

线性代数第三次作业答案

一、判断题

七. 综合习题

a. ~~✗~~ ✓ $\det(A)=0$ 不可逆. 那 2×2 矩阵

b. ✓ 两行相等 有一行可以被行变换为全0. 肯定不^行等价于单位阵.

c. ✗

d. ✗

e. ✗

f. ✗

g. ✓

h. ✓

i. ✗

j. ✗

k. ✓

l. ✗

m. ✗

n. ✓

o. ✗

p. ✓

$$\det(A^T A) = \det^2(A) \neq 0$$

二、计算、证明题

1、

To check whether the matrix A has the inverse matrix and to find the inverse matrix if exist at once, we consider the augmented matrix $[A \mid I]$, where I is the 3×3 identity matrix.

We apply the elementary row operations as follows.

$$\begin{aligned}
 [A \mid I] &= \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array} \right] \\
 &\xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right] \\
 &\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_1 - R_3 \\ R_2 - R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right].
 \end{aligned}$$

The left 3×3 matrix part became the identity matrix I , thus A is invertible (since it is row equivalent to I), and the inverse matrix A^{-1} is given by the right 3×3 matrix. Thus we have

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

2、

We first use the properties of transpose matrices and inverse matrices and simplify the expression.

Note that we have

$$(A^T - B)^T = (A^T)^T - B^T = A - B$$

since the double transpose $(A^T)^T = A$ and B is a symmetric matrix.

Also, note that we have

$$(B^{-1}C)^{-1} = C^{-1}(B^{-1})^{-1} = C^{-1}B$$

since $(B^{-1})^{-1} = B$. Care must be taken when you distribute the inverse sign in the first equality. We needed to switch the order of the product.

Then we have

$$C(B^{-1}C)^{-1} = CC^{-1}B = IB = B,$$

where I is the 3×3 identity matrix.

Therefore, the given expression can be simplified into

$$(A^T - B)^T + C(B^{-1}C)^{-1} = A - B + B = A.$$

Hence we have

$$(A^T - B)^T + C(B^{-1}C)^{-1} = A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

- 解本题的常犯错误举例

Common Mistakes

Here are two common mistakes.

The first one is

$$(B^{-1}C)^{-1} = (B^{-1})^{-1}C^{-1} \quad (\text{This is wrong!!}).$$

Note that in general, we have

$$(AB)^{-1} = B^{-1}A^{-1} \quad (\text{Note the order of products}).$$

The second common mistake is that

$$CBC^{-1} = CC^{-1}B = B \quad (\text{The first equality is wrong!!}).$$

Recall that in general matrix multiplication is not commutative, meaning that

$$AB \neq BA.$$

And surprisingly, if you combine these two mistakes, you get the right answer.

$$C(B^{-1}C)^{-1} = CBC^{-1} = B. \quad (\text{This is wrong!!}).$$

However, these mistakes show that you didn't understand matrix operations including transpose and inverse matrices.

3、

Let $P = A + B$. Then $B = P - A$.

Using these, we express the given expression in terms of only A and P .

On one hand, we have

$$A(A + B)^{-1}B = AP^{-1}(P - A) = AP^{-1}P - AP^{-1}A = A - AP^{-1}A$$

On the other hand we have

$$B(A + B)^{-1}A = (P - A)P^{-1}A = PP^{-1}A - AP^{-1}A = A - AP^{-1}A$$

Thus these are equal.

This completes the proof.

4、

Suppose that there are two inverse matrices B and C of the matrix A . Then they satisfy

$$AB = BA = I$$

and

$$AC = CA = I$$

To show that the uniqueness of the inverse matrix, we show that $B = C$ as follows. Let I be the $n \times n$ identity matrix.

We have

$$\begin{aligned}
B &= BI \\
&= B(AC) \\
&= (BA)C \\
&= IC \\
&= C
\end{aligned}$$

Thus, we must have $B = C$, and there is only one inverse matrix of A .

5、

(1) We prove $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ by induction on n .

The base case $n = 1$ is true by definition.

Suppose that $A^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}$. Then we have

$$A^{k+1} = AA^k = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{bmatrix}$$

Here we used the induction hypothesis in the second equality.

Hence the inductive step holds. This completes the proof.

(2) We show that $B^n = S^{-1}A^nS$ by induction on n .

When $n = 1$, this is just the definition of B

For induction step, assume that $B^k = S^{-1}A^kS$.

Then we have

$$B^{k+1} = BB^k = (S^{-1}AS)(S^{-1}A^kS) = S^{-1}AA^kS = S^{-1}A^{k+1}S$$

where we used the induction hypothesis in the second equality and the third equality follows by canceling $SS^{-1} = I_2$ in the middle.

