Car on a Washboard Road Surface

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1. **Modelling**
   In order to model this system, we must consider all of the forces acting on the car. It's relatively safe to assume that the car is in an inertial reference frame, therefore we know that $\vec{F}_{net} = m\vec{a}$. On the other hand, we know that the only forces acting on the car are the force of gravity $\vec{F_G} = m\vec{g}$ and the force of the "spring", which can be modelled as $\vec{F_S} = k\Delta\vec{y}$ where $\Delta\vec{y}$ is the distance from the equilibrium position to the current position of the mass. Now if we consider the system when it's at rest, in other words when the spring is at a relative equilibrium position, there is a force being applied on the car by the spring in order to perfectly balance the force of gravity, in which case we can effectively ignore the force of gravity and choose the new position from which to measure $\Delta\vec{y}$, leaving us with

$$\vec{F}_{net} = k\Delta\vec{y} \tag{1}$$

   Now we must consider the dashpot, which applies a force on the mass in proportion to the velocity of the mass in the form $\vec{F_D} = c\vec{v}$. Adding this into our equation for $\vec{F_{net}}$, we find

$$\vec{F}_{net} = k\Delta\vec{y} + c\vec{v} \tag{2}$$

   With a usual mass on a spring system, this is as far as it goes as the only thing that moves is the mass, but in this case both the mass and the connection point of the "spring" are moving and they're not necessarily moving in sync with each other. In order to account for this we need to modify the $\Delta\vec{y}$ and $\vec{v}$ terms as they will not be changing in a simple manner. For the $\Delta\vec{y}$ term, this isn't too hard to do. We just need to consider the effect that different values of $\Delta\vec{y}$ will have on $\vec{F_S}$. From this we find

$$\vec{F}_S = k(y(t) - Y(t)) \tag{3}$$

where $Y(t)$ is the upward displacement of the car and $y(t)$ is the upward displacement of the connection point of the "spring", given by

$$\begin{aligned} y(t) &= a \sin \frac{2\pi x}{\lambda} \\ &= a \sin \frac{2\pi v t}{\lambda} \end{aligned} \tag{4}$$

as $x = vt$. Now to consider the $\vec{F}_D$ term. Through some careful consideration of the force depending on which way the mass and the connection point are moving, we can find that

$$\vec{F}_D = c(\dot{y}(t) - \dot{Y}(t)) \tag{5}$$

Putting this all together, we can see that

$$\begin{aligned} \vec{F}_{net} &= k(y - Y) + c(\dot{y} - \dot{Y}) \\ m\vec{a} &= k(y - Y) + c(\dot{y} - \dot{Y}) \\ m\ddot{Y} &= ky - kY + c\dot{y} - c\dot{Y} \\ m\ddot{Y} + c\dot{Y} + kY &= ky + c\dot{y} \end{aligned} \tag{6}$$

which is the final form of the equation of motion we are looking for.

2. **Analysis**

$$\begin{aligned} &\frac{a(-(cz)^2 - k^2 + kmz^2)}{-(cz)^2 - k^2 - 2kmz^2 + (mz^2)^2}(\sin(zt)(k - mz^2) + \cos(zt)(cz)) \\ &+ \frac{acmz^3}{-(cz)^2 - k^2 - 2kmz^2 + (mz^2)^2}(\cos(zt)(k - mz^2) + \sin(zt)(cz)) \\ &= \cos(zt)(acz) + \sin(zt)(ak) \end{aligned} \tag{7}$$