Control Barrier Function based Robust Collision-Free Formation Control for Wheeled Robots*

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Abstract— **In this work, we extend the original well-studied swarm control strategy based on CBF-Quadratic Programming (CBF-QP) for double integrators to wheeled robots using dynamic feedback linearization. The proposed cascaded controller primarily models the robots as mass points, i.e. double integrators and applies the vanilla CBF-QP to leverage the collision avoidance maneuver in Euclidean space at the minimal risk of deadlock. The generated euclidean acceleration command was then transformed into longitudinal acceleration and steering angular velocity for robots using dynamic feedback linearization. The transformed control input will serve as the reference for the low-level CBF-QP controller which directly works with robots dynamic and state/input constraints. To handle the potential safety violation due to cascaded structure, unmodeled dynamics and external disturbances, a headway time based control barrier function was proposed, which enables robust control. The greatest strength of our controller is the compatibility for numerous top level task-dependent controllers including the simple but very practical Propotional-Derivative controller. Simulation verifies the validity of our controller under different scenarios.**

Index Terms— **Control Barrier Function, Formation Control, Collision Avoidance, Wheeled Robots, Dynamic Feedback Linearization**

I. INTRODUCTION

Formation control for swarms has attracted great interest from both academia and industry during past two decades for its huge potential application in search, salvage, reconnaissance, maintenance or even artistic performance. Formation control problems can be defined by a shape or a relative state, as well as an assignment component [1]. In reality, the formation control scheme must take into account the collision avoidance including those between agents and obstacles. In the context of robots formation control, each agent stands for a robot. Although there are many omnidirectional robots such as those equipped with mecanum wheels, typical wheeled robots are still most commonly used due to their robustness. However, these typical wheeled robots are subject to nonholonomic constraints including pure rolling and nonslipping conditions of wheels. The wheeled robots (referred as WRs later) are often more challenging to control. Early attempts for formation control of WRs (specifically for unicycle-type WRs) considering collision avoidance were mainly built upon some constructed potential functions [2], [3]. Although mathematically sound, these control strategies involving complex tracking control and safety guarantee are much difficult to be adapted for other formation tasks. More recently, model predictive control (MPC) for multi-objective flocking received more attention [4], [5]. The MPC takes advantages of global optimization to leverage optimal control input with respect to stage and input cost. Although many works have shown the competitive performance of MPC, the strong nonconvexity induced by nonconvex mutually exclusive constraints renders the problem NP-hard, which, unless practically efficient optimizer for these models would be found, apparently restricts its viability in many applications especially in the case of large swarm with limited computation resource.

An elegant method based on Control Barrier Function (CBF), Control Lyapunov Function (CLF) and Quadratic Programming (QP) was extensively studied by Ames et al [6], [7], [8] and were adapted to multiple applications including multi-agents safe swarm control [6], [9], [10], [11], bipedal walking [12] and adaptive cruise control [8], etc. The methodology consists in unifying Control Lyapunov Functions with Control Barrier Functions by QP using soft constraints technique. In this way safety and stabilization were mediated where safety is prioritized, namely whenever two requirements conflict, safety should be assured and both requirements would be satisfied as long as they were not in conflict. In the case where a stabilizing feedback control law is known, a min-norm controller can be found within the safety set defined by the control barrier function [13]. The problem then turns out to be a Linear Constrained Quadratic Program (LCQP) which can be very efficiently implemented on computation limited platforms. Decentralized control in this routine is also very straightforward and the scalability is considerable [11]. Moreover, since the pre-designed coordination control laws are basically not affected by the low-level controller, trying out different top-level controllers including the basic relative state based Proportional-Derivative (PD) controller [1] is possible. Despite of these merits, the CBF-QP controller can also suffer from slow convergence [9] and even deadlock in particular circumstances [14], [15]. More importantly, the dynamic model used in the works above is the simple double-integrator, which obviously cannot be used readily for WRs. Indeed, WRs can be controlled in a linear manner using a variety of *linerization* techniques. For instance, The *dynamic feedback linearization* [16] enables to control WRs "like" a mass point as long as the robot is not still, which is used in [17]. However, this one-to-one control input mapping doesn't respect any state/input constraints of WRs such as max. acceleration and velocity, which might cause unexpected violation for safety of WRs due to saturation. In this case, it is desirable to maintain safety for WRs directly. Fortunately, the barrier certificates make no assumptions

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about the linearity of the system therefore it also works for general nonlinear (control affine) systems such as the WRs. Considering the favorable efficiency and flexibility of CBF-QP over many classical schemes for the same task, it is apparently meaningful to design a CBF-QP based strategy for WRs formation control.

The outline of this paper is: In Section II we comprehensively review the key concepts and theorems about CBF and CBF-QP. In Section III, we propose a headway time based CBF which enjoys the robust feature and compare it with the widely-used velocity based CBF. The main contribution, a cascaded controller based on CBF-QP and proposed CBF are presented in Section IV. Then we validate our controller by simulation in Section V. Finally, we draw the conclusion and address the future direction.

