

# Poisson Statistics Analysis

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## Abstract

We test the hypothesis that count data from a Geiger counter detecting radiation from a  $^{60}\text{Co}$  source follows a Poisson distribution. This includes deriving the equations for quantities that describe the distribution, such as mean and variance.

## 1 Introduction

The underlying goal of any experiment is to collect data on some phenomenon or process and decide, by some rigorous mathematical method, whether or not the data supports the theory describing that phenomenon or process. Theory often predicts outcomes to an exact precision, but due to the nature of measurement, experiments will never be able to exactly confirm theories, they will only ever be able to agree within an accepted experimental uncertainty. The method by which these uncertainties are estimated is explored in this experiment, as well as some implementation of those methods to attempt to confirm a hypothesis.

The hypothesis in question is that the ionising radiation emitted by some radioactive source is emitted with a varying frequency that is distributed according to a Poisson distribution. We will test this hypothesis by using a Geiger counter to see a representation of the amount of radiation emitted in a given time period and then using some mathematical methods, outlined in the first part of section 3, to test whether this data actually follows a Poisson distribution in section 3.

We also hypothesise that background radiation, as detected by a Geiger counter, follows a Poisson frequency distribution. We will test this at the end of section 3.

## 2 Method

Due to the limitations of equipment, we are not able to detect every single emission from the source but since there is no bias for radiation to be emitted in a certain direction, we can simply place our detector, a Geiger counter, pointing at the source and this should give a representative sample of the emitted radiation. Since we are not concerned with the quantitative data, such as number of emissions per second, but only with the qualitative distribution of the data this limitation is no issue.

We began by letting the Geiger counter, set at 600 V, run for 10 seconds, 100 times, with no radioactive source nearby in order to determine the background radiation in the lab. After this we retrieved our  $^{60}\text{Co}$  source and placed it in front of the Geiger counter. We wanted to analyse the data from multiple approximate count rates or frequencies: 4/10s, 10/10s, 30/10s, and 100/10s. To achieve different rates the source could be moved closer to or further away from the Geiger counter.

Each student was assigned a count rate from the 3 higher rates and we shared our data among ourselves in order to have a complete set, as well as used the example data supplied for the 4/10s count rate. For each rate we ran the experiment for 100 trials of 10 seconds each.

### 3 Analysis

Our sample of  $^{60}\text{Co}$  has a number of nuclei on the order of Avagadro's number,  $\sim 10^{23}$ , and since the half life of  $^{60}\text{Co}$  is 5.27 years [1] while the time taken collecting data was around 30 minutes, the total number of nuclei throughout the course of the experiment was approximately constant:  $N_n$ . The number of nuclei that decay in any one time step is in general given by the binomial distribution  $B(n; p, N_n)$  but since in our case  $N_n$  is huge and  $p$  is tiny, we can safely approximate this distribution by the Poisson distribution  $P(n; \mu)$ , where  $\mu = pN_n$  (). To justify this, we can compute the probability  $p$  of any one nucleus decaying during one trial of 10 seconds:

The probability that any one nucleus will decay in a trial of length 5.27 years is 0.5. Thus the probability that any one nucleus will decay in a trial of length 10 seconds is

$$\begin{aligned} p &= 0.5 \frac{10s}{5.27y} \\ &= 0.5 \frac{10s}{166308552s} \\ &= 3.006 \times 10^{-8} \end{aligned}$$

Since the Poisson distribution gives  $p = \frac{\mu}{N_n}$  and  $N_n$  is very large,  $p$  being of order  $1 \times 10^{-8}$  is reasonable.

Our assumption is that the data we have collected is distributed according to a Poisson distribution, that is it has a mean value  $\mu$ . In order to extract the value of  $\mu$  for our data we use the Method of Maximum Likelihood (MML), which states that the estimator, in our case  $\hat{\mu}$ , for a given distribution is found at the maximum of the joint probability distribution function (pdf), so it's a solution to

$$\partial_{\hat{\mu}} \prod_{i=1}^N P(x_i; \hat{\mu}, \vec{\theta}) [2] \quad (3.1)$$

where  $N$  is the number of data points or trials and  $\vec{\theta}$  is the other parameters that the distribution could depend on. For a Poisson distribution this  $\vec{\theta}$  is 0, and for a Gaussian distribution it's  $\sigma$ . We find the value of this estimator  $\hat{\mu}$  for both the Poisson and Gaussian distributions by first noticing that the maximum of the joint pdf will occur at the same point as the maximum of the log of the joint pdf since log is monotonic. So we solve:

$$\begin{aligned}
\text{Poisson: } 0 &= \partial_{\hat{\mu}} \ln \left( \prod_{i=1}^N P(x_i; \hat{\mu}) \right) \\
&= \partial_{\hat{\mu}} \sum_{i=1}^N \ln(P(x_i; \hat{\mu})) \\
&= \sum_{i=1}^N \partial_{\hat{\mu}} \ln \left( \frac{\hat{\mu}^{x_i} e^{-\hat{\mu}}}{x_i!} \right) \\
&= \sum_{i=1}^N \left( \frac{\hat{\mu}^{x_i} e^{-\hat{\mu}}}{x_i!} \right)^{-1} \left( \frac{x_i \hat{\mu}^{x_i-1} e^{-\hat{\mu}}}{x_i!} - \frac{\hat{\mu}^{x_i} e^{-\hat{\mu}}}{x_i!} \right) \\
&= \sum_{i=1}^N \left( \frac{1}{\hat{\mu}^{x_i}} \right) \hat{\mu}^{x_i} (x_i \hat{\mu}^{-1} - 1) \\
&= \sum_{i=1}^N \left( \frac{x_i}{\hat{\mu}} - 1 \right) = 0 \\
\implies \sum_{i=1}^N x_i &= \sum_{i=1}^N \hat{\mu} \\
\implies \hat{\mu} &= \frac{1}{N} \sum_{i=1}^N x_i \equiv \bar{x}
\end{aligned}$$

