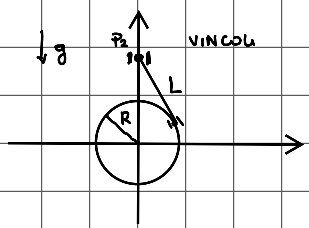
LAGRANGE



VINCOLI

$P_1(x_1, z_1)\ m_1,\quad P_2(x_2, z_2)\ m_2$

1) $x_1^2 + z_1^2 - R^2 = 0$

2) $x_2 = 0$

3) $(x_1 - x_2)^2 + (z_1 - z_2)^2 - L^2 = 0$

Eq Newton

$m_1 \underline{a_1} = \underline{F_1}^{TOT}, \quad m_2 \underline{a_2} = \underline{F_2}^{TOT}$

VINCOLI N punti materiali in $\mathbb{R}^3$

$\underline{X} = (\overbrace{x_1, x_2, x_3}^{x,y,z}, \dots, x_{3N})$

DEF. VINCOLO OLONOMO (non dipende da velocità)

- $f(\underline{x}, t) = 0 \rightarrow$ BILATERO
- $f(\underline{x}, t) \geq 0 \rightarrow$ UNILATERO

Se vincolo non dipende da tempo $\Rightarrow$ FISSO, altrimenti MOBILE

$$\begin{cases} f_1(\underline{x}) = 0 \\ f_2(\underline{x}) = 0 \\ \vdots \\ f_r(\underline{x}) = 0 \end{cases} \rightarrow \text{ogni vincolo elimina un grado di libertà} \Rightarrow 3N - r \text{ gradi di libertà}$$

$$3N - r = m \rightarrow \underset{\substack{\downarrow \\ \text{COORDINATE LIBERE LAGRANGIANE}}}{(q_1, \dots, q_m)} \quad \begin{cases} x_1 = \varphi_1(q_1, \dots, q_m) \\ x_2 = \varphi_2(q_1, \dots, q_m) \\ \vdots \\ x_{3N} = \varphi_{3N}(q_1, \dots, q_m) \end{cases}$$

(i) descrizione locale

(ii) Ben definita $\Rightarrow$ rango $\begin{pmatrix} \frac{\partial \varphi_1}{\partial q_1} & & \frac{\partial \varphi_1}{\partial q_m} \\ \vdots & & \\ \frac{\partial \varphi_{3N}}{\partial q_1} & \cdot & \frac{\partial \varphi_{3N}}{\partial q_m} \end{pmatrix} = m$

$P = \underline{x}(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_m)$

$$\lim_{h \to 0} \frac{\underline{x}(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_j + h, \dots, \tilde{q}_m) - \underline{x}(\tilde{q}_1, \dots, \tilde{q}_m)}{h} = \frac{\partial \underline{x}(\tilde{q})}{\partial q_j}$$ vettore tangente alla superficie

$(q_1(t), q_2(t), \dots, q_m(t)) \rightarrow$ Curva in $U \subseteq \mathbb{R}^m \quad t \in I$

$\downarrow \varphi_1, \dots, \varphi_{3N}$

$x(t)$ sulla varietà

$t_0 \in I \rightarrow q_1(t_0), \dots, q_m(t_0)$

$\varphi \hookrightarrow \underline{x}(t_0)$

$$\underline{v}(t_0) = \lim_{h \to 0} \frac{\underline{x}(t_0 + h) - \underline{x}(t_0)}{h} = \frac{d}{dx} x(t) \Big|_{t = t_0} = \left(\frac{d}{dt} x_1(q_1(t), \dots, q_m(t)), \frac{dx_2}{dt}(), \dots, \frac{dx_{3N}}{dt}() \right)$$

$\underline{x}(t) = \underline{x}(q_1(t), \dots, q_m(t))$

$$\Rightarrow \frac{d}{dt} x_1(q_1(t), \dots, q_m(t)) = \frac{\partial x_1}{\partial q_1} \dot{q}_1 + \frac{\partial x_1}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x_1}{\partial q_m} \dot{q}_m$$

$$\rightarrow = \sum_{j=1}^{m} \frac{\partial x_1}{\partial q_j} \cdot \dot{q}_j$$

$$\underline{V}(t_0) = \sum_{j=1}^{m} \frac{\partial \underline{x}(q_1(t), \dots, q_m(t))}{\partial q_j} \underbrace{\dot{q}_j}_{\text{VELOCITÀ GENERALIZZATA}} \Big|_{t = t_0}$$

Se la potenza delle reazioni vincolari è lungo spostamenti compatibili con il vincolo $\rightarrow$ VINCOLO LISCIO

$K \rightarrow$ energia cinetica $\Rightarrow \frac{dK}{dt} = \Pi_{TOT}$ $\hookrightarrow$ potenza

- $(x(q), y(q), z(q)) \rightarrow$ velocità compatibili con vincolo $\left(\overset{v_x}{\frac{dx}{dq} \cdot \dot{q}}, \overset{v_y}{\frac{dy}{dq} \dot{q}}, \overset{v_z}{\frac{dz}{dq} \dot{q}} \right)$

• $\vec{F}_{attive}$ conservative $\Rightarrow$ $\vec F_{attive} = -\vec\nabla U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$

• vincolo fisso e liscio $\Rightarrow$ $\Pi^{TOT} = \Pi^{attive} \Rightarrow \vec F_{attive}\cdot\vec v = (-\vec\nabla U)\cdot\vec v = -\left(\frac{\partial U}{\partial x} v_x + \frac{\partial U}{\partial y}\cdot v_y + \frac{\partial U}{\partial z}\cdot v_z\right) = -\left(\frac{\partial U}{\partial x}\frac{dx}{dq}\dot q + \frac{\partial U}{\partial y}\frac{dy}{dq}\dot q + \frac{\partial U}{\partial z}\frac{dz}{dq}\cdot\dot q\right)$

$\Pi^{TOT} = -\dot q\left[\frac{\partial U}{\partial x}\frac{dx}{dq} + \frac{\partial U}{\partial y}\frac{dy}{dq} + \frac{\partial U}{\partial z}\frac{dz}{dq}\right] = -\frac{dU}{dq}\cdot\dot q$

$\frac{dK}{dt} = -\frac{dU}{dq}\cdot\dot q \Rightarrow \frac{dU}{dq}\dot q = \frac{dU}{dt} \to \frac{dK}{dt}+\frac{dU}{dt}=0 \Rightarrow \frac{d}{dt}(U+K)=0$ → conservazione energia per sistemi conservativi in presenza di vincoli lisci

EQUAZIONE EULERO-LAGRANGE

$q \mapsto x(q), y(q), z(q)$. posizioni compatibili con il vincolo

⇝ velocità compatibili con il vincolo $v = (x'(q)\dot q, y'(q)\dot q, z'(q)\dot q)$ $\frac{d}{dq}x = x'$

