线性代数第三次作业答案

一、判断题

七. 综合习题

a. ~~✗~~ ✓ $\det(A)=0$ 不可逆. 那 2x2 矩阵

b ✓ 两行相等 有一行可以行初等变换为全0. 肯定不行等价于单位阵.

c ✗

d ✗

e ✗

f ✗

g ✓

h ✓

i ✗

j ✗

k ✓ $\det(A^TA) = \det^2(A) \geq 0$

l ✗

m ✗

n ✓

o ✗

p ✓

二、计算、证明题

1、

To check whether the matrix A has the inverse matrix and to find the inverse matrix if exist at once, we consider the augmented matrix $[A \mid I]$, where I is the 3×3 identity matrix.
We apply the elementary row operations as follows.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 - R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

The left 3×3 matrix part became the identity matrix I, thus A is invertible (since it is row equivalent to I), and the inverse matrix A^{-1} is given by the right 3×3 matrix. Thus we have

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

2、

We first use the properties of transpose matrices and inverse matrices and simplify the expression.
Note that we have

$$(A^{\mathrm{T}} - B)^{\mathrm{T}} = (A^{\mathrm{T}})^{\mathrm{T}} - B^{\mathrm{T}} = A - B$$

since the double transpose $(A^{\mathrm{T}})^{\mathrm{T}} = A$ and B is a symmetric matrix.

Also, note that we have

$$(B^{-1}C)^{-1} = C^{-1}(B^{-1})^{-1} = C^{-1}B$$

since $(B^{-1})^{-1} = B$. Care must be taken when you distribute the inverse sign in the first equality. We needed to switch the order of the product.
Then we have

$$C(B^{-1}C)^{-1} = CC^{-1}B = IB = B,$$

where I is the 3×3 identity matrix.

Therefore, the given expression can be simplified into

$$(A^{\mathrm{T}} - B)^{\mathrm{T}} + C(B^{-1}C)^{-1} = A - B + B = A.$$

Hence we have

$$(A^{\mathrm{T}} - B)^{\mathrm{T}} + C(B^{-1}C)^{-1} = A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

- 解本题的常犯错误举例

Common Mistakes

Here are two common mistakes.

The first one is

$$(B^{-1}C)^{-1} = (B^{-1})^{-1}C^{-1} \quad \text{(This is wrong!!).}$$

Note that in general, we have

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{(Note the order of products).}$$

The second common mistake is that

$$CBC^{-1} = CC^{-1}B = B \quad \text{(The first equality is wrong!!).}$$

Recall that in general matrix multiplication is not commutative, meaning that

$$AB \neq BA.$$

And surprisingly, if you combine these two mistakes, you get the right answer.

$$C(B^{-1}C)^{-1} = CBC^{-1} = B. \quad \text{(This is wrong!!).}$$

However, these mistakes show that you didn't understand matrix operations including transpose and inverse matrices.

3、

Let $P = A + B$. Then $B = P - A$.

Using these, we express the given expression in terms of only A and P.
On one hand, we have

$$A(A+B)^{-1}B = AP^{-1}(P-A) = AP^{-1}P - AP^{-1}A = A - AP^{-1}A$$

On the other hand we have

$$B(A+B)^{-1}A = (P-A)P^{-1}A = PP^{-1}A - AP^{-1}A = A - AP^{-1}A$$

Thus these are equal.
This completes the proof.

4、

Suppose that there are two inverse matrices B and C of the matrix A. Then they satisfy

$$AB = BA = I$$

and

$$AC = CA = I$$

To show that the uniqueness of the inverse matrix, we show that $B = C$ as follows. Let I be the $n \times n$ identity matrix.
We have

$$
\begin{aligned}
B &= BI \\
&= B(AC) \\
&= (BA)C \\
&= IC \\
&= C
\end{aligned}
$$

Thus, we must have $B = C$, and there is only one inverse matrix of A.

5、

(1) We prove $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ by induction on n.

The base case $n = 1$ is true by definition.

Suppose that $A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$. Then we have

$$
A^{k+1} = AA^k = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{bmatrix}
$$

Here we used the induction hypothesis in the second equality.
Hence the inductive step holds. This completes the proof.

