Verjetnost - teorija 17. 3. 2023

1. Vsak dogodek iz $\{A, B, A\cap B\}$ neodvisen od $\{C, D, C\cap D\}$ $\Rightarrow$ $A\cup B$, $C\cup D$ neodvisna

$$\mathbb{P}((A\cup B)\cap(C\cup D)) = \mathbb{P}((A\cap C)\cup(A\cap D)\cup(B\cap C)\cup(B\cap D)) =$$

$$= \mathbb{P}(A\cap C) + \mathbb{P}(A\cap D) + \mathbb{P}(B\cap C) + \mathbb{P}(B\cap D) - \mathbb{P}(A\cap C\cap D) - \mathbb{P}(A\cap B\cap C) - \mathbb{P}(A\cap C\cap B\cap D) - \mathbb{P}(A\cap D\cap B\cap C) - \mathbb{P}(A\cap D\cap B)$$
$$- \mathbb{P}(B\cap C\cap D) + 4\,\mathbb{P}(A\cap B\cap C\cap D) - \mathbb{P}(A\cap B\cap C\cap D),$$

$$= \mathbb{P}(A)\mathbb{P}(C) + \mathbb{P}(A)\mathbb{P}(D) + \mathbb{P}(B)\mathbb{P}(C) + \mathbb{P}(B)\mathbb{P}(D) - \mathbb{P}(A)\mathbb{P}(C\cap D) - \mathbb{P}(A\cap B)\mathbb{P}(C) - 2\mathbb{P}(A\cap B)\mathbb{P}(C\cap D) - \mathbb{P}(A\cap B)\mathbb{P}(D)$$
$$- \mathbb{P}(B)\mathbb{P}(C\cap D) + 3\mathbb{P}(A\cap B)\mathbb{P}(C\cap D)$$

$$= \mathbb{P}(A)(\mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(C\cap D)) + \mathbb{P}(B)(\mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(C\cap D)) - \mathbb{P}(A\cap B)(\mathbb{P}(C) + \mathbb{P}(D) - \mathbb{P}(C\cap D))$$

$$= \mathbb{P}(A)\mathbb{P}(C\cup D) + \mathbb{P}(B)\mathbb{P}(C\cup D) - \mathbb{P}(A\cap B)\mathbb{P}(C\cup D) = \underline{\underline{\mathbb{P}(A\cup B)\mathbb{P}(C\cup D)}}$$

2. $X_1, X_2, \dots, X_n$ zvezne omejene gostote, $X_1 \sim U(0,1)$, $f_{X_{k+1}}(x) = \int_0^1 f_{X_k}(x-u)\,du$, $\mathbb{E}(X_k) = ?$

$$\mathbb{E}(X_k) = \int_0^1 x f_{X_k}(x)\,dx = \int_0^1 x\,dx \int_0^1 f_{X_{k-1}}(x-u)\,du \overset{\text{Fubini}}{=} \int_0^1 du \int_0^1 x f_{X_{k-1}}(x-u)\,dx \overset{\substack{t = x-u \\ dt = dx}}{=}$$

$$= \int_0^1 du \int_0^1 (u+t) f_{X_{k-1}}(t)\,dt = \int_0^1 u\,du \underbrace{\int_0^1 f_{X_{k-1}}(t)\,dt}_{1} + \int_0^1 u\,du \underbrace{\int_0^1 t f_{X_{k-1}}(t)\,dt}_{\mathbb{E}(X_{k-1})}$$

$$= \int_0^1 u\,du + \int_0^1 \mathbb{E}(X_{k-1})\,u\,du = \underline{\underline{1 + \mathbb{E}(X_{k-1})}} = 1 + 1 + \mathbb{E}(X_{k-2}) = \dots =$$

$$= k-1 + \mathbb{E}(X_1) = k-1 + \int_0^1 x\,dx = k-1+1 = \underline{\underline{k}}$$

3. $I_1, \dots, I_n$ idikatorji, $\mathbb{P}(I_1 = i_1, \dots, I_n = i_n) = \frac{s!(n-s)!}{(n+1)!}$, $i_k \in \{0,1\}$, $s = \sum_{k=1}^n i_k$; $(I_1, I_2) \sim ?$

$$\mathbb{P}(I_1 = i_1, I_2 = i_2) = \sum_{\substack{(i_3,\dots,i_n)\in\{0,1\}^{n-2} \\ \sum_{k=3}^n i_k = s-i_1-i_2}} \mathbb{P}(I_3 = i_3, \dots, I_n = i_n) = \sum_{\substack{(i_3,\dots,i_n)\in\{0,1\}^{n-2} \\ \sum i_k = s-i_1-i_2}} \frac{(s-i_1-i_2)!(n-s+i_1+i_2)!}{(n-1)!}$$

$$= (s-i_1-i_2)\frac{(s-i_1-i_2)!(n-s+i_1+i_2)!}{(n-1)!} = n(s-i_1-i_2)\frac{1}{\binom{n}{s-i_1-i_2}}$$

preveri

4. $X, Y, Z \sim \exp(1)$ neodvisne, $U = \frac{X}{X+Y}$, $V = \frac{Z}{X+Y+Z}$ neodvisni

$$\Phi(x,y,z) = \left(x, \frac{x}{x+y}, \frac{z}{x+y+z}\right), \quad \Phi^{-1}(x,u,\vartheta) = \left(x, x(1-u), \frac{(2-u)x\vartheta}{1-\vartheta}\right); \qquad \Phi: (0,\infty)^3 \to (0,\infty)\times(0,1)^2$$

