An Explicit Solution to the Stokes Equations on the Half-Space: Undergraduate Research in Math

Miles Smith, 2023

April 14, 2022

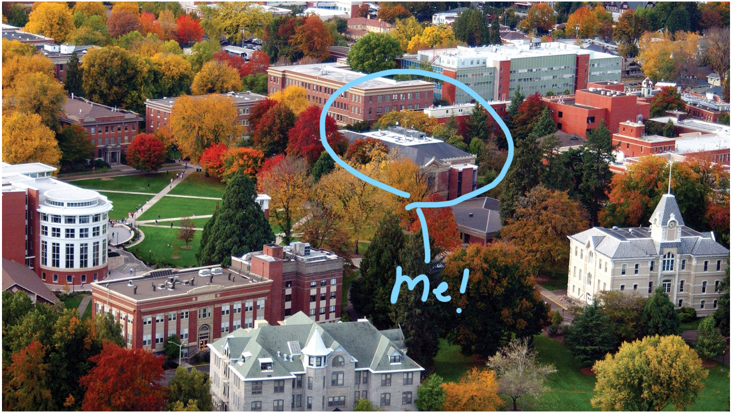
Structure

1. My REU program
2. What are the Navier-Stokes Equations
3. What are the Stokes Equations
4. My research goal
5. Soft-derivation of research
6. Wrap-up

Goals

1. Learn something about math
2. Learn something about research
3. Learn something about me, an Oxy math student

Research Institution: Oregon State University



OSU REU

Absolutely Terrifying

Part 2: Absolutely Terrifying

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \nu \triangle \vec{v} - \nabla \pi + \vec{f}$$

Math Lesson 1

1. **Be okay being uncomfortable: aka, be discomfortable**

Navier-Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \nu \triangle \vec{v} - \nabla \pi + \vec{f}$$

What? They describe the motion of a fluid in terms of *velocity (v)*, *pressure (π)*, *viscosity (ν)*, and *external forces (f)*.

Navier-Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \nu \triangle \vec{v} - \nabla \pi + \vec{f}$$

What? They describe the motion of a fluid in terms of *velocity (v)*, *pressure (π)*, *viscosity (ν)*, and *external forces (f)*.

Gradient: $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
Laplacian: $\triangle = \nabla^2$

Newton's Second Law:

$$F = m \cdot a$$

Newton's Second Law:

$$F = m \cdot a$$

$$F/m = a$$

The Navier-Stokes Equations

Newton's Second Law:

$$F = m \cdot a$$
$$F/m = a$$
$$a = \frac{d}{dt}(v)$$

Newton's Second Law:

$$F = m \cdot a$$
$$F/m = a$$
$$a = \frac{d}{dt}(v)$$

a special derivative...

Newton's Second Law:

$$F = m \cdot a$$

$$F/m = a$$

$$a = \frac{d}{dt}(v)$$

a special derivative... $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla)$

$$\left(\frac{\partial}{\partial t} + (\vec{v} \cdot \nabla)\right) \cdot \vec{v} = \nu \triangle \vec{v} - \nabla \pi + \vec{f}$$

The Navier-Stokes equation in $\mathbb{R}^3$,

$$\begin{cases} \frac{\partial \vec{v_i}}{\partial t} + (\vec{v_i} \cdot \nabla)\vec{v_i} = \nu \triangle \vec{v_i} - \nabla \pi + \vec{f_i}, \quad i \in [1, 2, 3] \\ \nabla \cdot \vec{v} = 0 \end{cases}$$

The Navier-Stokes Equations

assets/example1.jpg

(a) Compressible Fluid (Gases)

(b) Incompressible Fluids (water)

All Together...

