# HAMILTONIANA

$q \qquad p = \frac{\partial \mathcal{L}}{\partial \dot q} \overset{\text{inverto}}{\longleftrightarrow} \dot q = \phi(q, p, t)$

$$\begin{cases} \dot q = \phi(q, p, t) \\ \dot p = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} = \frac{\partial \mathcal{L}}{\partial q}(q, p, t) \end{cases}$$

## TRASFORMATA DI LEGENDRE

Sia $f(x) : \mathbb{R} \to \mathbb{R}$ di classe $C^2$ e convessa

Definisco $p(x) = \frac{df}{dx}$

$f$ convessa $\Rightarrow p(x)$ è invertibile $\rightsquigarrow x(p)$

Voglio mostrare che $x(p) = \frac{d\, g(p)}{dp}$ con $g(p) = [xp - f(x)]_{x = x(p)}$

↳ TRASFORMATA DI LEGENDRE $f(x)$

Infatti

$g(p) = [x(p)p - f(x(p))]$

$\frac{d}{dp} g(p) = \cancel{\frac{d\, x(p)}{dp} p} + x(p) - \cancel{\underbrace{\frac{df}{dx}}_{p} \frac{dx(p)}{dp}} \Rightarrow x(p) = \frac{d}{dp} g(p)$

NOTA: Se $g(p)$ è la trasformata di Legendre di $f(x)$ allora $f(x)$ è la Trasformata di Legendre di $g(p)$

$f(x) = [xp - g(p)]_{p(x)}$

es $f(x) = \frac{1}{2}\alpha x^2 \qquad p(x) = \alpha x \to x = \frac{p}{\alpha}$

$g(x) = [xp - f(x)]_{x = x(p)} = \frac{p}{\alpha} \cdot p - \frac{1}{2}\alpha \frac{p^2}{\alpha^2} = \frac{p^2}{\alpha} - \frac{1}{2}\frac{p^2}{\alpha} = \frac{1}{2}\frac{p^2}{\alpha}$

Se $f(x, y) \to$ definisco $p(x, y) = \frac{\partial f}{\partial x} \to$ inverto $x = \phi(p, y)$

$$x = \frac{\partial g}{\partial p}(p, y) \quad \text{con} \quad g(p, y) = \left[ xp - f(x, y) \right]_{x=\phi(x,y)}$$

Infatti $\quad g(p, y) = \left[ \phi(p, y)\, p - f(\phi(p, y), y) \right]$

$$\frac{\partial g}{\partial y} = p \cancel{\frac{\partial \phi}{\partial y}} - \underbrace{\cancel{\frac{\partial f}{\partial x}}}_{p} \cancel{\frac{\partial \phi}{\partial y}} - \frac{\partial f}{\partial y} \quad \Rightarrow \quad \frac{\partial g}{\partial y} = - \left. \frac{\partial f}{\partial y} \right|_{x=\phi(p,y)}$$

$\mathcal{L}(q, \dot{q}, t)$ convessa in $\dot{q}$

$$\boxed{\begin{array}{lcl} f & \leftrightarrow & \mathcal{L} \\ x & \leftrightarrow & \dot{q} \\ y & \leftrightarrow & q, t \end{array}}$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$H(q, p, t) = \left[ \dot{q} p - \mathcal{L} \right]_{\dot{q}=\phi(q,p,t)} \quad \text{HAMILTONIANA} \quad 1 \text{ gdl}$$

$H$ è la trasformata di Legendre di $\mathcal{L}$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \overset{\text{E.L.}}{=} \left. \frac{\partial \mathcal{L}}{\partial q} \right|_{\dot{q}=\phi(q,p,t)} = - \frac{\partial H}{\partial q}$$

Data $H(q, p, t)$ le equazioni del moto sono

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\[2mm] \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \quad \text{EQUAZIONI HAMILTON} \quad (\equiv \text{EQ. E-L})$$

Data $f(x_1, \dots, x_d)$ t.c. $\det \left( \dfrac{\partial^2 f}{\partial x_i x_j} \right) \neq 0$

$$p_i = \frac{\partial f}{\partial x_i} \qquad i = 1, \dots, d$$

$$g(p_1, \dots, p_d) = \left[ \overbrace{\sum_{i=1}^{d} (x_i p_i)}^{\underline{x} \cdot \underline{p}} - f(x_1, \dots, x_d) \right]_{\underline{x}=\phi(\underline{p})}$$

$$H(q,p,t) = \left[ \dot{q}\cdot p - \mathcal{L}(q,p,t) \right] \Big|_{\dot{q} = \phi(q,p,t)}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\frac{\partial H}{\partial t} = - \frac{\partial \mathcal{L}}{\partial t} \Big|_{\dot{q} = \phi(p,q,t)}$$

lungo il moto

$$- \frac{\partial \mathcal{L}}{\partial t} = \frac{d}{dt}(\text{integrale di Jacobi}) = \frac{d}{dt} H$$