$$\begin{aligned}
\text{Gaussian: } 0 &= \partial_{\hat{\mu}} \ln \left( \prod_{i=1}^N P(x_i; \hat{\mu}, \hat{\sigma}) \right) \\
&= \sum_{i=1}^N \partial_{\hat{\mu}} \ln \left( \frac{1}{\hat{\sigma} \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x_i - \hat{\mu}}{\hat{\sigma}} \right)^2} \right) \\
&= \sum_{i=1}^N \partial_{\hat{\mu}} \left[ \ln \left( \frac{1}{\hat{\sigma} \sqrt{2\pi}} \right) + \ln \left( e^{-\frac{1}{2} \left( \frac{x_i - \hat{\mu}}{\hat{\sigma}} \right)^2} \right) \right] \\
&= \sum_{i=1}^N \left[ 0 + \left( -\frac{x_i - \hat{\mu}}{\hat{\sigma}} \frac{1}{\hat{\sigma}} \right) \right] \\
&= \sum_{i=1}^N \frac{x_i - \hat{\mu}}{\hat{\sigma}^2} = 0 \\
\implies \sum_{i=1}^N x_i &= \sum_{i=1}^N \hat{\mu} \\
\implies \hat{\mu} &= \frac{1}{N} \sum_{i=1}^N x_i \equiv \bar{x}
\end{aligned}$$

So we can say that our estimator to the mean of the Poisson and Gaussian distributions,  $\hat{\mu}$  is just  $\bar{x}$ , the arithmetic mean of our data. Now we ask whether this estimator is biased or not, that is to say whether the expectation of the estimator is the parameter itself, i.e  $E[\hat{\mu}] = \mu$ . Noting

that the data is assumed to be Poisson distributed, so  $E[x_i] = \mu$ :

$$\begin{aligned}
E[\hat{\mu}] &= E[\bar{x}] = E\left[\frac{1}{N} \sum_{i=1}^N x_i\right] \\
&= \frac{1}{N} E\left[\sum_{i=1}^N x_i\right] \\
&= \frac{1}{N} \sum_{i=1}^N E[x_i] \\
&= \frac{1}{N} \sum_{i=1}^N \mu \\
&= \frac{N}{N} \mu = \mu
\end{aligned}$$

Now we consider the uncertainty related to the estimator  $\hat{\mu}$ , which is the square root of the variance  $V[\hat{\mu}]$ . This is motivated by the Gaussian distribution  $P(x; \mu, \sigma)$  where the uncertainty on  $x$  is  $V[x] \equiv E[(x - \mu)^2]$ :

$$\begin{aligned}
V[x] &= E[x^2] - E[x]^2 \\
&= E[x^2] - \mu^2 \\
&= \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx - \mu^2 \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx - \mu^2 \\
\text{let } t &= \frac{x - \mu}{\sqrt{2}\sigma}, \quad dt = \frac{dx}{\sqrt{2}\sigma} \\
\Rightarrow V[x] &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)^2 e^{-t^2} \sqrt{2}\sigma dt - \mu^2 \\
&= \frac{1}{\sqrt{\pi}} \left( \int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2} dt + \int_{-\infty}^{\infty} 2\sqrt{2}\sigma t \mu e^{-t^2} dt + \int_{-\infty}^{\infty} \mu^2 e^{-t^2} dt \right) - \mu^2 \\
&= \frac{1}{\sqrt{\pi}} \left( 2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + 2\sqrt{2}\sigma\mu \left(-\frac{1}{2}e^{-t^2}\right) \Big|_{-\infty}^{\infty} + \mu^2 \sqrt{\pi} \right) - \mu^2 \\
&= \frac{1}{\sqrt{\pi}} \left( 2\sigma^2 \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + 2\sqrt{2}\sigma\mu \cdot 0 \right) + \mu^2 - \mu^2 \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \left[ \left(-\frac{t}{2}e^{-t^2}\right) \Big|_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-t^2} dt \right] \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} = \sigma^2
\end{aligned}$$

And now we want to find the uncertainty on the mean value parameter that we extract from the data, which is the square root of the variance  $V[\hat{\mu}] \equiv E[(\hat{\mu} - \mu)^2]$ . Note the data is assumed

to be Poisson distributed.

$$\begin{aligned}
E[(\hat{\mu} - \mu)^2] &= E[(\bar{x} - \mu)^2] \\
&= E\left[\left(\frac{1}{N} \sum_{i=1}^N x_i - \mu\right)^2\right] \\
&= E\left[\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N x_i x_j - \frac{2\mu}{N} \sum_{i=1}^N x_i + \mu^2\right] \\
&= \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N E[x_i x_j] - \frac{2\mu}{N} \sum_{i=1}^N E[x_i] + E[\mu^2]
\end{aligned}$$

Now  $E[x_i x_j] = E[x^2] = \mu^2 + \mu$  if  $i = j$  and  $E[x_i]E[x_j] = \mu^2$  if  $i \neq j$ . So

$$\begin{aligned}
V[\hat{\mu}] &= \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N [(\mu^2 + \mu)\delta_{ij} + \mu^2(1 - \delta_{ij})] - 2\mu\bar{x} + \mu^2 \\
&= \frac{1}{N^2} [N\mu^2 + N\mu + N^2\mu^2 - N\mu^2] - 2\mu^2 + \mu^2 \\
&= \frac{\mu}{N} + \mu^2 - \mu^2 \\
&= \frac{\mu}{N} \approx \frac{\bar{x}}{N}
\end{aligned}$$