The Navier-Stokes equation in $\mathbb{R}^3$,

$$\begin{cases} \frac{\partial \vec{v_i}}{\partial t} + (\vec{v_i} \cdot \nabla)\vec{v_i} = \nu \triangle \vec{v_i} - \nabla \pi + \vec{f_i}, \quad i \in [1, 2, 3] \\ \nabla \cdot \vec{v} = 0 \end{cases}$$

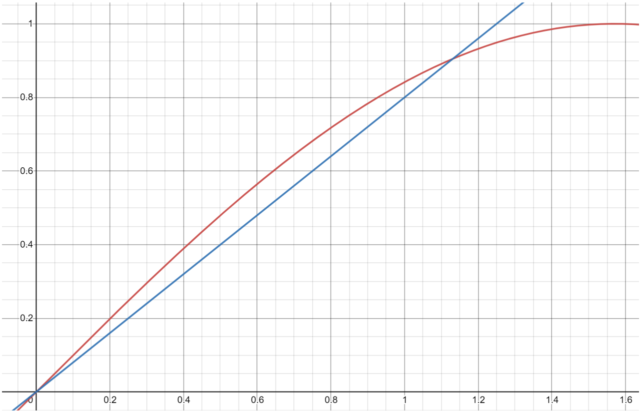
What are our unknowns?

Velocity and Pressure

Solutions

Flow Velocity: A vector field of the velocity of a fluid at any given point on our space

Linearity



Part 3: Stokes Equations

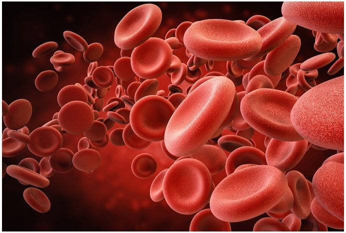
Linearize to get

Our Stokes Equations

$$\frac{\partial \vec{v}}{\partial t} = \triangle \vec{v} - \nabla p.$$

When is this useful?

(a) Lava Flow



(b) Vaccine Diffusion

Our Research Question

Can we solve this equation on the Half-Space?

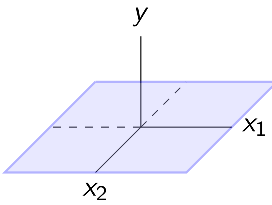


Figure: The Half Space, $\mathbb{R}^3_+$

1. Be Discomfortable
2. **Why?**

Why the Half-Space?

- Well-known
- Many Applications
- Proof of concept

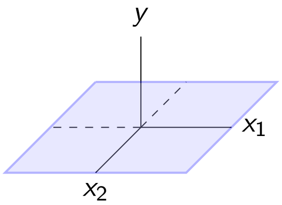


Figure: The Half Space, $\mathbb{R}^3_+$

Why Boundary and Initial Data?

Part 4: Our Research Goal

Our Domain

$\frac{\partial \vec{v}}{\partial t} = \vec{v} - \nabla p$ Given:

- $\vec{v}_0(x, y)$
- $\vec{b}(x, t)$, where $x = \langle x_1, x_2 \rangle$.

Can we solve this using undergraduate theory?

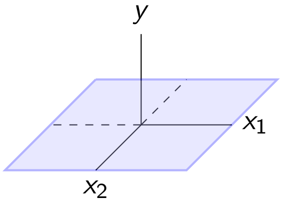


Figure: The Half Space, $\mathbb{R}^3_+$

Problem Set Up

$$\begin{cases} \frac{\partial \vec{v}}{\partial t} = \triangle \vec{v} - \nabla p \quad \text{on} \quad \mathbb{R}^3_+ \\ \nabla \cdot \vec{v} = 0 \end{cases}$$

Two Cases

1. Initial data $\neq 0$, boundary data $= 0$.
2. Initial data $= 0$, boundary data $\neq 0$.

By linearity, add together solutions for $\vec{v}$ and p for Cases 1 and 2 to get general solution for $\vec{v}$ and p.