EQ. HAMILTON

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = - \dfrac{\partial H}{\partial q_i} \end{cases} \quad i = 1, \dots, d$$

## HAMILTONIANA

$(q,p)$ $\quad H(q,p,t) = \left[ p\dot{q} - \mathcal{L}(q,\dot{q},t) \right]$ $\quad \dot{q} = \phi(q,p,t)$

$$\dot{p} = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)}_{} \begin{pmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{pmatrix}$$

$\hookrightarrow E$ : Matrice simplettica standard

Se $n=1$ $\quad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix}$

NOTA: $\quad E^T = -E \; ; \; E^{-1} = -E \; ; \; E^2 = \mathbb{1}$

$z = \begin{pmatrix} q \\ p \end{pmatrix}$ $\quad \nabla_z H = \left( \dfrac{\partial H}{\partial q_1}, \dots, \dfrac{\partial H}{\partial q_n}, \dfrac{\partial H}{\partial p_1}, \dots, \dfrac{\partial H}{\partial p_n} \right)$

$$\dot{z} = E \nabla_z H \rightarrow v(z)$$

**OSS.** $\quad \operatorname{div}(v(z)) = 0$

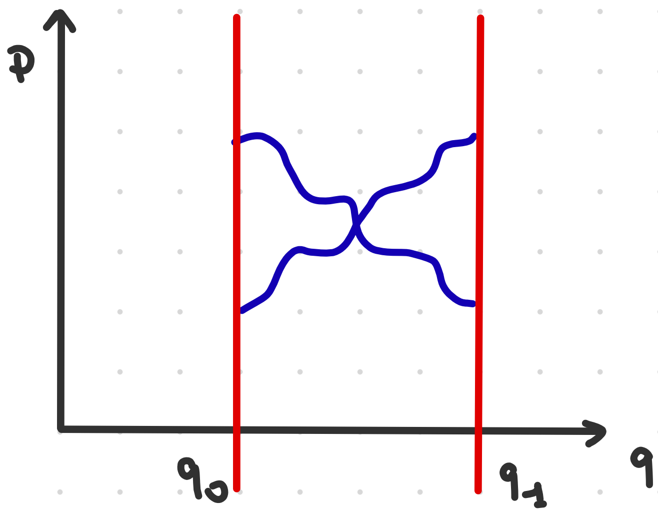
$$\operatorname{div}(v(z)) = \sum_{k=1}^{2n} \frac{\partial}{\partial z_k} v(z)_k = \sum_{k=1}^{2n} \frac{\partial}{\partial z_k} (E \nabla_z H)_k =$$

$$= \sum_{k=1}^{2n} \frac{\partial}{\partial z_k} \left( \sum_{j=1}^{2n} E_{kj} (\nabla_z H)_j \right) \qquad \rightarrow (\nabla_z H)_j = \frac{\partial H}{\partial z_j}$$

$$= \sum_{k=1}^{2n} \sum_{j=1}^{2n} E_{kj} \frac{\partial^2 H}{\partial z_k \partial z_j} \qquad \text{con } E_{kj} = -E_{jk} \Rightarrow \operatorname{div}(v(z)) = 0$$

## FORMULAZIONE VARIAZIONALE DELLE EQUAZIONI DI HAMILTON

$$S[q,p] = \int_{t_0}^{t_1} (p\dot{q} - H)\, dt$$



$$q(t) \to q(t) + \varepsilon h(t)$$

$$p(t) \to p(t) + \varepsilon k(t)$$

$$\begin{matrix} q(t_0) = q_0 \\ q(t_1) = q_1 \end{matrix} \quad \Rightarrow \quad h(t_0) = h(t_1) = 0$$

$$\lim_{\varepsilon \to 0} \frac{S[q+\varepsilon h, p+\varepsilon k] - S[q,p]}{\varepsilon}$$

$$\lim_{\varepsilon \to 0} \int_{t_0}^{t_1} \frac{\left[(p+\varepsilon k)(\dot q + \varepsilon \dot h) - H(q+\varepsilon h, p+\varepsilon k) - (p\dot q - H(q,p))\right]}{\varepsilon} dt$$

Espando al 1° ordine in $\varepsilon$

$$\lim_{\varepsilon \to 0} \int_{t_0}^{t_1} \frac{\left[\varepsilon(k\dot q + p\dot h) - \varepsilon\left(\frac{\partial H}{\partial q} h + \frac{\partial H}{\partial p} k\right) + o(\varepsilon)\right]}{\varepsilon} dt$$

$$= \int_{t_0}^{t_1} \left[k\left(\dot q - \frac{\partial H}{\partial p}\right) - h\left(\dot p + \frac{\partial H}{\partial q}\right)\right] dt$$

