The Man Who Flew Into Space

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MAM2046W 2OD
EmplID: 1669971

1 Abstract

In this project our aim is to compare the approaches of two people swinging on a swing set, regarding their aim of achieving growth in amplitudes, one aiming for exponential growth, the other for linear. The motion of a person on a swing driving themselves can be modelled with a differential equation, which we can analyse and work out how each person needs to tune their approach in order to obtain the kind of growth they are looking for.

1. **Analysis**

   (a) We aim to analyse the behaviour of the differential equation

   $$\frac{d^2\theta}{d\tau^2} + \nu\theta + \epsilon\cos(2\tau)\theta = 0 \tag{1}$$

   for $\nu \approx 1$ and determine if this θ will grow exponentially or linearly. To do this we use an expansion of $\nu = \nu_0 + \epsilon\nu_1 + \epsilon^2\nu_2 \dots$ where in this case $\nu_0 = 1$, as well as the method of multiple time scales, to find the leading order solution to Equation 1. The amplitude of this solution will show us the behaviour of this system.

   First, we define an operator

   $$\begin{aligned}
   \frac{d}{d\tau} &= (D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots) \\
   \implies \frac{d^2}{d\tau^2} &= (D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots) \\
   &= D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0D_2) + \dots
   \end{aligned}$$

where $D_n = \frac{\partial}{\partial T_n}$ and $T_0 = \tau, T_1 = \epsilon\tau, T_2 = \epsilon^2\tau$ etc. We can also expand $\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \dots$ and then we can rewrite Equation 1 as

$$\begin{gathered}(D_0^2 + 2\epsilon D_0 D_1 + \dots)(\theta_0 + \epsilon\theta_1 + \dots) + (\nu_0 + \epsilon\nu_1 + \dots)(\theta_0 + \epsilon\theta_1 + \dots) \\ +\epsilon\cos(2\tau)(\theta_0 + \epsilon\theta_1 + \dots) = 0\end{gathered} \tag{2}$$

We can then multiply these brackets out and set coefficients of powers of ϵ to 0, starting with ϵ^0, which gives us

$$D_0^2\theta_0 + \theta_0\cancelto{1}{\nu_1} = 0$$

which has the solution $\theta_0 = Ae^{iT_0} + A^*e^{-iT_0}$ where A^* is the complex conjugate of A. Now the coefficients of ϵ^1:

$$\begin{aligned}D_0^2\theta_1 + \theta_1 =& -2D_0D_1\theta_0 - \cos(2T_0)\theta_0 - \nu_1\theta_0 \\ =& -2D_0D_1(Ae^{iT_0} + A^*e^{-iT_0}) - \cos(2T_0)(Ae^{iT_0} + A^*e^{-iT_0}) \\ & -\nu_1(Ae^{iT_0} + A^*e^{-iT_0}) \\ =& -2D_0D_1(Ae^{iT_0} + A^*e^{-iT_0}) - \left(\frac{e^{2iT_0} + e^{-2iT_0}}{2}\right)(Ae^{iT_0} + A^*e^{-iT_0}) \\ & -\nu_1(Ae^{iT_0} + A^*e^{-iT_0}) \\ =& -2D_1(iAe^{iT_0} - iA^*e^{-iT_0}) - \nu_1(Ae^{iT_0} + A^*e^{-iT_0}) \\ & -\frac{1}{2}(Ae^{3iT_0} + Ae^{-iT_0} + A^*e^{iT_0} + A^*e^{-3iT_0})\end{aligned}$$

At this point, we kill the secular terms, those terms with frequency 1, by setting each of them to 0:

$$\begin{aligned}-2iD_1A - \nu_1 A - \frac{1}{2}A^* = 0 \\ 2iD_1A^* - \nu_1 A^* - \frac{1}{2}A = 0\end{aligned} \tag{3}$$

