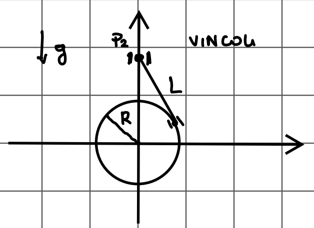
LAGRANGE



VINCOLI

$P_1(x_1, y_1)$ m_1, $P_2(x_2, y_2)$ m_2

1) $x_1^2 + z_1^2 - R^2 = 0$

2) $x_2 = 0$

3) $(x_1 - x_2)^2 + (z_1 - z_2)^2 - L^2 = 0$

Eq Newton

$m_1 \underline{a_1} = \underline{F_1}^{TOT}$, $m_2 \underline{a_2} = \underline{F_2}^{TOT}$

VINCOLI N punti materiali in $\mathbb{R}^3$

$\underline{X} = (\overbrace{x_1, x_2, x_3}^{x,y,z}, \dots, x_{3N})$

DEF. VINCOLO OLONOMO (non dipende da velocità)

- $f(\underline{x}, t) = 0 \rightarrow$ BILATERO
- $f(\underline{x}, t) \geq 0 \rightarrow$ UNILATERO

Se vincolo non dipende da tempo $\Rightarrow$ FISSO, altrimenti MOBILE

$$\begin{cases} f_1(\underline{x}) = 0 \\ f_2(\underline{x}) = 0 \\ \vdots \\ f_r(\underline{x}) = 0 \end{cases} \rightarrow \text{ogni vincolo elimina un grado di libertà} \Rightarrow 3N - r \text{ gradi di libertà}$$

$3N - r = m \rightarrow (q_1, \dots, q_m)$
$$\begin{cases} x_1 = \varphi_1(q_1, \dots, q_m) \\ x_2 = \varphi_2(q_1, \dots, q_m) \\ \vdots \\ x_{3N} = \varphi_{3N}(q_1, \dots, q_m) \end{cases}$$

$\downarrow$ COORDINATE LIBERE LAGRANGIANE

(i) descrizione locale

(ii) Ben definita $\Rightarrow$ rango $\begin{pmatrix} \frac{\partial \varphi_1}{\partial q_1} & \frac{\partial \varphi_1}{\partial q_m} \\ \vdots & \\ \frac{\partial \varphi_{3N}}{\partial q_1} & \cdot \frac{\partial \varphi_{3N}}{\partial q_m} \end{pmatrix} = m$

$P = \underline{x}(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_m)$

$$\lim_{h \to 0} \frac{\underline{x}(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_j + h, \dots, \tilde{q}_m) - \underline{x}(\tilde{q}_1, \dots, \tilde{q}_m)}{h} = \frac{\partial \underline{x}(\tilde{q})}{\partial q_j}$$ vettore tangente alla superficie

$(q_1(t), q_2(t), \dots, q_m(t)) \rightarrow$ Curva in $U \subseteq \mathbb{R}^m$ $t \in I$

$\downarrow \varphi_1, \dots, \varphi_{3N}$

$x(t)$ sulla varietà

$t_0 \in I \rightarrow q_1(t_0), \dots, q_m(t_0)$

$\varphi \hookrightarrow \underline{x}(t_0)$

$$\underline{v}(t_0) = \lim_{h \to 0} \frac{\underline{x}(t_0 + h) - \underline{x}(t_0)}{h} = \frac{d}{dx} x(t) \Big|_{t = t_0} = \left(\frac{d}{dt} x_1(q_1(t), \dots, q_m(t)), \frac{dx_2}{dt}(), \dots, \frac{dx_{3N}}{dt}() \right)$$

$\underline{x}(t) = \underline{x}(q_1(t), \dots, q_m(t))$

$$\Rightarrow \frac{d}{dt} x_1(q_1(t), \dots, q_m(t)) = \frac{\partial x_1}{\partial q_1} \dot{q}_1 + \frac{\partial x_1}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x_1}{\partial q_m} \dot{q}_m$$

$$\rightarrow = \sum_{j=1}^{m} \frac{\partial x_1}{\partial q_j} \cdot \dot{q}_j$$

$$\underline{v}(t_0) = \sum_{j=1}^{m} \frac{\partial \underline{x}(q_1(t), \dots, q_m(t))}{\partial q_j} \dot{q}_j \Big|_{t = t_0}$$

$\dot{q}_j \rightarrow$ VELOCITÀ GENERALIZZATA

Se la potenza delle reazioni vincolari è lungo spostamenti compatibili con il vincolo $\rightarrow$ VINCOLO LISCIO

$K \rightarrow$ energia cinetica $\Rightarrow \frac{dK}{dt} = \Pi_{TOT}$
$\quad \hookrightarrow$ potenza

- $(x(q), y(q), z(q)) \rightarrow$ velocità compatibili con vincolo $\left(\underbrace{\frac{dx}{dq} \cdot \dot{q}}_{v_x}, \underbrace{\frac{dy}{dq} \dot{q}}_{v_y}, \underbrace{\frac{dz}{dq} \dot{q}}_{v_z} \right)$

- $\vec{F}_{\text{ATTIVE}}$ CONSERVATIVE $\Rightarrow$ $\vec{F}_{\text{ATTIVE}} = -\vec{\nabla} U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right)$
- VINCOLO FISSO e LISCIO $\Rightarrow$ $\Pi^{TOT} = \Pi^{\text{ATTIVE}} \Rightarrow \vec{F}_{\text{ATTIVE}} \vec{v} = (-\vec{\nabla} U)\vec{v} = -\left(\frac{\partial U}{\partial x} v_x + \frac{\partial U}{\partial y} \cdot v_y + \frac{\partial U}{\partial z} \cdot v_z\right) = -\left(\frac{\partial U}{\partial x}\frac{dx}{dq}\dot{q} + \frac{\partial U}{\partial y}\frac{dy}{dq} q + \frac{\partial U}{\partial z}\frac{dz}{dq} \cdot \dot{q}\right)$

$$\Pi^{TOT} = -\dot{q}\left[\frac{\partial U}{\partial x}\frac{dx}{dq} + \frac{\partial U}{\partial y}\frac{dy}{dq} + \frac{\partial U}{\partial z}\frac{dz}{dq}\right] = -\frac{dU}{dq} \cdot q$$

$\frac{dK}{dt} = -\frac{dU}{dq} \cdot \dot{q} \Rightarrow \frac{dU}{dq}\dot{q} = \frac{dU}{dt} \rightarrow \frac{dK}{dt} + \frac{dU}{dt} = 0 \Rightarrow \frac{d}{dt}(U+K) = 0$ $\rightarrow$ conservazione energia per sistemi conservativi in presenza di vincoli lisci

EQUAZIONE EULERO - LAGRANGE

$q \mapsto x(q), y(q), z(q)$. posizioni compatibili con il vincolo

$\rightsquigarrow$ velocità compatibili con il vincolo $\underline{v} = (x'(q)\, q, y'(q)\, q, z'(q)\, \dot{q})$ $\quad \frac{d}{dq}x = x'$

$$K(q, q) = \frac{1}{2} m \underbrace{(x'(q)^2 + y'(q)^2 + z'(q)^2)}_{G(q)} \dot{q}^2$$

$G(q)$ sempre positivo $\Rightarrow K = 0 \Leftrightarrow q = 0$

$\frac{d}{dt}K(q(t), \dot{q}(t)) = \frac{\partial K}{\partial q}\dot{q} + \frac{\partial K}{\partial \dot{q}} \cdot \frac{d}{dt}\dot{q} = \frac{\partial K}{\partial q} q + \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}} \cdot q\right) - \left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right)\right] \cdot q = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}} \cdot \dot{q}\right) + q\left(\frac{\partial K}{\partial q} - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}}\right)$ $\quad K = \frac{1}{2}G(q)\dot{q}^2 \Rightarrow \dot{q}\frac{\partial K}{\partial q} = G(q) \cdot \dot{q}^2 = 2K$

$\frac{d}{dt}K = \frac{d}{dt}2K + \dot{q}\left(\frac{\partial K}{\partial q} - \frac{d}{dt}\frac{\partial K}{\partial q}\right) \Rightarrow \frac{d}{dt}K = \dot{q}\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} - \frac{\partial K}{\partial q}\right)$

$\frac{dK}{dt} = \Pi^{\text{ATTIVE}} = \underline{f}^{\text{ATTIVE}}\underline{v}$

vincolo liscio

$$\frac{d}{dt}\frac{\partial K}{\partial q} - \frac{\partial K}{\partial q} = \underline{f}^{\text{ATTIVE}}(x', y', z')$$

se le $\underline{f}^{\text{ATTIVE}}$ sono conservative $\Rightarrow \exists U(x, y, z) \rightsquigarrow U(x(q), y(q), z(q))$

$\Pi^{\text{ATTIVE}} = -\frac{dU}{dq} \cdot \dot{q} \Rightarrow \frac{d}{dt}K = \Pi^{\text{ATTIVE}} = q\left[\frac{d}{dt}\frac{\partial K}{\partial q} - \frac{\partial K}{\partial q}\right] = -\frac{dU}{dq} q \rightarrow \frac{d}{dt}\frac{\partial K}{\partial q} = \frac{\partial}{\partial q}(K - U)$ $\quad \frac{\partial}{\partial \dot{q}}U = 0 \rightsquigarrow \frac{\partial}{\partial \dot{q}}(K - U) = \frac{\partial}{\partial q}K$

$\mathcal{L}(q, q) = K(q, q) - U(q) \Rightarrow$ EQUAZIONE EULERO-LAGRANGE $\rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$