NOTA:

$$\int_{t_0}^{t_1} p\dot h \, dt = \underset{0}{\underset{\parallel}{ph}} \int_{t_0}^{t_1} dt - \int_{t_0}^{t_1} \dot p h \, dt$$

$$\Rightarrow = \int_{t_0}^{t_1} \left[k\left(\dot q - \frac{\partial H}{\partial p}\right) - h\left(\dot p + \frac{\partial H}{\partial q}\right)\right] dt$$

$q(t), p(t)$ soluzioni EQ. H $\Leftrightarrow$ rendono stazionario $S[q,p]$

## PARENTESI DI POISSON ($n$ gradi di libertà)

Date $f(q(t),p(t),t)$ e $g(q(t),p(t),t)$; Definisco PARENTESI DI POISSON:

$$\{f,g\} := \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)$$

$$\underline{z} = (\underline{q}, \underline{p})^T \qquad E = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right) \qquad \nabla_z f = \frac{\partial f}{\partial \underline{z}} = \begin{pmatrix} \frac{\partial f}{\partial \underline{q}} \\ \frac{\partial f}{\partial \underline{p}} \end{pmatrix}$$

$$\{f,g\} = \langle \nabla_z f, E \nabla_z g \rangle$$

$$E \cdot \nabla_z g = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \partial_q g \\ \partial_p g \end{pmatrix} = \begin{pmatrix} \partial_p g \\ -\partial_q g \end{pmatrix}$$

$$\langle \nabla_z f, E \cdot \nabla_z g\rangle = \left(\frac{\partial f}{\partial q}, \frac{\partial f}{\partial p}\right)\cdot\begin{pmatrix}\partial_p g\\ -\partial_q g\end{pmatrix} = \frac{\partial f}{\partial q}\cdot\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q}$$

PROPRIETA' $(f, g, h;\ \alpha \in \mathbb{R})$

1) ANTISIMMETRICA $\{f,g\} = -\{g,f\} \Rightarrow \{f,f\} = 0$

Dim

- $n=1$ $\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = -\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right) = -\{g,f\}$
- $n \geq 1$ $\{f,g\} = \nabla f^T E \nabla g = (E\nabla g)^T \nabla f = -\nabla g^T E \nabla f = -\{g,f\}$

2) LINEARITA' $\{(f+h), g\} = \{f,g\} + \{h,g\}$

$\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$

Dim

- $n=1$

$$\frac{\partial(\alpha f+\beta g)}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial(\alpha f+\beta g)}{\partial p}\frac{\partial h}{\partial q} = \left(\alpha\frac{\partial f}{\partial q} + \beta\frac{\partial g}{\partial q}\right)\frac{\partial h}{\partial p} - \left(\alpha\frac{\partial f}{\partial p} + \beta\frac{\partial g}{\partial q}\right)\frac{\partial h}{\partial p} =$$

$= \alpha\{f,h\} + \beta\{g,h\}$

- $n \geq 1$

$$[\nabla(\alpha f+\beta g)]^T E \nabla h = (\alpha\nabla f + \beta\nabla g)^T E\cdot\nabla h = (\alpha\nabla f^T + \beta\nabla g^T) E \nabla h =$$

$= \alpha \nabla f^T E \nabla h + \beta \nabla g^T E \nabla h = \alpha\{f,h\} + \beta\{g,h\}$

3) LIEBNIZ $\{fg, h\} = f\{g,h\} + \{f,h\}g$

Dim

- $n=1$ $\frac{\partial}{\partial q}(fg)\frac{\partial h}{\partial p} - \frac{\partial(fg)}{\partial p}\frac{\partial h}{\partial q} = \left(f\frac{\partial g}{\partial q} + \frac{\partial f}{\partial q}g\right)\frac{\partial h}{\partial p} - \left(\frac{\partial f}{\partial p}g + f\frac{\partial g}{\partial p}\right)\frac{\partial h}{\partial q}$

$= f\{g,h\} + \{f,h\}g$

- $n \geq 1$ $\{fg, h\} = (\nabla fg)^T E \nabla_g = (f\nabla g^T + \nabla f^T g) E \nabla h =$

$= f\{g,h\} + \{f,h\}g$

4) IDENTITA' DI JACOBI $\{f,\underset{(a)}{\{g,h\}}\} + \{g,\underset{(b)}{\{h,f\}}\} + \{h,\underset{(c)}{\{f,g\}}\} = 0$

