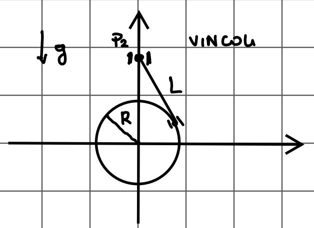
LAGRANGE



VINCOLI

$P_1(x_1, y_1)\ m_1, \quad P_2(x_2, y_2)\ m_2$

1) $x_1^2 + z_1^2 - R^2 = 0$

2) $x_2 = 0$

3) $(x_1 - x_2)^2 + (z_1 - z_2)^2 - L^2 = 0$

Eq Newton

$m_1 \underline{a_1} = \underline{F_1}^{TOT}, \quad m_2 \underline{a_2} = \underline{F_2}^{TOT}$

VINCOLI N punti materiali in $\mathbb{R}^3$

$\underline{X} = (\overbrace{x_1, x_2, x_3}^{x,y,z}, \dots, x_{3N})$

DEF. VINCOLO OLONOMO (non dipende da velocità)

- $f(\underline{x}, t) = 0 \to$ BILATERO
- $f(\underline{x}, t) \geq 0 \to$ UNILATERO

Se vincolo non dipende da tempo $\Rightarrow$ FISSO, altrimenti MOBILE

$$\begin{cases} f_1(\underline{x}) = 0 \\ f_2(\underline{x}) = 0 \\ \vdots \\ f_r(\underline{x}) = 0 \end{cases} \to \text{ogni vincolo elimina un grado di libertà} \Rightarrow 3N - r \text{ gradi di libertà}$$

$3N - r = m \to (q_1, \dots, q_m)$
$$\begin{cases} x_1 = \varphi_1(q_1, \dots, q_m) \\ x_2 = \varphi_2(q_1, \dots, q_m) \\ \vdots \\ x_{3N} = \varphi_{3N}(q_1, \dots, q_m) \end{cases}$$

$\downarrow$ COORDINATE LIBERE LAGRANGIANE

(i) descrizione locale

(ii) Ben definita $\Rightarrow$ rango $\begin{pmatrix} \frac{\partial \varphi_1}{\partial q_1} & \frac{\partial \varphi_1}{\partial q_m} \\ \vdots & \\ \frac{\partial \varphi_{3N}}{\partial q_1} & \cdot\ \frac{\partial \varphi_{3N}}{\partial q_m} \end{pmatrix} = m$

$P = \underline{x}(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_m)$

$$\lim_{h \to 0} \frac{\underline{x}(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_j + h, \dots, \tilde{q}_m) - \underline{x}(\tilde{q}_1, \dots, \tilde{q}_m)}{h} = \frac{\partial \underline{x}(q)}{\partial q_j} \quad \text{vettore tangente alla superficie}$$

$(q_1(t), q_2(t), \dots, q_m(t)) \to$ Curva in $U \subseteq \mathbb{R}^m \quad t \in I$

$\downarrow \varphi_1, \dots, \varphi_{3N}$

$x(t)$ sulla varietà

$t_0 \in I \to q_1(t_0), \dots, q_m(t_0)$

$\varphi \hookrightarrow \underline{x}(t_0)$

$$\underline{v}(t_0) = \lim_{h \to 0} \frac{\underline{x}(t_0 + h) - \underline{x}(t_0)}{h} = \frac{d}{dx} x(t) \Big|_{t = t_0} = \left(\frac{d}{dt} x_1(q_1(t), \dots, q_m(t)), \frac{dx_2}{dt}(), \dots, \frac{dx_{3N}}{dt}() \right)$$

$\underline{x}(t) = \underline{x}(q_1(t), \dots, q_m(t))$

$$\Rightarrow \frac{d}{dt} x_1(q_1(t), \dots, q_m(t)) = \frac{\partial x_1}{\partial q_1} \dot{q}_1 + \frac{\partial x_1}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x_1}{\partial q_m} \dot{q}_m$$

$$\to = \sum_{j=1}^{m} \frac{\partial x_1}{\partial q_j} \cdot \dot{q}_j$$

$$\underline{V}(t_0) = \sum_{j=1}^{m} \frac{\partial \underline{x}(q_1(t), \dots, q_m(t))}{\partial q_j} \dot{q}_j \Big|_{t = t_0}$$

$\dot{q}_j \to$ VELOCITA' GENERALIZZATA

Se la potenza delle reazioni vincolari è lungo spostamenti compatibili con il vincolo $\to$ VINCOLO LISCIO

$K \to$ energia cinetica $\Rightarrow \frac{dK}{dt} = \Pi_{TOT}$ $\hookrightarrow$ potenza

- $(x(q), y(q), z(q)) \to$ velocità compatibili con vincolo $\left(\underbrace{\frac{dx}{dq} \cdot \dot{q}}_{v_x}, \underbrace{\frac{dy}{dq} \dot{q}}_{v_y}, \underbrace{\frac{dz}{dq} \dot{q}}_{v_z} \right)$