CONSERVAZIONE ENERGIA

Supponiamo $\mathcal{L}(q, q, \cancel{t}) \Rightarrow \frac{\partial \mathcal{L}}{\partial t} = 0$

$$\frac{d}{dt}\mathcal{L}(q(t), q(t)) = \frac{\partial \mathcal{L}}{\partial q} q + \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{d}{dt}q = \frac{\partial \mathcal{L}}{\partial q} \cdot q + \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial q}\dot{q}\right] - q\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q} = q\underbrace{\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q}\right]}_{A} + \underbrace{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial q}\dot{q}\right]}_{B}$$

Se $q(t)$ è soluzione $\Rightarrow$ A $= 0 \Rightarrow \frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial q}\dot{q}\right]$ lungo il moto

$\frac{d}{dt}\left[\mathcal{L} - \frac{\partial \mathcal{L}}{\partial q}\dot{q}\right] = 0 \Rightarrow q\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$ è costante del moto

$$q\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = 2K - (K - U) = K + U = K(q, q) + U(q)$$

FORMA NORMALE EQUAZIONE EULERO-LAGRANGE (1 GRADO DI LIBERTA')

$$\mathcal{L}(q, q) = \frac{1}{2}G(q)\dot{q}^2 - U(q)$$

EQ E-L $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$

- $\frac{\partial \mathcal{L}}{\partial q} = \frac{1}{2}G'(q)\, q^2 - U'(q)$
- $\frac{\partial \mathcal{L}}{\partial \dot{q}} = G(q) \cdot q$
- $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = G'(q)\dot{q}^2 + G(q)q$

$\Rightarrow$ EQ E-L $\rightarrow G'(q)\dot{q}^2 + G(q)q = \frac{1}{2}G'(q)\dot{q}^2 - U'(q) \Rightarrow G(q)\dot{q} = -\frac{1}{2}G'(q)q^2 - U(q) \Rightarrow q = -\frac{1}{2}\frac{G'(q)}{G(q)}q^2 - \frac{U'(q)}{G(q)}$ $\quad G(q) > 0$

$$y = \dot{q} \Rightarrow \begin{cases} q = y \\ y = -\frac{1}{2}\frac{G'(q)}{G(q)}y^2 - \frac{U'(q)}{G(q)} \end{cases}$$

PUNTI STAZIONARI $\rightarrow (\bar{q}, 0)$ t.c. $U'(q) = 0$

- $E(q, y) = K(q, y) + U(q) = \frac{1}{2}G(q)y^2 + U(q)$ costante del moto

STABILITA' PUNTO EQUILIBRIO $(\bar{q}, 0)$

Teo Si consideri un sistema lagrangiano (1 GRADO LIBERTA') e sia $(\bar{q}, 0)$ P.TO di equilibrio, $\bar{q}$ t c $U'(\bar{q}) = 0$, e sia $U''(q) > 0 \Rightarrow (\bar{q}, 0)$ è STABILE

Dim II LYAPUNOV con $E(q, y)$

(ii) è vera perchè E è costante del moto

(i) $(\bar{q}, 0)$ è di minimo? $\quad \vec{\nabla}E = \left(\frac{\partial E}{\partial q}, \frac{\partial E}{\partial y}\right) = \left(\frac{1}{2}G'(q)y^2 + U'(q), G(q)y\right)$ $\quad \vec{\nabla}E\Big|_{(\bar{q}, 0)} = (0, 0) \Rightarrow (\bar{q}, 0)$ è punto stazionario per E

$$H_E(q, y) = \begin{pmatrix} \frac{\partial^2 E}{\partial q^2} & \frac{\partial^2 E}{\partial q \partial y} \\ \frac{\partial^2 E}{\partial y \partial q} & \frac{\partial^2 E}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}G''(q)y^2 + U''(q) & G'(q)y' \\ G'(q)y' & G(q) \end{pmatrix}\Bigg|_{(\bar{q}, 0)} = \begin{pmatrix} U''(\bar{q}) & 0 \\ 0 & G(\bar{q}) \end{pmatrix} \Rightarrow (\bar{q}, 0) \text{ è un punto di minimo}$$

$\hookrightarrow$ H_E definita positiva

PICCOLE OSCILLAZIONI

$$\begin{cases} q = \bar{q} + \varepsilon u(t) \\ y = 0 + \varepsilon v(t) = \varepsilon u(t) \end{cases} \qquad \dot{q} = \varepsilon \dot{u}$$

$$\tilde{\mathcal{L}}(u, \tilde{u}) = \mathcal{L}(\bar{q} + \varepsilon u, \varepsilon \dot{u}) = \frac{1}{2} G(\bar{q} + \varepsilon u)\, \varepsilon^2 u^2 + U(\bar{q} + \varepsilon u)$$

ESPANDO ATTORNO A $\varepsilon = 0$ FINO AL SECONDO ORDINE $\Rightarrow \tilde{\mathcal{L}}(u, \tilde{u}) = \frac{1}{2} G(\bar{q}) \varepsilon^2 u^2 - \left[\cancel{U(\bar{q})} + \cancel{U'(\bar{q}) u \varepsilon} + \frac{1}{2} U''(\bar{q}) \varepsilon^2 u^2\right] + o(\varepsilon^2) \simeq \varepsilon^2 \left[\frac{1}{2} \underbrace{G(\bar{q})}_{m} u^2 - \frac{1}{2} \underbrace{U''(\bar{q})}_{K} \varepsilon^2 u^2\right]$

$$w = \sqrt{\frac{K}{m}} = \sqrt{\frac{U''(\bar{q})}{G(\bar{q})}}$$

VINCOLI DIPENDENTI DAL TEMPO

es pto materiale su circonferenza $r(t) = r_0 e^t$

$$\begin{cases} r(\theta, t) = r_0 e^t \cos\theta \\ y(\theta, t) = r_0 e^t \sin\theta \end{cases}$$

$$\frac{dr}{dt} = r_0 e^t (-\sin\theta)\, \theta + r_0 e^t \cos\theta$$

$$\begin{cases} x(q,t) \\ y(q,t) \\ z(q,t) \end{cases} \rightsquigarrow \begin{cases} \frac{dx}{dt} = \frac{\partial x}{\partial q} q + \frac{\partial x}{\partial t} \\ \frac{dy}{dt} = \frac{\partial y}{\partial q} \cdot q + \frac{\partial y}{\partial t} \\ \frac{dz}{dt} = \frac{\partial z}{\partial q} q + \frac{\partial z}{\partial t} \end{cases}$$

VELOCITA' VIRTUALI

$(v_x, v_y, v_z) = v^{VIRT}$ tempo come congelato

$$\begin{cases} v_x = \frac{dx}{dt} - \frac{\partial x}{\partial t} = \frac{\partial x}{\partial q} q \\ v_y = \frac{dy}{dt} - \frac{\partial y}{\partial t} = \frac{\partial y}{\partial q} \dot{q} \\ v_z = \frac{dz}{dt} - \frac{\partial z}{\partial t} = \frac{\partial z}{\partial q} \cdot q \end{cases}$$

VINCOLI LISCI (dipendente da t) → Potenza reazioni vincolari nulla lungo le velocità virtuali $\Rightarrow \phi^{VINC}\, v^{VIRT} = 0$

$$m a = F^{ATT} + \phi^{VINC} \Rightarrow m a\, v^{VIRT} = F^{ATT} v^{VIRT} + \cancel{\phi^{VINC} v^{VIRT}}$$

NEL PIANO → $m\left(\ddot{x} \frac{\partial x}{\partial q} \cancel{\dot{q}}\right) + m\left(\ddot{y} \frac{\partial y}{\partial q} \cancel{\dot{q}}\right) = f_x^{ATT} \frac{\partial x}{\partial q} \cdot \cancel{\dot{q}} + f_y^{ATT} \frac{\partial y}{\partial q} \cancel{q} \quad \forall q$

Se F^{ATT} CONSERVATIVE $\Rightarrow (f_x, f_y) = \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}\right) \longrightarrow m\left[\ddot{x} \frac{\partial x}{\partial q} + \ddot{y} \frac{\partial y}{\partial q}\right] = -\left[\frac{\partial U}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial q}\right]$ (A) (B)

(B) $\rightarrow \frac{\partial U}{\partial q}(x(q,t), y(q,t))$

(A) $\ddot{x} \frac{\partial x}{\partial q} = \frac{d}{dt} \dot{x} \frac{\partial x}{\partial q} = \frac{d}{dt}\left(\dot{x} \frac{\partial x}{\partial q}\right) - \dot{x} \frac{d}{dt} \frac{\partial x}{\partial q} \Rightarrow m\left[\frac{d}{dt}\left(\dot{x} \frac{\partial \dot{x}}{\partial \dot{q}} + \dot{y} \frac{\partial \dot{y}}{\partial \dot{q}}\right) - \left(\dot{x} \frac{d}{dt} \frac{\partial x}{\partial q} + \dot{y} \frac{d}{dt} \frac{\partial y}{\partial q}\right)\right] = -\frac{\partial U}{\partial q}$

LEMMA 1 → $\frac{\partial x}{\partial q} = \frac{\partial \dot{x}}{\partial \dot{q}}$

dim $\frac{\partial}{\partial \dot{q}} \frac{d}{dt} x = \frac{\partial}{\partial \dot{q}}\left[\frac{\partial x}{\partial q} \dot{q} + \frac{\partial x}{\partial t}\right] = \frac{\partial x}{\partial q}$

LEMMA 2 → $\frac{d}{dt} \frac{\partial x}{\partial q} = \frac{\partial}{\partial q} \frac{d}{dt} x = \frac{\partial}{\partial q} \dot{x}$