$K(q,\dot q) = \frac12 m \underbrace{(x'(q)^2+y'(q)^2+z'(q)^2)}_{G(q)}\dot q^2$

$G(q)$ sempre positivo $\Rightarrow K=0 \Leftrightarrow \dot q = 0$

$\frac{d}{dt}K(q(t),\dot q(t)) = \frac{\partial K}{\partial q}\dot q + \frac{\partial K}{\partial \dot q}\cdot\frac{d}{dt}\dot q = \frac{\partial K}{\partial q}\dot q + \frac{d}{dt}\left(\frac{\partial K}{\partial\dot q}\cdot\dot q\right) - \left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\right)\right]\cdot\dot q = \frac{d}{dt}\left(\frac{\partial K}{\partial\dot q}\cdot\dot q\right) + \dot q\left(\frac{\partial K}{\partial q} - \frac{d}{dt}\frac{\partial K}{\partial\dot q}\right)$ $K = \frac12 G(q)\dot q^2 \Rightarrow \dot q\frac{\partial K}{\partial\dot q} = G(q)\cdot\dot q^2 = 2K$

$\frac{d}{dt}K = \frac{d}{dt}2K + \dot q\left(\frac{\partial K}{\partial q} - \frac{d}{dt}\frac{\partial K}{\partial \dot q}\right) \Rightarrow \frac{d}{dt}K = \dot q\left(\frac{d}{dt}\frac{\partial K}{\partial\dot q} - \frac{\partial K}{\partial q}\right)$

$\frac{dK}{dt} \underset{\uparrow}{=} \Pi^{attive} = f^{attive}\, v$

vincolo liscio

$\frac{d}{dt}\frac{\partial K}{\partial\dot q} - \frac{\partial K}{\partial q} = f^{attive}\,(x',y',z')$

se le f^{attive} sono conservative $\Rightarrow \exists U(x,y,z) \leadsto U(x(q),y(q),z(q))$

$\Pi^{attive} = -\frac{dU}{dq}\cdot\dot q \Rightarrow \frac{d}{dt}K = \Pi^{attive} = \dot q\left[\frac{d}{dt}\frac{\partial K}{\partial \dot q} - \frac{\partial K}{\partial q}\right] = -\frac{dU}{dq}\dot q \to \frac{d}{dt}\frac{\partial K}{\partial\dot q} = \frac{\partial}{\partial q}(K-U)$ $\frac{\partial}{\partial\dot q}U = 0 \leadsto \frac{\partial}{\partial\dot q}(K-U) = \frac{\partial}{\partial \dot q}K$

$\mathcal L(q,\dot q) = K(q,\dot q) - U(q)$ $\Rightarrow$ EQUAZIONE EULERO-LAGRANGE → $\frac{d}{dt}\frac{\partial\mathcal L}{\partial\dot q} = \frac{\partial \mathcal L}{\partial q}$

CONSERVAZIONE ENERGIA

Supponiamo $\mathcal L(q,\dot q,\cancel{t}) \Rightarrow \frac{\partial\mathcal L}{\partial t}=0$

$\frac{d}{dt}\mathcal L(q(t),\dot q(t)) = \frac{\partial\mathcal L}{\partial q}\dot q + \frac{\partial \mathcal L}{\partial\dot q}\frac{d}{dt}\dot q = \frac{\partial\mathcal L}{\partial q}\cdot\dot q + \frac{d}{dt}\left[\frac{\partial\mathcal L}{\partial\dot q}\dot q\right] - \dot q\frac{d}{dt}\frac{\partial\mathcal L}{\partial\dot q} = \dot q\overset{A}{\left[\frac{\partial\mathcal L}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal L}{\partial\dot q}\right]} + \overset{B}{\frac{d}{dt}\left[\frac{\partial\mathcal L}{\partial\dot q}\dot q\right]}$

Se $q(t)$ è soluzione $\Rightarrow A=0 \Rightarrow \frac{d\mathcal L}{dt} = \frac{d}{dt}\left[\frac{\partial\mathcal L}{\partial\dot q}\dot q\right]$ lungo il moto

$\frac{d}{dt}\left[\mathcal L - \frac{\partial\mathcal L}{\partial\dot q}\dot q\right] = 0 \Rightarrow \dot q\frac{\partial\mathcal L}{\partial\dot q} - \mathcal L$ è costante del moto

$\|$

$2K - (K-U) = K+U = K(q,\dot q) + U(q)$

FORMA NORMALE EQUAZIONE EULERO-LAGRANGE (1 GRADO DI LIBERTÀ)

$\mathcal L(q,\dot q) = \frac12 G(q)\dot q^2 - U(q)$

EQ E-L $\frac{d}{dt}\frac{\partial\mathcal L}{\partial\dot q} = \frac{\partial\mathcal L}{\partial q}$

• $\frac{\partial\mathcal L}{\partial q} = \frac12 G'(q)\dot q^2 - U'(q)$
• $\frac{\partial\mathcal L}{\partial\dot q} = G(q)\cdot\dot q$
• $\frac{d}{dt}\frac{\partial\mathcal L}{\partial\dot q} = G'(q)\dot q^2 + G(q)\ddot q$

$\Rightarrow$ EQ E-L → $G'(q)\dot q^2 + G(q)\ddot q = \frac12 G'(q)\dot q^2 - U'(q) \Rightarrow G(q)\ddot q = -\frac12 G'(q)\dot q^2 - U'(q) \Rightarrow \ddot q = -\frac12\frac{G'(q)}{G(q)}\dot q^2 - \frac{U'(q)}{G(q)}$ $G(q)>0$

$y = \dot q \Rightarrow \begin{cases}\dot q = y\\ \dot y = -\frac12\frac{G'(q)}{G(q)}y^2 - \frac{U'(q)}{G(q)}\end{cases}$

PUNTI STAZIONARI → $(\bar q, 0)$ t.c. $U'(\bar q)=0$

• $E(q,y) = K(q,y)+U(q) = \frac12 G(q)y^2 + U(q)$ costante del moto

STABILITÀ PUNTO EQUILIBRIO $(\bar q,0)$

TEO Si consideri un sistema lagrangiano (1 GRADO LIBERTÀ) e sia $(\bar q,0)$ p.to di equilibrio, $\bar q$ t.c. $U'(\bar q)=0$, e sia $U''(\bar q)>0 \Rightarrow (\bar q,0)$ è STABILE

DIM II LYAPUNOV con $E(q,y)$

(ii) è vera perché E è costante del moto

(i) $(\bar q,0)$ è di minimo? $\vec\nabla E = \left(\frac{\partial E}{\partial q}, \frac{\partial E}{\partial y}\right) = \left(\frac12 G'(q)y^2 + U'(q), G(q)y\right)$ $\vec\nabla E|_{(\bar q,0)} = (0,0) \Rightarrow (\bar q,0)$ è punto stazionario per E

$H_E(q,y) = \begin{pmatrix} \frac{\partial^2 E}{\partial q^2} & \frac{\partial^2 E}{\partial q\partial y}\\ \frac{\partial^2 E}{\partial y\partial q} & \frac{\partial^2 E}{\partial y^2}\end{pmatrix} = \begin{pmatrix}\frac12 G''(q)y^2+U''(q) & G'(q)y\\ G'(q)y & G(q)\end{pmatrix}\Bigg|_{(\bar q,0)} = \begin{pmatrix}U''(\bar q)&0\\0&G(\bar q)\end{pmatrix} \Rightarrow (\bar q,0)$ è un punto di minimo