- $\vec{F}_{ATTIVE}$ CONSERVATIVE $\Rightarrow$ $\vec{F}_{ATTIVE} = -\vec{\nabla}U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$
- VINCOLO FISSO e LISCIO $\Rightarrow$ $\Pi^{TOT} = \Pi^{ATTIVE}$ $\Rightarrow$ $\vec{F}_{ATTIVE}\,\vec{v} = (-\vec{\nabla}U)\,\vec{v} = -\left(\frac{\partial U}{\partial x} v_x + \frac{\partial U}{\partial y}\cdot v_y + \frac{\partial U}{\partial z}\cdot v_z\right) = -\left(\frac{\partial U}{\partial x}\frac{dx}{dq}\dot q + \frac{\partial U}{\partial y}\frac{dy}{dq}\dot q + \frac{\partial U}{\partial z}\frac{dz}{dq}\cdot \dot q\right)$

$\Pi^{TOT} = -\dot q\left[\frac{\partial U}{\partial x}\frac{dx}{dq} + \frac{\partial U}{\partial y}\frac{dy}{dq} + \frac{\partial U}{\partial z}\frac{dz}{dq}\right] = -\frac{dU}{dq}\cdot \dot q$

$\frac{dK}{dt} = -\frac{dU}{dq}\cdot\dot q \Rightarrow \frac{dU}{dq}\dot q = \frac{dU}{dt} \rightarrow \frac{dK}{dt} + \frac{dU}{dt} = 0 \Rightarrow \frac{d}{dt}(U+K) = 0$ $\rightarrow$ conservazione energia per sistemi conservativi in presenza di vincoli lisci

EQUAZIONE EULERO-LAGRANGE

$q \mapsto x(q), y(q), z(q)$. posizioni compatibili con il vincolo

$\rightsquigarrow$ velocità compatibili con il vincolo $v = (x'(q)\,\dot q,\, y'(q)\,\dot q,\, z'(q)\,\dot q)$ $\quad \frac{d}{dq}x = x'$

$K(q,\dot q) = \frac{1}{2}m\underbrace{\left(x'(q)^2 + y'(q)^2 + z'(q)^2\right)}_{G(q)}\dot q^2$

$G(q)$ sempre positivo $\Rightarrow K=0 \Leftrightarrow \dot q = 0$

$\frac{d}{dt}K(q(t),\dot q(t)) = \frac{\partial K}{\partial q}\dot q + \frac{\partial K}{\partial \dot q}\cdot\frac{d}{dt}\dot q = \frac{\partial K}{\partial q}\dot q + \frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\cdot\dot q\right) - \left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\right)\right]\cdot\dot q = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\cdot\dot q\right) + \dot q\left(\frac{\partial K}{\partial q} - \frac{d}{dt}\frac{\partial K}{\partial \dot q}\right)$ $\quad K = \frac{1}{2}G(q)\dot q^2 \Rightarrow \dot q\frac{\partial K}{\partial \dot q} = G(q)\cdot\dot q^2 = 2K$

$\frac{d}{dt}K = \frac{d}{dt}2K + \dot q\left(\frac{\partial K}{\partial q} - \frac{d}{dt}\frac{\partial K}{\partial \dot q}\right) \Rightarrow \frac{d}{dt}K = \dot q\left(\frac{d}{dt}\frac{\partial K}{\partial \dot q} - \frac{\partial K}{\partial q}\right)$

$\frac{dK}{dt} = \Pi^{ATTIVE} = f^{ATTIVE}\,v$

vincolo liscio

$\frac{d}{dt}\frac{\partial K}{\partial \dot q} - \frac{\partial K}{\partial q} = f^{ATTIVE}\,(x', y', z')$

se le f^{ATTIVE} sono conservative $\Rightarrow \exists U(x,y,z) \rightsquigarrow U(x(q), y(q), z(q))$

$\Pi^{ATTIVE} = -\frac{dU}{dq}\cdot\dot q \Rightarrow \frac{d}{dt}K = \Pi^{ATTIVE} = \dot q\left[\frac{d}{dt}\frac{\partial K}{\partial \dot q} - \frac{\partial K}{\partial q}\right] = -\frac{dU}{dq}\dot q \rightarrow \frac{d}{dt}\frac{\partial K}{\partial \dot q} = \frac{\partial}{\partial q}(K-U)$ $\quad \frac{\partial}{\partial \dot q}U = 0 \rightsquigarrow \frac{\partial}{\partial \dot q}(K-U) = \frac{\partial}{\partial \dot q}K$

$\mathcal{L}(q,\dot q) = K(q,\dot q) - U(q)$ $\Rightarrow$ EQUAZIONE EULERO-LAGRANGE $\rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} = \frac{\partial \mathcal{L}}{\partial q}$

CONSERVAZIONE ENERGIA

Supponiamo $\mathcal{L}(q,\dot q,\cancel{t}) \Rightarrow \frac{\partial \mathcal{L}}{\partial t} = 0$

$\frac{d}{dt}\mathcal{L}(q(t),\dot q(t)) = \frac{\partial \mathcal{L}}{\partial q}\dot q + \frac{\partial \mathcal{L}}{\partial \dot q}\cdot\frac{d}{dt}\dot q = \frac{\partial \mathcal{L}}{\partial q}\cdot\dot q + \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot q}\dot q\right] - \dot q\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} = \dot q\left[\underbrace{\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q}}_{A}\right] + \underbrace{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot q}\dot q\right]}_{B}$

Se $q(t)$ è soluzione $\Rightarrow$ (A) $= 0 \Rightarrow \frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot q}\dot q\right]$ lungo il moto