Dim $\frac{d}{dt}\left[\frac{\partial x}{\partial q}\right] = \frac{\partial}{\partial q}\left(\left[\frac{\partial x}{\partial q}\right] q\right) + \frac{\partial}{\partial t}\left[\frac{\partial x}{\partial q}\right] = \frac{\partial}{\partial q}\left(\left[\frac{\partial x}{\partial q}\right] \dot{q}\right) + \frac{\partial}{\partial q} \frac{\partial x}{\partial t} = \frac{\partial}{\partial q}\left[\frac{\partial x}{\partial q} q + \frac{\partial x}{\partial t}\right] = \frac{\partial}{\partial q} \frac{d}{dt} x$

$$\Rightarrow m\left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}} \underbrace{\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2)\right]}_{K}\right] - \left(\frac{\partial}{\partial q}\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2)\right]\right] = -\frac{\partial U}{\partial q}$$

$$\rightarrow \frac{d}{dt} \frac{\partial}{\partial \dot{q}} K - \frac{\partial K}{\partial q} = -\frac{\partial U}{\partial q} \Rightarrow \boxed{\frac{d}{dt} \frac{\partial}{\partial \dot{q}}(K - U) - \frac{\partial}{\partial q}(K - U) = 0}$$

d GRADI DI LIBERTA'

$\underline{x} = (x_1, x_2, x_3, \ldots, x_{3N})$

$$\begin{cases} x_1(q_1, q_2, \ldots, q_d, t) \\ x_2(q_1, q_2, \ldots, q_d, t) \\ \vdots \\ x_{3N}(q_1, q_2, \ldots, q_d, t) \end{cases}$$

$\underline{\dot{x}} = (\dot{x}_1, \dot{x}_2, \dot{x}_3, \ldots, \dot{x}_{3N}) \rightarrow \begin{cases} \dot{x}_1 = \sum_{\delta=1}^{d} \frac{\partial x_1}{\partial q_\delta} \dot{q}_\delta + \frac{\partial x_1}{\partial t} \\ \vdots \\ \dot{x}_{3N} = \sum_{\delta=1}^{d} \frac{\partial x_{3N}}{\partial q_\delta} \dot{q}_\delta + \frac{\partial x_{3N}}{\partial t} \end{cases}$ $\rightsquigarrow$ vettore velocità virtuali $v^{VIRT} = (v_1, \ldots, v_{3N})$

$$v_i = \frac{dx_i}{dt} - \frac{\partial x_i}{\partial t} = \sum_{\delta=1}^{d} \frac{\partial x_i}{\partial q_\delta} \cdot \dot{q}_\delta$$

PRINCIPIO DI D'ALAMBERT (VINCOLO liscio) d gradi di libertà

$\phi^{VINC} = (\phi_1, \ldots, \phi_{3N}) \Rightarrow \phi^{VINC}\, v^{VIRT} = 0 = \sum_{K=1}^{m} \phi^{(K)}\, v^{(K)}_{VIRT}$ → vincolo liscio $(\phi_1, \phi_2, \phi_3) = \phi^{(1)}$

$\to \sum_{k=1}^N v_{virt}^{(k)}\left(m^{(k)} a^{(k)}\right) = F_{TOT}^{(k)} \cdot v_{virt}^{(k)} = \sum_{k=1}^N \left(F_{ATT}^{(k)} + \phi^{(k)}\right) \cdot v_{virt}^{(k)}$

vincolo liscio $\Rightarrow \sum_{k=1}^N \phi^{(k)} v^{(k)} = 0 \Rightarrow \sum_{k=1}^N m^{(k)} a^{(k)} v_{virt}^{(k)} = \sum_{k=1}^N F^{(k)} \cdot v_{virt}^{(k)}$

$$\sum_{k=1}^N m^{(k)} \ddot{x}^{(k)} \left(\sum_{\gamma=1}^d \frac{\partial x^{(k)}}{\partial q_\gamma} \cdot \dot q_\gamma\right) = \sum_{\gamma=1}^d \dot q_\gamma \left(\sum_{k=1}^N m^{(k)} \ddot x^{(k)} \frac{\partial x^{(k)}}{\partial q_\gamma}\right) = \sum_{\gamma=1}^d \dot q_\gamma \left(\sum_{k=1}^N F^{(k)} \frac{\partial x^{(k)}}{\partial q_\gamma}\right)$$ vera $\forall$ scelta $\dot q(\dot q_1, .., \dot q_d)$

scelgo $\dot q = (0, .., \dot q_r, 0, .., 0)$

$\rightsquigarrow \cancel{\dot q_r}\left(\sum_{k=1}^N m^{(k)} \ddot x^{(k)} \frac{\partial x^{(k)}}{\partial q_r}\right) = \cancel{\dot q_r}\left(\sum_{k=1}^N F^{(k)}_{ATTIVA} \frac{\partial x^{(k)}}{\partial q_r}\right)$ $r = 1, ..., d$

$$\underbrace{\sum_{i=1}^{3N} m_i \ddot x_i \frac{\partial x_i}{\partial q_r}}_{A} = \underbrace{\sum_{i=1}^{3N} F_i^{ATTIVE} \frac{\partial x_i}{\partial q_r}}_{B}$$

A $\to \sum_{i=1}^{3N} m_i \left(\frac{d}{dt}\dot x_i\right)\frac{\partial x_i}{\partial q_r} = \sum_{i=1}^{3N} m_i\left(\frac{d}{dt}\left(\dot x_i \frac{\partial x_i}{\partial q}\right) - \dot x_i \frac{d}{dt}\frac{\partial x_i}{\partial q_r}\right) = \sum_{i=1}^{3N} m_i \left(\frac{d}{dt}\left(\dot x_i \frac{\partial \dot x_i}{\partial \dot q_r}\right) - \dot x_i \frac{\partial \dot x_i}{\partial q_r}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot q_r}\left(\frac{1}{2}\sum_{i=1}^{3N} m_i \dot x_i^2\right) - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\sum_{i=1}^{3N} m_i \dot x_i^2\right) = \frac{d}{dt}\frac{\partial}{\partial \dot q_r}K - \frac{\partial}{\partial q_r}K$

B $\to \sum_{i=1}^{3N} F_i^{AT} \frac{\partial x_i}{\partial q_r}$ se F^{ATT} CONSERVATIVE $\Rightarrow F_i^{ATT} = -\frac{\partial}{\partial x_i} U(x_1, .., x_{3N}) \to = -\sum_{i=1}^{3N} \frac{\partial U}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_r} = -\frac{\partial U}{\partial q_r}$

$\to \frac{d}{dt}\frac{\partial K}{\partial \dot q_r} - \frac{\partial}{\partial q_r} K = -\frac{\partial U}{\partial q_r} \to \frac{d}{dt}\frac{\partial}{\partial \dot q_r}(K - U) = \frac{\partial}{\partial q_r}(K - U)$

$\mathcal{L}(q_1, .., q_d, \dot q_1, .., \dot q_d) = K - U \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_r} = \frac{\partial \mathcal{L}}{\partial q_r}$ $r = 1, ..., d$ SISTEMA EQUAZIONI EULERO-LAGRANGE

STRUTTURA FORMALE ENERGIA CINETICA

$(d \geq 1)$ N pti materiali

se $d = 1 \Rightarrow K(q, \dot q) = \frac{1}{2} G(q) \dot q^2$

se $d > 1$

$\underline{q} = (q_1, ..., q_d)$ $\underline{x} = (x_1, ..., x_{3N})$

$K(\underline q, \dot{\underline q}, t) = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot x_i^2$ $\dot x_i = \frac{d}{dt} x_i(q_1, .., q_d, t) = \sum_{\gamma=1}^d \frac{\partial x_i}{\partial q_\gamma} \cdot \dot q_\gamma + \frac{\partial x_i}{\partial t}$

$K = \frac{1}{2}\sum_{i=1}^{3N} m_i \left(\sum_{\gamma=1}^d \frac{\partial x_i}{\partial q_\gamma}\dot q_\gamma + \frac{\partial x_i}{\partial t}\right)\left(\sum_{k=1}^d \frac{\partial x_i}{\partial q_k}\dot q_k + \frac{\partial x_i}{\partial t}\right) = \frac{1}{2}\left(\left[\sum_{\gamma,k=1}^d \sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_\gamma}\frac{\partial x_i}{\partial q_k} \dot q_\gamma \dot q_k\right] + G_1 + G_0\right)$

$G_1 \to$ lineare nelle $\dot q$
$G_0 \to$ non dipende da $\dot q$ $\Big] = 0 \to$ vincoli fissi

$\rightsquigarrow$ VINCOLI FISSI $\Rightarrow \underline x = \underline x(q_1, .., q_d, \cancel{t}) \Rightarrow K(\underline q, \dot{\underline q}) = \frac{1}{2}\sum_{k,\gamma=1}^d G_{\gamma k}(q_1, .., q_d)\dot q_\gamma \dot q_k$

$\to K = \frac{1}{2}(\dot q_1, .., \dot q_d)\underbrace{G(q_1, .., q_d)}_{\text{matrice } d \times d}\begin{pmatrix}\dot q_1 \\ \vdots \\ \dot q_d\end{pmatrix}$ FORMA BILINEARE

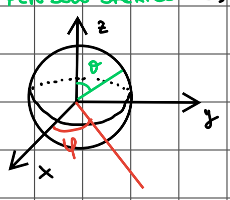
G matrice simmetrica definita positiva con entrate $G_{\gamma k}(\underline q) = \sum_{i=1}^{3N}\frac{\partial x_i}{\partial q_\gamma}\frac{\partial x_i}{\partial q_k} = G_{k\gamma}$