$$J\Phi^{-1} = \begin{bmatrix} 1 & 1-u & \frac{(2-u)\vartheta}{1-\vartheta} \\ 0 & -x & -\frac{x\vartheta}{1-\vartheta} \\ 0 & 0 & \frac{(2-u)x(1-\vartheta)+(2-u)x\vartheta}{(1-\vartheta)^2} \end{bmatrix} \Rightarrow |\det J\Phi^{-1}| = \left| -x \, \frac{(2-u)x - \cancel{(2-u)x\vartheta} + \cancel{(2-u)x\vartheta}}{(1-\vartheta)^2} \right|$$

$$\Rightarrow |\det J\Phi^{-1}| = \frac{(2-u)x^2}{(1-\vartheta)^2}$$

$$\Rightarrow f_{X,U,V}(x,u,\vartheta) = f_{X,Y,Z}(\Phi^{-1}(x,u,\vartheta))\,|\det J\Phi^{-1}(x,u,\vartheta)|$$

$$= f_X(x)\, f_Y(x(1-u))\, f_Z\!\left(\frac{2-u}{1-\vartheta}x\vartheta\right) \cdot \frac{(2-u)x^2}{(1-\vartheta)^2}$$

$$= e^{-x} e^{-x(1-u)} e^{-x\vartheta\frac{2-u}{1-\vartheta}} \cdot \frac{(2-u)x^2}{(1-\vartheta)^2} = e^{-x\left(1+1-u+\vartheta\frac{2-u}{1-\vartheta}\right)} \cdot \frac{2-u}{(1-\vartheta)^2}x^2$$

$$= e^{-x\cdot\frac{2-\cancel{2\vartheta}-u+\cancel{u\vartheta}+\cancel{2\vartheta}-\cancel{u\vartheta}}{1-\vartheta}} \cdot \frac{2-u}{(1-\vartheta)^2}x^2 = e^{-x\frac{1-u}{1-\vartheta}} \cdot \frac{2-u}{(1-\vartheta)^2}x^2$$

$$f_{U,V}(u,\vartheta) = \int_0^\infty f_{X,U,V}(x,u,\vartheta)\,dx = \int_0^\infty e^{-x\frac{1-u}{1-\vartheta}} \cdot \frac{2-u}{(1-\vartheta)^2}x^2\,dx \overset{t=\frac{x}{1-\vartheta},\ dt=\frac{dx}{1-\vartheta}}{=}$$

$$= (2-u)\int_0^\infty e^{-t(1-u)}(1-\vartheta)t^2\,dt \underset{\substack{t(1-u)=s \\ dt(1-u)=ds}}{=} (2-u)(1-\vartheta)\int_0^\infty \frac{1}{1-u+\vartheta} e^{-s}s^2\,ds$$

$$= \frac{(2-u)(1-\vartheta)}{1-u}\underbrace{\Gamma(1)}_{1} = \frac{(2-u)(1-\vartheta)}{1-u} = \underbrace{\frac{2-u}{1-u}}_{f(u)} \cdot \underbrace{(1-\vartheta)}_{g(\vartheta)} \Rightarrow \underline{U, V \text{ neodvisni}}$$

5. $I_1, \dots, I_n$ indikatorji, $\mathbb{P}(I_1 = i_1, \dots, I_n = i_n) = \frac{s!(n-s)!}{(n+1)!}$, $i_k \in \{0,1\}$, $\sum_{k=1}^n i_k = s$; $E(I_m) = ?$

$$E(I_1) = \mathbb{P}(I_1 = 1) = \sum_{\substack{(i_2,\dots,i_n)\in\{0,1\}^{n-1} \\ \sum_{k=2} i_k = s-1}} \mathbb{P}(I_2 = i_2, \dots, I_n = i_n) = \sum_{\substack{(i_2,\dots,i_n)\in\{0,1\}^{n-1} \\ \sum i_k = s-1}} \frac{(s-1)!(n-s+1)!}{n!}$$

$$= (s-1)\frac{(s-1)!(n-s+1)!}{n!} = \underline{\frac{s-1}{\binom{n}{s-1}}}$$

proveri

6. X, Y pozitivni, celoštevilski, $E(Y \mid X = l) = \begin{cases} 2 & ; l > 2 \\ l + E(Y) & ; l \le 2 \end{cases}$, $X \sim \text{Geom}(\frac{1}{2})$, $E(Y) = ?$

Dogodki $\{X=1\}, \{X=2\}, \dots$ tvorijo popoln sistem dogodkov:

$$E(Y) = \sum_{l=1}^\infty E(Y \mid X = l)\,\mathbb{P}(X = l) = (E(Y)+1)\cdot\frac{1}{2} + (E(Y)+2)\frac{1}{4} + \sum_{l=2}^\infty 2\cdot\left(\frac{1}{2}\right)^l$$

$$= \frac{3}{4}E(Y) + 1 + \sum_{k=0}^\infty \left(\frac{1}{2}\right)^k = \frac{3}{4}E(Y) + 1 + \frac{1}{1-\frac{1}{2}} = \frac{3}{4}E(Y) + 3$$

$$\Rightarrow E(Y) - \frac{3}{4}E(Y) = 3 \Rightarrow \frac{1}{4}E(Y) = 3 \Rightarrow \underline{E(Y) = 12}$$