Thus we have an estimate for the uncertainty of our mean:

$$\Delta\bar{x} = \sqrt{\frac{\bar{x}}{N}} \quad (3.2)$$

And for completeness, the mean count for a data set is given by

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i \quad (3.3)$$

### 3.1 Running Mean

In order to visualise how the arithmetic mean converges to a value and how the uncertainty for it shrinks, we plot a running mean

$$r_c(j) \equiv \bar{x}_j = \frac{1}{j} \sum_{i=1}^{i=j} x_i \quad (3.4)$$

with the uncertainty for each running mean given by

$$\Delta\bar{x}_j = \sqrt{\frac{\bar{x}_j}{j}} \quad (3.5)$$

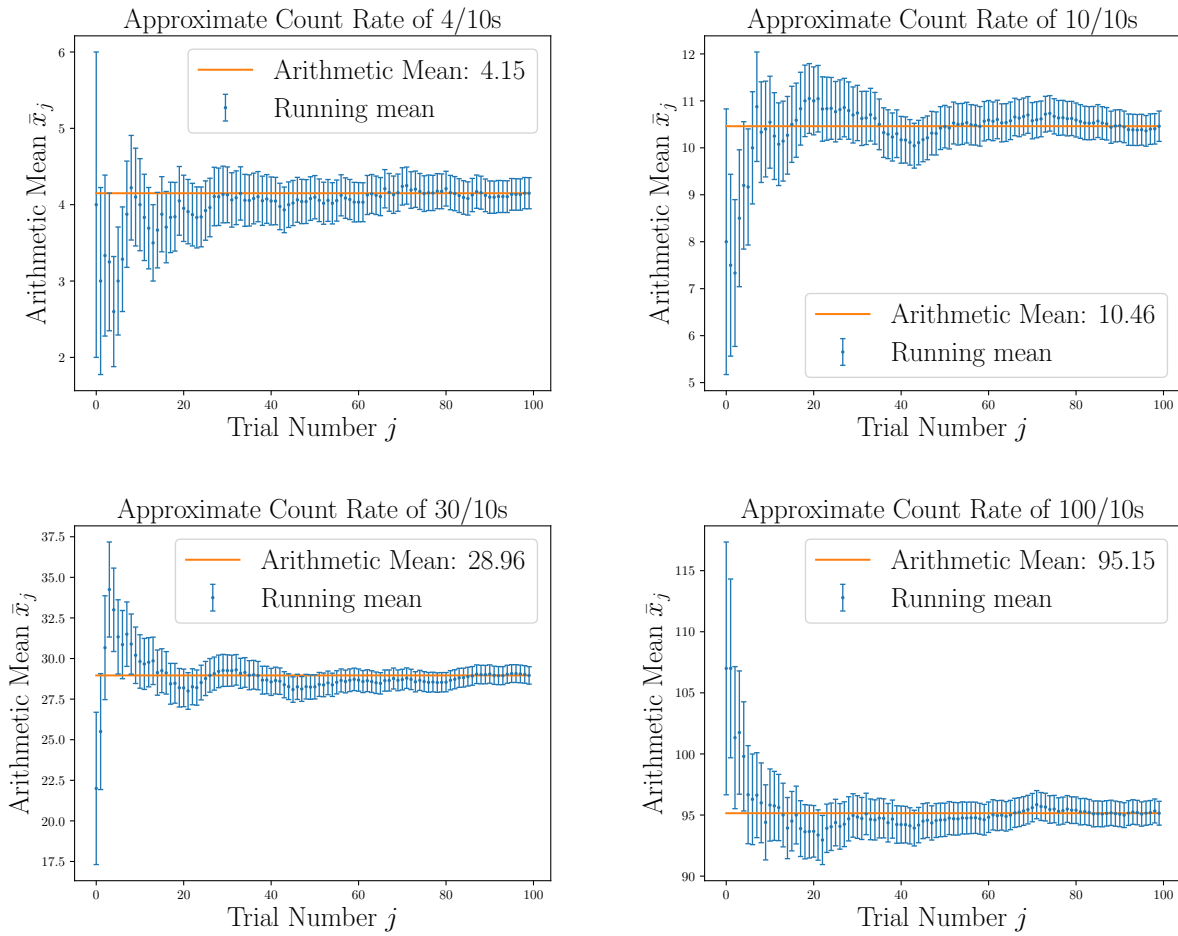


Figure 3.1: Running Means for all 4 approximate count rates. Means calculated using Equation 3.4 and uncertainties calculated using Equation 3.5.

We can clearly see the value of the mean tending to a value for each count rate, even if that value is not the expected value, as well as the uncertainty for each mean shrinking with each step. We also see that in the first 10 or so trials, the estimate for the uncertainty on the mean is very large as the estimation we use depends on a large number of trials, so for small  $j$  the estimate is inaccurate.

### 3.2 Arithmetic Mean and Sample Variance of the Data

In order to test with more certainty that our measured distribution is Poisson or not we must look at the variance of the data. A Poisson distribution has precisely the same mean and variance, as proved in the pre-lab questions, and so now we look for an unbiased estimator for the variance of a Poisson distribution. The sample variance is defined as

$$s^2 \equiv \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2 \quad (3.6)$$

and we can check that this is unbiased for a Poisson distribution, i.e.  $E[s^2] = \mu$ :

$$\begin{aligned}
E[s^2] &= E \left[ \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2 \right] \\
&= \frac{1}{N-1} E \left[ \sum_{i=1}^N (x_i - \bar{x})^2 \right] \\
&= \frac{1}{N-1} E \left[ \sum_{i=1}^N x_i^2 - 2 \sum_{i=1}^N x_i \bar{x} + \sum_{i=1}^N \bar{x}^2 \right] \\
&= \frac{1}{N-1} E \left[ \sum_{i=1}^N x_i^2 - 2\bar{x}N\bar{x} + N\bar{x}^2 \right] \\
&= \frac{1}{N-1} E \left[ \sum_{i=1}^N x_i^2 - N\bar{x}^2 \right] \\
&= \frac{1}{N-1} \left( \sum_{i=1}^N E[x_i]^2 - E[N\bar{x}^2] \right) \\
&= \frac{1}{N-1} \left( \sum_{i=1}^N (\mu^2 + \mu) - N \left( \frac{\mu}{N} + \mu^2 \right) \right) \\
&= \frac{1}{N-1} (N\mu^2 + N\mu - \mu - N\mu^2) \\
&= \frac{1}{N-1} (\mu(N-1)) = \mu
\end{aligned}$$