Dim

Lemma $\nabla(\{g,h\}) = H_g E \nabla h - H_h E \nabla g$

Dim

Componente j-esima $\nabla(\{g,h\})$

$$\frac{\partial}{\partial z_j}(\{g,h\}) = \frac{\partial}{\partial z_j}(\nabla^T g\, E \nabla h) = \frac{\partial}{\partial z_j}\left(\sum_{\alpha=1}^{2n}\frac{\partial g}{\partial z_\alpha}(E\nabla h)_\alpha\right) =$$

$$= \frac{\partial}{\partial z_j}\left(\sum_{\alpha=1}^{2n}\frac{\partial g}{\partial z_\alpha}\left(\sum_{\beta=1}^{2n} E_{\alpha\beta}\frac{\partial h}{\partial z_\beta}\right)\right) = \sum_{\alpha,\beta=1}^{2n}\left[\frac{\partial}{\partial z_j}\left(\frac{\partial g}{\partial z_\alpha}E_{\alpha\beta}\frac{\partial h}{\partial z_\beta}\right)\right] =$$

$$= \sum_{\alpha,\beta=1}^{2n}\left[\underbrace{\frac{\partial^2 g}{\partial z_j\,\partial z_\alpha}}_{(H_g)_{j\alpha}} E_{\alpha\beta}\frac{\partial h}{\partial z_\beta} + \frac{\partial g}{\partial z_\alpha}\underbrace{E_{\alpha\beta}}_{-E_{\beta\alpha}}\underbrace{\frac{\partial^2 h}{\partial z_j\,\partial z_\beta}}_{(H_h)_{j\beta}}\right] =$$

$$= \underbrace{\sum_{\beta=1}^{2n}(H_g E)_{j\beta}(\nabla h)_\beta}_{[(H_g E)\nabla h]_j} - [H_h E \nabla g]_j$$

(a) $(\nabla f)^T E[H_g E\nabla h - H_h E\nabla g]$

(b) $(\nabla g)^T E[H_h E\nabla f - H_f E\nabla h]$

(c) $(\nabla h)^T E[H_f E\nabla g - H_g E\nabla f]$

$\Rightarrow (\nabla f)^T E H_g E\nabla h - (\nabla f)^T E H_h \nabla g + (\nabla g)^T E H_h E \nabla f - (\nabla g)^T E H_f E\nabla h +$

$+ (\nabla h)^T E H_f E\nabla g - (\nabla h)^T E H_g E \nabla f$

$(\nabla f)^T E H_g E\nabla h - (\nabla h)^T E H_g E\nabla f$

NOTA: $(EHE)^T = E^T H^T E^T = (-E)H(-E) = EHE$

$(\nabla f)^T E H_g E \nabla h - ((\nabla f)^T E H_g E\nabla h) = 0$

## EVOLUZIONE DI UNA VARIABILE DINAMICA

$(q(t), p(t))$ soluzioni equazioni Hamilton

$f(q(t), p(t), t)$

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt}(q,p,t) = \sum_{j=1}^{m} \frac{\partial f}{\partial q_j} \dot{q}_j + \sum_{j=1}^{m} \frac{\partial f}{\partial p_j} \dot{p}_j + \frac{\partial f}{\partial t}$$

$q(t), p(t)$ sol. E.H. $\Rightarrow$ $$\frac{df}{dt} = \underbrace{\sum_{j=1}^{m} \left( \frac{\partial f}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial H}{\partial q_j} \right)}_{\{f, H\}} + \frac{\partial f}{\partial t}$$

NOTA 1 : $f(q, p, \cancel{t}) \Rightarrow \dot{f} = \{f, H\}$

NOTA 2 : $f(q, p)$ se $\{f, H\} \Rightarrow \dot{f} = 0 \iff f$ COSTANTE DEL MOTO

NOTA 3 : se $H(q, p, \cancel{t}) \Rightarrow H$ è COSTANTE DEL MOTO

**Teo POISSON** $f(q, p, \cancel{t})$ e $g(q, p, \cancel{t})$ COSTANTI DEL MOTO per Hamiltoniana

$\Rightarrow h(q) = \{f, g\}(q, p)$ è una costante del moto

**Dim**

IDENTITA' DI JACOBI $\rightarrow$ $\{H, \{f, g\}\} + \{f, \{g, H\}\} + \{g, \{H, f\}\} = 0$

$f, g$ costanti del moto $\rightarrow \{g, H\} = 0$ ; $\{H, f\} = 0 \Rightarrow \{0, f\} = \{g, 0\} = 0$

$\Rightarrow \{H, \{f, g\}\} = 0 \Rightarrow \{f, g\}$ è costanti del moto

## TRASFORMAZIONI CANONICHE

$H(q, p, t)$

**DEF.** TRASFORMAZIONE CANONICA

$(q, p) \longmapsto (Q(q, p, t), P(q, p, t))$ $[\det J \neq 0]$ È CANONICA se

$\forall H(q, p, t)\ \exists K(Q, P, t)$ t.c.

