Verjetnost - teoretični izpit   7.7.2023

1. A, B_1, B_2 neodvisni $\Rightarrow A, B_1 \cup B_2$ neodvisna

$\mathbb{P}(A\cap B_1 \cap B_2) = \mathbb{P}(A)\mathbb{P}(B_1)\mathbb{P}(B_2)$ def. neodvisnosti

~~B_1, B_2 neodvisna $\Rightarrow \mathbb{P}(B_1\cup B_2) = \mathbb{P}(B_1)+\mathbb{P}(B_2)$, $\mathbb{P}(B_1\cap B_2)=\emptyset \Rightarrow \mathbb{P}(A\cap B_1 \cap B_2)=0$~~

$$\mathbb{P}(A\cap (B_1\cup B_2)) = \mathbb{P}((A\cap B_1)\cup(A\cap B_2)) = \mathbb{P}(A\cap B_1)+\mathbb{P}(A\cap B_2) - \mathbb{P}((A\cap B_1)\cap(A\cap B_2))$$
$$= \mathbb{P}(A)\mathbb{P}(B_1)+\mathbb{P}(A)\mathbb{P}(B_2)-\mathbb{P}(A)\mathbb{P}(B_1)\mathbb{P}(B_2)$$
$$= \mathbb{P}(A)(\mathbb{P}(B_1)+\mathbb{P}(B_2)-\mathbb{P}(B_1)\mathbb{P}(B_2))$$
$$= \mathbb{P}(A)(\mathbb{P}(B_1)+\mathbb{P}(B_2)-\mathbb{P}(B_1\cap B_2)) = \underline{\underline{\mathbb{P}(A)\mathbb{P}(B_1\cup B_2)}}$$

2. $\mathbb{P}(X=k) = \frac{6}{\pi^2 k^2}$, $k=1,2,\dots \Rightarrow \{X\in\{2,4,6,\dots\}\}$ in $\{X\in\{3,6,9,\dots\}\}$ neodvisna

$A := \{2,4,6,\dots\}$, $B := \{3,6,9,\dots\}$, $A\cap B = \{6,12,18,\dots\}$

$$\mathbb{P}(X\in A\cap B) = \mathbb{P}\Big(X\in \bigcup_{k=1}^{\infty}\{6k\}\Big) = \sum_{k=1}^{\infty}\mathbb{P}(X=6k) = \sum_{k=1}^{\infty}\frac{6}{\pi^2 36k^2}\cdot\frac{6\pi^2k^2}{6\pi^2k^2} =$$

$$\mathbb{P}(X\in A)\mathbb{P}(X\in B) = \sum_{k=1}^{\infty}\mathbb{P}(X=2k)\cdot\sum_{l=1}^{\infty}\mathbb{P}(X=3l) = \sum_{k=1}^{\infty}\frac{6}{\pi^2 4k^2}\sum_{l=1}^{\infty}\frac{6}{\pi^2 9l^2}$$
$$= \frac{36}{\pi^4\cdot 36}\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{l=1}^{\infty}\frac{1}{l^2}.$$

$$\mathbb{P}(X\in A, X\in B) = \mathbb{P}\Big(\bigcup_{k=1}^{\infty}\{2k\}, \bigcup_{l=1}^{\infty}\{3l\}\Big) = \mathbb{P}\Big(X\in\bigcup_{k=1}^{\infty}\{2k\}\cap\{3k\}\Big) = \sum_{k=1}^{\infty}\mathbb{P}(X\in\{2k\}\cap\{3k\})$$
$$= \sum_{k=1}^{\infty}\frac{6}{\pi^2 4k^2}\cdot\frac{6}{\pi^2 9k^2} = \sum_{k=1}^{\infty}\mathbb{P}(X\in\{2k\})\mathbb{P}(X\in\{3k\})$$
$$= \mathbb{P}\Big(X\in\bigcup_{k=1}\{2k\}\Big)\mathbb{P}\Big(X\in\bigcup_{k=1}\{3k\}\Big) = \mathbb{P}(X\in A)\mathbb{P}(X\in B)$$ *preveri*

3. X, Y celoštevilski; $X, -X$ enako porazdeljeni, $Y, -Y$ enako porazdeljeni

$|X|, |Y|$ neodvisni, ali sta X in Y neodvisni?