We can do the same for a Gaussian distribution, i.e. show  $E[s^2] = \sigma^2$ . We can skip the first few lines since they are identical no matter the distribution:

$$\begin{aligned}
E[s^2] &= \frac{1}{N-1} \left( \sum_{i=1}^N E[x_i]^2 - E[N\bar{x}^2] \right) \\
&= \frac{1}{N-1} \left( \sum_{i=1}^N (\sigma^2 + \mu^2) - N \left( \frac{\sigma^2}{N} + \mu^2 \right) \right) \\
&= \frac{1}{N-1} (N\sigma^2 + N\mu^2 - \sigma^2 - N\mu^2) \\
&= \frac{1}{N-1} (\sigma^2(N-1)) = \sigma^2
\end{aligned}$$

Thus we have an unbiased estimator for the variance of our data, but we need an uncertainty for it, so we can find  $V[s^2] = E[(s^2 - \mu)^2]$ . We begin with

$$\begin{aligned}
V[s^2] &= \frac{1}{N} \left( \mu(1 + 3\mu) - \frac{N-3}{N-1} \mu^2 \right) [1] \\
&= \frac{1}{N} \left( \mu + 3\mu^2 - \frac{N-3}{N-1} \mu^2 \right) \\
&= \frac{1}{(N-1)N} ((N-1)\mu + 3(N-1)\mu^2 - (N-3)\mu^2) \\
&= \frac{1}{(N-1)N} ((N-1)\mu + 3N\mu - 3\mu^2 - N\mu^2 + 3\mu^2) \\
&= \frac{(N-1)\mu + 2N\mu^2}{(N-1)N}
\end{aligned}$$

Thus the uncertainty for  $s^2$  is

$$\Delta s^2 = \sqrt{\frac{(N-1)\mu + 2N\mu^2}{(N-1)N}} \quad (3.7)$$

Now we can finally report a sample mean and variance, with uncertainty, for each approximate count rate in Table 3.1.

Approximate Count Rate	Sample Mean	Sample Variance
4/10s	$4.15 \pm 0.20$	$5.12 \pm 0.62$
10/10s	$10.46 \pm 0.32$	$9.8 \pm 1.5$
30/10s	$28.96 \pm 0.54$	$31.2 \pm 4.2$
100/10s	$95.15 \pm 0.98$	$81 \pm 14$

Table 3.1: Sample Mean and Sample Variance for each approximate count rate, calculated using Equation 3.3, Equation 3.2, Equation 3.6, and Equation 3.7

And finally we can perform our test to see whether the data collected is in fact Poisson distributed, plotting the value of  $s^2/\bar{x}$  as a function of  $\bar{x}$  for each approximate count rate. The uncertainty on the  $s^2/\bar{x}$  value is calculated using

$$\Delta \left( \frac{s^2}{\bar{x}} \right) = \frac{s^2}{\bar{x}} \sqrt{\left( \frac{\Delta s^2}{s^2} \right)^2 + \left( \frac{\Delta \bar{x}}{\bar{x}} \right)^2} \quad (3.8)$$



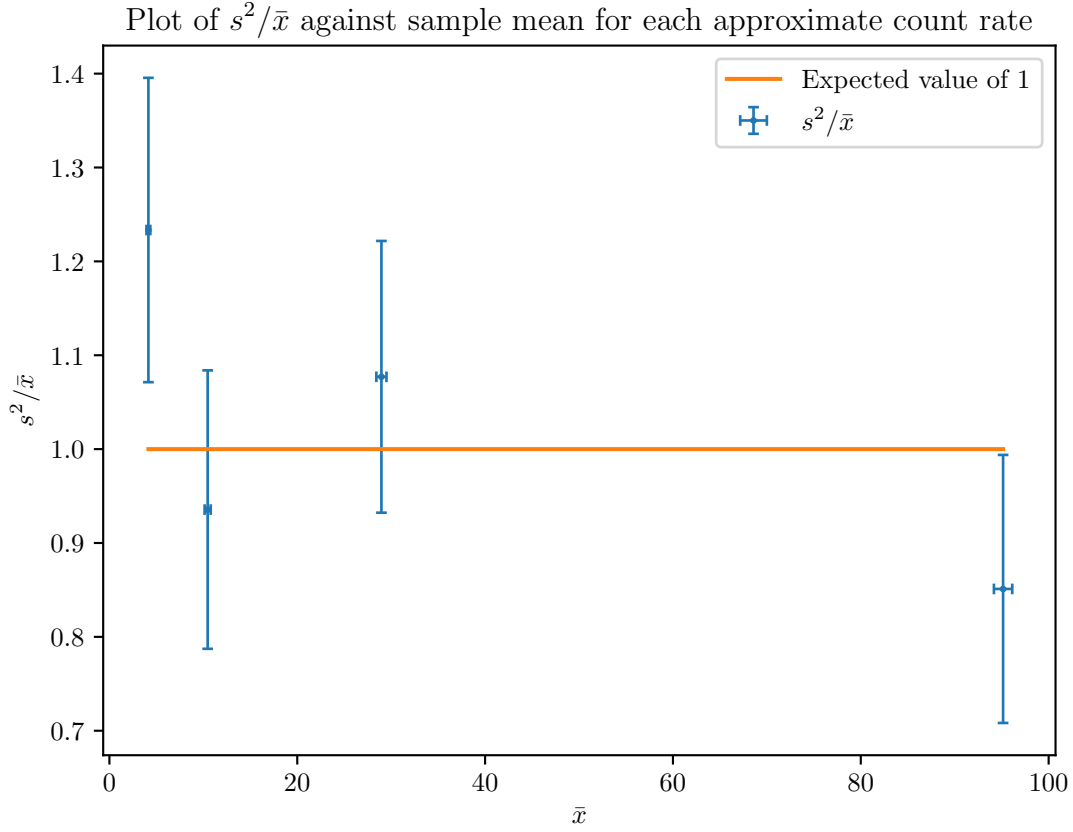


Figure 3.2: The ratio between the sample variance and the sample mean of our data, for each approximate count rate, plotted as a function of the sample mean. Uncertainty on  $s^2/\bar{x}$  found from Equation 3.8 and on  $\bar{x}$  found from Equation 3.2. The expected value of 1 is shown.