↳ H_E definita positiva

PICCOLE OSCILLAZIONI

$$\begin{cases} q = \bar{q} + \varepsilon u(t) \\ y = 0 + \varepsilon v(t) = \varepsilon u(t) \end{cases} \qquad \dot{q} = \varepsilon \dot{u}$$

$$\tilde{\mathcal{L}}(u, \tilde{u}) = \mathcal{L}(\bar{q} + \varepsilon u, \varepsilon \dot{u}) = \frac{1}{2} G(\bar{q} + \varepsilon u)\, \varepsilon^2 \dot{u}^2 + U(\bar{q} + \varepsilon u)$$

ESPANDO ATTORNO A $\varepsilon = 0$ FINO AL SECONDO ORDINE $\Rightarrow \tilde{\mathcal{L}}(u, \tilde{u}) = \frac{1}{2} G(\bar{q}) \varepsilon^2 u^2 - \left[\cancel{U(\bar{q})} + \cancel{U'(\bar{q})\, u \varepsilon} + \frac{1}{2} U''(\bar{q}) \varepsilon^2 u^2 \right] + o(\varepsilon^2) \simeq \varepsilon^2 \left[\frac{1}{2} \underbrace{G(\bar{q})}_{m} u^2 - \frac{1}{2} \underbrace{U''(\bar{q})}_{K} \varepsilon^2 u^2 \right]$

$$w = \sqrt{\frac{K}{m}} = \sqrt{\frac{U''(\bar{q})}{G(\bar{q})}}$$

VINCOLI DIPENDENTI DAL TEMPO

es p.to materiale su circonferenza $r(t) = r_0 e^t$

$$\begin{cases} r(\theta, t) = r_0 e^t \cos\theta \\ y(\theta, t) = r_0 e^t \sin\theta \end{cases}$$

$$\frac{dr}{dt} = r_0 e^t (-\sin\theta)\, \dot\theta + r_0 e^t \cos\theta$$

$$\begin{cases} x(q,t) \\ y(q,t) \\ z(q,t) \end{cases} \rightsquigarrow \begin{cases} \frac{dx}{dt} = \frac{\partial x}{\partial q} \dot q + \frac{\partial x}{\partial t} \\ \frac{dy}{dt} = \frac{\partial y}{\partial q} \cdot \dot q + \frac{\partial y}{\partial t} \\ \frac{dz}{dt} = \frac{\partial z}{\partial q} \dot q + \frac{\partial z}{\partial t} \end{cases}$$

VELOCITA' VIRTUALI

$(v_x, v_y, v_z) = v^{VIRT}$ tempo come congelato

$$\begin{cases} v_x = \frac{dx}{dt} - \frac{\partial x}{\partial t} = \frac{\partial x}{\partial q} \dot q \\ v_y = \frac{dy}{dt} - \frac{\partial y}{\partial t} = \frac{\partial y}{\partial q} \dot q \\ v_z = \frac{dz}{dt} - \frac{\partial z}{\partial t} = \frac{\partial z}{\partial q} \cdot \dot q \end{cases}$$

VINCOLI LISCI (dipendente da t) $\to$ Potenza reazioni vincolari nulla lungo le velocità virtuali $\Rightarrow \phi^{VINC}\, v^{VIRT} = 0$

$$m\underline{a} = \underline{F}^{ATT} + \underline{\phi}^{VINC} \Rightarrow m\, \underline{a}\, \underline{v}^{VIRT} = \underline{F}^{ATT} \underline{v}^{VIRT} + \cancel{\underline{\phi}^{VINC} \underline{v}^{VIRT}}$$

NEL PIANO $\to \quad m\left(\ddot x \frac{\partial x}{\partial q} \cancel{\dot q}\right) + m\left(\ddot y \frac{\partial y}{\partial q} \cancel{\dot q}\right) = f_x^{ATT} \frac{\partial x}{\partial q} \cancel{\dot q} + f_y^{ATT} \frac{\partial y}{\partial q} \cancel{\dot q} \qquad \forall q$

Se $\underline{F}^{ATT}$ CONSERVATIVE $\Rightarrow (f_x, f_y) = \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}\right) \longrightarrow m\left[\overset{A}{\ddot x \frac{\partial x}{\partial q}} + \ddot y \frac{\partial y}{\partial q}\right] = -\left[\overset{B}{\frac{\partial U}{\partial x} \frac{\partial x}{\partial q}} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial q}\right]$

B $\to \frac{\partial U}{\partial q} (x(q,t), y(q,t))$

A $\ddot x \frac{\partial x}{\partial q} = \frac{d}{dt} \dot x \frac{\partial x}{\partial q} = \frac{d}{dt}\left(\dot x \frac{\partial x}{\partial q}\right) - \dot x \frac{d}{dt} \frac{\partial x}{\partial q} \Rightarrow m\left[\frac{d}{dt}\left(\dot x \frac{\partial \dot x}{\partial \dot q} + \dot y \frac{\partial \dot y}{\partial \dot q}\right) - \left(\dot x \frac{d}{dt} \frac{\partial x}{\partial q} + \dot y \frac{d}{dt} \frac{\partial y}{\partial q}\right)\right] = -\frac{\partial U}{\partial q}$

LEMMA 1 $\to \frac{\partial x}{\partial q} = \frac{\partial \dot x}{\partial \dot q}$

dim $\frac{\partial}{\partial \dot q} \frac{d}{dt} x = \frac{\partial}{\partial \dot q}\left[\frac{\partial x}{\partial q} \dot q + \frac{\partial x}{\partial t}\right] = \frac{\partial x}{\partial q}$

LEMMA 2 $\to \frac{d}{dt} \frac{\partial x}{\partial q} = \frac{\partial}{\partial q} \frac{d}{dt} x = \frac{\partial}{\partial q} \dot x$

Dim $\frac{d}{dt}\left[\frac{\partial x}{\partial q}\right] = \frac{\partial}{\partial q}\left(\left[\frac{\partial x}{\partial q}\right] \dot q\right) + \frac{\partial}{\partial t}\left[\frac{\partial x}{\partial q}\right] = \frac{\partial}{\partial q}\left(\left[\frac{\partial x}{\partial q}\right] \dot q\right) + \frac{\partial}{\partial q} \frac{\partial x}{\partial t} = \frac{\partial}{\partial q}\left[\frac{\partial x}{\partial q} \dot q + \frac{\partial x}{\partial t}\right] = \frac{\partial}{\partial q} \frac{d}{dt} x$

$$\Rightarrow m\left[\frac{d}{dt} \frac{\partial}{\partial \dot q}\left[\underbrace{\frac{1}{2}(\dot x^2 + \dot y^2)}_{K}\right] - \left(\frac{\partial}{\partial q}\left[\frac{1}{2}(\dot x^2 + \dot y^2)\right]\right)\right] = -\frac{\partial U}{\partial q}$$

$$\to \frac{d}{dt} \frac{\partial}{\partial \dot q} K - \frac{\partial K}{\partial q} = -\frac{\partial U}{\partial q} \Rightarrow \boxed{\frac{d}{dt} \frac{\partial}{\partial \dot q} (K - U) - \frac{\partial}{\partial q} (K - U) = 0}$$

d GRADI DI LIBERTA'