$\frac{d}{dt}\left[\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot q}\dot q\right] = 0 \Rightarrow \dot q\frac{\partial \mathcal{L}}{\partial \dot q} - \mathcal{L}$ è costante del moto

$= 2K - (K-U) = K+U = K(q,\dot q) + U(q)$

FORMA NORMALE EQUAZIONE EULERO-LAGRANGE (1 GRADO DI LIBERTA')

$\mathcal{L}(q,\dot q) = \frac{1}{2}G(q)\dot q^2 - U(q)$

EQ E-L $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} = \frac{\partial \mathcal{L}}{\partial q}$

- $\frac{\partial \mathcal{L}}{\partial q} = \frac{1}{2}G'(q)\,\dot q^2 - U'(q)$
- $\frac{\partial \mathcal{L}}{\partial \dot q} = G(q)\cdot\dot q$
- $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} = G'(q)\,\dot q^2 + G(q)\ddot q$

$\Rightarrow$ EQ E-L $\rightarrow G'(q)\,\dot q^2 + G(q)\ddot q = \frac{1}{2}G'(q)\,\dot q^2 - U'(q) \Rightarrow G(q)\ddot q = -\frac{1}{2}G'(q)\dot q^2 - U(q) \Rightarrow \ddot q = -\frac{1}{2}\frac{G'(q)}{G(q)}\dot q^2 - \frac{U'(q)}{G(q)}$ $\quad G(q) > 0$

$y = \dot q \Rightarrow \begin{cases} \dot q = y \\ \dot y = -\frac{1}{2}\frac{G'(q)}{G(q)}y^2 - \frac{U'(q)}{G(q)} \end{cases}$

PUNTI STAZIONARI $\rightarrow (\bar q, 0)$ t.c. $U'(q) = 0$

- $E(q,y) = K(q,y) + U(q) = \frac{1}{2}G(q)\,y^2 + U(q)$ costante del moto

STABILITA' PUNTO EQUILIBRIO $(\bar q, 0)$

TEO Si consideri un sistema lagrangiano (1 GRADO LIBERTA') e sia $(\bar q, 0)$ p.to di equilibrio, $\bar q$ t.c. $U'(\bar q) = 0$, e sia $U''(q) > 0 \Rightarrow (\bar q, 0)$ è STABILE

DIM II LYAPUNOV con $E(q,y)$

(ii) è vera perché E è costante del moto

(i) $(\bar q, 0)$ è di minimo? $\quad \vec\nabla E = \left(\frac{\partial E}{\partial q}, \frac{\partial E}{\partial y}\right) = \left(\frac{1}{2}G'(q)y^2 + U'(q),\; G(q)\,y\right)$ $\quad \vec\nabla E\Big|_{(\bar q,0)} = (0,0) \Rightarrow (\bar q, 0)$ è punto stazionario per E

$H_E(q,y) = \begin{pmatrix} \frac{\partial^2 E}{\partial q^2} & \frac{\partial E}{\partial q\partial y} \\ \frac{\partial E}{\partial y\partial q} & \frac{\partial^2 E}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}G''(q)y^2 + U''(q) & G'(q)\,y' \\ G'(q)\,y' & G(q) \end{pmatrix}\Bigg|_{(\bar q,0)} = \begin{pmatrix} U''(\bar q) & 0 \\ 0 & G(\bar q) \end{pmatrix} \Rightarrow (\bar q, 0)$ è un punto di minimo

$\hookrightarrow H_E$ definita positiva

PICCOLE OSCILLAZIONI

$$\begin{cases} q = \bar{q} + \varepsilon u(t) \\ y = 0 + \varepsilon v(t) = \varepsilon u(t) \end{cases} \qquad \dot q = \varepsilon \dot u$$

$$\tilde{\mathcal{L}}(u, \dot{u}) = \mathcal{L}(\bar q + \varepsilon u, \varepsilon \dot u) = \frac{1}{2} G(\bar q + \varepsilon u)\,\varepsilon^2 \dot u^2 + U(\bar q + \varepsilon u)$$

ESPANDO ATTORNO A $\varepsilon = 0$ FINO AL SECONDO ORDINE $\Rightarrow \tilde{\mathcal{L}}(u,\dot u) = \frac{1}{2} G(\bar q)\varepsilon^2 \dot u^2 - \left[\cancel{U(\bar q)} + \cancel{U'(\bar q)\, u\varepsilon} + \frac{1}{2} U''(\bar q)\varepsilon^2 u^2\right] + o(\varepsilon^2) \simeq \varepsilon^2\left[\frac{1}{2}\underbrace{G(\bar q)}_{m}\dot u^2 - \frac{1}{2}\underbrace{U''(\bar q)}_{K}\varepsilon^2 u^2\right]$

$$w = \sqrt{\frac{K}{m}} = \sqrt{\frac{U''(\bar q)}{G(\bar q)}}$$

VINCOLI DIPENDENTI DAL TEMPO

es p.to materiale su circonferenza $r(t) = r_0 e^t$

$$\begin{cases} r(\theta, t) = r_0 e^t \cos\theta \\ y(\theta, t) = r_0 e^t \sin\theta \end{cases}$$

$$\frac{dr}{dt} = r_0 e^t(-\sin\theta)\,\dot\theta + r_0 e^t \cos\theta$$

$$\begin{cases} x(q,t) \\ y(q,t) \\ z(q,t) \end{cases} \rightsquigarrow \begin{cases} \frac{dx}{dt} = \frac{\partial x}{\partial q}\dot q + \frac{\partial x}{\partial t} \\ \frac{dy}{dt} = \frac{\partial y}{\partial q}\cdot \dot q + \frac{\partial y}{\partial t} \\ \frac{dz}{dt} = \frac{\partial z}{\partial q}\dot q + \frac{\partial z}{\partial t} \end{cases}$$