We see that 2 of the 4 count rates produce values of  $s^2/\bar{x}$  that agree with the expected value within their uncertainty. This partially confirms our hypothesis but more tests are required.

### 3.3 Poisson Plots

Now we visualise the Poisson nature of our data in a different way. We will plot a histogram of our data where the x-axis shows the number of counts per trial and the y-axis shows how many trials had that many counts. We generally describe binned data such as this with a binomial distribution  $B(n_i; p_i, N)$  where  $n_i$  is the number of counts in bin  $i$ ,  $N$  is the total number of trials, and  $p_i$  is the probability of a single measurement giving a result in the given bin  $i$ . The binomial distribution says that  $E[n_i] = Np_i$  and the variance is  $V[n_i] = Np_i(1 - p_i)$ , so we can estimate the uncertainty of the measured value  $n_i$  as

$$\sqrt{n_i \left(1 - \frac{n_i}{N}\right)} \quad (3.9)$$

We could bin our data where each bin has width of 1, but this leads to some inaccurate estimates of means and uncertainties for bins, especially when we have bins with 0 trials. For this reason we tweak our bins in order to get better statistics. Sometimes tweaking is not necessary

however. We also made sure that, on either side, we never had a bin with fewer than 5 trials. Finally we plotted the Poisson distribution with the arithmetic mean from the data as we expect the data to fit to the distribution if our hypothesis that this data obeys a Poisson distribution is correct. Note that the tails of the smooth curves are discontinuous. This is to compensate for the fact that the Poisson distribution does not terminate at the limits of our data. We used a Cumulative Distribution Function (CDF) to calculate the cumulative value of the curve from the edge to its limit on either side, 0 or  $\infty$ . We also combined the outer bins with small numbers of trials in them so that no outer bin had less than 5 trials in it. This is also in the interest of better statistics.

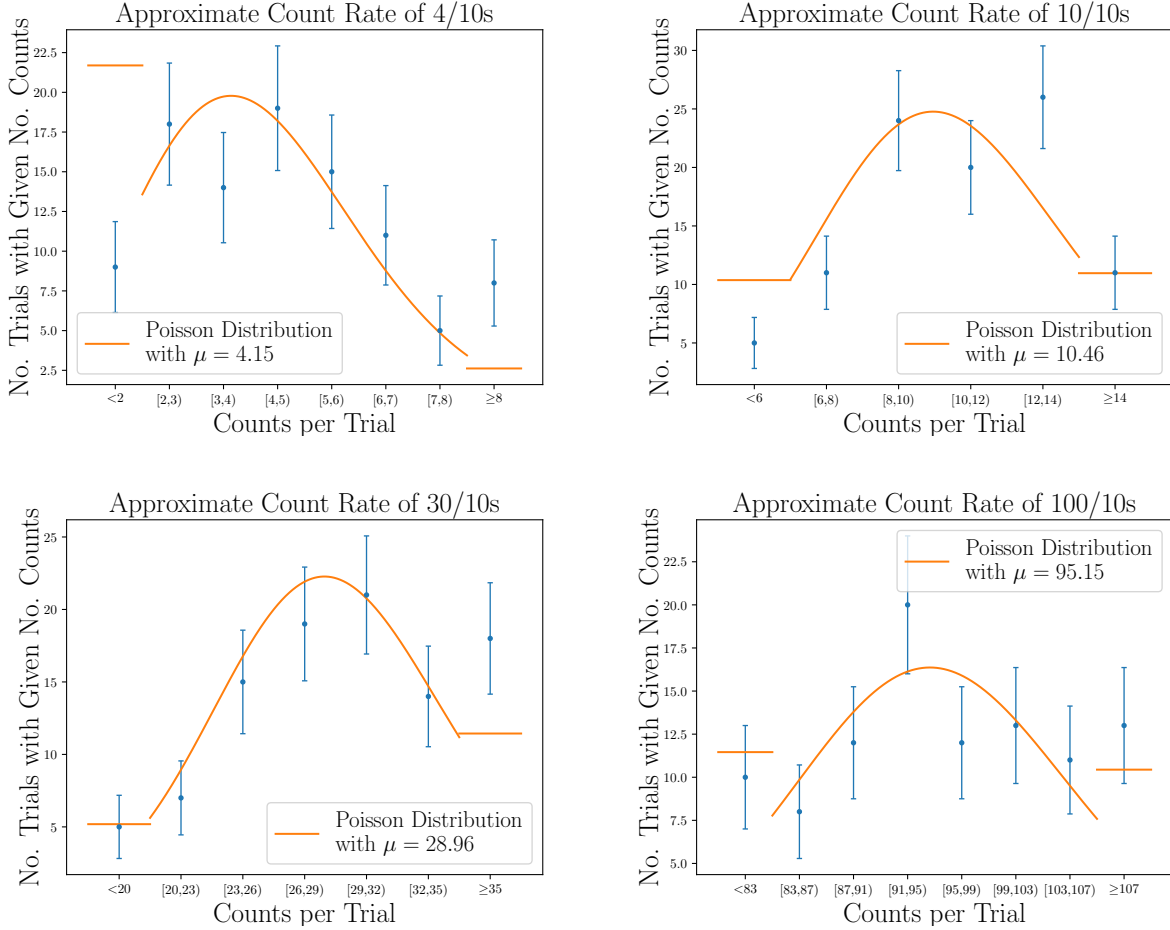


Figure 3.3: Frequency distribution of observed number of counts for all 4 approximate count rates. Uncertainty of Number of Trials given by Equation 3.9

We see that to some degree our data matches the Poisson distribution but a more rigorous approach is needed to confirm or reject our hypothesis.

### 3.4 Fraction of Bins Not Described by Poisson

We want to have a mathematical test that tells us if our estimates for our error bars was correct, in other words we are checking if our underlying hypothesis is correct as those estimates are based on that hypothesis. Our error bars really do represent the probability that 68% of future measurements will lie within their bounds, then there should be a 68% chance that the Poisson

prediction made using the arithmetic mean of the data lies within the error bars of each bin. Thus in order to check our hypothesis we can check to see the percentage of bins whose error bars enclose the Poisson curve and see if it is close to 68%.