$\underline{x} = (x_1, x_2, x_3, \ldots, x_{3N})$

$$\begin{cases} x_1(q_1, q_2, \ldots, q_d, t) \\ x_2(q_1, q_2, \ldots, q_d, t) \\ \vdots \\ x_{3N}(q_1, q_2, \ldots, q_d, t) \end{cases}$$

$\dot{\underline{x}} = (\dot x_1, \dot x_2, \dot x_3, \ldots, \dot x_{3N}) \to \begin{cases} \dot x_1 = \sum_{\delta=1}^{d} \frac{\partial x_1}{\partial q_\delta} \dot q_\delta + \frac{\partial x_1}{\partial t} \\ \vdots \\ \dot x_{3N} = \sum_{\delta=1}^{d} \frac{\partial x_{3N}}{\partial q_\delta} \dot q_\delta + \frac{\partial x_{3N}}{\partial t} \end{cases} \rightsquigarrow$ vettore velocità virtuali $\quad v^{VIRT} = (v_1, \ldots, v_{3N})$

$$v_i = \frac{dx_i}{dt} - \frac{\partial x_i}{\partial t} = \sum_{\delta=1}^{d} \frac{\partial x_i}{\partial q_\delta} \cdot \dot q_\delta$$

PRINCIPIO DI D'ALAMBERT (VINCOLO liscio) d gradi di libertà

$\phi^{VINC} = (\phi_1, \ldots, \phi_{3N}) \Rightarrow \phi^{VINC}\, v^{VIRT} = 0 = \sum_{K=1}^{m} \phi^{(K)}\, v^{(K)}_{VIRT} \to$ vincolo liscio $\quad (\phi_1, \phi_2, \phi_3) = \phi^{(1)}$

$$\to \sum_{k=1}^N v_{VIRT}^{(k)}\left(m^{(k)} a^{(k)}\right) = F_{TOT}^{(k)} \cdot v_{VIRT}^{(k)} = \sum_{k=1}^N \left(F_{ATT}^{(k)} + \phi^{(k)}\right) \cdot v_{VIRT}^{(k)}$$

VINCOLO LISCIO $\Rightarrow \sum_{k=1}^N \phi^{(k)} v^{(k)} = 0 \Rightarrow \sum_{k=1}^N m^{(k)} a^{(k)} v_{VIRT}^{(k)} = \sum_{k=1}^N F^{(k)} \cdot v_{VIRT}^{(k)}$

$$\sum_{k=1}^N m^{(k)} \ddot{x}^{(k)} \left(\sum_{\gamma=1}^d \frac{\partial x^{(k)}}{\partial q_\gamma}\cdot \dot q_\gamma\right) = \sum_{\gamma=1}^d \dot q_\gamma \left(\sum_{k=1}^N m^{(k)}\ddot x^{(k)} \frac{\partial x^{(k)}}{\partial q_\gamma}\right) = \sum_{\gamma=1}^d \dot q_\gamma\left(\sum_{k=1}^N F^{(k)}\frac{\partial x^{(k)}}{\partial q_\gamma}\right)$$ vera $\forall$ scelta $\dot q(\dot q_1, .. , \dot q_d)$

scelgo $\dot q = (0,..,\dot q_r, 0,..,0)$

$\rightsquigarrow \cancel{\dot q_r}\left(\sum_{k=1}^N m^{(k)}\ddot x^{(k)} \frac{\partial x^{(k)}}{\partial q_r}\right) = \cancel{\dot q_r}\left(\sum_{k=1}^N F^{(k)}_{ATTIVA}\frac{\partial x^{(k)}}{\partial q_r}\right)$ $r=1,...,d$

$$\underbrace{\sum_{i=1}^{3N} m_i \ddot x_i \frac{\partial x_i}{\partial q_r}}_{A} = \underbrace{\sum_{i=1}^{3N} F_i^{ATTIVE}\frac{\partial x_i}{\partial q_r}}_{B}$$

A $\to \sum_{i=1}^{3N} m_i \left(\frac{d}{dt}\dot x_i\right)\frac{\partial x_i}{\partial q_r} = \sum_{i=1}^{3N} m_i \left(\frac{d}{dt}\left(\dot x_i \frac{\partial x_i}{\partial q}\right) - \dot x_i \frac{d}{dt}\frac{\partial x_i}{\partial q_r}\right) = \sum_{i=1}^{3N} m_i\left(\frac{d}{dt}\left(\dot x_i \frac{\partial \dot x_i}{\partial \dot q_r}\right) - \dot x_i\frac{\partial \dot x_i}{\partial q_r}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot q_r}\left(\frac12 \sum_{i=1}^{3N} m_i \dot x_i^2\right) - \frac{\partial}{\partial q_r}\left(\frac12\sum_{i=1}^{3N} m_i\dot x_i^2\right) = \frac{d}{dt}\frac{\partial}{\partial \dot q_r}K - \frac{\partial}{\partial q_r}K$

B $\to \sum_{i=1}^{3N} F_i^{AT}\frac{\partial x_i}{\partial q_r}$ se F^{ATT} CONSERVATIVE $\Rightarrow F_i^{ATT} = -\frac{\partial}{\partial x_i}U(x_1,..,x_{3N}) \to = -\sum_{i=1}^{3N} \frac{\partial U}{\partial x_i}\cdot\frac{\partial x_i}{\partial q_r} = -\frac{\partial U}{\partial q_r}$

$\to \frac{d}{dt}\frac{\partial K}{\partial \dot q_r} - \frac{\partial}{\partial q_r}K = -\frac{\partial U}{\partial q_r} \to \frac{d}{dt}\frac{\partial}{\partial \dot q_r}(K-U) = \frac{\partial}{\partial q_r}(K-U)$

$\mathcal L(q_1,..,q_d,\dot q_1,..,\dot q_d) = K-U \Rightarrow \frac{d}{dt}\frac{\partial \mathcal L}{\partial \dot q_r} = \frac{\partial\mathcal L}{\partial q_r}$ $r=1,...,d$ SISTEMA EQUAZIONI EULERO-LAGRANGE

STRUTTURA FORMALE ENERGIA CINETICA ($d>1$) N pti materiali

se $d=1 \Rightarrow K(q,\dot q) = \frac12 G(q)\dot q^2$

se $d>1$

$\underline q = (q_1,...,q_d)$ $\underline x = (x_1,...,x_{3N})$

$K(\underline q,\dot{\underline q},t) = \frac12\sum_{i=1}^{3N} m_i \dot x_i^2$ $\dot x_i = \frac{d}{dt}x_i(q_1,..,q_d,t) = \sum_{\gamma=1}^d \frac{\partial x_i}{\partial q_\gamma}\cdot\dot q_\gamma + \frac{\partial x_i}{\partial t}$

