Emergent Fractal Growth and Chaos in Circular Cellular Automata

Milind Sarkar MS22268

Abstract

Cellular automata provide a powerful framework for studying how complex spatial structures emerge from simple local rules. In this work, we investigate a circular cellular automaton (CCA) growth model introduced by Sun et al. in Physical Review E, which evolves outward in concentric rings under an intermediate crowding rule. We reproduce the fractal growth patterns reported in the literature and systematically explore sensitivity to the growth parameter δ, finite-size effects, and update-sequence dependence. Quantitative characterization using fractal dimension and ring-resolved occupation statistics reveals a transition from regular self-similar geometry to chaotic morphology. Our results demonstrate how deterministic microscopic interactions generate scale-invariant and chaotic structures in discrete dynamical systems.

1 Introduction

Understanding how complex patterns arise from simple local interactions is a central problem in nonlinear dynamics and complex systems. Cellular automata (CA) offer minimal models where global structure emerges without centralized control. Classical examples include Conway's Game of Life, reaction-diffusion automata, and growth models exhibiting fractal geometry.

Fractal structures are of particular interest due to their scale invariance and appearance in physical systems such as crystal growth, diffusion-limited aggregation, and biological morphogenesis. Chaos and sensitivity to initial conditions further enrich such systems, linking discrete models to nonlinear dynamical behavior.

Sun, Wang, and Wu proposed a circular cellular automaton (CCA) that grows outward ring by ring instead of updating an entire lattice in parallel. Despite its simplicity, the model generates intricate fractal patterns and chaotic sensitivity to a single control parameter δ [1].

In this paper, we reproduce the CCA model and extend the analysis by systematically exploring parameter space, measuring scaling behavior, testing finite-size robustness, and investigating sensitivity to microscopic update ordering.

Detailed simulation code and analysis routines are provided in the accompanying Jupyter notebook for full reproducibility.

2 Model Description

The automaton is defined on a two-dimensional square lattice initialized with a single occupied cell at the center. Growth proceeds outward in successive circular shells indexed by radius i.

For each ring, sites are visited sequentially according to a prescribed order. A site becomes occupied if and only if exactly one of its four nearest neighbors is already occupied:

$$n_{nn}(i,j) = 1$$

This intermediate crowding rule prevents both isolated growth and over-crowding, producing branching structures.

The parameter δ controls the thickness and location of each growth ring, effectively perturbing the growth front and introducing sensitivity to spatial discretization.

Core Growth Algorithm

```
def grow_ring(grid, radius, delta):
    sites = ring_sites(radius, delta)
    sites.sort()   # line scanning order
    for i, j in sites:
        if grid[i,j] == 0:
            if count_neighbors(grid,i,j) == 1:
                grid[i,j] = 1
```

This sequential update mechanism is responsible for the strong sensitivity observed in the resulting structures.

Two update sequences are considered:

- Line-scanning (top-to-bottom, left-to-right)
- Clockwise angular ordering

Implementation Note. All simulations, data analysis, and figure generation were performed using a Jupyter notebook developed for this study. While representative algorithmic steps are included here for clarity, the complete implementation, parameter settings, and reproducible workflow are provided in the accompanying `CircularCA.ipynb` notebook, which should be consulted for full computational details.

3 Fractal Pattern Formation

For $\delta = 0$, the growth rings align perfectly with the lattice structure, leading to highly regular self-similar fractal geometry.

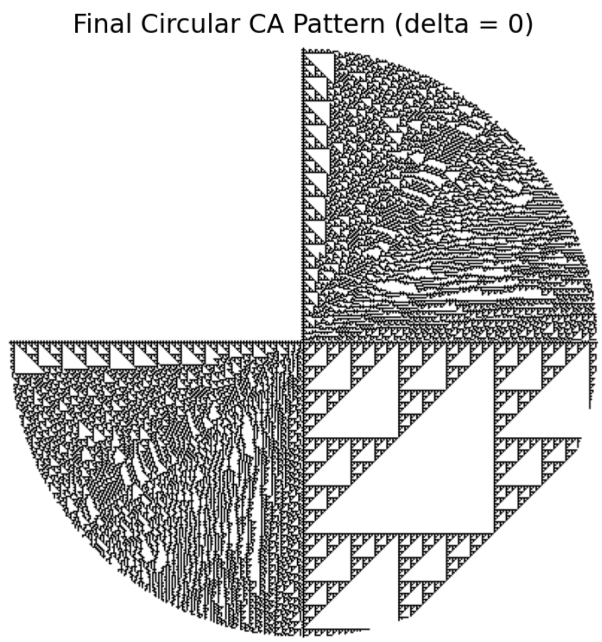

Figure 1: Final CCA pattern for $\delta = 0$. The structure exhibits clear self-similarity and triangular fractal motifs.