$$\mathbb{P}(|X|=k, |Y|=l) \underset{k,l>0}{=} \mathbb{P}(|X|=k)\mathbb{P}(|Y|=l) = \mathbb{P}(\{X=k\}\cup\{-X=k\})\mathbb{P}(\{Y=l\}\cup\{-Y=l\})$$
$$= (\mathbb{P}(X=k)+\mathbb{P}(-X=k))(\mathbb{P}(Y=l)+\mathbb{P}(-Y=l))$$
$$= \mathbb{P}(X=k)\mathbb{P}(Y=l)+\mathbb{P}(X=-k)\mathbb{P}(Y=l)+\mathbb{P}(X=k)\mathbb{P}(Y=-l)+\mathbb{P}(X=-k)\mathbb{P}(Y=-l)$$
$$= \mathbb{P}(X=k)(\mathbb{P}(Y=l)+\mathbb{P}(Y=-l))+\mathbb{P}(X=-k)(\mathbb{P}(Y=l)+\mathbb{P}(Y=-l))$$
$$\underset{a\in\mathbb{Z}}{=} \mathbb{P}(X=k)\mathbb{P}(Y=a)+\mathbb{P}(X=-k)\mathbb{P}(Y=a) \underset{b\in\mathbb{Z}}{=} \mathbb{P}(X=b)\mathbb{P}(Y=a)$$

$$\mathbb{P}(|X|=k, |Y|=l) = \mathbb{P}(\{X=k\}\cup\{X=-k\}, \{Y=l\}\cup\{Y=-l\}) =$$

$= \mathbb{P}\left(\{X=k\}\cap\{Y=l\} \cup \{X=k\}\cap\{Y=-l\} \cup \{X=-k\}\cap\{Y=l\} \cup \{X=-k\}\cap\{Y=-l\}\right)$

$= \mathbb{P}\left(\{X=k\}\cap(\{Y=l\}\cup\{Y=-l\}) \cup \{X=-k\}(\{Y=l\}\cup\{Y=-l\})\right)$

$= \mathbb{P}\left(\{X=k\}\cap\{Y=a\} \cup \{X=-k\}\{Y=a\}\right) = \mathbb{P}(X=b, Y=a)$

4. $X>0$, $f_X(x)$ gostota, $f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi x}} f_X(x) e^{-\frac{(y-x)^2}{2x}}$, $Z = \frac{Y-X}{\sqrt{X}}$, X, Z neodvisni

$\Phi(x,y) = \left(x, \frac{y-x}{\sqrt{x}}\right)$, $\Phi^{-1}(x,z) = (x, z\sqrt{x}+x)$

$J\Phi^{-1}(x,z) = \begin{bmatrix} 1 & \frac{z}{2\sqrt{x}}+1 \\ 0 & \sqrt{x} \end{bmatrix} \Rightarrow \det(J\Phi^{-1}(x,z)) = \sqrt{x}$

$f_{X,Z}(x,z) = f_{X,Y}(\Phi^{-1}(x,z))|\det J\Phi^{-1}(x,z)| = \frac{1}{\sqrt{2\pi x}} f_X(x) e^{-\frac{(z\sqrt{x}+x-x)^2}{2x}} \sqrt{x}$

$= \frac{1}{\sqrt{2\pi}} f_X(x) e^{-\frac{z^2 x}{2x}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} f_X(x) = g(z) f(x)$

$f(x) = f_X(x)$, $g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = f_Z(z) \Rightarrow$ X in Z sta neodvisni

5. $U, V \sim U(0,1)$ neodvisni, $X = \log((1-U)(1-V)) \sim ?$

$f_U(u) = 1$, $f_V(v) = 1$ za vse $u, v > 0$; $f_{U,V}(u,v) = f_U(u) f_V(v)$

$X = \ln((1-U)(1-V)) \Rightarrow e^x = (1-U)(1-V) \Rightarrow 1-U = \frac{e^x}{1-V} \Rightarrow U = \frac{1-V-e^x}{1-V}$

$\Phi(U,V) = (\ln((1-U)(1-V)), V)$, $\Phi^{-1}(x,V) = \left(\frac{1-v-e^x}{1-v}, V\right)$

$J\Phi^{-1}(x,v) = \begin{bmatrix} \frac{-e^x}{1-v} & 0 \\ \frac{e^x}{(1-v)^2} & 1 \end{bmatrix} \Rightarrow |\det J\Phi(x,v)| = \frac{e^x}{1-v}$

$f_{X,V}(x,v) = \frac{e^x}{1-v} \Rightarrow f_X(x) = e^x$ za $x \in (-\infty, 0)$ *preveri*

6. X, Y z gostoto $f_{X,Y}(x,y)$; X, $Y-aX$ neodvisni za določena; $E(Y|X=x) = ?$ za $f_X(x) > 0$

Velja $f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

$\Phi(x,y) = (x, y-ax)$, $\Phi^{-1}(x,z) = (x, z+ax)$, $J\Phi^{-1} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \Rightarrow |\det J\Phi^{-1}| = 1$

$f_{X,Z}(x,z) = f_{X,Y}(x, z+ax) = f_X(x) f_Z(z) \Rightarrow f_X(x) f_{Y-aX}(y-ax) = f_{X,Y}(x,y)$