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \rightsquigarrow \begin{cases} \dot{Q} = \frac{\partial K}{\partial P} \\ \dot{P} = -\frac{\partial K}{\partial Q} \end{cases}$$

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \iff S[p,q] = \int_{t_0}^{t_1} \left(p\dot{q} - H(q,p)\right) dt \quad \text{stazionarietà}$$

$$\begin{cases} \dot{Q} = \dfrac{\partial K}{\partial P} \\ \dot{P} = -\dfrac{\partial K}{\partial Q} \end{cases} \iff S[Q,P] = \int_{t_0}^{t_1} \left(P\dot{Q} - H(Q,P)\right) dt \quad \text{stazionarietà}$$

$$\rightarrow \; p\dot{q} - H = P\dot{Q} - K + \frac{dF}{dt}$$

$$\frac{dF}{dt} = (p\dot{q} - H) - (P\dot{Q} - K)$$

$$(q,p) \longmapsto q, Q$$

$$\frac{dF}{dt}(q,Q,t) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t} = \left(p(q,Q,t)\dot{q} - H\right) - \left(P\dot{Q} - K\right)$$

$F(q,Q,t) \rightarrow$ FUNZIONE GENERATRICE DI I SPECIE

$$\begin{cases} p(q,Q,t) = \dfrac{\partial F}{\partial q} \\ P(q,Q,t) = -\dfrac{\partial F}{\partial Q} \\ K = H + \dfrac{\partial F}{\partial t} \end{cases}$$

$$F(q_1, \dots, q_n, Q_1, \dots, Q_N, t)$$

$$\sum_{i=1}^{n} p_i \dot{q}_i - H = \sum_{i=1}^{n} P_i \dot{Q}_i - K + \frac{dF}{dt}$$

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$$

$$\rightarrow \begin{cases} p_i = \dfrac{\partial F}{\partial q_i} \\ P_i = \dfrac{\partial F}{\partial Q_i} \\ K = H + \dfrac{\partial F}{\partial t} \end{cases}$$

$F(q,P,t) \rightarrow$ FUNZIONE GENERATRICE DI II SPECIE

$$p\dot{q} - H = -Q\dot{P} - K + \frac{dF}{dt}$$

$$\frac{d}{dt}F(q,P,t) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial t} = p\dot{q} + Q\dot{P} + K - H \quad \forall \dot{q},\dot{p}$$

$$\begin{cases} Q(q,P,t) = \dfrac{\partial F}{\partial P} \\ p(q,P,t) = \dfrac{\partial F}{\partial q} \\ K = H + \dfrac{\partial F}{\partial t} \end{cases}$$

$$\begin{cases} Q = f(q) \rightarrow \dfrac{df}{dq} \neq 0 \\ P = \dfrac{p}{f'(q)} \end{cases}$$ TRASFORMAZIONE CANONICA

FUNZIONE GENERATRICE DI III SPECIE $(p,Q)$

$$-q\dot{p} - H = P\dot{Q} - K + \frac{dF}{dt} = P\dot{Q} - K + \left(\frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t}\right)$$

$$\begin{cases} q = -\dfrac{\partial F}{\partial p} \\ P = -\dfrac{\partial F}{\partial Q} \\ K = H + \dfrac{\partial F}{\partial t} \end{cases}$$

FUNZIONE GENERATRICE DI IV SPECIE $(p,P)$

$$-q\dot{p} - H = -Q\dot{P} - K + \frac{dF}{dt}$$

$$\begin{cases} q = -\dfrac{\partial F}{\partial p} \\ Q = \dfrac{\partial F}{\partial P} \\ K = H + \dfrac{\partial F_4}{\partial t} \end{cases}$$

## TEOREMA DELLE TRASFORMAZIONI DI CONTATTO

$Q_j = f(q_1, \dots, q_n)$

$$\mathcal{P}_j = \sum_{k=1}^{n} A_{jk}(q_1, \dots, q_n)\, \mathcal{P}_k \qquad (\mathcal{P} = A(\ )\mathcal{P})$$

è canonica $\iff A^{-1} = J^T$

$$\det \underset{(q,p)\mapsto(Q,\mathcal{P})}{J} = \left(\begin{array}{c|c} \frac{\partial f_i}{\partial q_i} & \phi \\ \hline (\ \ ) & A \end{array}\right)$$

$$\begin{cases} \det_{q\to Q} \neq 0 \\ \det A \neq 0 \end{cases}$$

Data $(q,p) \mapsto (Q,\mathcal{P})$ quali specie di generatrici esistono?

$$J = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial \mathcal{P}}{\partial q} & \frac{\partial \mathcal{P}}{\partial p} \end{pmatrix}$$

