Class Test 1

KDSMIL001 MAM2046W 2BP

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1. **Heat Distribution**
Given the following:

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0,\ 0 < r < 30,\ 0 < \theta < \frac{\pi}{2}$$

$$\begin{aligned} T(30,\theta) &= 50,\ 0<\theta<\frac{\pi}{2} \\ |T(0,\theta)| &< \infty,\ 0<\theta<\frac{\pi}{2} \\ T(r,0) &= 0,\ 0<r<30 \\ T(r,\frac{\pi}{2}) &= 0,\ 0<r<30 \end{aligned}$$

(a) We separate $T(r,\theta)$ into two functions of only r and θ:

$$T(r,\theta) = R(r)\Theta(\theta)$$

Our ICs and BCs become

$$\begin{aligned} R(30) &= 50 \\ |R(0)| &< \infty \\ \Theta(0) &= 0 \\ \Theta(\frac{\pi}{2}) &= 0 \end{aligned}$$

and our initial PDE then becomes

$$\begin{aligned} 0 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial \theta^2} \\ \implies \frac{\Theta}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= -\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} \\ \implies \frac{1}{R}\left[r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right] &= -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} \end{aligned}$$

We then set each ODE to the same constant λ, since they are equal, giving us

$$\frac{d^2\Theta}{d\theta^2} = -\lambda\Theta$$
$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0$$

Starting with Θ, we know the solutions to this type of equation. For $\lambda > 0$ we have

$$\Theta(\theta) = A\sin\left(\sqrt{\lambda}\theta\right) + B\cos\left(\sqrt{\lambda}\theta\right)$$

Applying the BCs we get

$$\begin{aligned}
\Theta(0) &= A * 0 + B * 1 = 0 \\
\implies B &= 0 \\
\Theta(\frac{\pi}{2}) &= A\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0 \\
\implies \sqrt{\lambda}\frac{\pi}{2} &= n\pi,\ n \in \mathbb{N} \\
\implies \lambda &= 4\pi^2,\ n \in \mathbb{N}
\end{aligned}$$

And so we have a solution to the Θ ODE for $\lambda > 0$:

$$\Theta(\theta) = A\sin(2n\theta),\ n \in \mathbb{N}$$

We know that this equation has no solution for $\lambda < 0$, and the solution for $\lambda = 0$ is $\Theta = C,\ C \in \mathbb{R}$.
Moving on to the R ODE, we have

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0$$

which is an Euler ODE, the standard ansatz for which is $R = r^s,\ r > 0$, so it's always valid in our case. Using this ansatz we find

$$\begin{aligned}
r^2[s(s-1)r^{s-2}] + r[sr^{s-1}] - 4n^2r^s &= 0 \\
\implies s^2 - 4n^2 &= 0 \\
\implies s &= \pm 2n
\end{aligned}$$

For an Euler ODE the solutions usually depend on n, but in our case $n > 0$ so we only examine that case. For $n \neq 0$, Euler ODEs have the solution:

$$R(r) = Dr^{2n} + Er^{-2n}$$

Here is where our boundedness condition comes in. If we examine the solution as $r \to 0$, we can see that it explodes to ∞, but since we know that $|R(0)| < \infty$, we must have that $E = 0$ to stop this blowing up. So for $n \neq 0$

$$R(r) = Dr^{2n}$$

Recall we had $T = R\Theta$, so our solution is

$$\begin{aligned} T_n(r, \theta) &= A \sin(2n\theta) D r^{2n} \\ &= C \sin(2n\theta) r^{2n} \end{aligned}$$

And we have the infinite series representation:

$$T(r, \theta) = \sum_{n=1}^{\infty} C_n \sin(2n\theta) r^{2n}$$

(b) Applying the final BC, we have

$$\begin{aligned} T(30, \theta) = 50 &= \sum_{n=1}^{\infty} C_n \sin(2n\theta) 30^{2n} \\ \implies 50 \sin(2m\theta) &= \sum_{n=1}^{\infty} C_n 30^{2n} \sin(2n\theta) \sin(2m\theta) \\ \implies \int_0^{\frac{\pi}{2}} 50 \sin(2m\theta) d\theta &= \sum_{n=1}^{\infty} C_n 30^{2n} \int_0^{\frac{\pi}{2}} \sin(2n\theta) \sin(2m\theta) d\theta \end{aligned}$$

