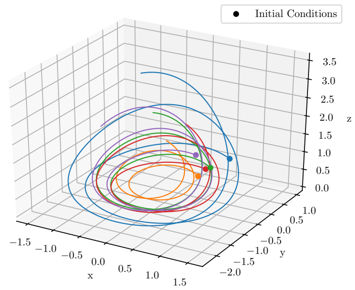
Expressing the eigenvalues of this matrix with respect to b is extremely messy, so we have done you the service of calculating them for a wide range of values of b in Table 4.1.

b	$\vec{x}_+$	$\vec{x}_-$
0.01	$0.451 - 1.12j$	$0.451 + 1.12j$
	$-0.451 + 1.12j$	$-0.451 - 1.12j$
	$0.0951 + 0.099j$	$0.0951 - 0.099j$
0.02	$0.442 - 1.14j$	$0.442 + 1.14j$
	$-0.442 + 1.14j$	$-0.442 - 1.14j$
	$0.0902 + 0.0986j$	$0.0902 - 0.0986j$
0.1	$0.344 - 1.27j$	$0.344 + 1.27j$
	$-0.345 + 1.27j$	$-0.345 - 1.27j$
	$0.0505 + 0.0863j$	$0.0505 - 0.0863j$
0.2	$1.41j$	$1.41j$
	$-1.41j$	$-1.41j$
	-4.79×10^{-19}	-4.79×10^{-19}
1	$-4.16 \times 10^{-6} + 3.30j$	$0.0234 + 1.02j$
	$-4.16 \times 10^{-6} - 3.30j$	$0.0234 - 1.02j$
	8.99×10^{-2}	-0.937
2	$-1.14 \times 10^{-6} + 4.56j$	$0.0398 + 1.003j$
	$-1.14 \times 10^{-6} - 4.56j$	$0.0398 - 1.003j$
	9.5×10^{-2}	-1.97
10	$-4.9 \times 10^{-8} + 10.05j$	$0.0495 + 0.999j$
	$-4.9 \times 10^{-8} - 10.05j$	$0.0495 - 0.999j$
	9.9×10^{-2}	-9.9
20	$-1.24 \times 10^{-8} + 14.2j$	$0.0499 + 0.999j$
	$-1.24 \times 10^{-8} - 14.2j$	$0.0499 - 0.999j$
	9.95×10^{-2}	-19.9

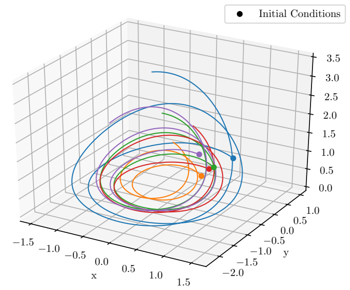
Table 4.1: Eigenvalues of the system in Equation 4.1 for various values of b.

Note that these values have been rounded in order to be displayed easily but this doesn't affect our analysis, since all we are interested in is the sign of the real part of the eigenvalues as that tells us the stability of the respective eigenvector. What we can see from the eigenvalues is that for $b < 0.2$ the real part of the first two eigenvalues for a given fixed point have opposite sign but equal magnitude, which is some sort of unstable spiral. For $b > 0.2$ the two eigenvalues have the same real part for a given fixed point, indicating a stable spiral. For $b = 0.2$ we expect stable orbits, at least in some sense. Any instability will come from the third eigenvalue. Speaking of, the third eigenvalue is always different in magnitude from the first two, but we notice that for $b < 0.2$ it always has positive real part, indicating instability for both fixed points. For $b > 0.2$ we see that it is stable for one fixed point and unstable for the other.

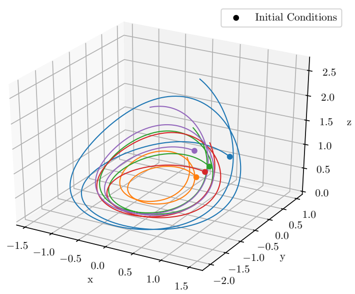
Now we get to the exciting part, visualising some lovely trajectories.



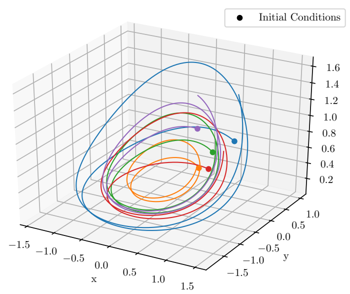
(a) $b = 0.01$



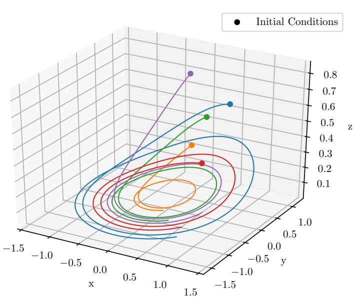
(b) $b = 0.02$



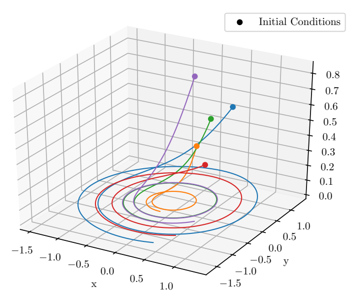
(c) $b = 0.1$



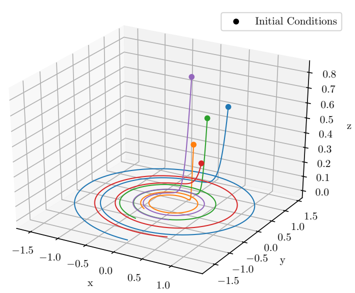
(d) $b = 0.2$



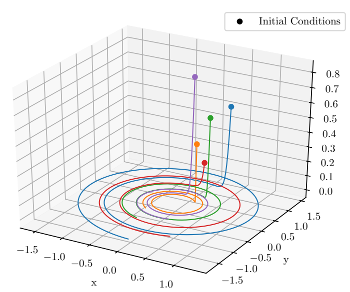
(e) $b = 1$



(f) $b = 2$



(g) $b = 10$



(h) $b = 20$

Figure 4.1: Trajectories starting from the same random initial conditions near to the origin, for the system in Equation 4.1, for various values of b.

Figure 4.1 shows the trajectories of the systems we found eigenvalues for earlier. We can clearly see that for $b > 0.2$, everything looks nice and regular with z collapsing to its fixed point pretty quickly and then leading to unstable spirals around the origin of the xy-plane. For $b < 0.2$ however, things get a little crazy, with chaos-like behaviour being seen. The reason that we attribute to this behaviour is the fact that, for $b < 0.2$, the fixed points of the system become complex, which leads to all sorts of mishaps and mistakes.

5 Conclusion

We found that simply introducing a third dimension doesn't automatically create a chaotic system. This makes sense as pretty much all systems in the real world are three dimensional and don't devolve into chaos immediately. In fact they take millions of years to do it properly. Either way, it takes some careful consideration of the constructions of the system to allow for chaos, and even then it only seems to arise for, in our case, some values of b.