Counting the number of bins that don't agree with the Poisson curve is easy, we can do it by inspection, but we want to have some kind of uncertainty associated with that number and since each bin has a probability  $p_a$  of agreeing with the curve and there are  $N_{bins}$  bins, the number of bins that agree with the curve  $n_a$  is given by the binomial distribution and thus the uncertainty of  $n_a$  is the square root of the variance of the binomial distribution:  $N_{bins}p_a(1 - p_a)$ . Thus we report the uncertainty of  $n_a$  as

$$\sqrt{n_a \left(1 - \frac{n_a}{N_{bins}}\right)} \quad (3.10)$$

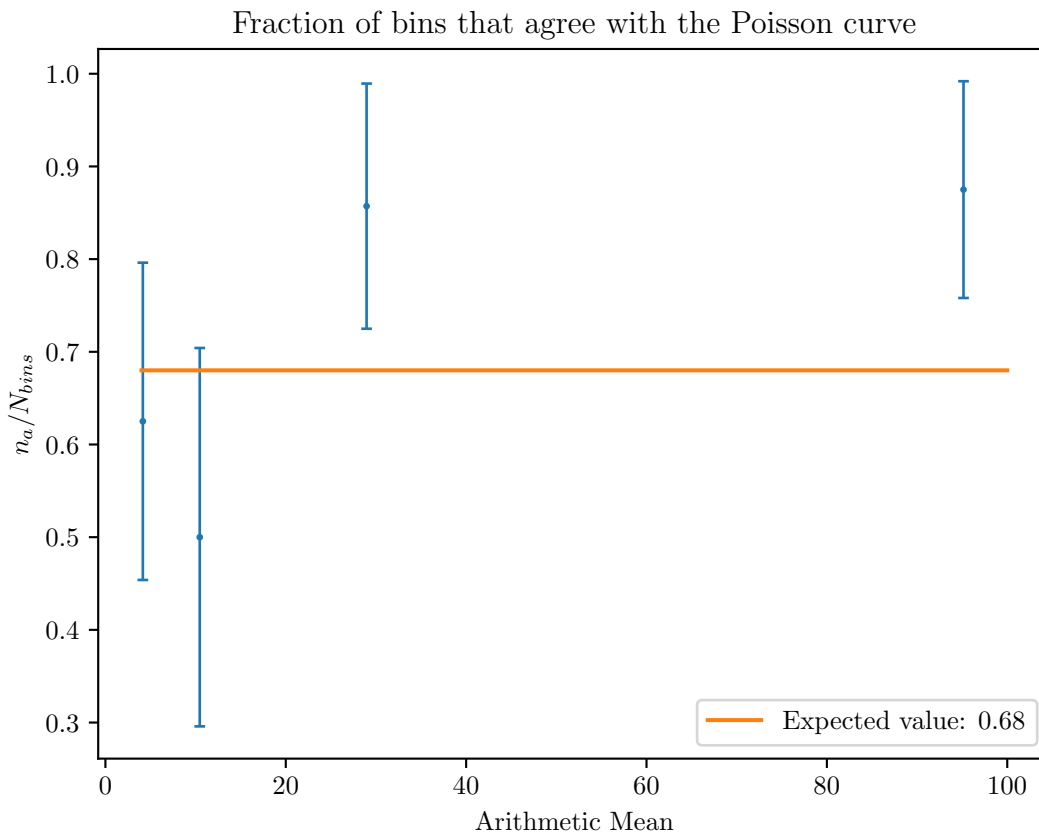


Figure 3.4: Fraction of bins that agree with the Poisson curve within their uncertainty. The expected value of 0.68 is shown. The uncertainty of each fraction is given by Equation 3.10/ $N_{bins}$ .

We see that 2 of our 4 runs have a fraction of agreeing bins that agrees with the expected value of 0.68 within  $1\sigma$ , and the other two within  $2\sigma$ . This definitely supports our hypothesis. The mean of the fraction of bins that agree across all 4 runs is  $0.714 \pm 0.059$  with the uncertainty given by the square root of the sample variance as given by Equation 3.6. This agrees within its uncertainty with 0.68 and so is further support for our hypothesis.

We see that for every single test we tried; the behaviour of the running mean,  $s^2/\bar{x}$ , the fraction of bins which agree with the Poisson curve, our data gave us values that for the most part

agreed with the expectation within experimental uncertainty. For that reason we can say that our hypothesis is, at least provisionally, confirmed.

### 3.5 Background

The process of testing our hypothesis regarding the frequency distribution of background radiation is essentially identical to that done for the radioactive source. The graphical results are found in Figure 3.5 and the sample mean and variance in Table 3.2.

