Waves

KDSMIL001 PHY2004W PHYLAB2

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1 Introduction

Electromagnetic waves can propagate on a transmission line as well as in free space. One example of such a transmission line is the coaxial cable, most commonly known as the cable that connects a TV set to an aerial.

2 Aim

The standard for propagating electromagnetic (EM) waves is those propagating in a vacuum, the speed of which is well established as $c = 299792458\,\text{m/s}$, but in this experiment we are more interested in the speed of propagation for EM waves in the coaxial cable. We will also determine the characteristic impedance of the cable.

3 Theory

Coaxial cables are a special type of Transmission Line, with an inner and outer conductor separated by some dielectric. These cables have some intrinsic properties, namely a capacitance and inductance per unit length, which are

$$C = \frac{2\pi\epsilon}{\ln(r_{out}/r_{in})} \qquad \text{and} \qquad L = \frac{\mu}{2\pi}\ln\frac{r_{out}}{r_{in}}$$

where we have ϵ and μ are the permittivity and permeability of the dielectric in the cable respectively. For our case, we will actually take $\mu = \mu_0 = 4\pi \times 10^{-7}\,\text{H/m}$. Finally r_{out} and r_{in} are the outer and inner radius of the cable, that is the outer radius of the outer and inner conductor respectively. For and EM wave in the cable, the electric and magnetic energy densities are equal, given by

$$E_E = \frac{1}{2}CV^2 \qquad \text{and} \qquad E_B = \frac{1}{2}LI^2$$

where V and I are the the potential and current at a point in the cable.
Exercise 1: We want to find an expression for the characteristic impedance Z_0 of the cable, so we have

$$\begin{aligned}
Z_0 &= \frac{V}{I} = \sqrt{\frac{E_E 2}{C}}\sqrt{\frac{L}{2E_B}} \\
E_E &= E_B \\
\implies Z_0 &= \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{2\pi}\ln\frac{r_{out}}{r_{in}}\frac{\ln(r_{out}/r_{in})}{2\pi\epsilon}} \\
&= \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\frac{r_{out}}{r_{in}}
\end{aligned}$$

Another characteristic of the coaxial cable is that EM waves will propagate through it at a speed

$$v = \frac{c}{\sqrt{\epsilon\mu}} = \frac{c}{n} \tag{3.1}$$

where we call n the refractive index of the dielectric (at the specific frequency of propagation).
For these cables, if the characteristic impedance changes at some point, the propagation of the wave will be affected, namely the wave will be partially reflected. The reflection coefficient for a purely resistive load R is given by

$$\mathcal{R} = \frac{R - Z_0}{R + Z_0} \tag{3.2}$$

Note that **Exercise 2:** We want to show that $\lim_{R\to\infty} \mathcal{R} = 1$:

$$\begin{aligned}
\mathcal{R} &= \frac{R - Z_0}{R + Z_0} \\
\lim_{R\to\infty} \frac{R - Z_0}{R + Z_0} &= \lim_{R\to\infty} \frac{1 - \frac{Z_0}{R}}{1 + \frac{Z_0}{R}} \\
&= \frac{1-0}{1+0} = 1
\end{aligned}$$

By measuring the time it takes for reflections of pulses to return to the start of the cable, we will be able to determine the speed of propagation for the waves in the cable with a known length. A similar technique can be used for more extended waves, such as sine waves.

For the pulses we would expect that, depending on the conditions at the end of the cable, we will have different types of reflection. For an open circuit, that is the cable is not connected to anything on the other side, as we have shown in Exercise 2, $\mathcal{R} = 1$, so there will be perfect reflection of the pulse. For a short circuit, $R = 0$, so we will have $\mathcal{R} = -1$, resulting in a perfectly reflected wave, but with negative voltage.