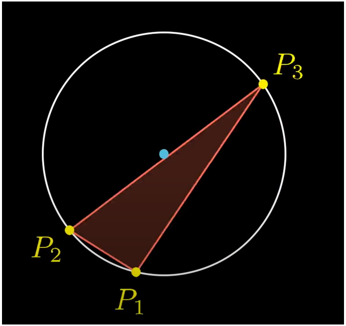
Math Lesson 3

1. Be okay being discomfortable
2. Why?
3. **When Math is hard, make it easy**

Making Easier, example

assets/3b1b1.png

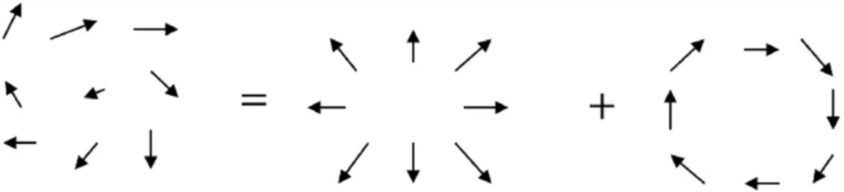
(a) 3D problem



(b) 2D problem

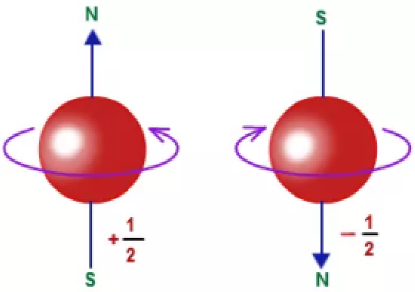
Helmholtz Decomposition

$$\boldsymbol{\xi} = \nabla \boldsymbol{E} + \nabla \times \boldsymbol{W}.$$



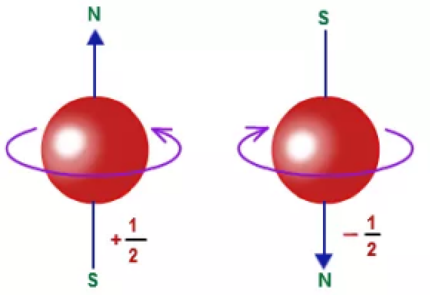
Vorticity

The *spin* of a particle:



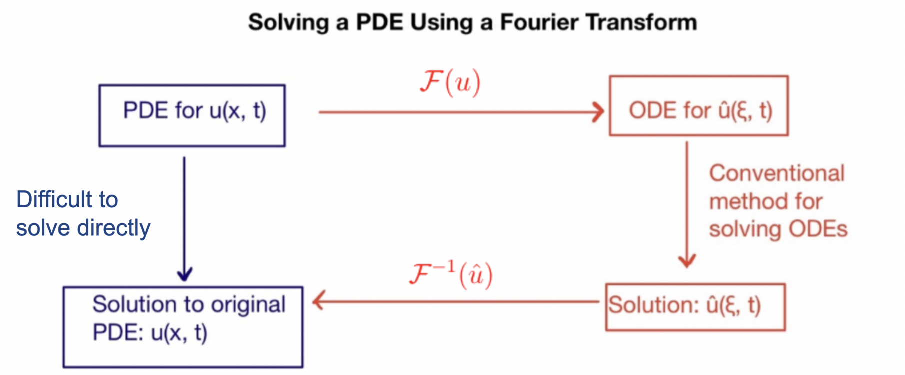
Vorticity

The *spin* of a particle:



Vorticity: $\vec{w} = \nabla \times \vec{v}$

Making it Easy(..er) Part 2: Transforming it



Part 5: *Soft* Derivations of Research

Case 1 and 2: Set-Up

1. Find w
2. Take Fourier
3. Take Laplace
4. Use a special identity to then come back
5. Use decomposition to find velocity

Case 1 and 2: Set-Up

1. Find w
2. Take Fourier
3. Take Laplace
4. Use a special identity to then come back
5. Use decomposition to find velocity

Case 1

We wish to solve the Stokes Equation when we have **some initial data** but **no boundary data**

$$\vec{v}(x, y, 0) \neq 0$$
$$\vec{v}(x, 0, t) = 0.$$

Want to make it easier using decomposition...

$$\nabla \times \left(\frac{\partial \vec{v}}{\partial t} = \triangle \vec{v} - \nabla p \right)$$

Equation for Vorticity $\vec{w}$

$$\frac{\partial}{\partial t} \vec{w} = \triangle \vec{w}$$

5.2: Fourier Transform

Vorticity Equation

$$\frac{\partial}{\partial t}\vec{w} = \triangle \vec{w} = \frac{\partial^2}{(\partial x)^2}\vec{w}$$