VELOCITA' VIRTUALI

$(v_x, v_y, v_z) = v^{VIRT}$ tempo come congelato

$$\begin{cases} v_x = \frac{dx}{dt} - \frac{\partial x}{\partial t} = \frac{\partial x}{\partial q}\dot q \\ v_y = \frac{dy}{dt} - \frac{\partial y}{\partial t} = \frac{\partial y}{\partial q}\dot q \\ v_z = \frac{dz}{dt} - \frac{\partial z}{\partial t} = \frac{\partial z}{\partial q}\cdot \dot q \end{cases}$$

VINCOLI LISCI (dipendente da t) $\rightarrow$ Potenza reazioni vincolari nulla lungo le velocità virtuali $\Rightarrow \underline{\phi}^{VINC}\, \underline{v}^{VIRT} = 0$

$$m\underline{a} = \underline{F}^{ATT} + \underline{\phi}^{VINC} \Rightarrow m\,\underline{a}\,\underline{v}^{VIRT} = \underline{F}^{ATT}\,\underline{v}^{VIRT} + \cancel{\underline{\phi}^{VINC}\,\underline{v}^{VIRT}}$$

NEL PIANO $\rightarrow$ $m\left(\ddot x \frac{\partial x}{\partial q}\cancel{\dot q}\right) + m\left(\ddot y \frac{\partial y}{\partial q}\cancel{\dot q}\right) = f_x^{ATT}\frac{\partial x}{\partial q}\cancel{\dot q} + f_y^{ATT}\frac{\partial y}{\partial q}\cancel{\dot q} \quad \forall q$

Se $\underline{F}^{ATT}$ CONSERVATIVE $\Rightarrow (f_x, f_y) = \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}\right) \longrightarrow m\left[\overset{A}{\ddot x \frac{\partial x}{\partial q}} + \ddot y \frac{\partial y}{\partial q}\right] = -\left[\overset{B}{\frac{\partial U}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q}}\right]$

B $\rightarrow \frac{\partial U}{\partial q}(x(q,t), y(q,t))$

A $\ddot x \frac{\partial x}{\partial q} = \frac{d}{dt}\dot x \frac{\partial x}{\partial q} = \frac{d}{dt}\left(\dot x \frac{\partial x}{\partial q}\right) - \dot x \frac{d}{dt}\frac{\partial x}{\partial q} \Rightarrow m\left[\frac{d}{dt}\left(\dot x \frac{\partial \dot x}{\partial \dot q} + \dot y \frac{\partial \dot y}{\partial \dot q}\right) - \left(\dot x \frac{d}{dt}\frac{\partial x}{\partial q} + \dot y \frac{d}{dt}\frac{\partial y}{\partial q}\right)\right] = -\frac{\partial U}{\partial q}$

LEMMA 1 $\rightarrow \frac{\partial x}{\partial q} = \frac{\partial \dot x}{\partial \dot q}$

dim $\frac{\partial}{\partial \dot q}\frac{d}{dt}x = \frac{\partial}{\partial \dot q}\left[\frac{\partial x}{\partial q}\dot q + \frac{\partial x}{\partial t}\right] = \frac{\partial x}{\partial q}$

LEMMA 2 $\rightarrow \frac{d}{dt}\frac{\partial x}{\partial q} = \frac{\partial}{\partial q}\frac{d}{dt}x = \frac{\partial}{\partial q}\dot x$

Dim $\frac{d}{dt}\left[\frac{\partial x}{\partial q}\right] = \frac{\partial}{\partial q}\left(\left[\frac{\partial x}{\partial q}\right]\dot q\right) + \frac{\partial}{\partial t}\left[\frac{\partial x}{\partial q}\right] = \frac{\partial}{\partial q}\left(\left[\frac{\partial x}{\partial q}\right]\dot q\right) + \frac{\partial}{\partial q}\frac{\partial x}{\partial t} = \frac{\partial}{\partial q}\left[\frac{\partial x}{\partial q}\dot q + \frac{\partial x}{\partial t}\right] = \frac{\partial}{\partial q}\frac{d}{dt}x$

$$\Rightarrow m\left[\frac{d}{dt}\frac{\partial}{\partial \dot q}\Big[\underbrace{\tfrac{1}{2}(\dot x^2 + \dot y^2)}_{K}\Big]\right] - \left(\frac{\partial}{\partial q}\left[\frac{1}{2}(\dot x^2 + \dot y^2)\right]\right) = -\frac{\partial U}{\partial q}$$

$$\rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot q}K - \frac{\partial K}{\partial q} = -\frac{\partial U}{\partial q} \Rightarrow \boxed{\frac{d}{dt}\frac{\partial}{\partial \dot q}(K - U) - \frac{\partial}{\partial q}(K - U) = 0}$$

d GRADI DI LIBERTA'