For the extended sine wave, we expect to see particular resonant frequencies depending on the length of the cable, resulting in standing waves. Again these frequencies depend on the condition at the end of the cable. For an open circuit, the resonant frequencies will be

$$f_n = n\frac{v}{4L}, \quad n = 1, 3, 5\ldots \tag{3.3}$$

For a short circuit, we expect

$$f_n = n\frac{v}{2L}, \quad n = 1, 2, 3\ldots \tag{3.4}$$

With both of these, we obviously expect a fundamental frequency with harmonics on top of that, hence the multiple nodes.

4 Apparatus

- Signal generator (with duty cycle control).
- Oscilloscope.
- Digital multimeter.
- Coaxial cable of length (55.845 ± 0.005) m with switch to switch termination end between short circuit, open circuit, and variable resistance.

We set up the signal generator to feed into the coaxial cable, measuring the voltage at that point with the oscilloscope.

5 Method

5.1 Propagation Speed of the Wave

In this section we aimed to determine the speed at which EM waves propagate through the coaxial cable. We did this by two methods.

Firstly we looked at the pulses. We know the length of the cable, so all we needed to do was time how long it took for the pulse to travel from the start of the cable to the end, be reflected, and return to the start. The velocity will simply be given by

$$v = \frac{2L}{\Delta t} \tag{5.1}$$

To measure measure this Δt we set the signal generator to a square wave with the duty cycle to a minimum (20%), and set the frequency to 1MHz. We measured Δt for both the open and short circuit by looking at the oscilloscope. Using these values we could find v for each.

The second method we used to find v was by using Equation 3.3 and Equation 3.4, varying the frequency of a sine wave supplied by the signal generator to try find the first and second resonant frequencies of both the open and closed circuit. We could then find v for both.

5.2 Characteristic Impedance

In this section we aimed to find the characteristic impedance Z_0 of the coaxial cable. For this we use the variable resistance setting for the termination end of the cable. We know from Equation 3.2 that $\mathcal{R} = 0$ when $R = Z_0$, and we can find this resistance by looking at the oscilloscope for when the reflections vanish completely while varying the resistance.

6 Results

6.1 Propagation Speed of the Wave

First, the pulses.

Figure 6.1: Oscilloscope display showing voltage across the start of an open circuit coaxial cable with 1 MHz square wave signal with duty cycle 20%.

Figure 6.2: Oscilloscope display showing voltage across the start of a short circuit coaxial cable with 1 MHz square wave signal with duty cycle 20%.

Measuring from the peak of the first spike of the larger signal to the peak of the first spike of the smaller signal we find a Δt of 12 ± 1 small divisions. Looking at the time scale of the oscilloscope we see that each small division is 50 ns. The uncertainty of this measurement comes from 2% of the time scale, as well as from the triangular pdf associated with analogue measurements. We have, for both the open and short circuits:

$$\begin{aligned}
\Delta t &= 600 \times 10^{-9}\,\mathrm{s} \\
u(\Delta t) &= \sqrt{\left(\frac{2 \cdot 50 \times 10^{-9}}{2\sqrt{6}}\right)^2 + (0.02 \cdot 250 \times 10^{-9})^2} \\
\implies \Delta t &= (600 \pm 21) \times 10^{-9}\,\mathrm{s}
\end{aligned}$$

Using Equation 5.1, we can find v. We have $L = (55.845 \pm 0.005)\,\mathrm{m}$ so $2L = (111.69 \pm 0.01)\,\mathrm{m}$. The uncertainty comes from the uncertainty propagation for multiplication, giving us

$$\begin{aligned}
v &= \frac{111.69}{600 \times 10^{-9}} = 186150000\,\mathrm{m \cdot s^{-1}} \\
u(v) &= v\sqrt{\left(\frac{u(\Delta t)}{\Delta t}\right)^2 + \left(\frac{u(2L)}{2L}\right)^2} \\
&= 186150000\sqrt{\left(\frac{21 \times 10^{-9}}{600 \times 10^{-9}}\right)^2 + \left(\frac{0.01}{111.69}\right)^2} \\
&= 6515271.317 \\
\implies v_{open} = v_{short} &= (186200000 \pm 6500000)\,\mathrm{m \cdot s^{-1}}
\end{aligned}$$