1ª specie: $(q,p) \mapsto (q, Q)$ $\qquad J = \begin{pmatrix} 1 & 0 \\ \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \end{pmatrix}$ $\exists F_I \iff \frac{\partial Q}{\partial p} \neq 0$

2ª specie: $(q,p) \mapsto (q,\mathcal{P})$ $\qquad J = \begin{pmatrix} 1 & 0 \\ \frac{\partial \mathcal{P}}{\partial q} & \frac{\partial \mathcal{P}}{\partial p} \end{pmatrix}$ $\exists F_{II} \iff \frac{\partial \mathcal{P}}{\partial p} \neq 0$

3ª specie: $\exists F_{III} \iff \frac{\partial Q}{\partial q} \neq 0$

4ª specie: $\exists F_{IV} \iff \frac{\partial \mathcal{P}}{\partial q} \neq 0$

<u>Teo</u> Sia $(q,p) \mapsto (Q,\mathcal{P})$ una trasformazione di coordinate che non dipende esplicitamente dal tempo

$\to$ È CANONICA $\iff$ preserva le parentesi di Poisson fondamentali

$Q_j(q,p)$, $P_j(q,p)$

- $\{Q_j, Q_k\} = 0$
- $\{P_j, P_k\} = 0$
- $\{Q_j, P_k\} = \delta_{jk} \quad \forall\, j,k = 1,\dots,n$

Dim

$- n = 1$

$\rightarrow \{Q, Q\} = 0 = \{P, P\}$ sempre

$$\dot{Q} = \{Q, H\} = \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q}$$

$$H = K(Q(q,p), P(q,p)) \qquad \frac{\partial H}{\partial q} = \frac{\partial K}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p}$$

$$\Rightarrow \dot{Q} = \frac{\partial Q}{\partial q}\left(\frac{\partial K}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial Q}{\partial p}\left(\frac{\partial K}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial K}{\partial P}\frac{\partial P}{\partial p}\right)$$

$$\begin{cases} \dot{Q} = \frac{\partial K}{\partial Q}\left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) + \frac{\partial K}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) = \frac{\partial K}{\partial P}\{Q, P\} \\ \dot{P} = \{P, H\} = \frac{\partial K}{\partial q}\{P, Q\} \end{cases}$$

$- n \geq 1 \qquad \underline{z} = (z_1, \dots, z_n) = (Q_1, \dots, Q_n, P_1, \dots, P_n)$

$$\{z_\alpha, z_\beta\} = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right) = E_{\alpha\beta}$$

$$\dot{z}_\alpha = \{z_\alpha, H\} = \langle \nabla_z, E \nabla_z H \rangle = \sum_{j,k=1}^{2n} \frac{\partial z_\alpha}{\partial z_j} E_{jk} \frac{\partial H}{\partial z_k}$$

$$\frac{\partial H}{\partial z_k} = \sum_{l=1}^{2n} \frac{\partial K}{\partial z_l}\frac{\partial z_l}{\partial z_k}$$

$$\Rightarrow \dot{z}_\alpha = \sum_{j,k=1}^{2n} \frac{\partial z_\alpha}{\partial z_k} E_{kj} \left(\sum_{l=1}^{2n} \frac{\partial K}{\partial z_l}\frac{\partial z_l}{\partial z_k}\right) = \sum_{l=1}^{2n} \frac{\partial K}{\partial z_l}\left(\sum_{j,k=1}^{2n} \frac{\partial z_\alpha}{\partial z_k} E_{kj} \frac{\partial z_l}{\partial z_j}\right)$$

$$\dot{z}_\alpha = \sum_{l=1}^{2n} \frac{\partial K}{\partial z_l}\{z_\alpha, z_l\} \underset{\substack{\downarrow \\ \text{canonica}}}{\overset{?}{=}} \dot{\underline{z}} = E \nabla_z H \iff \{z_\alpha, z_l\} = E$$

## TRASFORMAZIONI CANONICHE INFINITESIME

$$\begin{cases} Q = q + \varepsilon A(q,p) + o(\varepsilon) \\ P = p + \varepsilon B(q,p) + o(\varepsilon) \end{cases}$$

$F = qP \Rightarrow F = qP + \varepsilon F_1(q,P) + o(\varepsilon)$

$$\begin{cases} Q = \dfrac{\partial F}{\partial P} = q + \varepsilon \dfrac{\partial F_1}{\partial P}(q,P) + o(\varepsilon) \\ p = \dfrac{\partial F}{\partial q} = P + \varepsilon \dfrac{\partial F_1}{\partial q}(q,P) + o(\varepsilon) \end{cases}$$

oss. $p - P = O(\varepsilon) = \varepsilon \dfrac{\partial F_1}{\partial q} + o(\varepsilon)$