The repeating triangular voids resemble Sierpinski-type fractals, indicating scale-invariant branching behavior.

When δ is increased, the precise alignment of growth rings is perturbed. This breaks the regular fractal symmetry and introduces increasingly irregular structures.

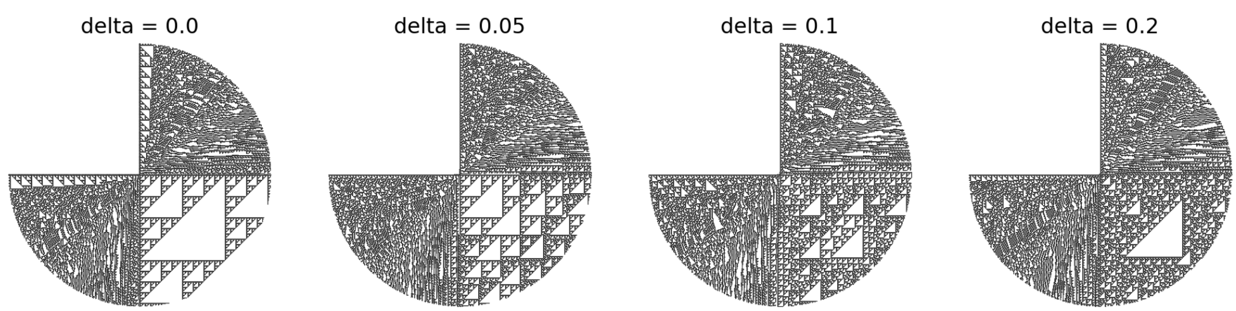

Figure 2: CCA patterns for increasing δ. Regular fractal symmetry gradually gives way to chaotic morphology.

Even small changes in δ produce visibly distinct geometries, highlighting strong sensitivity to growth conditions.

4 Quantitative Fractal Dimension

To quantify geometric complexity, we employ the box-counting method. The number of occupied boxes $N(\epsilon)$ of size ϵ follows:

$$N(\epsilon) \propto \epsilon^{-D}$$

where D is the fractal dimension.

Algorithmic Extraction of D

```
def fractal_dimension(Z):
    sizes = 2**np.arange(1, n)
    counts = [box_count(Z,s) for s in sizes]
    D = np.polyfit(np.log(1/sizes), np.log(counts), 1)[0]
```

The slope of the scaling relation yields D.

4.1 Dependence on Growth Parameter

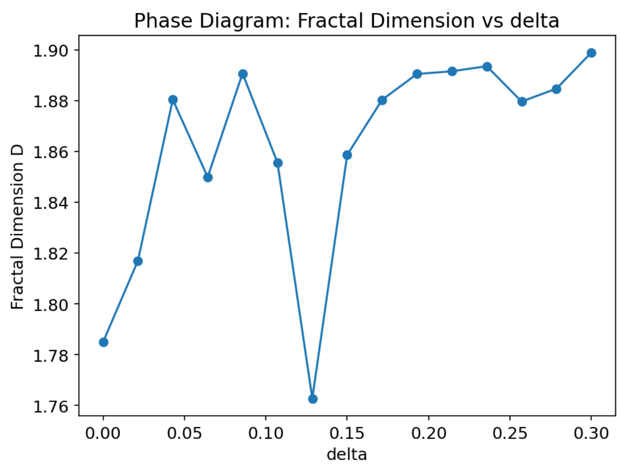

Figure 3: Fractal dimension D as a function of growth parameter δ.

For $\delta = 0$, the fractal dimension corresponds to sparse self-similar geometry produced by highly ordered branching. As δ increases, D generally increases, indicating progressively more space-filling and irregular growth.