$f_{Y|X=x}(y) = \frac{f_X(x) f_{Y-aX}(y-ax)}{f_X(x)} = f_{Y-aX}(y-ax)$

$\Rightarrow E(Y|X=x) = \int_{-\infty}^{\infty} y f_{X-aX}(y-ax)\, dy$

7. X, Y, Z nenegativne celoštevilske, $\mathbb{P}(X=i, Y=j, Z=k) = \mathbb{P}(X=i, Y=k, Z=j)$ za $\forall (i,j,k)$

$\forall i \in \{0,1,\dots\}$. $\mathbb{E}(Y|X=i) = \mathbb{E}(Z|X=i)$

$$\mathbb{P}(Y=j) = \sum_{i,k} \mathbb{P}(X=i, Y=j, Z=k) = \sum_{i,k} \mathbb{P}(X=i, Y=k, Z=k) = \mathbb{P}(Z=j)$$

$\Rightarrow$ Y in ~~X~~ sta enako porazdeljeni

$$\mathbb{E}(Y|X=i) = \sum_j j\, \mathbb{P}(Y=j|X=i) = \sum_j j \frac{\mathbb{P}(X=i, Y=j)}{\mathbb{P}(X=i)} \overset{*}{=} \sum_j j \frac{\mathbb{P}(X=i, Z=j)}{\mathbb{P}(X=i)} = \mathbb{E}(Z|X=i)$$

$$\mathbb{P}(X=i, Y=j) = \sum_k \mathbb{P}(X=i, Y=j, Z=k) = \sum_k \mathbb{P}(X=i, Y=k, Z=j) = \mathbb{P}(X=i, Y=j)$$

preveri

8. $X_1, X_2, \dots, X_n$ neodvisne, enako porazdeljene, celoštevilske; $S_n := X_1 + X_2 + \dots + X_n$

$$\frac{k}{n} = \mathbb{E}(X_1|S_n=k) = \sum_{j=0}^{k} \frac{j\, \mathbb{P}(X_1=j)\mathbb{P}(S_{n-1}=k-j)}{\mathbb{P}(S_n=k)}$$

$$\mathbb{E}(X_1|S_n=k) = \sum_{j=0}^{k} j\, \mathbb{P}(X_1=j|S_n=k) = \sum_{j=0}^{k} j \frac{\mathbb{P}(X_1=j, S_n=k)}{\mathbb{P}(S_n=k)} =$$

$$= \sum_{j=0}^{k} j \frac{\mathbb{P}(X_1=j, X_2+\dots+X_n=k-j)}{\mathbb{P}(S_n=k)} \overset{\text{neodvisnost}}{=} \sum_{j=0}^{k} j \frac{\mathbb{P}(X_1=j)\mathbb{P}(X_2+\dots+X_n=k-j)}{\mathbb{P}(S_n=k)}$$

$$= \sum_{j=0}^{k} j \frac{\mathbb{P}(X_1=j)\mathbb{P}(S_{n-1}=k-j)}{\mathbb{P}(S_n=k)} = \frac{1}{\mathbb{P}(S_n=k)} \sum_{j=0}^{k} j\, \mathbb{P}(X_1=j)\mathbb{P}(S_{n-1}=k-j)$$

kaj zdaj ?!!

9. $X_1, X_2, \dots, X_n$; $X_1, X_2-\rho X_1, X_3-\rho X_2, \dots, X_n-\rho X_{n-1}$ nekorelirane ≟ neodvisne

$\mathbb{E}(X_1)=0$, $\mathrm{var}(X_1)=1$, $\mathrm{cov}(X_1, X_n)=?$

preveri

Označimo: $Y_1 = X_1$, $Y_2 = X_2 - \rho X_1, \dots, Y_n = X_n - \rho X_{n-1}$

Vse Y_i so neodvisne. Torej je $\mathrm{cov}(Y_i, Y_j) = 0$ za $i \neq j$.

$$X_n = Y_n - \rho X_{n-1} = Y_n - \rho Y_{n-1} + \rho^2 X_{n-2} = \sum_{i=0}^{n-1} (-\rho)^i Y_{n-i}$$

$$\Rightarrow \mathrm{cov}(X_1, X_n) = \mathrm{cov}\Big(Y_1, \sum_{i=0}^{n-1} (-\rho)^i Y_{n-i}\Big) \overset{\text{bilinearnost}}{=} \sum_{i=0}^{n-1} (-\rho)^i \mathrm{cov}(Y_1, Y_{n-i})$$

$$= (-\rho)^{n-1} \mathrm{var}(Y_1) + \sum_{i=0}^{n-2} (-\rho)^i \underbrace{\mathrm{cov}(Y_1, Y_{n-i})}_{0} \Rightarrow \underline{\underline{\mathrm{cov}(X_1, X_n) = (-\rho)^{n-1}}}$$