Infinitesima è canonica se $\exists F_1(q,P)$ t.c. $A(q,p) = \dfrac{\partial F_1}{\partial p}$ ; $B(q,p) = -\dfrac{\partial F_1}{\partial q}$

$F = qP + \varepsilon F_I(q,p) + o(\varepsilon)$

$$\begin{cases} Q = q + \varepsilon \dfrac{\partial F_I}{\partial p}(q,p) + o(\varepsilon) =: q(\varepsilon) \\ P = p - \varepsilon \dfrac{\partial F_I}{\partial q}(q,p) + o(\varepsilon) \end{cases}$$

$$\lim_{\varepsilon \to 0} \frac{Q-q}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{q(\varepsilon) - q(0)}{\varepsilon} = \frac{d\, q(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{\partial F_I}{\partial p}$$

$$\begin{cases} \dfrac{d\, q(\varepsilon)}{d\varepsilon} = \dfrac{\partial F_I}{\partial p} \\ \dfrac{d\, p(\varepsilon)}{d\varepsilon} = -\dfrac{\partial F_I}{\partial q} \end{cases}$$

$\varepsilon \leftrightarrow t$
$F_I \leftrightarrow H$

## TEOREMA LIOUVIOLLE Conservazione volume nello spazio delle fasi

$\phi^t : (q_0, p_0) \overset{H}{\curvearrowright} (q_t, p_t)$

$Vol(D_t) = \int_{D_t} dq_t \, dp_t = \int_{D_0} dp_0 \, dq_0 \, |\det J_{\phi_t}|$

$Vol(D_t) = Vol(D_0) \Leftrightarrow \det J_{\phi_t} = \pm 1$

**LEMMA:** Se $(Q,P)$ è trasformazione canonica $\Rightarrow \det J_{(q,p)\to(Q,P)} = \pm 1$

Dim $\{Z_\alpha, Z_\beta\} = E_{\alpha\beta}$ se tr. canonica

$$\rightarrow \sum_{j,k=1}^{2n} \frac{\partial Z_\alpha}{\partial z_k} E_{kj} \frac{\partial Z_\beta}{\partial z_j} = \left( J_{z\to Z}\, E\, J^T_{z\to Z} \right)_{\alpha\beta}$$

Tr. canonica $\Rightarrow J E J^T = E \Rightarrow \det(J E J^T) = \det(E) = 1$

$\rightarrow \det(J)\det(E)\det(J^T) = (\det(J))^2 = 1$

$\Rightarrow \det J = \pm 1$

La Composizione di due tr. canoniche è canonica

Se $A$ è canonica $\Rightarrow J_A E J_A^T = E$
Se $B$ è canonica $\Rightarrow J_B E J_B^T = E$

$A \circ B$ è canonico infatti $J_{A\circ B} = J_A \cdot J_B$

$$J_{A\circ B} E J^T_{A\circ B} = J_A (J_B E J_B^T) J_A^T = J_A E J_A^T = E$$

## TEOREMA NOETHER Formalismo Hamiltoniano

$G(q,p;s)$ famiglia da 1 parametro di tr. canoniche

fissato $s \in I \Rightarrow (q,p) \xrightarrow{G} (Q,P)$ è canonica

$G$ è detto GRUPPO ad un parametro di tr. canoniche se:

1) $G(q,p;0)$ è IDENTITA'

2) $G(q,p;-s) = G^{-1}(q,p;s)$

3) $G(G(q,p;s_1);s_2) = G(q,p;s_3 = s_2+s_1)$

$$\begin{tikzcd}
(q,p) \arrow[r, bend left, "s_1"] \arrow[rr, bend right, "s_3 = s_1+s_2"'] & (Q,P) = G(q,p;s_1) \arrow[r, bend left, "s_2"] & (Q',P') = G(Q,P;s_2)
\end{tikzcd}$$

4) $G(\ ;s)$ è derivabile rispetto ad $s$

per $s$ piccolo $\Rightarrow$ trasformazione canonica infinitesima

$\Rightarrow F$ funzione generatrice di II specie $\rightarrow F(q,P,s) = qP + s F_I(q,P) + o(s)$

$$\begin{cases} Q = q + s \dfrac{\partial F_I}{\partial p}(q,p) + o(s) \\ P = p - s \dfrac{\partial F_I}{\partial q}(q,p) + o(s) \end{cases}$$

