1 Average Distance of a Chord

Let $\theta_1, \theta_2 \sim \text{Unif}[0, 2\pi)$ and let the points on the unit circle $P, Q \in \partial B_1(0) \subset \mathbb{R}^2$ be defined by $P = (\cos\theta_1, \sin\theta_1), Q = (\cos\theta_2, \sin\theta_2)$. Their euclidean distance is then

$$
\begin{aligned}
D(P,Q) &= \sqrt{(\cos\theta_2 - \cos\theta_1)^2 + (\sin\theta_2 - \sin\theta_1)^2} \\
&= \sqrt{2 - 2\cos(\theta_2 - \theta_1)} \\
&= 2\sqrt{\frac{1 - cos(|\gamma|)}{2}} && \text{where } \gamma = \theta_2 - \theta_1 \\
&= 2\sin\frac{|\gamma|}{2}
\end{aligned}
$$

Note that, as the difference between uniform r.v.s, γ is another r.v. with symmetric triangular distribution and support $[-2\pi, 2\pi)$. That is,

$$
f_\gamma(x) = \frac{2\pi - |x|}{4\pi^2} \qquad \text{for } x \in [-2\pi, 2\pi)
$$

Now, we *could* attempt to find the probability density of D as a transformation of γ. But, for our goal this is not necessary. Instead, we have:

$$
\begin{aligned}
E[D] &= \int_0^2 x f_D(x)dx \\
&= \int_{-2\pi}^{2\pi} D(x) f_\gamma(x) dx \\
&= \int_{-2\pi}^{2\pi} \left(2\sin\frac{|x|}{2}\right)\left(\frac{2\pi - |x|}{4\pi^2}\right) dx \\
&= \frac{1}{\pi^2}\int_0^{2\pi} \sin\frac{x}{2}(2\pi - x)dx \\
&= \frac{2}{\pi}\int_0^{2\pi} \sin\frac{x}{2}dx - \frac{1}{\pi^2}\int_0^{2\pi} x\sin\frac{x}{2}dx \\
&= \frac{4}{\pi} \approx 1.27324
\end{aligned}
$$

Which is exactly what our numerical approximation program converges to! $\square$