$\underline{x} = (x_1, x_2, x_3, \dots, x_{3N})$

$$\begin{cases} x_1(q_1, q_2, \dots, q_d, t) \\ x_2(q_1, q_2, \dots, q_d, t) \\ \vdots \\ x_{3N}(q_1, q_2, \dots, q_d, t) \end{cases}$$

$\dot{\underline{x}} = (\dot x_1, \dot x_2, \dot x_3, \dots, \dot x_{3N}) \rightarrow \begin{cases} \dot x_1 = \sum_{\gamma=1}^{d} \frac{\partial x_1}{\partial q_\gamma}\dot q_\gamma + \frac{\partial x_1}{\partial t} \\ \vdots \\ \dot x_{3N} = \sum_{\gamma=1}^{d} \frac{\partial x_{3N}}{\partial q_\gamma}\dot q_\gamma + \frac{\partial x_{3N}}{\partial t} \end{cases} \rightsquigarrow$ vettore velocità virtuali $\underline{v}^{VIRT} = (v_1, \dots, v_{3N})$

$$v_i = \frac{dx_i}{dt} - \frac{\partial x_i}{\partial t} = \sum_{\gamma=1}^{d} \frac{\partial x_i}{\partial q_\gamma}\cdot \dot q_\gamma$$

PRINCIPIO DI D'ALAMBERT (VINCOLO liscio) d gradi di libertà

$$\underline{\phi}^{VINC} = (\phi_1, \dots, \phi_{3N}) \Rightarrow \underline{\phi}^{VINC}\,\underline{v}^{VIRT} = 0 = \sum_{K=1}^{m} \underline{\phi}^{(K)}\,\underline{v}^{(K)}_{VIRT} \rightarrow \text{vincolo liscio} \quad (\phi_1, \phi_2, \phi_3) = \underline{\phi}^{(1)}$$

$\rightarrow \sum_{k=1}^{N} v_{VIRT}^{(k)} \left(m^{(k)} a^{(k)}\right) = F_{TOT}^{(k)} \cdot v_{VIRT}^{(k)} = \sum_{k=1}^{N} \left(F_{ATT}^{(k)} + \phi^{(k)}\right) \cdot v_{VIRT}^{(k)}$

VINCOLO LISCIO $\Rightarrow \sum_{k=1}^{N} \phi^{(k)} v^{(k)} = 0 \Rightarrow \sum_{k=1}^{N} m^{(k)} a^{(k)} v_{VIRT}^{(k)} = \sum_{k=1}^{N} F^{(k)} \cdot v_{VIRT}^{(k)}$

$$\sum_{k=1}^{N} m^{(k)} \ddot{x}^{(k)} \left(\sum_{\gamma=1}^{d} \frac{\partial x^{(k)}}{\partial q_\gamma} \cdot \dot q_\gamma\right) = \sum_{\gamma=1}^{d} \dot q_\gamma \left(\sum_{k=1}^{N} m^{(k)} \ddot{x}^{(k)} \frac{\partial x^{(k)}}{\partial q_\gamma}\right) = \sum_{\gamma=1}^{d} \dot q_\gamma \left(\sum_{k=1}^{N} F^{(k)} \frac{\partial x^{(k)}}{\partial q_\gamma}\right)$$ vera $\forall$ scelta $\dot q(\dot q_1, .., \dot q_d)$

scelgo $\dot q = (0, .., \dot q_r, 0, .., 0)$

$\rightsquigarrow \cancel{\dot q_r}\left(\sum_{k=1}^{N} m^{(k)} \ddot{x}^{(k)} \frac{\partial x^{(k)}}{\partial q_r}\right) = \cancel{\dot q_r}\left(\sum_{k=1}^{N} F^{(k)}_{ATTIVA} \frac{\partial x^{(k)}}{\partial q_r}\right)$ $r = 1, ..., d$

$$\underset{(A)}{\sum_{i=1}^{3N} m_i \ddot x_i \frac{\partial x_i}{\partial q_r}} = \underset{(B)}{\sum_{i=1}^{3N} F_i^{ATTIVE} \frac{\partial x_i}{\partial q_r}}$$

A $\rightarrow \sum_{i=1}^{3N} m_i \left(\frac{d}{dt}\dot x_i\right) \frac{\partial x_i}{\partial q_r} = \sum_{i=1}^{3N} m_i \left(\frac{d}{dt}\left(\dot x_i \frac{\partial x_i}{\partial q}\right) - \dot x_i \frac{d}{dt}\frac{\partial x_i}{\partial q_r}\right) = \sum_{i=1}^{3N} m_i \left(\frac{d}{dt}\left(\dot x_i \frac{\partial \dot x_i}{\partial \dot q_r}\right) - \dot x_i \frac{\partial \dot x_i}{\partial q_r}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot q_r}\left(\frac{1}{2}\sum_{i=1}^{3N} m_i \dot x_i^2\right) - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\sum_{i=1}^{3N} m_i \dot x_i^2\right) = \frac{d}{dt}\frac{\partial}{\partial \dot q_r}K - \frac{\partial}{\partial q_r}K$