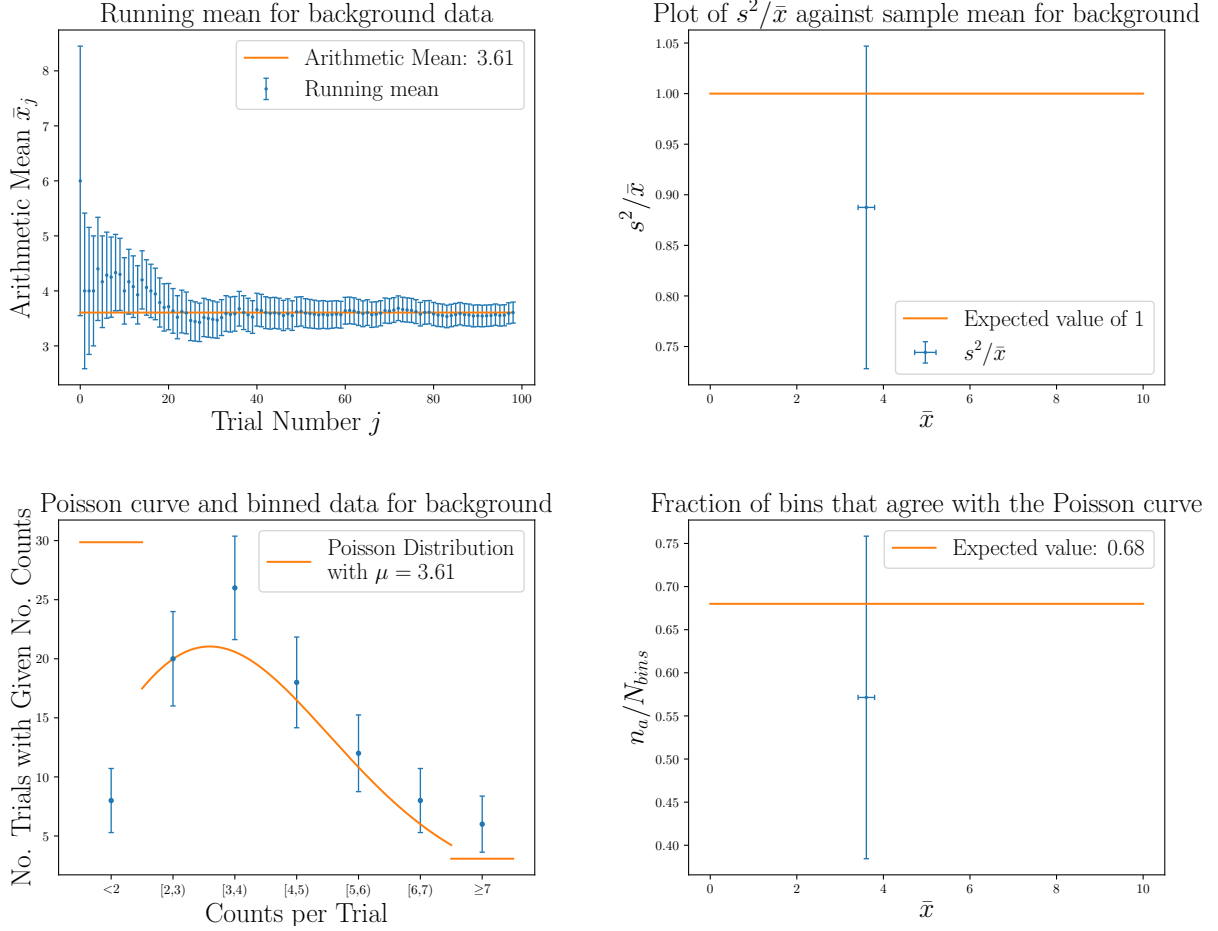


Figure 3.5: All 4 mathematical tests, outlined in subsection 3.1, Equation 3.8, subsection 3.3, subsection 3.4, applied to the data taken for background radiation.

	Sample Mean	Sample Variance
Background	$3.61 \pm 0.19$	$3.20 \pm 0.55$

Table 3.2: Sample Mean and Sample Variance for background data, calculated using Equation 3.3, Equation 3.2, Equation 3.6, and Equation 3.7

We can see that for all 4 tests we see either the expected behaviour such as the uncertainty on a running mean decreasing as  $j$  increases, or we see that numerical values extracted from the data agree within uncertainty with our expectations, such as  $s^2/\bar{x}$  and the fraction of bins which agree with the Poisson curve.

## 4 Conclusion

For both the background data and the data from a radioactive source, the tests applied to our hypothesis, that emitted radiation follows a Poisson frequency distribution, gave us very positive results. We are confident that in both cases our hypothesis is correct.

Improvements can be made in order to reduce the uncertainties estimated in this process, namely increasing the number of trials done for each count rate. We see from Equation 3.2 that the uncertainty goes by  $1/\sqrt{N}$  where  $N$  is the number of trials. Another improvement is to attempt to properly shield the Geiger counter from background radiation. No attempt was made in this case and we believe that for lower approximate count rates,  $\sim 4$  and  $\sim 10$ , the background radiation can have a considerable effect.

# A Pre-Lab Questions

## Poisson Lab Prelim Q's

1. A charged particle or photon enters the gas chamber of a geiger counter. The gas is inert and there are an anode and cathode inside the gas chamber, that do not touch, but across which a large voltage is applied. When the ~~ion~~ particle or photon enter the gas chamber they have a chance of ionising the gas, making it conductive for a moment. The circuit is then complete and the counter can detect a current pulse. ~~as~~ The pulses are independent since radiation can only ionise the gas once.

2. Let  $\mu = \lambda p$  be the average number of events per trial.

$$\Rightarrow p = \frac{\mu}{\lambda}$$

$$\text{Now } \lim_{\lambda \rightarrow \infty} P(x; p, \lambda) = \lim_{\lambda \rightarrow \infty} \frac{\lambda!}{x!(\lambda-x)!} \left(\frac{\mu}{\lambda}\right)^x \left(1 - \frac{\mu}{\lambda}\right)^{\lambda-x}$$

$$= \frac{\mu^x}{x!} \lim_{\lambda \rightarrow \infty} \frac{\lambda!}{(\lambda-x)!} \frac{1}{\lambda^x} \left(1 - \frac{\mu}{\lambda}\right)^{\lambda} \left(1 - \frac{\mu}{\lambda}\right)^{-x}$$

$$= \frac{\mu^x}{x!} (1) \cdot e^{-\mu} (1) = \frac{\mu^x}{x!} e^{-\mu}$$

$$\begin{aligned} \text{a) } \langle x \rangle &= \sum_{x=0}^{\infty} x \frac{\mu^x}{x!} e^{-\mu} = \sum_{x=0}^{\infty} \frac{\mu^x}{(x-1)!} e^{-\mu} = \mu \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!} e^{-\mu} \\ &= e^{-\mu} \mu \sum_{x=0}^{\infty} \frac{\mu^x}{x!} = e^{-\mu} \mu e^{\mu} = \mu \end{aligned}$$

$$\text{b) } \langle x^2 \rangle = \langle x(x-1) + x \rangle = \langle x(x-1) \rangle + \langle x \rangle = \mu + \langle x(x-1) \rangle$$