Interestingly, a pronounced dip in the fractal dimension appears near $\delta \approx 0.13$. This non-monotonic behavior reflects a temporary suppression of branching complexity caused by partial misalignment of growth rings with the lattice structure. In this regime, growth pathways interfere destructively, leading to more filamentary and less space-filling configurations before chaotic filling dominates at higher δ.

Beyond this region, increasing perturbation enhances overlap and local filling, producing denser chaotic morphology and a corresponding rise in fractal dimension.

Such non-monotonic transitions are characteristic of systems where competing ordering and disordering mechanisms coexist.

4.2 Finite-Size Scaling

To verify that scaling behavior is not an artifact of lattice size, we compute D for multiple system sizes.

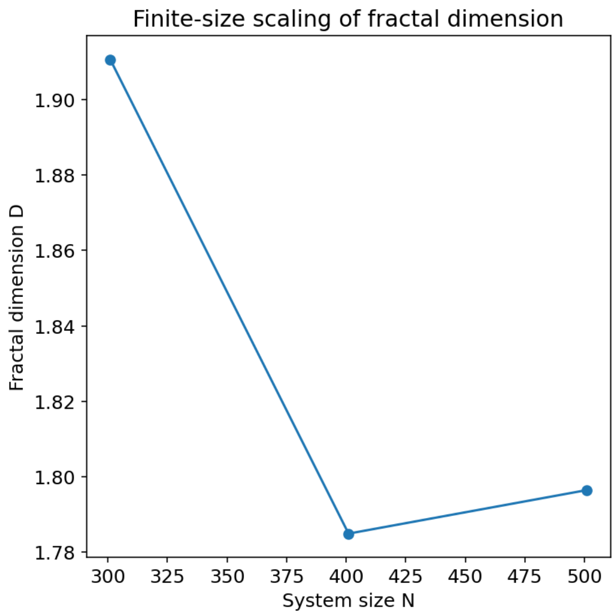

Figure 4: Fractal dimension convergence with increasing system size.

The convergence of D demonstrates robust scale-invariant behavior characteristic of fractal growth processes.

5 Ring-Resolved Occupation Statistics

Following Sun et al., we define the occupation fraction for each growth ring:

$$\eta(i) = \frac{N_{\text{occupied}}(i)}{N_{\text{total}}(i)}$$

This measures how densely each successive shell is populated.

Density Computation

```
def ring_occupation(grid, r):
    sites = ring_sites(r)
    return sum(grid[i,j] for i,j in sites) / len(sites)
```

