Homework 1: An option's fair price

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Abstract. In this first homework we will be computing using a Monte Carlo method the exercise value of a share, which we will then use to determine a fair price for a call option of that share.

1 Computing the share's value via a Wiener process

Firstly, we need to compute the value of the share itself after a certain amount of time. For this purpose we will solve the following stochastic differential equation, known as *Wiener process* [1]:

$$\frac{\mathrm{d}S}{S} = \sigma \mathrm{d}X + \mu \mathrm{d}t \quad , \tag{1}$$

where σ is the volatility, μ is the drift and $S(t)$ the value of the option as a function of time. Instead of solving it analytically, we can discretize it as in [2] to find

$$S_{i+1} = S_i \left(1 + \sigma \Delta W_i + \mu \Delta t_i\right) \quad , \tag{2}$$

where $\Delta W_i = \sqrt{\Delta t_i} N(0, 1)$, simulating the random walk. In our case the time step is constant, so every Δt_i will have the same value. The value after nt time steps is the value at expiration date, which we will use to determine the price of the option.

2 Determining fair prices for call options

Now that we can compute the exercise value of the share, we can use it to determine how much we can pay for a call option for that share. Since this is not a value that can be derived via an objective procedure, we will consider two approaches to compute the price: the *risk-free approach* and the *risky approach.*

2.1 Risk-free approach

For the **risk-free approach**, we consider the probability distribution of the value of the share at expiration date, which will be a normal $N(\mu, \sigma)$. Now, since buying the call option gives us the *right* to buy the share at the exercise price, we can choose not to use it if that means losing money. To represent this fact, instead of using the previous normal for our computations, we take every instance with share price lower than the exercise price and assign a profit of zero to it, instead of a negative value.

In the following plots we can clearly see that transformation:

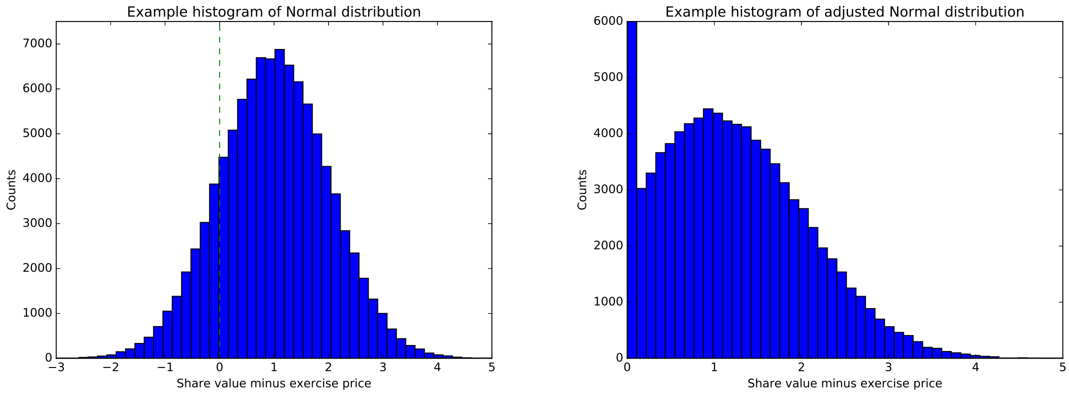

Figure 1: (a) Example of normal distribution $N(1,1)$. (b) Distribution after applying the mentioned transformation, $\tilde{N}(1,1)$.

Using this new probability distribution we can compute the risk-free price as the **average value** of it. This price is the minimum profit that on average we expect to make, so the fair price using this approach would be any amount smaller than the sum of the exercise price plus this value, as much as the writer of the option is willing to accept.

We can write this as

$$\langle oP\rangle_{rf} \equiv \left\langle \tilde{N}(\mu,\sigma)\right\rangle \quad , \tag{3}$$

where aE is the exercise price and $\langle oP\rangle_{rf}$ is the option price computed risk-free, and $\tilde{N}(\mu,\sigma)$ is the adjusted normal distribution.

2.2 Risky approach

On the other hand, we can take a **risky approach** and pay more than the risk-free price with a smaller probability of having profits. For this to be quantitative we have to assign a risk value to a price, in a way that zero corresponds to making the average profit and one corresponds to the impossible case of having profit paying an infinite amount of money.