By orthogonality relations, the $\int_0^{\frac{\pi}{2}} \sin(2n\theta) \sin(2m\theta) d\theta$ term goes to 0 for all $n \neq m$, which simplifies it down to

$$\begin{aligned} \int_0^{\frac{\pi}{2}} 50 \sin(2m\theta) d\theta &= C_m 30^{2m} \int_0^{\frac{\pi}{2}} \sin^2(2m\theta) d\theta \\ &= C_m 30^{2m} \frac{\pi}{4} \\ \implies C_m &= \frac{4}{30^{2m}\pi} \int_0^{\frac{\pi}{2}} 50 \sin(2m\theta) d\theta \end{aligned}$$

2. **Fourier Series**
Given the function:

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x^2 & 0 \leq x \leq \pi \end{cases}$$

(a) The Fourier Series of $f(x)$ will look like the following

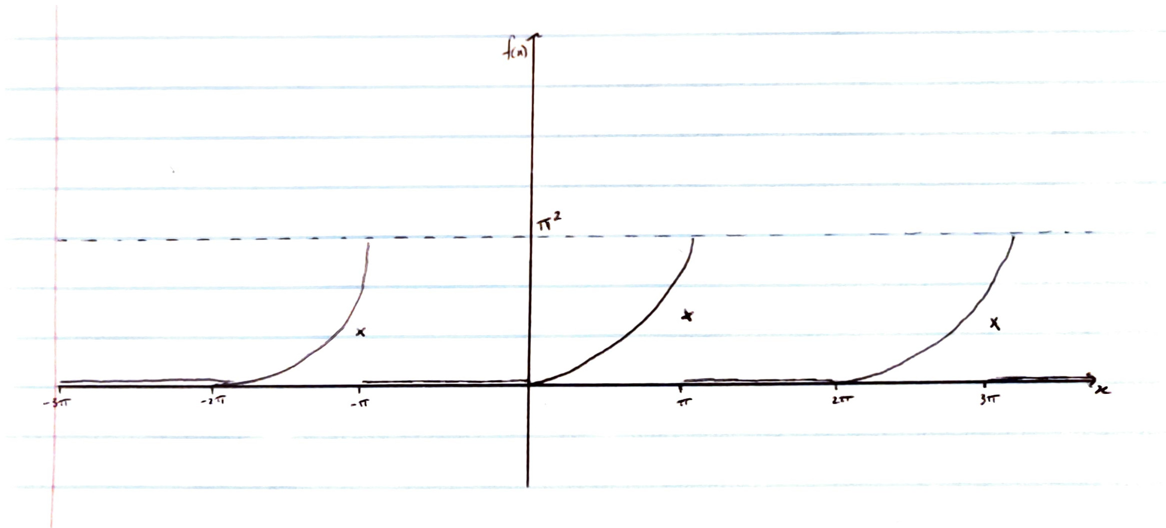

Figure 1: Fourier Series plot of $f(x)$

(b) The Fourier Coefficients are given by

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$
$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$
$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

Starting with A_0:

$$\begin{aligned} A_0 &= \frac{1}{4\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x^2dx\right] \\ &= \frac{1}{4\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \\ &= \frac{1}{4\pi}\frac{\pi^3}{3} \\ A_0 &= \frac{\pi^2}{12} \end{aligned}$$

Now

$$
\begin{aligned}
A_m &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x^2 \cos\left(\frac{mx}{2}\right)dx\right] \\
&= \frac{1}{2\pi}\int_{0}^{\pi} x^2 \cos\left(\frac{mx}{2}\right)dx \\
&= \frac{(\pi^2 m^2 - 8)\sin\left(\frac{\pi m}{2}\right) + 4\pi m\cos\left(\frac{\pi m}{2}\right)}{m^3\pi}
\end{aligned}
$$

And

$$
\begin{aligned}
B_m &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} x^2 \sin\left(\frac{mx}{2}\right)dx\right] \\
&= \frac{1}{2\pi}\int_{0}^{\pi} x^2 \sin\left(\frac{mx}{2}\right)dx \\
&= -\frac{((\pi^2 m^2 - 8)\cos\left(\frac{\pi m}{2}\right) - 4\pi m\sin\left(\frac{\pi m}{2}\right) + 8)}{m^3\pi}
\end{aligned}
$$

Since the original function is piecewise-smooth but has jump discontinuities, we know that the Fourier Series will converge to $f(x)$ within the bounds of the domain, and will converge to $\frac{pi^2}{2}$ at the discontinuities.

(c) The Fourier Series can not be differentiated term by term as

$$f(-\pi) = 0 \neq f(\pi) = \pi^2$$