$K = \frac12\sum_{i=1}^{3N} m_i\left(\sum_{\gamma=1}^d \frac{\partial x_i}{\partial q_\gamma}\dot q_\gamma + \frac{\partial x_i}{\partial t}\right)\left(\sum_{k=1}^d \frac{\partial x_i}{\partial q_k}\dot q_k + \frac{\partial x_i}{\partial t}\right) = \frac12\left(\left[\sum_{\gamma,k=1}^d\sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_\gamma}\frac{\partial x_i}{\partial q_k}\dot q_\gamma\dot q_k\right] + G_1 + G_0\right)$

$G_1 \to$ lineare nelle $\dot q$
$G_0 \to$ non dipende da $\dot q$ $\Big] = 0 \to$ vincoli fissi

$\rightsquigarrow$ VINCOLI FISSI $\Rightarrow \underline x = \underline x(q_1,..,q_d, \cancel{t}) \Rightarrow K(\underline q,\dot{\underline q}) = \frac12\sum_{k,\gamma=1}^d G_{\gamma k}(q_1,..,q_d)\dot q_\gamma\dot q_k$

$\to K = \frac12(\dot q_1,..,\dot q_d)\, G(q_1,..,q_d)\begin{pmatrix}\dot q_1\\ \vdots\\ \dot q_d\end{pmatrix}$ FORMA BILINEARE

matrice $d\times d$

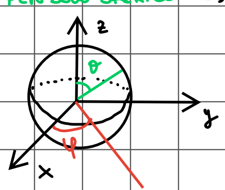
G matrice simmetrica definita positiva con entrate $G_{\gamma k}(\underline q) = \sum_{i=1}^{3N}\frac{\partial x_i}{\partial q_\gamma}\frac{\partial x_i}{\partial q_k} = G_{k\gamma}$

$\forall \dot q \neq 0 \Rightarrow \sum_{\gamma,k} G_{\gamma k}\dot q_\gamma \dot q_k > 0$

$v = \sum_{\gamma=1}^d \frac{\partial x_i}{\partial q_\gamma}\dot q_\gamma$ $v \cdot v \geq 0$ $(=0 \Leftrightarrow v=0)$

PENDOLO SFERICO

$\to$ pendolo vincolato a superficie sfera $\to$ 2 GRADI LIBERTA'



R : raggio $\quad \hookrightarrow x^2+y^2+z^2=R^2$

$\theta \in [0,\pi)$

$\varphi \in [0,2\pi)$

$$\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \rightsquigarrow \begin{cases} \dot x = R\cos\theta\dot\theta\cos\varphi - R\sin\theta\sin\varphi\dot\varphi \\ \dot y = R\cos\theta\dot\theta\sin\varphi + R\sin\theta\cos\varphi\dot\varphi \\ \dot z = -R\sin\theta\dot\theta \end{cases}$$

$$K = \frac12 m\left[\dot x^2+\dot y^2+\dot z^2\right] = \frac12 mR^2\left((\cos\theta\dot\theta\sin\varphi - \sin\theta\sin\varphi\dot\varphi)^2 + (\cos\theta\dot\theta\sin\varphi + \sin\theta\cos\varphi\dot\varphi)^2 + (\sin\theta\dot\theta)^2\right) = \frac12 mR^2\left(\dot\theta^2 + \sin^2\theta\dot\varphi^2\right) = \frac12(\dot\theta,\dot\varphi)\begin{pmatrix} mR^2 & 0 \\ 0 & mR^2\sin^2\theta\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\varphi\end{pmatrix}$$

$U(\theta,\varphi) = mgR\cos\theta$

$\mathcal L(\theta,\cancel{\varphi},\dot\theta,\dot\varphi) = \frac12 mR^2\left(\dot\theta^2 + \sin^2\theta\dot\varphi^2\right) - mgR\cos\theta$

DEF. VARIABILE CICLICA

Se $\mathcal{L}(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d)$ non dipende da $q_j \Rightarrow q_j$ è CICLICA

$\to \dfrac{\partial \mathcal{L}}{\partial q_j}=0 \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot q_j}\right)=0 \Rightarrow \dfrac{\partial \mathcal{L}}{\partial \dot q_j}$ è costante del moto

LAGRANGIANA RIDOTTA $\mathcal{L}^R(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\cancel{\dot q_j},\dots,\dot q_d,c)$

q_j ciclica $\Rightarrow$ $C=\dfrac{\partial \mathcal{L}}{\partial \dot q_j}(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d)$

$\dot q_j=f(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d,C)$

$\to E=K+U$ sostituisco con $\dot q_j=f$ $\to$ $E'=K|_{\dot q_j=f}+U \Rightarrow E'=K^{eff}-U^{eff}$

$\mathcal{L}^{RID}\neq \mathcal{L}|_{\dot q=f}$ $\quad \mathcal{L}^{RID}=\mathcal{L}^R(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\cancel{\dot q_j},\dots,\dot q_d,c)$

FREQUENZE PROPRIE E MODI NORMALI $(d>1$ gdl$)$

⇝ forze conservative, vincoli lisci e fissi

$$\underline{q}=(q_1,\dots,q_d) \quad \mathcal{L}(\underline{q},\underline{\dot q})=\frac{1}{2}\sum_{\alpha,\beta=1}^{d} G_{\alpha\beta}(\underline{q})\,\dot q_\alpha \dot q_\beta - U(\underline{q})$$

$\uparrow$ simmetrica e definita positiva

Eq. E-L per q_γ $\to$ $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot q_\gamma}=\dfrac{\partial \mathcal{L}}{\partial q_\gamma}$

$$\frac{\partial \mathcal{L}}{\partial \dot q_\gamma}=\frac{1}{2}\left[\sum_{\beta\neq\gamma} G_{\gamma\beta}(\underline{q})\dot q_\beta+\sum_{\alpha\neq\gamma} G_{\alpha\gamma}(\underline{q})\dot q_\alpha+2G_{\gamma\gamma}(\underline{q})\dot q_\gamma\right]$$

G simmetrica $\Rightarrow$ $G_{\alpha\gamma}(\underline{q})=G_{\gamma\alpha}(\underline{q})$

$$\sum_{\beta\neq\gamma} G_{\gamma\beta}(\underline{q})\dot q_\beta=\sum_{\alpha\neq\gamma} G_{\alpha\gamma}(\underline{q})\dot q_\alpha$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot q_\gamma}=\frac{1}{2}\left[2\sum_{\alpha\neq\gamma} G_{\alpha\gamma}(\underline{q})\dot q_\alpha+2G_{\gamma\gamma}\dot q_\gamma\right]=\sum_{\alpha=1}^{d} G_{\alpha\gamma}(\underline{q})\dot q_\alpha$$