5.1 Average Occupation vs Growth Parameter

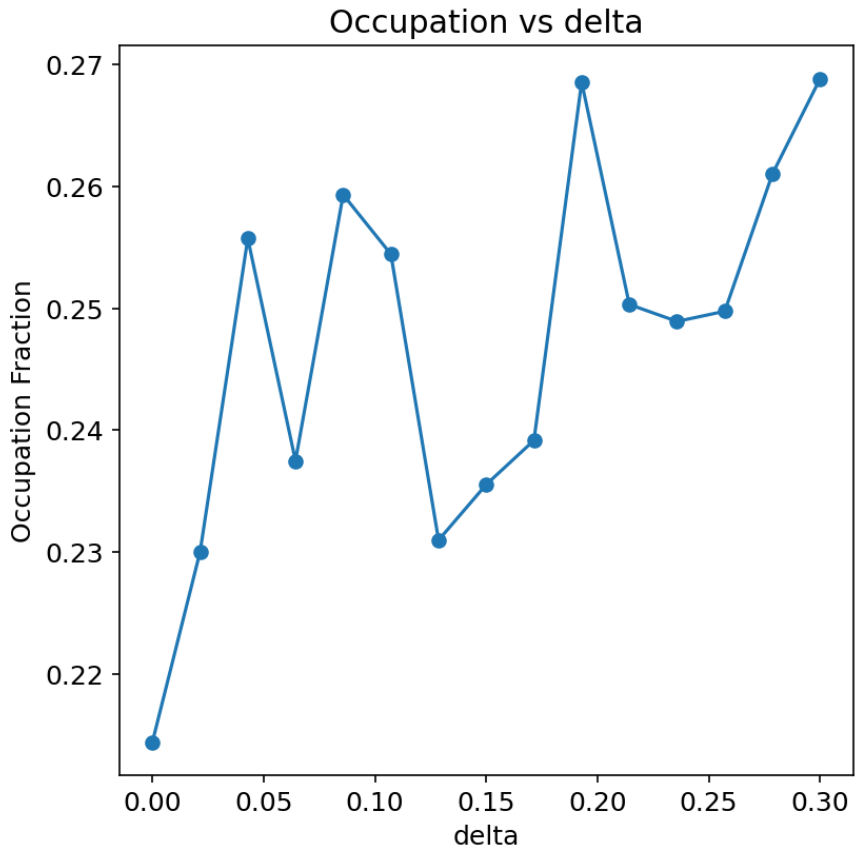

Figure 5: Average occupation fraction as a function of δ.

Higher δ generally leads to increased filling, consistent with more compact chaotic growth.

5.2 Occupation Fluctuations Across Radius

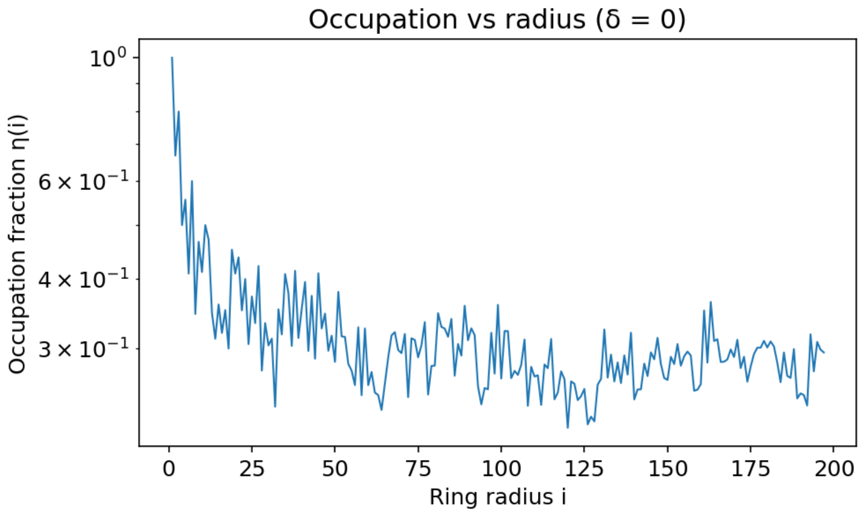

Figure 6: Ring occupation fraction for $\delta = 0$. Regular fluctuations reflect self-similar fractal scaling.

At $\delta = 0$, the occupation fraction exhibits structured peaks at characteristic radii associated with fractal hierarchy.

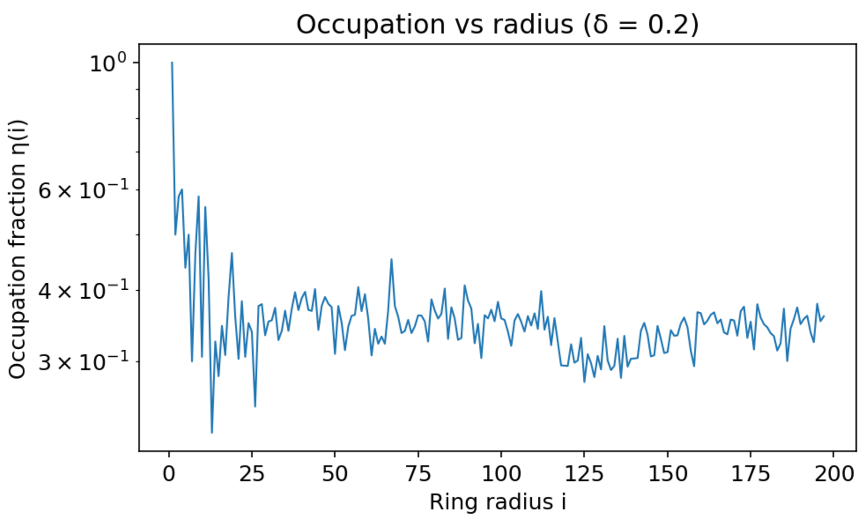

Figure 7: Ring occupation fraction for $\delta = 0.2$. Irregular chaotic fluctuations dominate.