We can solve this with an Ansatz. If we guess $A = a + ib; A^* = a - ib$ and substitute in, we end up with

$$\begin{aligned}-2iD_1a + 2D_1b - \nu_1(a + ib) - \frac{1}{2}(a - ib) &= 0 \\ 2iD_1a + 2D_1b - \nu_1(a - ib) - \frac{1}{2}(a + ib) &= 0 \\ \implies 2D_1b - i2D_1a &= a(\nu_1 + \frac{1}{2}) + ib(\nu_1 - \frac{1}{2}) \\ 2D_1b + i2D_1a &= a(\nu_1 + \frac{1}{2}) + ib(\frac{1}{2} - \nu_1)\end{aligned}$$

Now we can equate the real and imaginary parts of each equation, leaving us with

$$
\begin{aligned}
D_1 b &= \frac{a}{2}(\nu_1 + \frac{1}{2}) \\
D_1 b &= \frac{a}{2}(\nu_1 + \frac{1}{2}) \\
D_1 a &= \frac{b}{2}(\frac{1}{2} - \nu_1) \\
D_1 a &= \frac{b}{2}(\frac{1}{2} - \nu_1) \\
\implies a &= (\frac{1}{4} - \frac{\nu_1}{2}) \int b dT_1 \\
b &= (\frac{\nu_1}{2} + \frac{1}{4}) \int a dT_1
\end{aligned}
$$

we can solve for a and b

$$
\begin{aligned}
D_1 a &= (\frac{1}{16} - \frac{\nu_1^2}{4}) \int a dT_1 \quad |\frac{\partial}{\partial T_1} \\
\implies D_1^2 a &= (\frac{1}{16} - \frac{\nu_1^2}{4}) a \\
\implies 0 &= D_1^2 a + (\frac{\nu_1^2}{4} - \frac{1}{16}) a
\end{aligned}
$$

We can solve this with an Ansatz of $a = e^{kT_1}$

$$
\begin{aligned}
\implies k^2 &= (\frac{\nu_1^2}{4} - \frac{1}{16}) \\
\implies k &= \pm i \sqrt{(\frac{\nu_1^2}{4} - \frac{1}{16})} \\
\implies a &= C e^{\frac{i}{4}\sqrt{4\nu_1^2 - 1} T_1} + C^* e^{\frac{-i}{4}\sqrt{4\nu_1^2 - 1} T_1} \\
\implies b &= \frac{i(2\nu_1 + 1)}{\sqrt{4\nu_1^2 - 1}} \left[C^* e^{\frac{-i}{4}\sqrt{4\nu_1^2 - 1} T_1} - C e^{\frac{i}{4}\sqrt{1 - 4\nu_1^2} T_1} \right]
\end{aligned}
$$

Importantly, both a and b need to be real in order for A and A^* to be complex conjugates. Luckily for us, given a complex number $z = x + iy$ and its conjugate $z^* x - iy$, $z + z^* = 2x$ and $z - z^* = 2iy$ and so a is entirely real and the bracket in b is entirely imaginary, which means b is also entirely real. so we have found

$$
\begin{aligned}
A &= C e^{\frac{i}{4}\sqrt{4\nu_1^2 - 1} T_1} + C^* e^{\frac{-i}{4}\sqrt{4\nu_1^2 - 1} T_1} - \frac{(2\nu_1 + 1)}{\sqrt{4\nu_1^2 - 1}} \left[C^* e^{\frac{-i}{4}\sqrt{4\nu_1^2 - 1} T_1} - C e^{\frac{i}{4}\sqrt{4\nu_1^2 - 1} T_1} \right] \\
A^* &= C e^{\frac{i}{4}\sqrt{4\nu_1^2 - 1} T_1} + C^* e^{\frac{-i}{4}\sqrt{4\nu_1^2 - 1} T_1} + \frac{(2\nu_1 + 1)}{\sqrt{4\nu_1^2 - 1}} \left[C^* e^{\frac{-i}{4}\sqrt{4\nu_1^2 - 1} T_1} - C e^{\frac{i}{4}\sqrt{4\nu_1^2 - 1} T_1} \right]
\end{aligned}
$$