$$\begin{aligned} &= \mu + \sum_{x=0}^{\infty} (x(x-1)) \frac{\mu^x e^{-\mu}}{x!} = \mu + \sum_{x=1}^{\infty} (x(x-1)) \frac{\mu^x e^{-\mu}}{x!} = \mu + \mu e^{-\mu} \sum_{x=2}^{\infty} \frac{(x-1)}{(x-1)!} \frac{\mu^{x-1}}{(x-1)!} \\ &= \mu + \mu e^{-\mu} \sum_{x=2}^{\infty} \frac{(x-1)}{(x-1)!} \frac{\mu^{x-1}}{(x-1)!} = \mu + \mu^2 e^{-\mu} \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!} = \mu + e^{-\mu} \mu^2 \sum_{x=0}^{\infty} \frac{\mu^x}{x!} \\ &= \mu + e^{-\mu} \mu^2 e^{\mu} = \mu(1 + \mu) \end{aligned}$$

$$\begin{aligned} \text{c) } \langle (x-\mu)^2 \rangle &= \langle x^2 - 2\mu x + \mu^2 \rangle = \langle x^2 \rangle - 2\mu \langle x \rangle + \mu^2 \\ &= \mu(1 + \mu) - 2\mu^2 + \mu^2 = \mu + \mu^2 - 2\mu^2 = \mu \end{aligned}$$

3. Let  $X$  and  $Y$  be independent random variables distributed according to the Poisson distribution with  $\mu_x$  and  $\mu_y$ .

Then let  $Z = X + Y$ ,  $\mu = \mu_x + \mu_y$

Now

$$P(Z=z) = \sum_{i=0}^z P(X=i, Y=z-i) \quad \text{such that } X+Y=z$$

$$= \sum_{i=0}^z P(X=i) P(Y=z-i) \quad X, Y \text{ independent.}$$

$$= \sum_{i=0}^z \frac{e^{-\mu_x} \mu_x^i}{i!} \cdot \frac{e^{-\mu_y} \mu_y^{z-i}}{(z-i)!}$$

$$= \sum_{i=0}^z \frac{1}{i!(z-i)!} e^{-\mu_x} \mu_x^i e^{-\mu_y} \mu_y^{z-i}$$

$$= \sum_{i=0}^z \frac{z!}{i!(z-i)!} \frac{e^{-\mu_x} \mu_x^i e^{-\mu_y} \mu_y^{z-i}}{z!} \quad \left( \cdot \frac{z!}{z!} \right)$$

$$= \sum_{i=0}^z \binom{z}{i} \frac{e^{-\mu}}{z!} \mu_x^i \mu_y^{z-i} \quad (\mu_x + \mu_y = \mu)$$

$$\stackrel{\text{binomial expansion}}{=} \frac{e^{-\mu}}{z!} \sum_{i=0}^z \binom{z}{i} (\mu_x + \mu_y)^z \quad (\text{binomial expansion})$$

$$= \frac{e^{-\mu}}{z!} \mu^z, \text{ which is a Poisson distribution.}$$

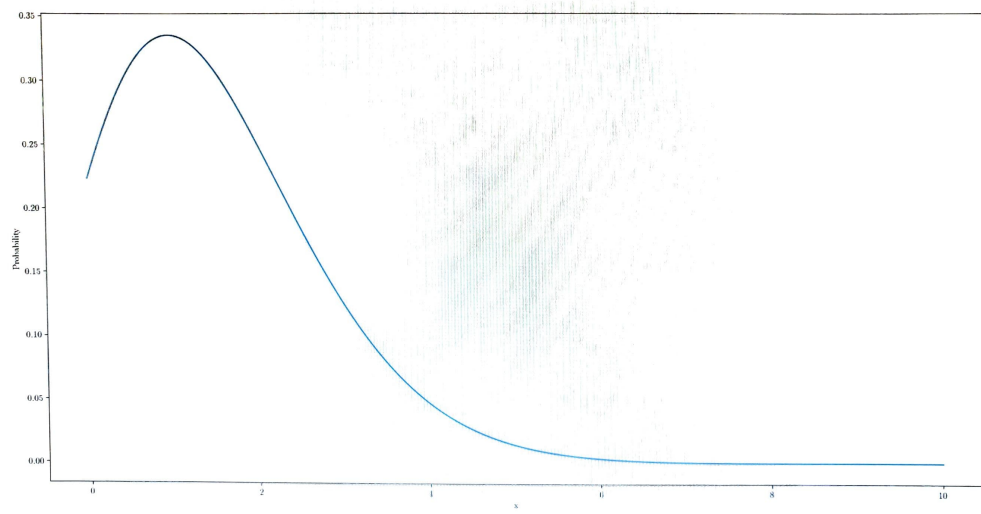
$$\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x} dx = \int_0^{\infty} e^{-x} dx = -e^{-x} \Big|_0^{\infty} = -e^{-\infty} + 1 = 1$$

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx = -e^{-x} x^{n-1} \Big|_0^{\infty} - \int_0^{\infty} -e^{-x} (n-1) x^{n-2} dx$$

$$= [-e^{-x} x^{n-1} + (1 \cdot 0)] + (n-1) \int_0^{\infty} e^{-x} x^{(n-1)-1} dx$$

$$= (n-1) \Gamma(n-1)$$

by the PMI...  $\Gamma(n) = (n-1)!$  guess





6.  $\mu = 2.1$ . so  $P(0; 2.1) = \frac{2.1^0}{0!} e^{-2.1} = 0.1224$

The mean interval between counts is  $\frac{1}{2.1}$  s

7. The mean counts per trial is 3, given the one trial of the control group. So the probability of having fewer than 3 sick people in a following trial is  $P(2; 3) + P(1; 3) + P(0; 3) = \frac{3^2}{2!} e^{-3} + \frac{3^1}{1!} e^{-3} + \frac{3^0}{0!} e^{-3} = 0.4232$

8. I don't have the tools to answer this.

## References

- [1] W.A. Horowitz, *Poisson Statistics*, (UCT, 2021)
- [2] G. Cowan, *Statistical Data Analysis, Ch. 6*, (Oxford Science Publications, 1998)