$\forall \dot q \neq 0 \Rightarrow \sum_{\gamma, k} G_{\gamma k}\dot q_\gamma \dot q_k > 0$

$v = \sum_{\gamma=1}^d \frac{\partial x_i}{\partial q_\gamma}\dot q_\gamma$ $v \cdot v \geq 0$ $(= 0 \Leftrightarrow v = 0)$

PENDOLO SFERICO

$\to$ pendolo vincolato a superficie sfera $\to$ 2 GRADI LIBERTA'



R : raggio $\hookrightarrow x^2 + y^2 + z^2 = R^2$

$\theta \in [0, \pi)$

$\varphi \in [0, 2\pi)$

$$\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \rightsquigarrow \begin{cases} \dot x = R\cos\theta\dot\theta\cos\varphi - R\sin\theta\sin\varphi\dot\varphi \\ \dot y = R\cos\theta\dot\theta\sin\varphi + R\sin\theta\cos\varphi\dot\varphi \\ \dot z = -R\sin\theta\dot\theta \end{cases}$$

$K = \frac{1}{2} m\left[\dot x^2 + \dot y^2 + \dot z^2\right] = \frac{1}{2} mR^2\left((\cos\theta\dot\theta\sin\varphi - \sin\theta\sin\varphi\dot\varphi)^2 + (\cos\theta\dot\theta\sin\varphi + \sin\theta\cos\varphi\dot\varphi)^2 + (\sin\theta\dot\theta)^2\right) = \frac{1}{2}mR^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) = \frac{1}{2}(\dot\theta, \dot\varphi)\begin{pmatrix} mR^2 & 0 \\ 0 & mR^2\sin^2\theta\end{pmatrix}\begin{pmatrix}\dot\theta \\ \dot\varphi\end{pmatrix}$

$U(\theta, \varphi) = mgR\cos\theta$

$\mathcal{L}(\theta, \cancel{\varphi}, \dot\theta, \dot\varphi) = \frac{1}{2}mR^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) - mgR\cos\theta$

DEF. VARIABILE CICLICA

Se $\mathcal{L}(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d)$ non dipende da $q_j \Rightarrow q_j$ è CICLICA

$\rightarrow \dfrac{\partial\mathcal{L}}{\partial q_j}=0 \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot q_j}\right)=0 \Rightarrow \dfrac{\partial\mathcal{L}}{\partial\dot q_j}$ è costante del moto

LAGRANGIANA RIDOTTA $\mathcal{L}^R(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\cancel{\dot q_j},\dots,\dot q_d,c)$

q_j ciclica $\Rightarrow$ $c=\dfrac{\partial\mathcal{L}}{\partial\dot q_j}(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d)$

$\dot q_j=f(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\dot q_d,c)$

$\rightarrow E=K+U$ sostituisco con $\dot q_j=f$ $\rightarrow E'=K|_{\dot q_j=f}+U \Rightarrow E'=K^{eff}-U^{eff}$

$\mathcal{L}^{RID}\neq\mathcal{L}|_{\dot q=f}$ $\quad\mathcal{L}^{RID}=\mathcal{L}^R(q_1,\dots,\cancel{q_j},\dots,q_d,\dot q_1,\dots,\cancel{\dot q_j},\dots,\dot q_d,c)$

FREQUENZE PROPRIE E MODI NORMALI ($d>1$ gdl)

$\rightsquigarrow$ forze conservative, vincoli lisci e fissi

$$q=(q_1,\dots,q_d)\quad \mathcal{L}(q,\dot q)=\frac{1}{2}\sum_{\alpha,\beta=1}^{d} G_{\alpha\beta}(q)\,\dot q_\alpha\dot q_\beta-U(q)$$

$\uparrow$ simmetrica e definita positiva

Eq. E-L per q_γ $\rightarrow \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot q_\gamma}=\dfrac{\partial\mathcal{L}}{\partial q_\gamma}$

$$\frac{\partial\mathcal{L}}{\partial\dot q_\gamma}=\frac{1}{2}\left[\sum_{\beta\neq\gamma}G_{\gamma\beta}(q)\dot q_\beta+\sum_{\alpha\neq\gamma}G_{\alpha\gamma}(q)\dot q_\alpha+2G_{\gamma\gamma}(q)\dot q_\gamma\right]$$

G simmetrica $\Rightarrow G_{\alpha\gamma}(q)=G_{\gamma\alpha}(q)$

$$\sum_{\beta\neq\gamma}G_{\gamma\beta}(q)\dot q_\beta=\sum_{\alpha\neq\gamma}G_{\alpha\gamma}(q)\dot q_\alpha$$

$$\Rightarrow\frac{\partial\mathcal{L}}{\partial\dot q_\gamma}=\frac{1}{2}\left[2\sum_{\alpha\neq\gamma}G_{\alpha\gamma}(q)\dot q_\alpha+2G_{\gamma\gamma}\dot q_\gamma\right]=\sum_{\alpha=1}^{d}G_{\alpha\gamma}(q)\dot q_\alpha$$

$d > 1$ gdl

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} G_{\alpha\beta}(q)\,\dot{q}_\alpha \dot{q}_\beta - U(q)$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_j} = \dfrac{\partial \mathcal{L}}{\partial q_j} \\ j = 1,\dots,d \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_\gamma} = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\,\dot{q}_\gamma = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\,\ddot{q}_\alpha + \sum_{\alpha=1}^{d} \dot{q}_\alpha \frac{d}{dt} G_{\alpha\gamma}(q)$$

$$\frac{d}{dt} G_{\alpha\gamma}(q) = \sum_{\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}\,\dot{q}_\beta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\gamma} = \sum_{\alpha=1}^{d} G_{\alpha\gamma}(q)\,\ddot{q}_\alpha + \sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial \dot{q}_\beta}\,\dot{q}_\alpha \dot{q}_\beta$$

$$\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial \dot{q}_\beta}\,\dot{q}_\alpha \dot{q}_\beta = \frac{1}{2}\left(\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}(q)\,\dot{q}_\alpha \dot{q}_\beta + \frac{\partial G_{\alpha\gamma}}{\partial q_\beta}(q)\,\dot{q}_\alpha \dot{q}_\beta\right)$$

$$\frac{\partial \mathcal{L}}{\partial q_\gamma} = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \frac{\partial G_{\alpha\beta}}{\partial q_\gamma}\,\dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma}$$

$$\Gamma_{\alpha\beta\gamma} := \frac{1}{2}\left(\frac{\partial}{\partial q_\gamma} G_{\alpha\beta} - \frac{\partial}{\partial q_\gamma} G_{\alpha\gamma} - \frac{\partial}{\partial q_\gamma} G_{\beta\gamma}\right)$$

$$\text{E-L} \to \begin{cases} \displaystyle\sum_{\alpha=1}^{d} G_{\alpha\gamma}\,\ddot{q}_\alpha = \sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma}\,\dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} \\ \gamma = 1,\dots,d \end{cases}$$

PUNTI DI EQUILIBRIO $\rightsquigarrow$ $\dot{q} = 0$ e $\displaystyle\sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma}\,\dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} = 0$

$$\Rightarrow \frac{\partial U}{\partial q_\gamma} = 0$$

q_0 t.c. $\dfrac{\partial U}{\partial q_\gamma}(q_0) = 0 \quad \forall \gamma = 1,\dots,d \quad \rightsquigarrow \quad \vec{\nabla} U(q_0) = 0$

Teo (LAGRANGE - DIRICHLET) $d \geq 1$

Sistema con forze attive conservative, vincoli lisci e fissi

Supponiamo che U abbia punti critici isolati

I MINIMI DI U (guardo Hessiana) $\rightarrow$ sono p.ti equilibrio stabile sistema

Dim Suppongo q_0 t.c. $\vec{\nabla} U(q_0) = 0$

$$H(q) = \begin{pmatrix} \frac{\partial^2 U}{\partial q_1^2} & \frac{\partial^2 U}{\partial q_2 q_1} & \cdots & \frac{\partial^2 U}{\partial q_d q_1} \\ \vdots & & & \\ \frac{\partial^2 U}{\partial q_1 q_d} & \frac{\partial^2 U}{\partial q_2 q_d} & \cdots & \frac{\partial^2 U}{\partial q_d^2} \end{pmatrix}$$

q_0 è di minimo se $H(q_0)$ è DEFINITA POSITIVA

$\underline{q}^*$ p.to di minimo per $U(\underline{q})$

$\underline{q}(t) = \underline{q}^* + \varepsilon \underline{u}(t) \qquad \dot{\underline{q}}(t) = \varepsilon \dot{\underline{u}}(t) \qquad \ddot{\underline{q}}(t) = \varepsilon \ddot{\underline{u}}(t)$

$$\hookrightarrow \begin{cases} q_1(t) = q_1^* + \varepsilon u_1(t) \\ \vdots \\ q_d(t) = q_d^* + \varepsilon u_d(t) \end{cases}$$

$$E-L \longrightarrow \begin{cases} \sum_{\alpha=1}^{d} G_{\alpha\gamma} \ddot{q}_\alpha = \sum_{\alpha,\beta=1}^{d} \Gamma_{\alpha\beta\gamma} \dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} \\ \gamma = 1, \ldots, d \end{cases}$$

$$\sum_\alpha G_{\alpha\gamma}(\underline{q}^* + \varepsilon \underline{u}) \varepsilon \ddot{u}_\alpha = \sum_{\alpha,\beta} \Gamma_{\alpha\beta\gamma}(\underline{q}^* + \varepsilon \underline{u}) \varepsilon^2 \dot{u}_\alpha \dot{u}_\beta - \frac{\partial}{\partial q_\gamma}(\underline{q}^* + \varepsilon \underline{u})$$