and finally

$$\theta_0 = e^{iT_0}\left(Ce^{\frac{i}{4}\sqrt{4\nu_1^2-1}T_1} + C^*e^{\frac{-i}{4}\sqrt{4\nu_1^2-1}T_1} - \frac{(2\nu_1+1)}{\sqrt{4\nu_1^2-1}}\left[C^*e^{\frac{-i}{4}\sqrt{4\nu_1^2-1}T_1} - Ce^{\frac{i}{4}\sqrt{4\nu_1^2-1}T_1}\right]\right)$$
$$+e^{iT_0}\left(Ce^{\frac{i}{4}\sqrt{4\nu_1^2-1}T_1} + C^*e^{\frac{-i}{4}\sqrt{4\nu_1^2-1}T_1} + \frac{(2\nu_1+1)}{\sqrt{4\nu_1^2-1}}\left[C^*e^{\frac{-i}{4}\sqrt{4\nu_1^2-1}T_1} - Ce^{\frac{i}{4}\sqrt{4\nu_1^2-1}T_1}\right]\right) \tag{4}$$

From our values of A and A^*, our amplitudes, we can tell that θ' growth will be governed by the $\sqrt{4\nu_1^2-1}$ term. If $\nu_1^2 > \frac{1}{4}$, the motion will be purely oscillatory as the exponential term will be entirely imaginary. If $\nu_1^2 < \frac{1}{4}$, the exponential term will be entirely real and thus the amplitude will grow exponentially. Thus we can return to our expansion of ν and find, in order to have exponential growth,

$$\nu < 1 + \frac{\epsilon}{2}; \nu > 1 - \frac{\epsilon}{2}$$
$$\implies \epsilon > |2\nu - 2|$$

This gives us Figure 1 below

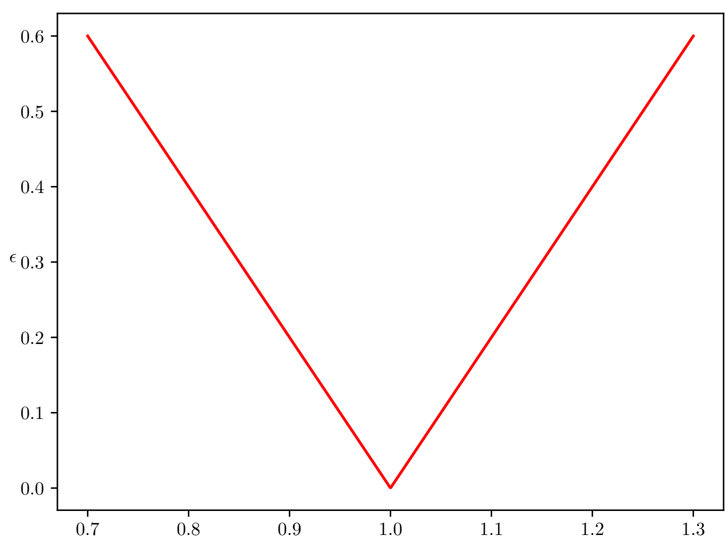

Figure 1: Analytical Determination of $\epsilon_1(\nu)$

where any combination of ϵ, ν above that line will produce exponential growth.

(b) Now we aim to analyse the same equation but for $\nu \approx 4$. We do the same analysis as before, but this time we expand ν as $\nu = 4 + \epsilon\nu_1 + \epsilon^2\nu_2 + \ldots$, which gives us

$$\begin{aligned} D_0^2\theta_0 + 4\theta_0 &= 0 \\ \implies \theta_0 &= Ae^{2iT_0} + A^*e^{-2iT_0} \end{aligned}$$

and then

$$D_0^2\theta_1+\theta_1 = -4iD_1(Ae^{2iT_0}-A^*e^{-2iT_0})-\frac{1}{2}(Ae^{4iT_0}+A^*e^{-4iT_0}+A+A^*)-\nu_1(Ae^{2iT_0}-A^*e^{-2iT_0})$$

To kill the secular terms in this case, we kill terms with frequency 2, which gives us

$$\begin{aligned} 4iD_1A - \nu_1 A &= 0 \\ -4iD_1A^* - \nu_1 A^* &= 0 \\ \implies A &= Ce^{-4i\nu_1T_1} \\ \implies \theta_0 &= Ce^{i(T_0-4\nu_1T_1)} + C^*e^{i(4\nu_1T_1-T_0)} \\ &= Ce^{i\tau(1-4\nu_1\epsilon)} + C^*e^{i\tau(4\nu_1\epsilon-1)} \end{aligned}$$

This solution gives purely oscillatory motion, which might not be correct as, intuitively, we would expect Miu Miu to have some kind of growth, whether it be exponential or linear. We aren't sure if finding the solution is possible. Maybe by finding θ_1 by looking at coefficients of ϵ^2 terms we could uncover a solution for $\epsilon_4(\nu)$.