$d > 1$ gdl

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \sum_{\alpha,\beta=1}^{d} G_{\alpha\beta}(q)\, \dot{q}_\alpha \dot{q}_\beta - U(q)$$

$$\begin{cases} \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{q}_j} = \dfrac{\partial \mathcal{L}}{\partial q_j} \\ j = 1, \dots, d \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_\gamma} = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\, \dot{q}_\gamma = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\, \ddot{q}_\alpha + \sum_{\alpha=1}^{d} \dot{q}_\alpha \frac{d}{dt} G_{\alpha\gamma}(q)$$

$$\frac{d}{dt} G_{\alpha\gamma}(q) = \sum_{\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial q_\beta} \dot{q}_\beta$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_\gamma} = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\, \ddot{q}_\alpha + \sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial \dot{q}_\beta} \dot{q}_\alpha \dot{q}_\beta$$

$$\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial \dot{q}_\beta} \dot{q}_\alpha \dot{q}_\beta = \frac{1}{2} \left(\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}(q)\, \dot{q}_\alpha \dot{q}_\beta + \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}(q)\, \dot{q}_\alpha \dot{q}_\beta \right)$$

$$\frac{\partial \mathcal{L}}{\partial q_\gamma} = \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\beta}}{\partial q_\gamma} \dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma}$$

$$\Gamma_{\alpha\beta\gamma} := \frac{1}{2} \left(\frac{\partial}{\partial q_\gamma} G_{\alpha\beta} - \frac{\partial}{\partial q_\gamma} G_{\alpha\gamma} - \frac{\partial}{\partial q_\gamma} G_{\beta\gamma} \right)$$

$$\text{E-L} \to \begin{cases} \displaystyle\sum_{\alpha=1}^{d} G_{\alpha\gamma}\, \ddot{q}_\alpha = \sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma}\, \dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} \\ \gamma = 1, \dots, d \end{cases}$$

PUNTI DI EQUILIBRIO $\rightsquigarrow$ $\dot{q} = 0$ e $\displaystyle\sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma}\, \dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} = 0$

$$\Rightarrow \frac{\partial U}{\partial q_\gamma} = 0$$

q_0 t.c. $\dfrac{\partial U}{\partial q_\gamma}(q_0) = 0 \quad \forall \gamma = 1, \dots, d \quad \rightsquigarrow \quad \vec{\nabla} U(q_0) = 0$

Teo (LAGRANGE - DIRICHLET) $d \geq 1$

Sistema con forze attive conservative, vincoli lisci e fissi

Supponiamo che U abbia punti critici isolati

I MINIMI DI U (guardo Hessiana) $\rightarrow$ sono p.ti equilibrio stabile sistema

Dim Suppongo q_0 t.c. $\vec{\nabla} U(q_0) = 0$

$$H(q) = \begin{pmatrix} \frac{\partial^2 U}{\partial q_1^2} & \frac{\partial^2 U}{\partial q_2 q_1} & \cdots & \frac{\partial^2 U}{\partial q_d q_1} \\ \vdots & & & \\ \frac{\partial^2 U}{\partial q_1 q_d} & \frac{\partial^2 U}{\partial q_2 q_d} & \cdots & \frac{\partial^2 U}{\partial q_d^2} \end{pmatrix}$$

q_0 è di minimo se $H(q_0)$ è DEFINITA POSITIVA

q^* p.to di minimo per $U(q)$

$$\underline{q}(t) = \underline{q}^* + \varepsilon \underline{u}(t) \qquad \underline{\dot{q}}(t) = \varepsilon \underline{\dot{u}}(t) \qquad \underline{\ddot{q}}(t) = \varepsilon \underline{\ddot{u}}(t)$$

$$\hookrightarrow \begin{cases} q_1(t) = q_1^* + \varepsilon u_1(t) \\ \vdots \\ q_d(t) = q_d^* + \varepsilon u_d(t) \end{cases}$$

$$\text{E-L} \longrightarrow \begin{cases} \sum_{\alpha=1}^{d} G_{\alpha\gamma} \ddot{q}_\alpha = \sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma} \dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} \\ \gamma = 1, \dots, d \end{cases}$$

$$\sum_\alpha G_{\alpha\gamma}(\underline{q}^* + \varepsilon \underline{u}) \varepsilon \ddot{u}_\alpha = \sum_{\alpha,\beta} \Gamma_{\alpha\beta\gamma}(\underline{q}^* + \varepsilon \underline{u}) \varepsilon^2 \dot{u}_\alpha \dot{u}_\beta - \frac{\partial}{\partial q_\gamma}(\underline{q}^* + \varepsilon \underline{u})$$

ESPANSIONE AL PRIMO ORDINE IN ε

$$\sum_\alpha G_{\alpha\gamma}(\underline{q}^*) \varepsilon \ddot{u}_\alpha = -\left(\cancel{\frac{\partial U}{\partial q_\gamma}(\underline{q}^*)} + \sum_{\beta=1}^{d} \left[\frac{\partial}{\partial q_\beta} \left(\frac{\partial U}{\partial q_\gamma} \right) \right] \Bigg|_{(\underline{q}^*)} \varepsilon u_\beta \right)$$

$$\begin{cases} \sum_\alpha G_{\alpha\gamma}(\underline{q}^*) \cancel{\varepsilon} \ddot{u}_\alpha = -\sum_{\beta=1}^{d} \frac{\partial^2 U}{\partial q_\beta \partial q_\gamma}(\underline{q}^*) \cancel{\varepsilon} u_\beta \\ \gamma = 1, \dots, d \end{cases}$$

$$\left[G(\underline{q}^*) \underline{\ddot{u}} \right]_\gamma = -\left[H_U(\underline{q}^*) \underline{u} \right]_\gamma \Rightarrow \left[G(\underline{q}^*) \underline{\ddot{u}} \right] = -\left[H_U(\underline{q}^*) \underline{u} \right]$$

$\downarrow$

E-L approssimata

$$\mathcal{L}^{LIN}(\underline{u}, \underline{\dot{u}}) = \frac{1}{2} \left(\underline{\dot{u}} \cdot G(\underline{q}^*) \underline{\dot{u}} \right) - \frac{1}{2} \left(\underline{u} \cdot H_U(\underline{q}^*) \underline{u} \right)$$

$$G_0 := G(\underline{q}^*) \qquad H_0 := H_U(\underline{q}^*)$$

Cerco soluzioni del tipo $\underline{u}(t) = \cos(wt+\varphi)\cdot \underline{w}$ $\quad \underline{w} \neq 0 \in \mathbb{R}^d$

$$\dot{\underline{u}} = -w \sin(wt+\varphi)\,\underline{w}$$
$$\ddot{\underline{u}} = -w^2 \cos(wt+\varphi)\,\underline{w} = -w^2 \underline{u}$$

$-w^2 G_0 \underline{u} = -H_0 \underline{u}$ $\quad (w^2 G_0 - H_0)\underline{u} = 0 \to (w^2 G_0 - H_0)\underline{w} \cos(wt+\varphi) = 0$

$\Rightarrow \forall t \to (w^2 G_0 - H_0)\underline{w} = 0$

$\det(\lambda G_0 - H_0) = 0 \quad \lambda = w^2 \rightsquigarrow \underline{w}$ t.c. $(\lambda G_0 - H_0)\cdot \underline{w} = 0$

w : pulsazione propria $\quad \underline{w}$: modo normale corrispondente a w

Teo G_0 è simmetrica e definita positiva
H_0 è simmetrica

1) $\exists$ una base di $\mathbb{R}^d$ formata da autovettori di H_0 rispetto a G_0
2) Autovettori corrispondenti ad autovalori diversi sono ortogonali a G_0
3) Se H_0 è definita positiva $\Rightarrow$ λ_i sono tutti positivi