Two Variables: t and our x.
Make it easier: Transform our x variable first using a *Fourier Transformation*

5.2: Fourier Transform

Vorticity Equation

$$\frac{\partial}{\partial t}\vec{w} = \triangle \vec{w} = \frac{\partial^2}{(\partial x)^2}\vec{w}$$

Two Variables: t and our x.
Make it easier: Transform our x variable first using a *Fourier Transformation*

Fourier Transformation

$$\frac{\partial}{\partial t}\hat{w} = -|\xi|^2\hat{w} + \frac{\partial^2}{\partial y^2}\hat{w},$$

What did this transform?

1. x_1, x_2 into ξ_1, ξ_2
2. w into $\hat{w}$

5.3: Laplace Transform

Laplace Transformation

$$s\mathcal{L}[\hat{w}] - \hat{w}\Big|_{t=0} = -|\xi|^2\mathcal{L}[\hat{w}] + \frac{\partial^2}{\partial y^2}\mathcal{L}[\hat{w}],$$

where $\mathcal{L}[\hat{w}]$ is a function of (ξ, y, s).

What did this transform?

1. t into s
2. $\hat{w}$ into $\mathcal{L}[\hat{w}]$

5.2: Make it look easier

let

$$\vec{g}(\xi, y, s) = g(y) = \mathcal{L}[\hat{w}(x, y, t)],$$

and

$$\vec{f}(\xi, y, 0) = f(y) = -\hat{w}(x, y, t)\Big|_{t=0},$$

and then

$$\alpha^2 = |\xi|^2 + s.$$

So we can rewrite it as...

$$g''(y) - \alpha^2 g(y) = f(y),$$

which is a second-order, nonhomogenous, constant-coefficient ordinary differential equation.

5.4: Solve Using ODE Theory

Using Variation of Parameters and Green's Function, we obtain

Proposition

$$g(y) = \mathcal{L}\mathcal{F}[w] = \int_0^\infty \frac{e^{-\sqrt{|\xi|+s}(y+y')} - e^{-\sqrt{|\xi|+s}|y+y'|}}{2\sqrt{|\xi|+s}} f(\xi, y')dy'.$$

Next, we take the inverse integral transforms to solve for vorticity.

5.4: Problem coming back

$$\mathcal{L}\mathcal{F}[w] = \int_0^\infty \frac{e^{-\sqrt{|\xi|+s}(y+y')}}{2\sqrt{|\xi|+s}} f(\xi, y')dy'.$$

5.4: Our New Tool!

Theorem

Let $f(x)$ be a continuous and integrable function on $x \in (-\infty, \infty)$, and let $a > 0$ and $b > 0$. Then,

$$\frac{1}{a}\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(ax - b/x)dx.$$

Corollary

Again, let $a > 0$, $b > 0$, and $x \in \mathbb{R}$.

$$e^{-2ab} = \frac{a}{\sqrt{\pi}}\int_0^{\infty} \frac{e^{-a^2\tau - \frac{b^2}{\tau}}}{\sqrt{\tau}} d\tau.$$

5.4: Using new tool

$$\mathcal{L}\mathcal{F}[w] = \int_0^\infty \frac{e^{-\sqrt{|\xi|+s}(y+y')}}{2\sqrt{|\xi|+s}} f(\xi, y')dy'.$$

Plugging variable in...

$$e^{-2ab} = \frac{a}{\sqrt{\pi}} \int_0^\infty \frac{e^{-a^2\tau - \frac{b^2}{\tau}}}{\sqrt{\tau}} d\tau.$$

$$e^{-2(y+y')(-\sqrt{|\xi|+s})} = \frac{(y+y')}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(y+y')^2\tau - \frac{(-\sqrt{|\xi|+s})^2}{\tau}}}{\sqrt{\tau}} d\tau$$

After using this corollary and the Convolution Theorem, we can take the Inverse Fourier and Inverse Laplace transforms, giving a unique solution for **vorticity**,