2. **Numerical**

(a) To simulate this differential equation we use `scipy.integrate.odeint`, splitting up Equation 1 into two first order differential equations. Running that program for a series of values of ν around 1 and reasonably sized ϵ's, checking if θ goes to 100 before $\tau = 100$ and classifying that ϵ as being above ϵ_1. This gives us the plot in Figure 2 below.

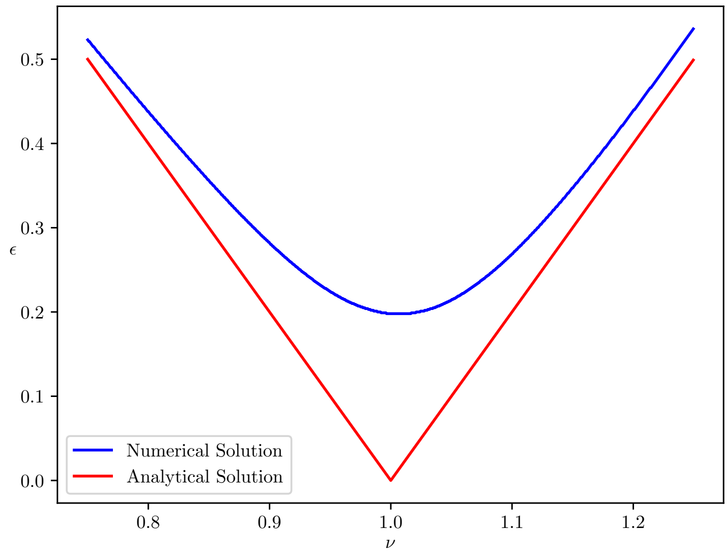

Figure 2: Numerical Determination of $\epsilon_1(\nu)$

So our numerical result agrees somewhat with our analytical solution, with some exceptions for numerical methods being imperfect.

(b) For $\nu \approx 4$, we do exactly the same thing but with values around 4, producing Figure 3 below.

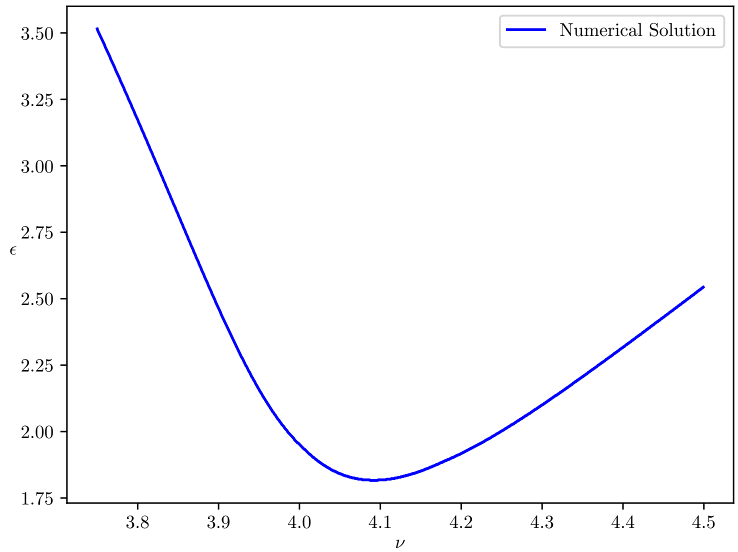

Figure 3: Numerical Determination of $\epsilon_4(\nu)$

This at least tells us that Miu Miu would obtain exponential growth with some combination of ν and ϵ but that solution has a rather strange form.