(2) We show that $B^n = S^{-1} A^n S$ by induction on n.
When $n = 1$, this is just the definition of B

For induction step, assume that $B^k = S^{-1} A^k S$.
Then we have

$$
B^{k+1} = BB^k = \left(S^{-1} AS\right)\left(S^{-1} A^k S\right) = S^{-1} AA^k S = S^{-1} A^{k+1} S
$$

where we used the induction hypothesis in the second equality and the third equality follows by canceling $SS^{-1} = I_2$ in the middle.

Thus the inductive step holds, and this competes the proof.

6、

(a).

The system can be written as

$$
A\mathbf{x} = \mathbf{b}
$$

where

$$
A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}
$$

is the coefficient matrix of the system and

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

(b).

We apply the elementary row operations to the augmented matrix $[A \mid I]$, where I is the 3×3 identity matrix.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$
$$\xrightarrow[\text{then } R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$
$$\xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

Now the left 3×3 part has become the identity matrix.
So the matrix A is invertible and the inverse is given by the right 3×3 part.
Hence we obtain

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

(c).

As noted in (a), the system can be written using matrices as

$$A\mathbf{x} = \mathbf{b}$$

Multiplying by the inverse A^{-1} on the left, we have

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

Therefore the solution $\mathbf{x}$ of the system is given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Thus, the solution of the system is

$$x = 2, y = -1, z = -2$$

7、

We use the fact that a matrix is invertible if and only if its determinant is nonzero.
So we compute the determinant of the matrix A.

We have

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 1 & x \\ 1 & x & x \\ x & x & x \end{vmatrix} \\ &= (1)\begin{vmatrix} x & x \\ x & x \end{vmatrix} - (1)\begin{vmatrix} 1 & x \\ x & x \end{vmatrix} + x\begin{vmatrix} 1 & x \\ x & x \end{vmatrix} && \text{by the first row cofactor expansion.} \\ &= (x^2 - x^2) - (x - x^2) + x(x - x^2) \\ &= (x-1)(x-x^2) \\ &= x(x-1)^2. \end{aligned}$$

Thus, the determinant $\det(A)$ is zero if and only if $x = 0, 1$.
Hence the matrix A is invertible if and only if $x \neq 0, 1$.

Next, we suppose that $x \neq 0, 1$ and find the inverse matrix of A.
We reduce the augmented matrix $[A \mid I]$ as follows.
We have

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & x & 1 & 0 & 0 \\ 1 & x & x & 0 & 1 & 0 \\ x & x & x & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - xR_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & x & 1 & 0 & 0 \\ 0 & x-1 & 0 & -1 & 1 & 0 \\ 0 & 0 & x-x^2 & -x & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{x-x^2}R_3]{\frac{1}{x-1}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & x & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{x-1} & \frac{1}{x-1} & 0 \\ 0 & 0 & 1 & \frac{-1}{1-x} & 0 & \frac{1}{x-x^2} \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & x & \frac{x}{x-1} & \frac{-1}{x-1} & 0 \\ 0 & 1 & 0 & \frac{-1}{x-1} & \frac{1}{x-1} & 0 \\ 0 & 0 & 1 & \frac{-1}{1-x} & 0 & \frac{1}{x-x^2} \end{array}\right] \xrightarrow{R_1 - xR_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{-1}{x-1} & \frac{-x}{x-x^2} \\ 0 & 1 & 0 & \frac{-1}{x-1} & \frac{1}{x-1} & 0 \\ 0 & 0 & 1 & \frac{-1}{1-x} & 0 & \frac{1}{x-x^2} \end{array}\right].$$

Now that we reduced the left 3×3 matrix into the identity matrix, the right 3×3 matrix is the inverse matrix of A.
(Note that when we applied elementary row operations, we divided by $x - 1$ and $x - x^2$, and this is where we needed to assume $x \neq 0, 1$.)