$(q_1, \dots, q_d) \rightsquigarrow (\tilde{q}_1, \dots, \tilde{q}_d)$

$$\begin{cases} q_j = q_j(\tilde{q}_1, \dots, \tilde{q}_d) \to \dot{q}_j = \dot{q}_j(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}}) \\ j = 1, \dots, d \end{cases} \qquad J_{kj} = \frac{\partial q_k}{\partial \tilde{q}_j} \quad J \text{ t.c. } \det J \neq 0$$

$$\begin{cases} \tilde{q}_j(q_1, \dots, q_d) \\ j = 1, \dots, d \end{cases}$$

$$\mathcal{L}(q_1, \dots, q_d, \dot{q}_1, \dots, \dot{q}_d) \to \tilde{\mathcal{L}}(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}}) = \mathcal{L}(\underline{q}(\tilde{\underline{q}}), \dot{q}(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}}))$$

PROP $q(t)$ è soluzione E-L per $\mathcal{L}(\underline{q}, \dot{\underline{q}}) \iff \tilde{q}(t)$ è soluzione E-L per $\tilde{\mathcal{L}}(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}})$

Dim $(\Rightarrow)$ $q(t)$ è soluzione E-L per $\mathcal{L}$

per lemma 1

$$\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\tilde{q}}_j} = \frac{d}{dt}\left(\sum_{i=1}^{d} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{\tilde{q}}_j}\right) = \frac{d}{dt}\left(\sum_{i=1}^{d} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \tilde{q}_j}\right) =$$

$$= \sum_{i=1}^{d}\left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \tilde{q}_j} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\frac{\partial q_i}{\partial \tilde{q}_j}\right] = \sum_{i=1}^{d}\left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \tilde{q}_j} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \tilde{q}_j}\right]$$

per lemma 2

uso ipotesi
$$\Rightarrow \quad = \sum_{i=1}^{d} \left(\frac{\partial \mathcal{L}}{\partial q_i} \frac{\partial q_i}{\partial \tilde{q}_j} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \dot{\tilde{q}}_j} \right) = \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{q}_j}$$

- Data una certa $\mathcal{L}(q,\dot{q},t)$, $\exists\ \mathcal{L}'(q,\dot{q},t)$ t.c. $q(t)$ è sol. per $\mathcal{L}$ e $\mathcal{L}'$

se $\mathcal{L}' = \alpha\mathcal{L} + \frac{d}{dt}F(q,t)$

PROP $\forall$ scelta di $F(q,t)$ e $\alpha \neq 0 \in \mathbb{R}$ allora

$\mathcal{L}(q,\dot{q},t)$ e $\mathcal{L}'(q,\dot{q},t) = \alpha\mathcal{L}(q,\dot{q},t) + \frac{d}{dt}F$

Conducono alla stessa soluzione per E-L

Dim $\quad L_0 = \frac{dF}{dt} = \dot{F} \qquad \mathcal{L}' = \alpha\mathcal{L} + L_0$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \alpha \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} + \frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j}$$

lemma 1, lemma 2

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j} = \frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}_j} \stackrel{\text{lemma 1}}{=} \frac{d}{dt}\frac{\partial F}{\partial q_j} \stackrel{\text{lemma 2}}{=} \frac{\partial \dot{F}}{\partial q_j}$$

$$= \alpha \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} + \frac{\partial L_0}{\partial q_j}$$

($\Rightarrow$) eq. E-L per $\mathcal{L}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \alpha \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial L_0}{\partial q_j}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \frac{\partial}{\partial q_j}(\alpha\mathcal{L} + L_0) = \frac{\partial}{\partial q_j}\mathcal{L}'$$

($\Leftarrow$) Eq. E-L per $\mathcal{L}'$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}'}{\partial q_j} = \frac{\partial}{\partial q_j}[\alpha\mathcal{L} + L_0] = \alpha\frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial L_0}{\partial q_j} = \alpha\frac{\partial \mathcal{L}}{\partial q_j} + \frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j}$$

CORPO RIGIDO

DEF $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^3$ si dicono **RIGIDAMENTE COLLEGATI** se

$\|\underline{x}_1 - \underline{x}_2\| = c \quad \to c$ costante nel tempo

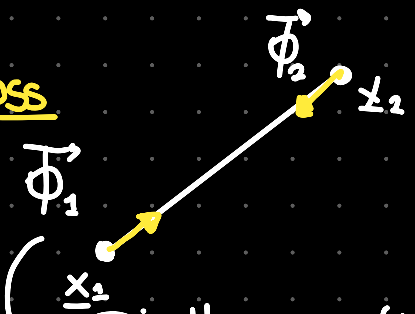
LEMMA IL VINCOLO DI RIGIDITA' è LISCIO

Dim $\frac{d}{dt}\|\underline{x}_1 - \underline{x}_2\|^2 = 0$

$$\frac{d}{dt}\left[(\underline{x}_1 - \underline{x}_2)\cdot(\underline{x}_1 - \underline{x}_2)\right] = 2(\dot{\underline{x}}_1 - \dot{\underline{x}}_2)(\underline{x}_1 - \underline{x}_2) = 0$$

Allora $(\dot{\underline{x}}_1 - \dot{\underline{x}}_2)$ è ortogonale a $(\underline{x}_1 - \underline{x}_2)$

OSS



Diretto come $(\underline{x}_2 - \underline{x}_1)$

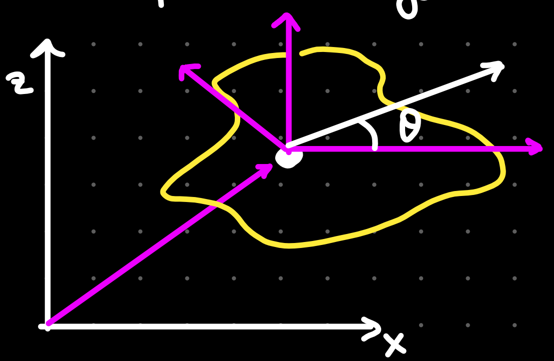
$$\underline{\Phi}_1 = \alpha(\underline{x}_1 - \underline{x}_2)$$