ESPANSIONE AL PRIMO ORDINE IN ε

$$\sum_\alpha G_{\alpha\gamma}(\underline{q}^*) \varepsilon \ddot{u}_\alpha = -\left(\cancel{\frac{\partial U}{\partial q_\gamma}(\underline{q}^*)} + \sum_{\beta=1}^{d} \left[\frac{\partial}{\partial q_\beta}\left(\frac{\partial U}{\partial q_\gamma} \right) \right] \Bigg|_{(\underline{q}^*)} \varepsilon u_\beta \right)$$

$$\begin{cases} \sum_\alpha G_{\alpha\gamma}(\underline{q}^*) \cancel{\varepsilon} \ddot{u}_\alpha = - \sum_{\beta=1}^{d} \frac{\partial^2 U}{\partial q_\beta \partial q_\gamma}(\underline{q}^*) \cancel{\varepsilon} u_\beta \\ \gamma = 1, \ldots, d \end{cases}$$

$$[G(\underline{q}^*) \ddot{\underline{u}}]_\gamma = -[H_U(\underline{q}^*) \underline{u}]_\gamma \Rightarrow \underbrace{[G(\underline{q}^*) \ddot{\underline{u}}] = -[H_U(\underline{q}^*) \underline{u}]}$$

$\downarrow$ E-L approssimata

$$\mathcal{L}^{LIN}(\underline{u}, \dot{\underline{u}}) = \frac{1}{2}(\dot{\underline{u}} \cdot G(\underline{q}^*) \dot{\underline{u}}) - \frac{1}{2}(\underline{u} \cdot H_U(\underline{q}^*) \underline{u})$$

$G_0 := G(\underline{q}^*) \qquad H_0 := H_U(\underline{q}^*)$

Cerco soluzioni del tipo $\underline{u}(t) = \cos(wt+\varphi)\cdot \underline{w}$ $\underline{w} \neq 0 \in \mathbb{R}^d$

$\dot{\underline{u}} = -w \sin(wt+\varphi)\,\underline{w}$
$\ddot{\underline{u}} = -w^2 \cos(wt+\varphi)\,\underline{w} = -w^2 \underline{u}$

$-w^2 G_0 \underline{u} = -H_0 \underline{u}$ $(w^2 G_0 - H_0)\underline{u} = 0 \rightarrow (w^2 G_0 - H_0)\underline{w}\cos(wt+\varphi) = 0$

$\Rightarrow \forall t \rightarrow (w^2 G_0 - H_0)\underline{w} = 0$

$\det(\lambda G_0 - H_0) = 0$ $\lambda = w^2 \rightsquigarrow \underline{w}$ t.c. $(\lambda G_0 - H_0)\cdot \underline{w} = 0$

w : pulsazione propria $\underline{w}$: modo normale corrispondente a w

Teo G_0 è simmetrica e definita positiva
H_0 è simmetrica

1) $\exists$ una base di $\mathbb{R}^d$ formata da autovettori di H_0 rispetto a G_0
2) Autovettori corrispondenti ad autovalori diversi sono ortogonali a G_0
3) Se H_0 è definita positiva $\Rightarrow$ λ_i sono tutti positivi

$(q_1, \dots, q_d) \rightsquigarrow (\tilde{q}_1, \dots, \tilde{q}_d)$

$$\begin{cases} q_j = q_j(\tilde{q}_1, \dots, \tilde{q}_d) \rightarrow \dot{q}_j = \dot{q}_j(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}}) \\ j = 1, \dots, d \end{cases} \qquad J_{kj} = \frac{\partial q_k}{\partial \tilde{q}_j} \quad J \text{ t.c. } \det J \neq 0$$

$$\begin{cases} \tilde{q}_j(q_1, \dots, q_d) \\ j = 1, \dots, d \end{cases}$$

$$\mathcal{L}(q_1, \dots, q_d, \dot{q}_1, \dots, \dot{q}_d) \rightarrow \tilde{\mathcal{L}}(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}}) = \mathcal{L}(\underline{q}(\tilde{\underline{q}}), \dot{q}(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}}))$$

PROP $q(t)$ è soluzione E-L per $\mathcal{L}(\underline{q}, \dot{\underline{q}}) \Leftrightarrow \tilde{q}(t)$ è soluzione E-L per $\tilde{\mathcal{L}}(\tilde{\underline{q}}, \dot{\tilde{\underline{q}}})$

Dim $(\Rightarrow)$ $q(t)$ è soluzione E-L per $\mathcal{L}$

per lemma 1

$$\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\tilde{q}}_j} = \frac{d}{dt}\left(\sum_{i=1}^{d} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{\tilde{q}}_j}\right) = \frac{d}{dt}\left(\sum_{i=1}^{d} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \tilde{q}_j}\right) =$$

$$= \sum_{i=1}^{d}\left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \tilde{q}_j} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{d}{dt}\frac{\partial q_i}{\partial \tilde{q}_j}\right] = \sum_{i=1}^{d}\left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \tilde{q}_j} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \tilde{q}_j}\right]$$

per lemma 2

uso ipotesi
$$\Rightarrow \quad = \sum_{i=1}^{d} \left(\frac{\partial \mathcal{L}}{\partial q_i} \frac{\partial q_i}{\partial \dot{q}_j} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \dot{\tilde{q}}_j} \right) = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{\tilde{q}}_j}$$

- Data una certa $\mathcal{L}(q,\dot{q},t)$, $\exists\, \mathcal{L}'(q,\dot{q},t)$ t.c. $q(t)$ è sol. per $\mathcal{L}$ e $\mathcal{L}'$

  se $\mathcal{L}' = \alpha \mathcal{L} + \frac{d F(q,t)}{dt}$

PROP $\forall$ scelta di $F(q,t)$ e $\alpha \neq 0 \in \mathbb{R}$ allora

$\mathcal{L}(q,\dot{q},t)$ e $\mathcal{L}'(q,\dot{q},t) = \alpha \mathcal{L}(q,\dot{q},t) + \frac{d}{dt}F$

Conducono alla stessa soluzione per E-L

Dim $\quad L_0 = \frac{dF}{dt} = \dot{F} \qquad \mathcal{L}' = \alpha\mathcal{L} + L_0$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \alpha \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} + \frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j}$$

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j} = \frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}_j} \overset{\text{lemma 1}}{=} \frac{d}{dt}\frac{\partial F}{\partial q_j} \overset{\text{lemma 2}}{=} \frac{\partial \dot{F}}{\partial q_j}$$

$$= \alpha \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} + \frac{\partial L_0}{\partial q_j}$$

($\Rightarrow$) eq. E-L per $\mathcal{L}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \alpha \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial L_0}{\partial q_j}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \frac{\partial}{\partial q_j}(\alpha\mathcal{L} + L_0) = \frac{\partial}{\partial q_j}\mathcal{L}'$$

($\Leftarrow$) Eq. E-L per $\mathcal{L}'$

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}'}{\partial q_j} = \frac{\partial}{\partial q_j}[\alpha\mathcal{L} + L_0] = \alpha\frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial L_0}{\partial q_j} = \alpha\frac{\partial \mathcal{L}}{\partial q_j} + \frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_j}$$

CORPO RIGIDO

DEF $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^3$ si dicono **RIGIDAMENTE COLLEGATI** se

$\|\underline{x}_1 - \underline{x}_2\| = c \quad \to c$ costante nel tempo

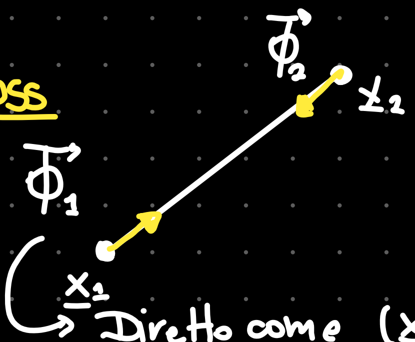
LEMMA IL VINCOLO DI RIGIDITA' è LISCIO

Dim $\frac{d}{dt}\|\underline{x}_1 - \underline{x}_2\|^2 = 0$

$$\frac{d}{dt}\left[(\underline{x}_1-\underline{x}_2)\cdot(\underline{x}_1-\underline{x}_2)\right] = 2(\dot{\underline{x}}_1-\dot{\underline{x}}_2)(\underline{x}_1-\underline{x}_2)=0$$

Allora $(\dot{\underline{x}}_1-\dot{\underline{x}}_2)$ è ortogonale a $(\underline{x}_1-\underline{x}_2)$

OSS



$\underline{\Phi}_1 = \alpha(\underline{x}_1-\underline{x}_2)$

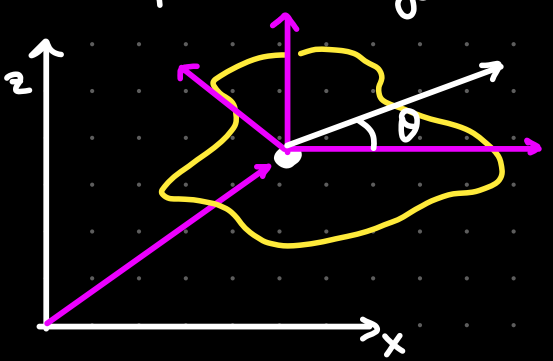
$\underline{\Phi}_2 = -\alpha(\underline{x}_1-\underline{x}_2)$