Now moving on to the resonant frequencies:

Open	
f_1:	$(830 \pm 1) \times 10^3\,\mathrm{Hz}$
f_3:	$(2.5 \pm 0.1) \times 10^6\,\mathrm{Hz}$

Short	
f_1:	$(1.7 \pm 0.1) \times 10^6\,\mathrm{Hz}$
f_2:	$(3.4 \pm 0.1) \times 10^6\,\mathrm{Hz}$

Using these in Equation 3.3 and Equation 3.4, we find, for the open circuit

$$\begin{aligned}
v_1 &= \frac{4f_1L}{1} = 185405400 \\
v_3 &= \frac{4f_3L}{3} = 186150000
\end{aligned}$$

and for the short circuit

$$\begin{aligned}
v_1 &= \frac{2f_1L}{1} = 189873000 \\
v_2 &= \frac{2f_2L}{2} = 189873000
\end{aligned}$$

Uncertainty on these values comes from the multiplicative propagation formula, as well as propagation when multiplying by a constant. For the open circuit we find

$$\begin{aligned} u(v_1) &= v_1\sqrt{\left(\frac{u(f_1)}{f_1}\right)^2 + \left(\frac{u(L)}{L}\right)^2} \\ &= 223995.9473 \\ u(v_3) &= v_3\sqrt{\left(\frac{u(f_3)}{f_3}\right)^2 + \left(\frac{u(L)}{L}\right)^2} \\ &= 744698.653 \end{aligned}$$

And for the short circuit:

$$\begin{aligned} u(v_1) &= v_1\sqrt{\left(\frac{u(f_1)}{f_1}\right)^2 + \left(\frac{u(L)}{L}\right)^2} \\ &= 11169012.94 \\ u(v_2) &= v_2\sqrt{\left(\frac{u(f_2)}{f_2}\right)^2 + \left(\frac{u(L)}{L}\right)^2} \\ &= 5584506.469 \end{aligned}$$

And so we have

Open	
v_1:	$(185400000 \pm 220000)\,\mathrm{m\cdot s^{-1}}$
v_3:	$(186150000 \pm 740000)\,\mathrm{m\cdot s^{-1}}$

Short	
v_1:	$(189000000 \pm 11000000)\,\mathrm{m\cdot s^{-1}}$
v_2:	$(189870000 \pm 560000)\,\mathrm{m\cdot s^{-1}}$

6.2 Characteristic Impedance

After finding the frequency which results in having no reflection, we measured it to be

$$R = (47.1 \pm 4.0)\,\Omega$$

using a digital multimeter. This uncertainty is the uncertainty associated with a digital reading, $\frac{a}{2\sqrt{3}}$, combined with the 2% uncertainty rating of the multimeter, which was set to the 200Ω scale. From this we can simply see that

$$Z_0 = (47.1 \pm 4.0)\,\Omega$$

7 Discussion and Recommendations

Looking first at the propagation speeds of the EM waves in the coaxial cable, the likely most accurate method would be the first method: measuring the time it takes for a pulse to travel the length of the cable. We say this because the second method requires judgement of the resonant frequency simply by looking at the screen of the oscilloscope as well as tuning the frequency with a signal generator that only increases in relatively large increments. For this reason we will take the first result as being the most accurate and judging the rest off of that one.
All of the standing wave speeds agree with the pulse speed within their own experimental uncertainty, except for the second short circuit speed. It still, however, agrees within the uncertainty of the pulse speed.

Regarding the characteristic impedance of the coaxial cable, we do not have a theoretical estimate for Z_0, so we have nothing to compare our value to.

8 Conclusion

To conclude, we found that we could use two methods to determine the speed of propagation for EM waves in a given coaxial cable and that both of these methods are quite reliable, both supplying values that agree with each other within experimental uncertainty. We also found a value for the characteristic impedance of the coaxial cable, but we could not confirm whether it was a reasonable value.