For $\delta > 0$, this regular structure disappears, indicating chaotic sensitivity.

6 Sensitivity to Update Sequence

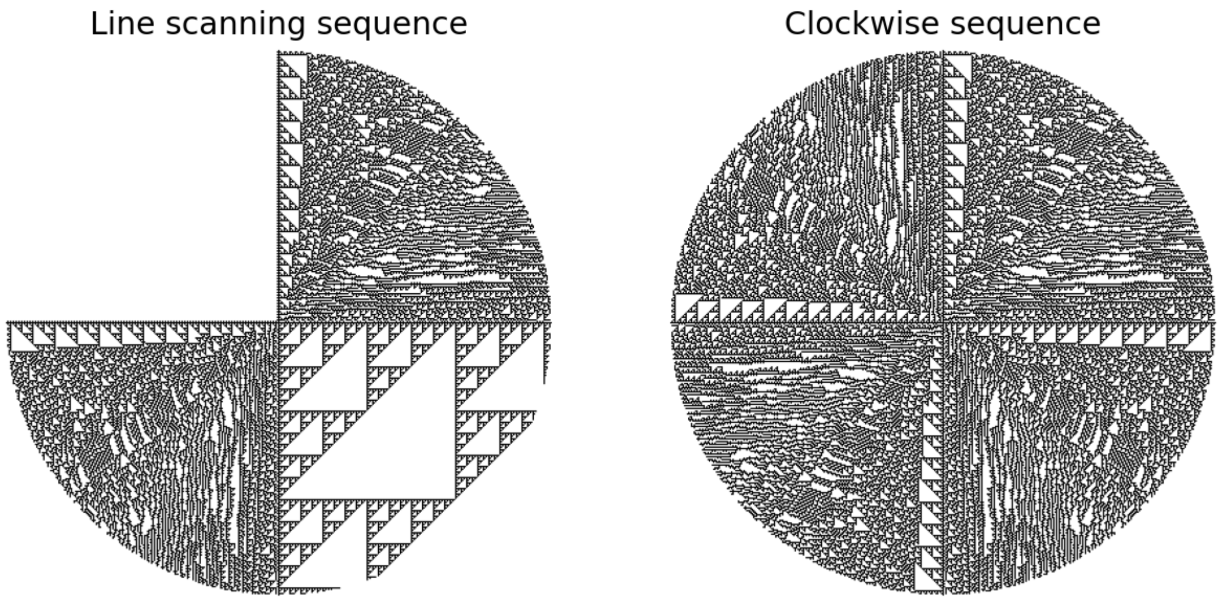

Figure 8: Comparison between line-scanning and clockwise growth sequences.

Different microscopic update orders produce distinct global geometries, despite identical growth rules. This highlights how local temporal ordering strongly influences macroscopic structure — a hallmark of nonlinear systems.

7 Discussion

The circular cellular automaton demonstrates how minimal local interactions can generate remarkably complex spatial organization. The intermediate crowding rule balances branching and suppression, naturally producing fractal growth.

For perfectly aligned growth rings ($\delta = 0$), deterministic self-similar geometry emerges. Small perturbations disrupt this symmetry, introducing chaotic sensitivity and increasingly space-filling morphology.

Fractal dimension analysis reveals a continuous transition from sparse fractal structures to dense chaotic growth. Ring-resolved occupation statistics further illustrate how regular scaling collapses into irregular fluctuations under perturbation.

Finite-size scaling confirms that observed behavior is intrinsic to the model rather than a lattice artifact. Sensitivity to update sequence demonstrates that microscopic temporal ordering plays a crucial role in emergent structure formation.

Together, these results illustrate how complexity, chaos, and scaling naturally arise from simple deterministic rules.

8 Conclusion

We reproduced and extended the circular cellular automaton model of Sun et al., demonstrating emergent fractal geometry, chaotic sensitivity, and scale-invariant behavior. Through systematic parameter exploration and quantitative analysis, we showed how simple local growth rules generate rich nonlinear spatial structures.

The CCA serves as a minimal model for understanding fractal growth, pattern formation, and chaos in discrete dynamical systems.

References

[1] X. Sun, D. Wang, and Z. Wu, "Fractal and chaotic behavior of circular cellular automata," *Physical Review E*, vol. 64, 036105, 2001.