DEF Un gruppo ad 1 parametro di tr. canoniche è detto SIMMETRIA per $H$ se:

$$H(G(q,p;s)) = H(q,p) \quad \forall s \in I$$

SIMMETRIA INFINITESIMA $\rightarrow H(G(q,p;s)) = H(q,p) + o(s)$

## TEOREMA NOETHER Formalismo Hamiltoniano

$G$ è una simmetria (infinitesima) $\Leftrightarrow F_I$ è una costante del moto

Dim

$$H(Q(q,p;s), P(q,p;s)) \underset{\text{svi luppo in } s=0}{=} H(q,p) + \frac{\partial H}{\partial q}\frac{\partial F_I}{\partial p} \cdot s - \frac{\partial H}{\partial p}\frac{\partial F_I}{\partial q} s + o(s)$$

$$H(Q,P;s) - H(q,p) = s\{H, F_I\} + o(s)$$

simmetria(infinitesima) $\Rightarrow \{H, F_I\} = 0 \Leftrightarrow \{F_I, H\} = 0 \Leftrightarrow F_I$ costante del moto

## EQUAZIONE DI HAMILTON-JACOBI

$$H(q,p,t) \xrightarrow{\text{tr. canonica}} K(Q,P,t) = 0 \qquad \begin{cases} \dot{Q} = 0 \\ \dot{P} = 0 \end{cases} \rightarrow \quad \begin{aligned} Q(t) &= Q_0 \\ P(t) &= P_0 \end{aligned}$$

$$K = H + \frac{\partial F}{\partial t} \Rightarrow H + \frac{\partial F}{\partial t} = 0$$

Se $F$ è di I specie $\Rightarrow F(q,P) \rightarrow p = \dfrac{\partial F}{\partial q} \Rightarrow H = H(q,p) = H\left(q, \dfrac{\partial F}{\partial q}\right)$

$$H\left(q, \frac{\partial F}{\partial q}\right) + \frac{\partial F}{\partial t}(q,p) = 0 \rightarrow \text{EQ. HAMILTON-JACOBI}$$

PIÙ GRADI DI LIBERTA'

$H(\underline{q},\underline{p},t) \to K(\underline{Q},\underline{P},t) = 0 \quad j=1,\dots,d \quad \begin{cases} \dot{Q}_j = 0 \\ \dot{P}_j = 0 \end{cases} \Rightarrow \begin{array}{l} Q_j(t) = Q_{j0} \\ P_j(t) = P_{j0} \end{array}$

$K = H(q_1,\dots,q_d,p_1,\dots,p_d,t) + \frac{\partial F}{\partial t}$

$F_{II} \Rightarrow p_j = \frac{\partial F}{\partial q_j} \Rightarrow H\left(q_1,\dots,q_d,\frac{\partial F}{\partial q_1},\dots,\frac{\partial F}{\partial q_d},t\right) + \frac{\partial F}{\partial t} = 0$ EQ. H-J

1) ipotesi di SEPARAZIONE $F(q,t) = W(q) + F(t) \rightsquigarrow P$ (famiglia ad 1 parametro)

2) Necessario $(q,P) \to (q,p)$ buono $\Rightarrow \frac{\partial^2 F}{\partial q \partial P} \neq 0$

$n \geq 1$ g.d.l. $F(q_1,\dots,q_d,t)$ soddisfa $H\left(\frac{\partial F}{\partial q_j} = p_j, q_j, t\right) + \frac{\partial F}{\partial t} = 0$

1) $F(q_1,\dots,q_d,t) = W(q_1,\dots,q_d) + G(t) \rightsquigarrow p_j$ come parametri

$\Leftrightarrow F_{\underline{P}}(\underline{q},t)$ a $d$ parametri

2) $(\underline{q},\underline{P}) \to (\underline{q},\underline{p})$ buona $\Rightarrow \det J \neq 0 \Leftrightarrow \det\left(\frac{\partial \underline{p}}{\partial \underline{P}}\right) \neq 0 \Leftrightarrow \det\left(\frac{\partial^2 F}{\partial \underline{q} \partial \underline{P}}\right) \neq 0$

DEF. Una soluzione $F$ dell'equazione H-J è detta INTEGRALE COMPLETO se dipende da $d$ parametri $P_1,\dots,P_d$ t.c.

$$\det\left(\frac{\partial^2 F}{\partial q_j \partial P_k}\right) \neq 0$$

OSS. (1 g.d.l.)

1) $p = \frac{\partial F}{\partial q}$

2) $H\left(q, \frac{\partial F}{\partial q}, t\right) = -\frac{\partial F}{\partial t}$

3) $\begin{cases} \dot{P} = 0 \\ \dot{Q} = 0 \end{cases}$

$$\frac{dF}{dt} \overset{\text{lungo moto}}{=} \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial t} = p\dot{q} + 0 - H(q,p,t)$$

lungo il moto -> $\frac{dF}{dt} = p\dot{q} - H(q,P,t)$ $F$ = integrale di azione valutato lungo il moto

$F(q,P,t)$ è l'azione vista come funzione del suo secondo estremo di integrazione calcolata lungo il moto

$$S[q,p] = \int_{t_0}^{t_1} (p\dot{q} - H)\,dt$$