B $\rightarrow \sum_{i=1}^{3N} F_i^{AT} \frac{\partial x_i}{\partial q_r}$ se F^{ATT} CONSERVATIVE $\Rightarrow F_i^{ATT} = -\frac{\partial}{\partial x_i} U(x_1, .., x_{3N}) \rightarrow = -\sum_{i=1}^{3N} \frac{\partial U}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_r} = -\frac{\partial U}{\partial q_r}$

$\rightarrow \frac{d}{dt}\frac{\partial K}{\partial \dot q_r} - \frac{\partial}{\partial q_r}K = -\frac{\partial U}{\partial q_r} \rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot q_r}(K-U) = \frac{\partial}{\partial q_r}(K-U)$

$\mathcal{L}(q_1, .., q_d, \dot q_1, .., \dot q_d) = K - U \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_r} = \frac{\partial \mathcal{L}}{\partial q_r}$ $r = 1, ..., d$ SISTEMA EQUAZIONI EULERO-LAGRANGE

STRUTTURA FORMALE ENERGIA CINETICA ($d \geq 1$) N pti materiali

se $d = 1 \Rightarrow K(q, \dot q) = \frac{1}{2} G(q) \dot q^2$

se $d > 1$

$q = (q_1, ..., q_d)$ $x = (x_1, ..., x_{3N})$

$K(q, \dot q, t) = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot x_i^2$ $\dot x_i = \frac{d}{dt} x_i(q_1, ..., q_d, t) = \sum_{\gamma=1}^{d} \frac{\partial x_i}{\partial q_\gamma} \cdot \dot q_\gamma + \frac{\partial x_i}{\partial t}$

$K = \frac{1}{2}\sum_{i=1}^{3N} m_i \left(\sum_{\gamma=1}^{d} \frac{\partial x_i}{\partial q_\gamma}\dot q_\gamma + \frac{\partial x_i}{\partial t}\right)\left(\sum_{k=1}^{d} \frac{\partial x_i}{\partial q_k}\dot q_k + \frac{\partial x_i}{\partial t}\right) = \frac{1}{2}\left(\left[\sum_{\gamma,k=1}^{d}\sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_\gamma}\frac{\partial x_i}{\partial q_k}\dot q_\gamma \dot q_k\right] + G_1 + G_0\right)$

$G_1 \rightarrow$ lineare nelle $\dot q$
$G_0 \rightarrow$ non dipende da $\dot q$ $] = 0 \rightsquigarrow$ vincoli fissi

$\rightsquigarrow$ VINCOLI FISSI $\Rightarrow x = x(q_1, .., q_d, \cancel{t}) \Rightarrow K(q, \dot q) = \frac{1}{2}\sum_{k,\gamma=1}^{d} G_{\gamma k}(q_1, ..., q_d)\dot q_\gamma \dot q_k$

$\rightarrow K = \frac{1}{2}(\dot q_1, .., \dot q_d)\, \underset{\text{matrice } d\times d}{G(q_1, ..., q_d)} \begin{pmatrix}\dot q_1 \\ \vdots \\ \dot q_d\end{pmatrix}$ FORMA BILINEARE

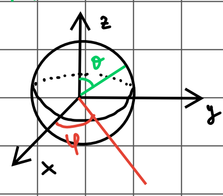
G matrice simmetrica definita positiva con entrate $G_{\gamma k}(q) = \sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_\gamma}\frac{\partial x_i}{\partial q_k} = G_{k\gamma}$

$\forall \dot q \neq 0 \Rightarrow \sum_{\gamma,k} G_{\gamma k} \dot q_\gamma \dot q_k > 0$

$v = \sum_{\gamma=1}^{d} \frac{\partial x_i}{\partial q_\gamma} \dot q_\gamma$ $v \cdot v \geq 0$ $(= 0 \Leftrightarrow v = 0)$

PENDOLO SFERICO

$\rightarrow$ pendolo vincolato a superficie sfera $\rightarrow$ 2 GRADI LIBERTA'



R: raggio $\;\hookrightarrow x^2 + y^2 + z^2 = R^2$

$\theta \in [0, \pi)$

$\varphi \in [0, 2\pi)$

$$\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \rightsquigarrow \begin{cases} \dot x = R\cos\theta\dot\theta\cos\varphi - R\sin\theta\sin\varphi\dot\varphi \\ \dot y = R\cos\theta\dot\theta\sin\varphi + R\sin\theta\cos\varphi\dot\varphi \\ \dot z = -R\sin\theta\dot\theta \end{cases}$$

$$K = \frac{1}{2}m\left[\dot x^2 + \dot y^2 + \dot z^2\right] = \frac{1}{2}mR^2\left((\cos\theta\dot\theta\sin\varphi - \sin\theta\sin\varphi\dot\varphi)^2 + (\cos\theta\dot\theta\sin\varphi + \sin\theta\cos\varphi\dot\varphi)^2 + (\sin\theta\dot\theta)^2\right) = \frac{1}{2}mR^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) = \frac{1}{2}(\dot\theta, \dot\varphi)\begin{pmatrix} mR^2 & 0 \\ 0 & mR^2\sin^2\theta\end{pmatrix}\begin{pmatrix}\dot\theta \\ \dot\varphi\end{pmatrix}$$

$U(\theta, \varphi) = mgR\cos\theta$

$\mathcal{L}(\theta, \cancel{\varphi}, \dot\theta, \dot\varphi) = \frac{1}{2}mR^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) - mgR\cos\theta$