We have

$$A^{-1} = \begin{bmatrix} 0 & \frac{-1}{x-1} & \frac{-x}{x-x^2} \\ \frac{-1}{x-1} & \frac{1}{x-1} & 0 \\ \frac{-1}{1-x} & 0 & \frac{1}{x-x^2} \end{bmatrix} = \frac{1}{x(1-x)} \begin{bmatrix} 0 & x & -x \\ x & -x & 0 \\ -x & 0 & 1 \end{bmatrix}.$$

8、

We use the following two properties of determinants.
Let C and D be $n \times n$ matrices. Then we have

$$\det(CD) = \det(C)\det(D)$$

and if C is invertible, then

$$\det(C^{-1}) = \det(C)^{-1} = \frac{1}{\det(C)}.$$

Using the properties of determinants, we compute

$$\begin{aligned}
&\det(A^2B^{-1}A^{-2}B^2) \\
&= \det(A)^2 \det(B)^{-1} \det(A)^{-2} \det(B)^2 \\
&= \det(A)^2 \det(A)^{-2} \det(B)^{-1} \det(B)^2 && \text{(determinants are just numbers)} \\
&= \det(B).
\end{aligned}$$

Hence it suffices to find the determinant of the matrix B.
Since the matrix B is an upper triangular matrix, its determinant is the product of diagonal entries, thus

$$\det(B) = 2 \cdot 3 \cdot 4 = 24.$$

As a result, we obtain

$$\det(A^2B^{-1}A^{-2}B^2) = 24.$$

9、

Note that the determinant does not change if the i-th row is added by a scalar multiple of the j-th row if $i \neq j$. We use this fact about the determinant and compute $\det(A)$ as follows.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 100 & 101 & 102 \\ 101 & 102 & 103 \\ 102 & 103 & 104 \end{vmatrix} \\
&= \begin{vmatrix} 100 & 101 & 102 \\ 101 & 102 & 103 \\ 1 & 1 & 1 \end{vmatrix} \qquad (\text{by } R_3 - R_2) \\
&= \begin{vmatrix} 100 & 101 & 102 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \qquad (\text{by } R_2 - R_1) \\
&= \begin{vmatrix} 100 & 101 & 102 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} \qquad (\text{by } R_3 - R_1) \\
&= 0 \quad (\text{by the third row cofactor expansion. })
\end{aligned}$$

Therefore the determinant $\det(A)$ is zero.

10、

3.1 先使用矩阵乘法的行列法则：

$$AB=\begin{bmatrix}2&-3&1&0&-4\\1&5&-2&3&-1\\0&-4&-2&7&-1\end{bmatrix}\begin{bmatrix}6&4\\-2&1\\-3&7\\-1&3\\5&2\end{bmatrix}=\begin{bmatrix}2\times6+(-3)(-2)+\cdots+(-4)5&\cdots\cdots\\6\times1+5(-2)+\cdots+(-1)5&\cdots\cdots\\0\times6+(-4)(-2)+\cdots+(-1)5&\cdots\cdots\end{bmatrix}=\begin{bmatrix}-5&4\\-6&2\\2&1\end{bmatrix}$$

使用分块矩阵计算：

$$A=\left[\begin{array}{ccc:cc}2&-3&1&0&-4\\1&5&-2&3&-1\\\hdashline 0&-4&-2&7&-1\end{array}\right]=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix}\quad B=\left[\begin{array}{cc}6&4\\-2&1\\-3&7\\\hdashline -1&3\\5&2\end{array}\right]=\begin{bmatrix}B_1\\B_2\end{bmatrix}$$

$$A\cdot B=\begin{bmatrix}A_{11}B_1+A_{12}B_2\\A_{21}B_1+A_{22}B_2\end{bmatrix}$$

$$A_{11}B_1=\begin{bmatrix}15&12\\2&-5\end{bmatrix}\qquad A_{12}B_2=\begin{bmatrix}-20&-8\\-8&7\end{bmatrix}$$

$$A_{21}B_1=\begin{bmatrix}14&-18\end{bmatrix}\qquad A_{22}B_2=\begin{bmatrix}-12&19\end{bmatrix}$$

$$AB=\begin{bmatrix}-5&4\\-6&2\\2&1\end{bmatrix}$$

这两个方法的结果是一样的. 分块矩阵计算矩阵乘积的时候,要注意分块：分块分的要合理（合理的意思是分块后的矩阵要满足乘法性质，即 AB 中对A列的分法要和对B行的分法要匹配）