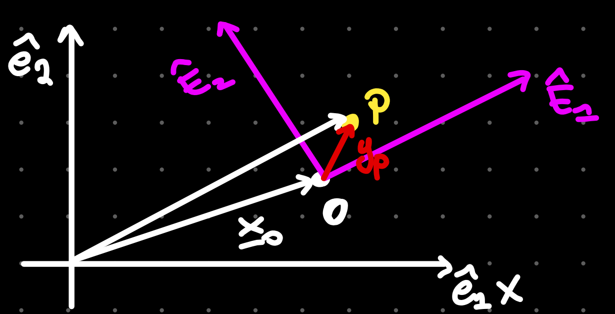
Diretto come $(\underline{x}_1-\underline{x}_2)$

$$\Pi_{RV}^{TOT} = \underline{\Phi}_1\dot{\underline{x}}_1 + \underline{\Phi}_2\dot{\underline{x}}_2 = \alpha(\underline{x}_1-\underline{x}_2) - \alpha(\underline{x}_1-\underline{x}_2)\cdot\dot{\underline{x}}_2 = \alpha(\underline{x}_1-\underline{x}_2)(\dot{\underline{x}}_1-\dot{\underline{x}}_2) = 0$$

Nel piano → 3 gdl : ORIGINE + SR solidale con corpo rigido
↳ 2 gdl ↳ 1 gdl

Nello spazio → 6 gdl





$\underline{x}_p = \underline{x}_0 + \underline{y}_p$

$\dot{\underline{x}}_p = \dot{\underline{x}}_0 + \dot{\underline{y}}_p$

$$\underline{y} = \underline{y}_p = y_1\hat{e}_1 + y_2\hat{e}_2 = Y_1\hat{E}_1 + Y_2\hat{E}_2$$

MATRICE ROTAZIONE $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $RR^T = 1 \rightarrow R^{-1} = R^T$

$$y_i = \sum_{k=1}^{2} R_{ik} Y_k$$

$$\hat{E}_i = \sum_{k=1}^{2} R_{ik}\, \hat{e}_k \qquad \hat{e}_i = \sum_{k=1}^{2} R_{ik}\, \hat{E}_k$$

$$\underline{\dot{y}} = \dot{y}_1 \hat{e}_1 + \dot{y}_2 \hat{e}_2 = \frac{d}{dt}\left(\sum_{k=1}^{2} R_{1k} Y_k\right)\hat{e}_1 + \frac{d}{dt}\left(\sum_{k=1}^{2} R_{2k} Y_k\right)\hat{e}_2$$

$$\underline{\dot{y}} = \sum_{i=1}^{2}\left(\sum_{k=1}^{2} \dot{R}_{ik} Y_k \hat{e}_i\right) = \sum_{i=1}^{2}\left[\sum_{k=1}^{2} \dot{R}_{ik} Y_k + \sum_{j=1}^{2} R_{ij} \hat{E}_j\right] = \sum_{k,j=1}^{2}\left(\sum_{i=1}^{2}\left(\dot{R}_{ik} R_{ij}\right)\right) Y_k \hat{E}_j$$

$$R_{ij} = (R^T)_{ji} \quad \Rightarrow \quad \sum_{i=1}^{2} (R^T_{ji} \dot{R}_{ik}) = (R^T \dot{R})_{jk}$$

$$\rightarrow \quad \underline{\dot{y}} = \sum_{k,j=1}^{2} (R^T \dot{R})_{jk}\, Y_k \hat{E}_j$$

$$R = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \qquad \dot{R} = \begin{pmatrix} -\sin\theta\,\dot{\theta} & -\cos\theta\,\dot{\theta} \\ \cos\theta\,\dot{\theta} & -\sin\theta\,\dot{\theta} \end{pmatrix}$$

$$R^T \dot{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\sin\theta\,\dot{\theta} & -\cos\theta\,\dot{\theta} \\ \cos\theta\,\dot{\theta} & -\sin\theta\,\dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix} = A \rightsquigarrow \text{ANTISIMMETRICA}$$

$$A + A^T = 0$$

$$\underline{\dot{y}} = \sum_{k,j=1}^{2} (R^T \dot{R})_{jk} Y_k \hat{E}_j = -\dot{\theta} Y_2 \hat{E}_1 + \dot{\theta} Y_1 \hat{E}_2$$

$$\| \underline{\dot{y}}^2 \| = \dot{\theta}^2 (Y_2^2 + Y_1^2) = \dot{\theta}^2 (y_1^2 + y_2^2) = \dot{\theta}\, \| \underline{y} \|^2$$

$$\| \underline{\dot{y}}_p^2 \| = \dot{\theta}\, \| \underline{y}_p \|^2$$

$\hat{E}_3 \rightarrow$ perpendicolare al piano del moto $\qquad \underline{y}_p = Y_1 \hat{E}_1 + Y_2 \hat{E}_2 + Y_3 \hat{E}_3$

$$\underline{\Omega} = \dot{\theta} \hat{E}_3$$

$$\underset{\substack{\uparrow \\ \text{VELOCITA' ANGOLARE}}}{\underline{\Omega}} \times \underline{y}_p = \det\begin{pmatrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ 0 & 0 & \dot{\theta} \\ Y_1 & Y_2 & Y_3 \end{pmatrix} = \hat{E}_1(-\dot{\theta} Y_2) - \hat{E}_2(-\dot{\theta} Y_1) + \hat{E}_3(0) = \underline{\dot{y}}_p$$

$$\underline{\dot{x}}_p = \underline{\dot{X}}_0 + \underline{\dot{y}}_p$$

$$K = \frac{1}{2}\sum_{p=1}^{m} m_p \|\underline{\dot{x}}_p\|^2 = \frac{1}{2}\sum_{p=1}^{m} m_p (\underline{\dot{X}}_0 + \underline{\dot{y}}_p)\cdot(\underline{\dot{X}}_0 + \underline{\dot{y}}_p)$$

$$K = \frac{1}{2}\sum_{p=1}^{N} m_p \left(\|\underline{\dot{X}}_0\|^2 + 2\underline{\dot{X}}_0 \underline{\dot{y}}_p + \|\underline{\dot{y}}_p\|^2 \right)$$

$$K = \frac{1}{2}\underbrace{\left(\sum_{p=1}^{N} m_p\right)}_{\to M_{TOT}} \|\dot{\underline{x}}_o\|^2 + \frac{1}{2}\sum_{p=1}^{N} m_p \|\dot{\underline{y}}_p\|^2 + \frac{1}{2}\left(\dot{\underline{x}}_o \cdot \sum_{p=1}^{N} m_p \dot{\underline{y}}_p\right)$$

→ SCELGO ORIGINE O DEL sistema solidale al corpo rigido come il suo centro massa

ALLORA

$$\sum_{p=1}^{N} m_p \dot{\underline{y}}_p = 0$$

$$\to K = \frac{1}{2} M_{TOT} v_{cm}^2 + \frac{1}{2}\sum_{p=1}^{N} m_p \dot{\theta}^2 \|\underline{y}_p\|^2 \to \frac{1}{2}\dot{\theta}^2 \underbrace{\sum_{p=1}^{N} m_p \|\underline{y}_p\|^2}_{I_{z,cm}}$$

$$I_{z,cm} = \sum_{p=1}^{N} m_p \|\underline{y}_p\|^2 \quad \to d^2(p, cm)$$

dipende solo da geometria del corpo

Nel caso di distribuzione continua massa $\rightsquigarrow \int_\Omega dx\,dy\, \rho(x,y)(x^2+y^2) = I_{z,cm}$

CORPO RIGIDO IN 3 DIMENSIONI

Energia cinetica rotazionale

$$\underline{y}_p = \sum_{i=1}^{3} y_i \underline{e}_i = \sum_{i=1}^{3} Y_i \hat{E}_i$$

$$y_i = \sum_{k=1}^{3} R_{ik} Y_k \qquad R \text{ matrice } 3\times3 \text{ ortogonale} \to R^T R = R R^T = \mathbb{1}$$

$$\dot{\underline{y}}_p = \sum_{i=1}^{3}\left(\sum_{k=1}^{3} \underbrace{(R^T \dot{R})}_{}{}_{jk} Y_k\right)\hat{E}_j$$

A matrice antisimmetrica $(A + A^T) = 0$

PROP. Sia $R(t)$ una famiglia di matrici ortogonali t.c. $\dot{R}(t)$ sia ben definita

Allora

$$(R^T\dot{R}) + (R^T\dot{R})^T = \underline{0} \quad \forall t$$

Dim

$$\frac{d}{dt}(R^T R) = \frac{d}{dt}(\mathbb{1}) = 0$$

$$\frac{d}{dt}(R^T R)_{ij} = \frac{d}{dt}\left(\sum_{k=1}^{D} R^T_{ik} R_{kj}\right) = \sum_{k=1}^{D} \dot{R}^T_{ik} R_{jj} + R^T_{ik}\dot{R}_{kj} = \dot{R}^T R + R^T \dot{R} = 0$$

A è antisimmetrica

↳ Diagonale $= 0 \rightarrow \frac{D^2 - D}{2}$ #entrate indipendenti

se A 3x3 $\Rightarrow \frac{9-3}{2} = 3 \rightarrow 3$ gdl rotazioni

$$A = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \qquad \underline{\Omega} = \Omega_1 \hat{E}_1 + \Omega_2 \hat{E}_2 + \Omega_3 \hat{E}_3$$

$$\dot{\underline{y}}_p = \sum_{j=1}^{3} \left(\sum_{k=1}^{3} (A_{jk} Y_k) \hat{E}_j \right) = \underline{\Omega} \times \underline{y}_p = \det \begin{pmatrix} \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\ \Omega_1 & \Omega_2 & \Omega_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix}$$