Thus the inductive step holds, and this completes the proof.

6、

(a).

The system can be written as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

is the coefficient matrix of the system and

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

(b).

We apply the elementary row operations to the augmented matrix $[A \mid I]$, where I is the 3×3 identity matrix.

$$\begin{aligned} [A \mid I] &= \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array} \right] \\ &\xrightarrow[\text{then } R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array} \right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array} \right] \\ &\xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array} \right] \end{aligned}$$

Now the left 3×3 part has become the identity matrix.

So the matrix A is invertible and the inverse is given by the right 3×3 part.

Hence we obtain

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

(c).

As noted in (a), the system can be written using matrices as

$$A\mathbf{x} = \mathbf{b}$$

Multiplying by the inverse A^{-1} on the left, we have

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Therefore the solution \mathbf{x} of the system is given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$\begin{aligned}
&= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}
\end{aligned}$$

Thus, the solution of the system is

$$x = 2, y = -1, z = -2$$

7、

We use the following two properties of determinants.

Let C and D be $n \times n$ matrices. Then we have

$$\det(CD) = \det(C) \det(D)$$

and if C is invertible, then

$$\det(C^{-1}) = \det(C)^{-1} = \frac{1}{\det(C)}.$$

Using the properties of determinants, we compute

$$\begin{aligned}
&\det(A^2 B^{-1} A^{-2} B^2) \\
&= \det(A)^2 \det(B)^{-1} \det(A)^{-2} \det(B)^2 \\
&= \det(A)^2 \det(A)^{-2} \det(B)^{-1} \det(B)^2 \quad (\text{determinants are just numbers}) \\
&= \det(B).
\end{aligned}$$

Hence it suffices to find the determinant of the matrix B .

Since the matrix B is an upper triangular matrix, its determinant is the product of diagonal entries, thus

$$\det(B) = 2 \cdot 3 \cdot 4 = 24.$$

As a result, we obtain

$$\det(A^2 B^{-1} A^{-2} B^2) = 24.$$

8、

We first use the properties of transpose matrices and inverse matrices and simplify the expression.

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Also, note that we have

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Then we have

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where I is the 3×3 identity matrix.

Therefore, the given expression can be simplified into

$$(A^T - B)^T + C(B^{-1}C)^{-1} = A - B + B = A.$$

Hence we have

$$(A^T - B)^T + C(B^{-1}C)^{-1} = A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

9、

Note that the determinant does not change if the i -th row is added by a scalar multiple of the j -th row if $i \neq j$.

We use this fact about the determinant and compute $\det(A)$ as follows.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 100 & 101 & 102 \\ 101 & 102 & 103 \\ 102 & 103 & 104 \end{vmatrix} \\ &= \begin{vmatrix} 100 & 101 & 102 \\ 101 & 102 & 103 \\ 1 & 1 & 1 \end{vmatrix} && (\text{by } R_3 - R_2) \\ &= \begin{vmatrix} 100 & 101 & 102 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} && (\text{by } R_2 - R_1) \\ &= \begin{vmatrix} 100 & 101 & 102 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} && (\text{by } R_3 - R_1) \\ &= 0 && (\text{by the third row cofactor expansion.}) \end{aligned}$$

Therefore the determinant $\det(A)$ is zero.

10、

3.1 先使用矩阵乘法的行列法则：

$$AB = \begin{bmatrix} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 6 + (-3) \times (-2) + \dots + (-4) \times 5 & \dots \\ 1 \times 6 + 5 \times (-2) + \dots + (-1) \times 5 & \dots \\ 0 \times 6 + (-4) \times (-2) + \dots + (-1) \times 5 & \dots \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \\ 2 & 1 \end{bmatrix}$$

使用分块矩阵计算：

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

$$A_{11}B_1 = \begin{bmatrix} 15 & 12 \\ 2 & 5 \end{bmatrix} \quad A_{12}B_2 = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

$$A_{21}B_1 = \begin{bmatrix} 14 & -8 \end{bmatrix} \quad A_{22}B_2 = \begin{bmatrix} -12 & 19 \end{bmatrix}$$

$$AB = \begin{bmatrix} -5 & 4 \\ -6 & 2 \\ 2 & 1 \end{bmatrix}$$

这两种方法的结果是一样的。分块矩阵计算矩阵乘积的时候，要注意分块：

分块分得要合理（合理的意思是分块后的矩阵要满足乘法性质，即 $A_{ij}B_{jk}$ 中对A列的分法要和对B行的分法要匹配）