DEF. VARIABILE CICLICA

Se $\mathcal{L}(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d)$ non dipende da $q_j \Rightarrow q_j$ è CICLICA

$\rightarrow \frac{\partial \mathcal{L}}{\partial q_j}=0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_j}\right)=0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot q_j}$ è costante del moto

LAGRANGIANA RIDOTTA $\mathcal{L}^R(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\cancel{\dot q_j},\dots,\dot q_d,c)$

q_j ciclica $\Rightarrow c=\frac{\partial \mathcal{L}}{\partial \dot q_j}(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d)$

$\dot q_j = f(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d,c)$

$\rightarrow E=K+U$ sostituisco con $\dot q_j=f \rightarrow E'=K|_{\dot q_j=f}+U \Rightarrow E'=K^{eff}-U^{eff}$

$\mathcal{L}^{RID} \neq \mathcal{L}|_{\dot q=f}$ $\quad \mathcal{L}^{RID}=\mathcal{L}^R(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\cancel{\dot q_j},\dots,\dot q_d,c)$

FREQUENZE PROPRIE E MODI NORMALI ($d>1$ gdl)

⇝ forze conservative, vincoli lisci e fissi

$q=(q_1,\dots,q_d)$ $\quad \mathcal{L}(q,\dot q)=\frac{1}{2}\sum_{\alpha,\beta=1}^{d} G_{\alpha\beta}(q)\,\dot q_\alpha \dot q_\beta - U(q)$

↑ simmetrica e definita positiva

Eq. E-L per $q_\gamma \rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_\gamma}=\frac{\partial \mathcal{L}}{\partial q_\gamma}$

$$\frac{\partial \mathcal{L}}{\partial \dot q_\gamma}=\frac{1}{2}\left[\sum_{\beta\neq\gamma} G_{\gamma\beta}(q)\dot q_\beta+\sum_{\alpha\neq\gamma} G_{\alpha\gamma}(q)\dot q_\alpha+2G_{\gamma\gamma}(q)\dot q_\gamma\right]$$

G simmetrica $\Rightarrow G_{\alpha\gamma}(q)=G_{\gamma\alpha}(q)$

$$\sum_{\beta\neq\gamma} G_{\gamma\beta}(q)\dot q_\beta=\sum_{\alpha\neq\gamma} G_{\alpha\gamma}(q)\dot q_\alpha$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot q_\gamma}=\frac{1}{2}\left[2\sum_{\alpha\neq\gamma} G_{\alpha\gamma}(q)\dot q_\alpha+2G_{\gamma\gamma}\dot q_\gamma\right]=\sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\dot q_\alpha$$

$d > 1$ gdl

$$\mathcal{L}(q,\dot q) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} G_{\alpha\beta}(q)\,\dot q_\alpha \dot q_\beta - U(q)$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot q_j} = \dfrac{\partial \mathcal{L}}{\partial q_j} \\ j = 1,\dots,d \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \dot q_\gamma} = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\,\dot q_\gamma = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\,\ddot q_\alpha + \sum_{\alpha=1}^{d} \dot q_\alpha \frac{d}{dt} G_{\alpha\gamma}(q)$$

$$\frac{d}{dt} G_{\alpha\gamma}(q) = \sum_{\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}\,\dot q_\beta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_\gamma} = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\,\ddot q_\alpha + \sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial \dot q_\beta}\,\dot q_\alpha \dot q_\beta$$

$$\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial \dot q_\beta}\,\dot q_\alpha \dot q_\beta = \frac{1}{2}\left(\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}(q)\,\dot q_\alpha \dot q_\beta + \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}(q)\,\dot q_\alpha \dot q_\beta\right)$$

$$\frac{\partial \mathcal{L}}{\partial q_\gamma} = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\beta}}{\partial q_\gamma}\,\dot q_\alpha \dot q_\beta - \frac{\partial U}{\partial q_\gamma}$$

$$\Gamma_{\alpha\beta\gamma} := \frac{1}{2}\left(\frac{\partial}{\partial q_\gamma} G_{\alpha\beta} - \frac{\partial}{\partial q_\gamma} G_{\alpha\gamma} - \frac{\partial}{\partial q_\gamma} G_{\beta\gamma}\right)$$

$$\text{E-L} \rightarrow \begin{cases} \displaystyle\sum_{\alpha=1}^{d} G_{\alpha\gamma}\,\ddot q_\alpha = \sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma}\,\dot q_\alpha \dot q_\beta - \frac{\partial U}{\partial q_\gamma} \\ \gamma = 1,\dots,d \end{cases}$$

PUNTI DI EQUILIBRIO $\rightsquigarrow$ $\dot q = 0$ e $\displaystyle\sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma}\,\dot q_\alpha \dot q_\beta - \frac{\partial U}{\partial q_\gamma} = 0$

$$\Rightarrow \frac{\partial U}{\partial q_\gamma} = 0$$

q_0 t.c. $\dfrac{\partial U}{\partial q_\gamma}(q_0) = 0 \quad \forall \gamma = 1,\dots,d \quad \rightsquigarrow \quad \vec\nabla U(q_0) = 0$