↳ velocità angolare

$$K_{ROTAZIONALE} = \frac{1}{2} \Sigma_p \, m_p \, \| \dot{\underline{y}}_p \|^2 = \frac{1}{2} \Sigma_p m_p \| \underline{\Omega} \times \underline{y}_p \|^2$$

$$= \frac{1}{2} \Sigma_p \, m_p \| \underline{\Omega}_p \|^2 \| \underline{y}_p \|^2 \sin^2 \alpha (\underline{\Omega}, \underline{y}_p)$$

$$= \frac{1}{2} \Sigma_p \, m_p \| \underline{\Omega} \|^2 \| \underline{y}_p \|^2 (1 - \cos^2 \alpha (\underline{\Omega}, \underline{y}_p))$$

$$= \frac{1}{2} \Sigma_p \, m_p (\| \underline{\Omega} \|^2 \| \underline{y}_p \|^2 - (\underline{\Omega} \cdot \underline{y}_p)^2)$$

⇝ in coordinate rispetto $\hat{E}_i$

$$\| \underline{\Omega} \|^2 = \sum_{j,k=1}^{3} \Omega_j \Omega_k \delta_{jk} \qquad \delta_{jk} = \begin{cases} 1 & \text{se } j = k \\ 0 & \text{se } j \neq k \end{cases}$$

$$(\underline{\Omega} \cdot \underline{y}_p)^2 = \sum_{j,k=1}^{3} \Omega_j Y_j \Omega_k Y_k$$

$$K_{ROT} = \frac{1}{2} \Sigma_p \, m_p \left(\sum_{j,k=1}^{3} \left(\Omega_j \Omega_k \delta_{jk} \| \underline{y}_p \|^2 - \Omega_j \Omega_k Y_j Y_k \right) \right)$$

$$= \frac{1}{2} \sum_{j,k} \Omega_j \Omega_k \underbrace{\boxed{\Sigma_p \, m_p (\delta_{jk} \| \underline{y}_p \|^2 - Y_k Y_j))}}_{\text{non dipende dal tempo}} = I_{jk} \rightarrow \text{Tensore di inerzia}$$

$$K_{ROT} = \frac{1}{2} \sum_{j,k} \Omega_j I_{jk} \Omega_k = \frac{1}{2} \underline{\Omega} \cdot (\underbrace{I}_{\text{matrice}} \underline{\Omega})$$

I :
- simmetrico → autovalori reali ≥ 0
- descrive solo la geometria delle masse (scelta base $\hat{E}$)

• I non diagonale

I diagonalizzabile → scelgo assi principali di inerzia

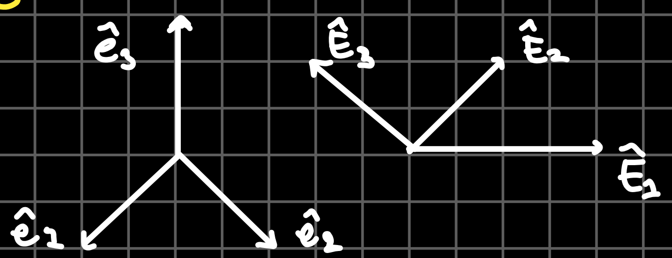
Nell'apposita base (ASSI PRINCIPALI DI INERZIA) ⇒ $I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ Diagonale

I_j : momenti principali di inerzia

$$K_{ROT} = \frac{1}{2}\left(\Omega_1^2 I_1 + \Omega_2^2 I_2 + \Omega_3^2 I_3\right)$$

ANGOLI DI EULERO

$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$



1) φ ruota attorno $\hat{e}_3$ $\left[\begin{array}{l} \hat{e}_1 \to \hat{e}_m \\ \hat{e}_3 \to \hat{e}_3 \end{array}\right]$

2) θ ruota attorno $\hat{e}_m$ $\left[\begin{array}{l} \hat{e}_3 \to \hat{E}_3 \\ \hat{e}_1 \to \hat{e}_1 \end{array}\right]$

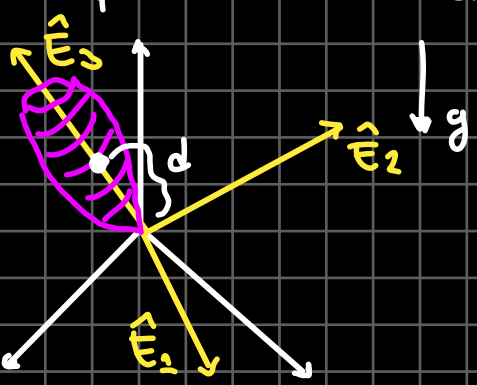
3) ψ ruota attorno $\hat{E}_3$ $\left[\begin{array}{l} \hat{e}_1 \to \hat{E}_1 \\ \hat{e}_3 \to \hat{e}_3 = \hat{E}_3 \\ \hat{e}_2 \to \hat{E}_2 \end{array}\right]$

TROTTOLA DI LAGRANGE

→ corpo rigido simmetrico attorno asse (ρ costante → solido rotazione)

→ punto fisso ($\neq$ baricentro)

→ potenziale forze simmetrico rispetto asse verticale (tipicamente asse gravitazionale)



3 gdl

usiamo angoli di Eulero

$U(\varphi,\theta,\psi) = Mg\,z_{cm} = Mg\,d\cos\theta$

$$\begin{cases} x_{cm} = d\sin\theta\cos\varphi \\ y_{cm} = d\sin\theta\sin\varphi \\ z_{cm} = d\cos\theta \end{cases} \qquad \begin{cases} \dot{x}_{cm} = d\cos\theta\cos\varphi\,\dot{\theta} - d\sin\theta\sin\varphi\,\dot{\varphi} \\ \dot{y}_{cm} = d\cos\theta\sin\varphi\,\dot{\theta} + d\sin\theta\cos\varphi\,\dot{\varphi} \\ \dot{z}_{cm} = -d\cos\theta\,\dot{\theta} \end{cases}$$

$$K_{cm} = \frac{1}{2}M(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 + \dot{z}_{cm}^2) = \frac{1}{2}Md^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)$$

$K = K_{cm} + K_{ROT}$

NOTA: $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$ sono gli assi principali di inerzia $\rightsquigarrow I = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

$$K_{ROT} = \frac{1}{2}I(\Omega_1^2 + \Omega_2^2) + \frac{1}{2}I_3\Omega_3^2$$

la rotazione R è la composizione di 3 rotazioni:

1) φ attorno $\hat{e}_3$ $\rightsquigarrow$ $\dot{\varphi}\hat{e}_3$
2) θ attorno $\hat{e}_n$
3) ψ attorno $\hat{E}_3$

$$\Omega = \underbrace{\dot{\varphi}\,\hat{e}_3 + \dot{\theta}\,\hat{e}_n}_{} + \dot{\psi}\,\hat{E}_3$$

$\hookrightarrow$ $\hat{e}_3, \hat{e}_n$ $\rightsquigarrow$ riscrivo nella base $(\hat{E}_1, \hat{E}_2, \hat{E}_3)$

Anziché fare il conto esplicito osservo che:

a) La trottola ha asse di simmetria $\hat{E}_3$ $\Rightarrow$ ψ è ciclica

b) Il problema è simmetrico rispetto asse di gravità $\hat{e}_3$ $\rightarrow$ φ ciclica

Allora posso calcolare K_{ROT} ($\mathcal{L}$ in generale) per valori di φ e ψ comodi

$$K_{ROT}(\theta, \cancel{\varphi}, \cancel{\psi}, \dot{\theta}, \dot{\psi}, \dot{\varphi})$$

Scelgo $\varphi = 0 \Rightarrow \hat{e}_1 = \hat{e}_n$

$\qquad\qquad\qquad\qquad \Rightarrow \hat{E}_1 = \hat{e}_1 = \hat{e}_n$

$\qquad\quad \psi = 0 \Rightarrow \hat{E}_1 = \hat{e}_n$

$$\Omega = \dot{\varphi}\,\hat{e}_3 + \dot{\theta}\,\hat{E}_1 + \dot{\psi}\,\hat{E}_3$$

$$\hat{e}_3 = \cos\theta\,\hat{E}_3 - \sin\theta\,\hat{E}_2 \Rightarrow \Omega = \dot{\varphi}(\cos\theta\,\hat{E}_3 - \sin\theta\,\hat{E}_2) + \dot{\theta}\hat{E}_1 + \dot{\psi}\hat{E}_3$$

$$\underline{\Omega} = \underbrace{\dot\theta}_{\Omega_1}\hat{E}_1 - \underbrace{\sin\theta\,\dot\varphi}_{\Omega_2}\hat{E}_2 + (\underbrace{\cos\theta\,\dot\varphi + \dot\psi}_{\Omega_3})\hat{E}_3$$

$$K_{ROT} = \frac{1}{2} I (\dot\theta^2 + \sin^2\theta\,\dot\varphi^2) + \frac{1}{2} I_3 (\cos\theta\,\dot\varphi + \dot\psi)^2$$

$$\mathcal{L} = K_{ROT} + K_{cm} - U =$$

$$= \frac{1}{2}\underbrace{(I + Md^2)}_{J_1}(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2) + \frac{1}{2} I_3 (\cos\theta\,\dot\varphi + \dot\psi)^2 - Mgd\cos\theta$$