So, assuming the expiry values of the share is a random variable with normal distribution, we can introduce the risk as a value $P \in [0,1)$ and compute the probability of making average profit using the cumulative distribution function (cdf) as

$$\mathrm{cdf}(\langle oP\rangle_{rf}, \mu, \sigma) \quad ,$$

and then computing the final probability using P as

$$\mathrm{cdf}(\langle oP\rangle_{rf}, \mu, \sigma)(1-P) + P \quad .$$

Now, what we are doing here is mapping the interval $\left[\mathrm{cdf}(\langle oP\rangle_{rf}, \mu, \sigma), 1\right)$ to $[0,1)$, so that by computing the inverse cdf of this probability we can map the expected profits interval, $[\langle oP\rangle_{rf}, \infty)$, to $P \in [0,1)$. We can write this calculation as

$$\langle oP\rangle_{r} \equiv \mathrm{cdf}^{-1}\left(\mathrm{cdf}(\langle oP\rangle_{rf}, \mu, \sigma)(1-P) + P\right) \quad . \tag{4}$$

3 Results

In this section we will use the previous theory with the following values: $\mu = 0.02$, $\sigma = 0.2$, $S(0) = 14$ and $aE = 15$, over a period of 101 days. In the following histogram we can see the results of performing $N = 10^5$ random walks to compute the price of the option at exercise date. From here we can compute the values of μ and σ mentioned in the previous section.

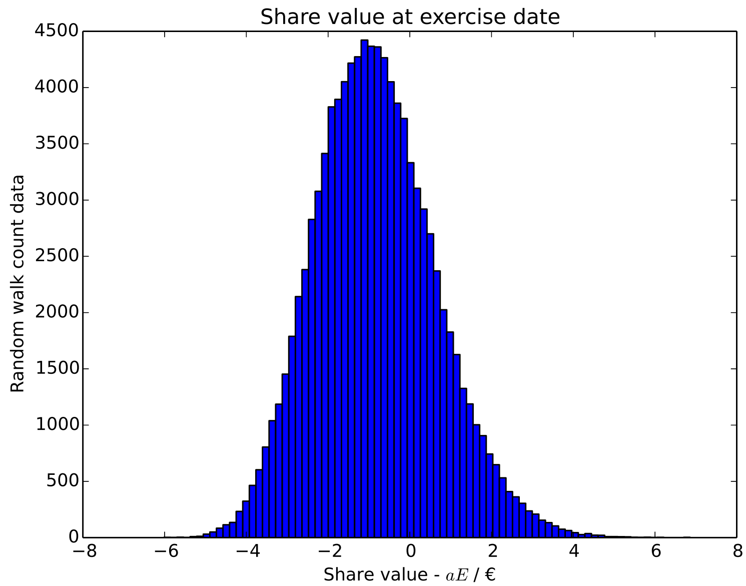

Figure 2: Histogram showing the distribution of share values after $N = 10^5$ random walks.

From this plot we can see that a drift ratio of 0.02 is not enough to increase the value of the share significantly, however the volatility increases the width of the normal, and therefore for some value there will be a profit. Now we can proceed to compute the risk-free and risky value for the option. The two graphs below show the profit-risk relation in absolute profit and in percentage given these values.

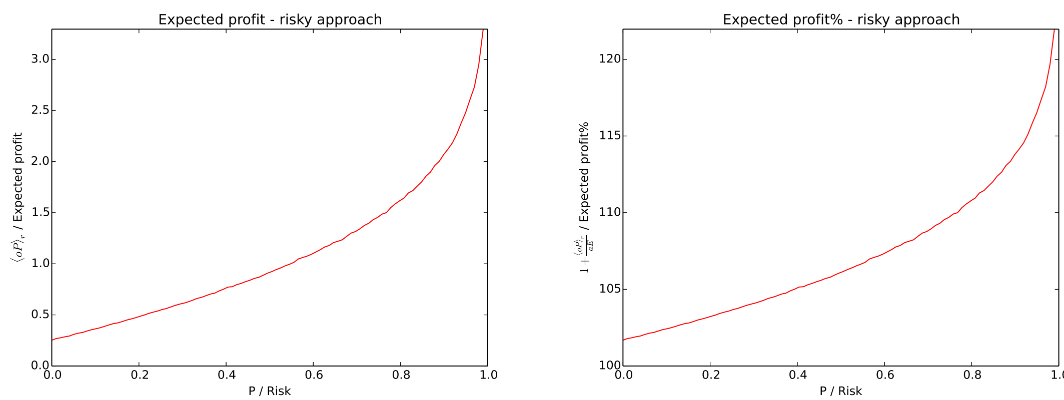

Figure 3: (a) Expected profit at the exercise date of the option. (b) Expected profit over exercise value of the option.

As we can see in Figure 3a, the risk-free approach gives a value for the option of ~ 0.25€, however it can be increased by taking more risk as we explained before. In this case we can consider a risk $P = 0.5$ acceptable for a profit% of around $\sim 5\%$, although still very small. If this were a real option, we would advise against buying it in the first place, and find others with greater expected profits.

References

[1] P. Wilmott et al, *The Mathematics of Financial Derivatives*, 1995.

[2] Timothy Sauer, *Handbook of Computational Statistics*, Springer, pp 529-550, July 2011.