Teo (LAGRANGE - DIRICHLET) $d \geq 1$

Sistema con forze attive conservative, vincoli lisci e fissi

Supponiamo che U abbia punti critici isolati

I MINIMI DI U (guardo Hessiana) $\rightarrow$ sono p.ti equilibrio stabile sistema

Dim Suppongo q_0 t.c. $\vec{\nabla} U(q_0) = 0$

$$H(q) = \begin{pmatrix} \frac{\partial^2 U}{\partial q_1^2} & \frac{\partial^2 U}{\partial q_2 q_1} & \dots & \frac{\partial^2 U}{\partial q_d q_1} \\ \vdots & & & \\ \frac{\partial^2 U}{\partial q_1 q_d} & \frac{\partial^2 U}{\partial q_2 q_d} & \dots & \frac{\partial^2 U}{\partial q_d^2} \end{pmatrix}$$

q_0 è di minimo se $H(q_0)$ è DEFINITA POSITIVA

$\underline{q}^*$ p.to di minimo per $U(\underline{q})$

$\underline{q}(t) = \underline{q}^* + \varepsilon \underline{u}(t) \qquad \dot{\underline{q}}(t) = \varepsilon \dot{\underline{u}}(t) \qquad \ddot{\underline{q}}(t) = \varepsilon \ddot{\underline{u}}(t)$

$$\hookrightarrow \begin{cases} q_1(t) = q_1^* + \varepsilon u_1(t) \\ \vdots \\ q_d(t) = q_d^* + \varepsilon u_d(t) \end{cases}$$

$$E-L \longrightarrow \begin{cases} \sum_{\alpha=1}^{d} G_{\alpha\gamma} \ddot{q}_\alpha = \sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma} \dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} \\ \gamma = 1, \dots, d \end{cases}$$

$$\sum_\alpha G_{\alpha\gamma}(\underline{q}^* + \varepsilon \underline{u}) \varepsilon \ddot{u}_\alpha = \sum_{\alpha,\beta} \Gamma_{\alpha\beta\gamma}(\underline{q}^* + \varepsilon \underline{u}) \varepsilon^2 \dot{u}_\alpha \dot{u}_\beta - \frac{\partial}{\partial q_\gamma}(\underline{q}^* + \varepsilon \underline{u})$$

ESPANSIONE AL PRIMO ORDINE IN ε

$$\sum_\alpha G_{\alpha\gamma}(\underline{q}^*) \varepsilon \ddot{u}_\alpha = -\left(\cancel{\frac{\partial U}{\partial q_\gamma}(\underline{q}^*)} + \sum_{\beta=1}^{d} \left[\frac{\partial}{\partial q_\beta}\left(\frac{\partial U}{\partial q_\gamma} \right) \right] \Bigg|_{(\underline{q}^*)} \varepsilon u_\beta \right)$$

$$\begin{cases} \sum_\alpha G_{\alpha\gamma}(\underline{q}^*) \cancel{\varepsilon} \ddot{u}_\alpha = - \sum_{\beta=1}^{d} \frac{\partial^2 U}{\partial q_\beta \partial q_\gamma}(\underline{q}^*) \cancel{\varepsilon} u_\beta \\ \gamma = 1, \dots, d \end{cases}$$

$$[G(\underline{q}^*) \ddot{\underline{u}}]_\gamma = -[H_U(\underline{q}^*) \underline{u}]_\gamma \Rightarrow \underbrace{[G(\underline{q}^*) \ddot{\underline{u}}] = -[H_U(\underline{q}^*) \underline{u}]}_{\text{E-L approssimata}}$$

$$\mathcal{L}^{LIN}(\underline{u}, \dot{\underline{u}}) = \frac{1}{2}(\dot{\underline{u}} \cdot G(\underline{q}^*) \dot{\underline{u}}) - \frac{1}{2}(\underline{u} \cdot H_U(\underline{q}^*) \underline{u})$$

$G_0 := G(\underline{q}^*) \qquad H_0 := H_U(\underline{q}^*)$

Cerco soluzioni del tipo $\underline{u}(t) = \cos(wt+\varphi) \cdot \underline{w}$ $\quad \underline{w} \neq 0 \in \mathbb{R}^d$

$\dot{\underline{u}} = -w \sin(wt+\varphi)\, \underline{w}$
$\ddot{\underline{u}} = -w^2 \cos(wt+\varphi)\, \underline{w} = -w^2 \underline{u}$

$-w^2 G_0 \underline{u} = -H_0 \underline{u}$ $\quad (w^2 G_0 - H_0)\underline{u} = 0 \rightarrow (w^2 G_0 - H_0)\underline{w} \cos(wt+\varphi) = 0$

$\Rightarrow \forall t \rightarrow (w^2 G_0 - H_0)\underline{w} = 0$

$\det(\lambda G_0 - H_0) = 0$ $\quad \lambda = w^2 \leadsto \underline{w}$ t.c. $(\lambda G_0 - H_0) \cdot \underline{w} = 0$

w : pulsazione propria $\quad \underline{w}$: modo normale corrispondente a w

Teo G_0 è simmetrica e definita positiva
H_0 è simmetrica

1) $\exists$ una base di $\mathbb{R}^d$ formata da autovettori di H_0 rispetto a G_0
2) Autovettori corrispondenti ad autovalori diversi sono ortogonali a G_0
3) Se H_0 è definita positiva $\Rightarrow$ λ_i sono tutti positivi