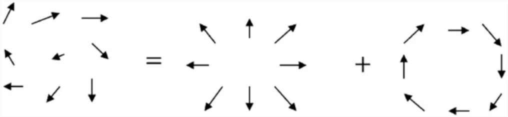
Theorem: Case 1 Equation for Vorticity

$$\vec{w} = \frac{1}{4\pi^2} \int_0^t \int_0^\infty \int_{\mathbb{R}^2} \left(\frac{e^{\frac{-|x-x'|^2}{4\tau}}}{4\pi\tau} \right) \cdot \frac{e^{\frac{-(y+y')^2}{4\tau}} - e^{\frac{-(y-y')^2}{4\tau}}}{\sqrt{4\pi\tau}} w_0(t-\tau) dx' dy' d\tau.$$

$$\vec{v} = -\nabla\phi + \nabla \times \vec{A} \qquad \in \mathbb{R}^3_+$$

Non-rotational component, $-\nabla\phi$
Rotational component $\nabla\times\vec{A}$

$\boldsymbol{\xi} = \nabla \boldsymbol{E} + \nabla \times \boldsymbol{W}.$



Synthesizing Results

By the Superposition Principle

Theorem: Velocity on Half-Space

$$\vec{v} = \nabla \frac{1}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^2} \left(\frac{e^{\frac{-y^2}{4\tau}}}{\sqrt{\tau}} \, e^{\frac{-|x|^2}{2\tau}} \right) \vec{a}_3(x', t) dx' d\tau$$

$$+ \nabla \times 2 \int_{\mathbb{R}^3_+} \vec{w}(x, y, t) \left[\frac{1}{4\pi\sqrt{|x-x'|^2+(y+y')^2}} - \frac{1}{4\pi\sqrt{|x-x'|^2+(y-y')^2}} \right] dx' dy'$$

Theorem: Pressure on the Half-Space

$$\nabla p = \left(\frac{\partial}{\partial t} - \triangle\right)\vec{v}$$

Part 6: What does it all mean?

Give me some data for velocity, I can tell you everything about the fluid

and

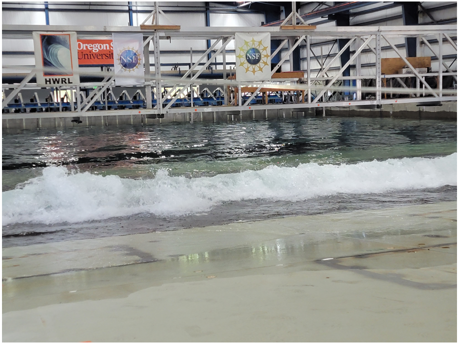
we can use undergraduate theory to solve this!

Takeaways

Math Lessons:

1. Be okay being discomfortable
2. Why?
3. When Math is hard, make it easy

My Research Experience:



Thank You

Questions?

Miles Smith: msmith3@oxy.edu

References

Kim, Hoewoon and Thomann, Enrique and Guenther, Ronald (2019)
A Representation of the Solution of the Stokes Equations in the Half Space $\mathbb{R}^3_+$: Application to Spatial and Temporal Estimates of the Pressure
Journal of Mathematical Fluid Mechanics

Duffy, Dean G. (2015)
Green's Functions with Applications
Chapman and Hall/CRC

Logan, J David (2014)
Applied partial differential equations
Springer

Cuicui Zhang and Hao Wei and Chongke Bi and Zhilei Liu (2019)
Helmholtz–Hodge decomposition-based 2D and 3D ocean surface current visualization for mesoscale eddy detection
Springer Science and Business Media LLC 231–243

References

Guenther, R.B. and Lee, J.W. (2012)
Partial Differential Equations of Mathematical Physics and Integral Equations
Dover Publications

Chorin, A.J. and Marsden, J.E. (2013)
A Mathematical Introduction to Fluid Mechanics
Springer New York

Phil Dyke (2014)
An Introduction to Laplace Transforms and Fourier Series
Springer-Verlag London

Trenberth, Kevin E and Trenberth, Kevin E (1992)
Climate system modeling
Cambridge University Press

Erik Cheever (2005)
Fourier Xform Properties
Cambridge University Press