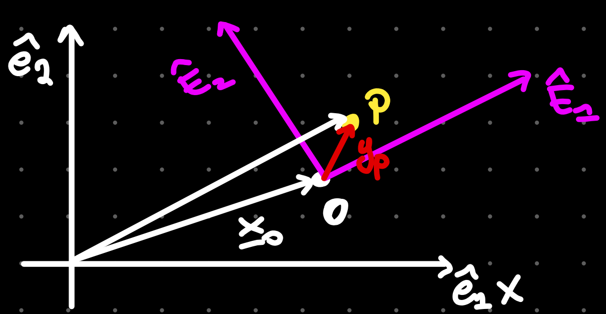
$$\underline{\Phi}_2 = -\alpha(\underline{x}_1 - \underline{x}_2)$$

$$\Pi_{RV}^{TOT} = \underline{\Phi}_1 \dot{\underline{x}}_1 + \underline{\Phi}_2 \dot{\underline{x}}_2 = \alpha(\underline{x}_1 - \underline{x}_2) - \alpha(\underline{x}_1 - \underline{x}_2)\cdot\dot{\underline{x}}_2 = \alpha(\underline{x}_1 - \underline{x}_2)(\dot{\underline{x}}_1 - \dot{\underline{x}}_2) = 0$$

Nel piano → 3 gdl : ORIGINE + SR solidale con corpo rigido
↳ 2 gdl ↳ 1 gdl

Nello spazio → 6 gdl





$$\underline{x}_p = \underline{x}_0 + \underline{y}_p$$

$$\dot{\underline{x}}_p = \dot{\underline{x}}_0 + \dot{\underline{y}}_p$$

$$\underline{y} = \underline{y}_p = \underline{y}_1\hat{e}_1 + \underline{y}_2\hat{e}_2 = Y_1\hat{E}_1 + Y_2\hat{E}_2$$

MATRICE ROTAZIONE $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad RR^T = 1 \rightarrow R^{-1} = R^T$

$$y_i = \sum_{k=1}^{2} R_{ik} Y_k$$

$$\hat{E}_i = \sum_{k=1}^{2} R_{ik}\, \hat{e}_k \qquad \hat{e}_i = \sum_{k=1}^{2} R_{ik}\, \hat{E}_k$$

$$\dot{\underline{y}} = \dot{y}_1 \hat{e}_1 + \dot{y}_2 \hat{e}_2 = \frac{d}{dt}\left(\sum_{k=1}^{2} R_{1k} Y_k\right)\hat{e}_1 + \frac{d}{dt}\left(\sum_{k=1}^{2} R_{2k} Y_k\right)\hat{e}_2$$

$$\dot{\underline{y}} = \sum_{i=1}^{2}\left(\sum_{k=1}^{2} \dot{R}_{ik} Y_k \hat{e}_i\right) = \sum_{i=1}^{2}\left[\sum_{k=1}^{2} \dot{R}_{ik} Y_k + \sum_{j=1}^{2} R_{ij}\hat{E}_j\right] = \sum_{k,j=1}^{2}\left(\sum_{i=1}^{2}\left(\dot{R}_{ik} R_{ij}\right)\right) Y_k \hat{E}_j$$

$$R_{ij} = (R^T)_{ji} \quad \Rightarrow \quad \sum_{i=1}^{2} \left(R^T_{ji}\, \dot{R}_{ik}\right) = (R^T \dot{R})_{jk}$$

$$\rightarrow \quad \dot{\underline{y}} = \sum_{k,j=1}^{2} (R^T\dot{R})_{jk}\, Y_k \hat{E}_j$$

$$R = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \qquad \dot{R} = \begin{pmatrix} -\sin\theta\,\dot{\theta} & -\cos\theta\,\dot{\theta} \\ \cos\theta\,\dot{\theta} & -\sin\theta\,\dot{\theta} \end{pmatrix}$$

$$R^T\dot{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\sin\theta\,\dot{\theta} & -\cos\theta\,\dot{\theta} \\ \cos\theta\,\dot{\theta} & -\sin\theta\,\dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix} = A \rightsquigarrow \text{ANTISIMMETRICA}$$

$$A + A^T = 0$$

$$\dot{\underline{y}} = \sum_{k,j=1}^{2} (R^T\dot{R})_{jk}\, Y_k \hat{E}_j = -\dot{\theta} Y_2 \hat{E}_1 + \dot{\theta} Y_1 \hat{E}_2$$

$$\| \dot{\underline{y}}^2 \| = \dot{\theta}^2 (Y_2^2 + Y_1^2) = \dot{\theta}^2 (y_1^2 + y_2^2) = \dot{\theta}\, \| \underline{y} \|^2$$

$$\| \dot{\underline{y}}_p^2 \| = \dot{\theta}\, \| \underline{y}_p \|^2$$

$\hat{E}_3 \rightarrow$ perpendicolare al piano del moto $\qquad \underline{y}_p = Y_1 \hat{E}_1 + Y_2 \hat{E}_2 + Y_3 \hat{E}_3$

$$\underline{\Omega} = \dot{\theta}\hat{E}_3$$

$$\underset{\substack{\uparrow \\ \text{VELOCITA' ANGOLARE}}}{\underline{\Omega}} \times \underline{y}_p = \det\begin{pmatrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ 0 & 0 & \dot{\theta} \\ Y_1 & Y_2 & Y_3 \end{pmatrix} = \hat{E}_1(-\dot{\theta} Y_2) - \hat{E}_2(-\dot{\theta} Y_1) + \hat{E}_3(0) = \dot{\underline{y}}_p$$

$$\dot{\underline{x}}_p = \dot{\underline{x}}_0 + \dot{\underline{y}}_p$$

$$K = \frac{1}{2}\sum_{p=1}^{m} m_p \|\dot{\underline{x}}_p\|^2 = \frac{1}{2}\sum_{p=1}^{m} m_p (\dot{\underline{x}}_0 + \dot{\underline{y}}_p)\cdot(\dot{\underline{x}}_0 + \dot{\underline{y}}_p)$$

$$K = \frac{1}{2}\sum_{p=1}^{N} m_p \left(\|\dot{\underline{x}}_0\|^2 + 2\dot{\underline{x}}_0 \dot{\underline{y}}_p + \|\dot{\underline{y}}_p\|^2\right)$$

$$K = \frac{1}{2}\underbrace{\left(\sum_{p=1}^{N} m_p\right)}_{\to M_{TOT}} \|\dot{\underline{x}}_0\|^2 + \frac{1}{2}\sum_{p=1}^{N} m_p \|\dot{\underline{y}}_p\|^2 + \frac{1}{2}\left(\dot{\underline{x}}_0 \cdot \sum_{p=1}^{N} m_p \dot{\underline{y}}_p\right)$$

→ SCELGO ORIGINE O DEL sistema solidale al corpo rigido come il suo centro massa

ALLORA
$$\sum_{p=1}^{N} m_p \dot{\underline{y}}_p = 0$$

$$\to K = \frac{1}{2} M_{TOT} v_{cm}^2 + \frac{1}{2}\sum_{p=1}^{N} m_p \dot{\theta}^2 \|\underline{y}_p\|^2 \to \frac{1}{2}\dot{\theta}^2 \underbrace{\sum_{p=1}^{N} m_p \|\underline{y}_p\|^2}_{I_{z,cm}}$$

dipende solo da geometria del corpo

$$I_{z,cm} = \sum_{p=1}^{N} m_p \underbrace{\|\underline{y}_p\|^2}_{\to d^2(p,cm)}$$

Nel caso di distribuzione continua massa $\rightsquigarrow \int_{\Omega} dx\,dy\, \rho(x,y)(x^2+y^2) = I_{z,cm}$