RIDUZIONE AD UN GRADO DI LIBERTÀ : perchè 2 variabili cicliche

$$\frac{\partial\mathcal{L}}{\partial\dot\psi} =: L_3 = I_3(\cos\theta\,\dot\varphi + \dot\psi)$$

$$\frac{\partial\mathcal{L}}{\partial\dot\varphi} =: L_z = J_1\sin^2\theta\,\dot\varphi + I_3(\cos\theta\,\dot\varphi + \dot\psi) = J_1\sin^2\theta\,\dot\varphi + L_3\cos\theta$$

$$\dot\psi = \frac{L_3}{I_3} - \cos\theta\,\dot\varphi = \frac{L_3}{I_3} - \cos\theta\left(\frac{L_z - L_3\cos\theta}{J_1\cos\theta}\right)$$

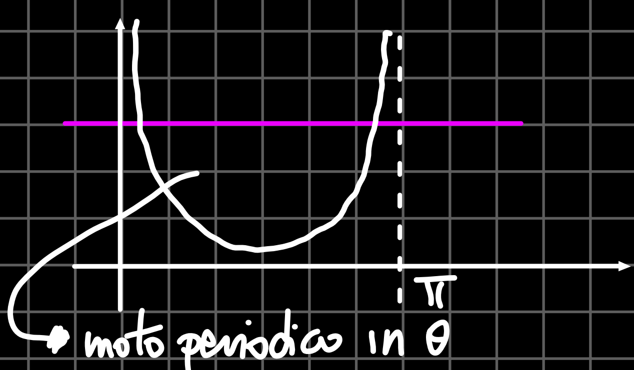
$$\dot\varphi = \frac{L_z - L_3\cos\theta}{J_1\cos\theta}$$

$$E = \frac{1}{2}J_1\dot\theta^2 + \frac{1}{2}J_1\sin^2\theta\left(\frac{L_z - L_3\cos\theta}{J_1\sin^2\theta}\right)^2 + \cancel{\frac{1}{2}I_3 L_3^2} + Mgd\cos\theta = K + U$$

↳ energia per un sistema ad 1 gdl con

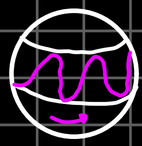
- Potenziale $U^{eff} = \frac{1}{2}\frac{(L_z - L_3\cos\theta)^2}{J_1\sin^2\theta} + Mgd\cos\theta$
- $K^{eff} = \frac{1}{2}J_1\dot\theta^2$

NOTA: $U^{eff}(\theta) \to +\infty$ per $\theta \to 0$, $\theta \to \pi$

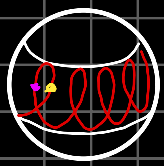


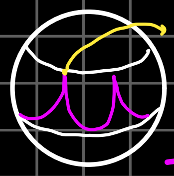
Il segno $\dot\varphi$ dipende da segno $L_z - L_3\cos\theta$

- $\dot{\varphi} > 0$ (φ non cambia segno)



- $\dot{\varphi} > 0$
  $\dot{\varphi} < 0$



- 

$\dot{\varphi} = 0 \quad \dot{\theta} = 0$

$\to \dot{\varphi} > 0$

TEOREMA DI NOETHER

DEF. GRUPPO AD UN PARAMETRO DI TRASFORMAZIONI DI COORDINATE
DIFFEOMORFISMI

$q_j \rightsquigarrow q'_j = Q_j(\underline{q}, s) \quad s \in \mathbb{R}$ se soddisfa

1) $\forall s$ fissato $\underline{Q}(\underline{q}, s)$ invertibile

2) fissato $\underline{q}$; $\underline{Q}(\underline{q}, s)$ sia differenziabile in s

3) per $s = 0$ $\underline{Q}(\underline{q}, 0) = \underline{q}$ (IDENTITA')

4) $\underline{q} \xrightarrow{s_1} \underline{Q}(\underline{q}, s_1) \xrightarrow{s_2} \underline{Q}(\underline{Q}(\underline{q}, s_1), s_2)$

$= \underline{Q}(\underline{q}, s_1 + s_2)$

$s_1 + s_2$

<u>esempi</u>

a) $\begin{cases} q'_j = q_j & \forall j \neq d \\ q'_d = q_d + s \end{cases}$

b) (x, y, z) in $\mathbb{R}^3$; rotazioni attorno z

$$\underline{x} \rightsquigarrow \underline{x}' \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_s \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad R_s R_s^T = \mathbb{1} \qquad R_s = \begin{pmatrix} \cos(s) & -\sin(s) & 0 \\ \sin(s) & \cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) Traslazioni in $\mathbb{R}^d \ni (q_1, \dots, q_d)$

$\underline{q}' = \underline{q} + s\underline{v} \qquad \underline{v} \in \mathbb{R}^d$ costante

DEF. GENERATORE INFINITESIMALE di un gruppo ad un parametro di diffeomorfismi

$$q'_j = Q_j(q,s) = Q_j(s,0) + \frac{d}{ds} Q_j(q,0)\cdot s + o(s)$$
$$= q_j + \underbrace{\frac{d}{ds} Q_j(q,0)}_{\text{generatore infinitesimale } Y_j(q)} s + o(s)$$

$$\rightsquigarrow (q'_j - q_j) = \delta q_j = Y_j(q)\cdot s + o(s)$$

esempio 2) $\underline{x}' = R_s \underline{x}$

$$\begin{cases} x' = \cos(s)x - \sin(s)y \\ y' = \sin(s)x + \cos(s)y \\ z' = z \end{cases} \rightsquigarrow \begin{cases} x' = x - sy + o(s) \\ y' = sx + y + o(s) \\ z' = z \end{cases}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + s \underbrace{\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}}_{\underline{Y}} + o(s)$$

DEF SIMMETRIA DI $\mathcal{L}$

Diciamo che un gruppo ad 1 parametro di trasformazioni è una simmetria di $\mathcal{L}$ se $\mathcal{L}$ è invariante in forma sotto tale trasformazione

$\mathcal{L}(q, \dot{q}, t) \longrightarrow$ esprimo $\underline{q}' = \underline{Q}(q,s)$
$\dot{\underline{q}}' = \frac{d}{dt} \underline{Q}(q,s)$

$$\tilde{\mathcal{L}}(q, \dot{q}, t) = \mathcal{L}(Q(q,s), \dot{\underline{Q}}(q,\dot{q},s), t) = \mathcal{L}(q, \dot{q}, t) \Rightarrow \text{invariante in forma}$$

esempio

$$\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}(q_1^2 + q_2^2)^4$$

$\to$ Le rotazioni sono una simmetria per $\mathcal{L}$

$$\begin{cases} q'_1 = q_1 \cos(s) - q_2 \sin(s) \\ q'_2 = q_1 \sin(s) + q_2 \cos(s) \end{cases}$$

$$\|\underline{q}'\|^2 = \|\underline{q}\|^2 \qquad \|\dot{\underline{q}}'\|^2 = \|\dot{\underline{q}}\|^2$$

Teo NOETHER

Sia $q_j \to Q_j(\underline{q}, s)$ una simmetria di $\mathcal{L}$

Sia $\underline{Y}(\underline{q})$ il suo generatore infinitesimale

Allora $I = \sum_{j=1}^{d} Y_j(\underline{q}) \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$ è una costante del moto

oss: se $\mathcal{L}$ è quadratica nelle $\dot{\underline{q}}$ $\Rightarrow$ I è lineare nelle $\dot{\underline{q}}$

Dim

- $Q_j(\underline{q}, s) = q_j + s Y_j(\underline{q}) + o(s)$
- $\dot{Q}_j(\underline{q}, \dot{\underline{q}}, s) = \dot{q}_j + s \frac{d}{dt} Y_j(\underline{q}) + o(s)$

$\tilde{\mathcal{L}}(q, \dot{q}, s) = \mathcal{L}(\underline{Q}(q, s), \dot{\underline{Q}}(\underline{q}, \dot{\underline{q}}, s), t)$ sviluppo attorno a $s = 0$

$$\mathcal{L}(\underline{Q}(\underline{q}, 0), \dot{\underline{Q}}(\underline{q}, \dot{\underline{q}}, 0), t) + s \sum_{j=1}^{d} \frac{\partial \mathcal{L}}{\partial q_j} Y_j + s \sum_{j=1}^{d} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{d}{dt} Y_j + o(s)$$

Per ipotesi $\tilde{\mathcal{L}}(\underline{q}, \dot{\underline{q}}, s) = \mathcal{L}(\underline{q}, \dot{\underline{q}}) \quad \forall s$

$$\tilde{\mathcal{L}}(\underline{q}, \dot{\underline{q}}, s) = \mathcal{L}(\underline{Q}(\underline{q}, s), \dot{\underline{Q}}(\underline{q}, \dot{\underline{q}}, s))$$

$$\Rightarrow \underbrace{\cancel{\mathcal{L}(\underline{Q}(\underline{q}, 0), \dot{\underline{Q}}(\underline{q}, \dot{\underline{q}}, 0))}}_{\mathcal{L}(\underline{q}, \dot{\underline{q}})} + s \sum_{j=1}^{d} \frac{\partial \mathcal{L}}{\partial q_j} Y_j + s \sum_{j=1}^{d} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{d}{dt} Y + o(s) = \cancel{\mathcal{L}(\underline{q}, \dot{\underline{q}})}$$

$$\Rightarrow 0 = \sum_{j=1}^{d} \left(\frac{\partial \mathcal{L}}{\partial q_j} Y_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{d}{dt} Y_j \right)$$

LUNGO IL MOTO VALGONO EQ. E.L.

$$0 = \sum_{j=1}^{d} \left[\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} Y_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \frac{d}{dt} Y_j \right] = \sum_{j=1}^{d} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} Y_j \right) = \frac{d}{dt} \underbrace{\sum_{j=1}^{d} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} Y_j \right)}_{I} = 0$$