II. BACKGROUND

A. Zeroing Control Barrier Function

We start with the essential tool: Control Barrier Function (CBF). Specifically, the type of CBF presented here is referred to Zeroing Control Barrier Function (ZCBF) to distinguish from the one which is precisely termed *reciprocal* CBF[8], [13]. This will be soon clear in this section. Now consider a general control affine nonlinear control system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, f and g are assumed to be locally Lipschitz. Instead of stabilizing the system to the origin whenever the initial condition is within some sublevel set, which is exactly the goal of CLF method, the safety set $\mathcal{C}$ only requires the *forward invariance* of $\mathcal{C}$, i.e., once started within the $\mathcal{C}$, the closed-loop state trajectory under the control law $u = u(x,t)$ will never leave $\mathcal{C}$. In particular, we consider $\mathcal{C}$ defined as the superlevel set of a continuously differentiable function $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ given by:

$$\begin{aligned} \mathcal{C} &= \{x \in \mathbb{R}^n : h(x) \geq 0\} \\ \partial\mathcal{C} &= \{x \in \mathbb{R}^n : h(x) = 0\} \\ \mathrm{Int}(\mathcal{C}) &= \{x \in \mathbb{R}^n : h(x) > 0\} \end{aligned} \tag{2}$$

Definition 1 A continuous function $\alpha : (-b, a) \to (-\infty, \infty)$ for some $a, b > 0$ belongs to extended class $\mathcal{K}_\infty$ function if it is strictly increasing and $\alpha(0) = 0$.

Definition 2 Consider the dynamic (1) and definition of $\mathcal{C}$, a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is called a *zeroing control barrier function* (ZCBF) if there exists an extended class-$\mathcal{K}_\infty$ function α such that [7]

$$\sup_{u \in U}[L_f h(x) + L_g h(x) + \alpha(h(x))u] \geqslant 0 \,, \forall x \in \mathcal{D} \tag{3}$$

Given a ZCBF h, define the set of admissible safe control input

$$K_{\text{zcbf}}(x) = \{u \in U : L_f h(x) + L_g h(x)u + \alpha(h(x)) \geqslant 0\} \tag{4}$$

The following theorem states safety when $u \in K_{\text{zcbf}}$:

Theorem 1 (Ames, et al) *Given a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2) for a continuously differentiable ZCBF h, any Lipschitz continuous controller $u(x) \in K_{zcbf}(x)$ will render the set $\mathcal{C}$ forward invariant. Additionally, the set $\mathcal{C}$ is asymptotically stable in $\mathcal{D}$.*

Remark 1 As explained in [11], the extended class-$\mathcal{K}_\infty$ function α determines the decaying rate of h when system (1) approaches $\partial\mathcal{C}$. For any $c > 0$ and positive odd integer p, $\alpha(r) = cr^p$ is indeed a valid extended class-$\mathcal{K}_\infty$ function. In particular, if $p = 3$, simple algebraic manipulation shows that the ZCBF can be equivalently stated as a *reciprocal* CBF $B(x) = 1/h(x)$ as the resulting admissible sets $K_{\text{rcbf}}(x)$ has exactly the same expression as $K_{\text{zcbf}}(x)$ for all $x \in \mathrm{Int}(\mathcal{C})$. The difference between two definition consists in the domain: $B(x)$ is only defined within $\mathrm{Int}(\mathcal{C})$ while $h(x)$ can be well defined over the entire $\mathcal{D}$.

A favorable property of ZCBF (not *reciprocal* CBF since it is not defined outside $\mathcal{C}$) is the robustness against disturbances. As shown in [7], the safety set $\mathcal{C}$ defined by h is asymptotically stable over the open set $\mathcal{D} \subset \mathbb{R}^n$. In the context of collision-free robots formation control, the ZCBF based controller is able to "push" away agents that accidentally moved too close to each other due to disturbances. Nevertheless the asymptotic stability relies on the profile of disturbances. Concretely, for closed loop system $\dot{x} = f(x)$ subject to disturbance $g(x)$ with known model, the following Theorem holds [7]:

Theorem 2

- *There exist $\epsilon > 0$ and class-$\mathcal{K}$ function $\sigma : [0, \epsilon] \to \mathbb{R}_0^+$* such that for any continuous function $g_1 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $||g_1(x)|| \leqslant \sigma(||x||_\mathcal{C}), \forall x \in \mathcal{D} \setminus \mathrm{Int}(\mathcal{C})$, the set $\mathcal{C}$ is still asymptotically stable for system $\dot{x} = f(x) + g_1(x)$.
- *There exist a constant $k > 0$ and a class-$\mathcal{K}$ function γ such that for any $||g_2||_\infty \leqslant k$, the set $\mathcal{C}_{\gamma(||g_2||_\infty)} \subseteq \mathcal{D}$ for the system $\dot{x} = f(x) + g_2(x)$ is locally asymptotically stable*

The notation $||x||_\mathcal{S} := \inf_{s\in\mathbb{S}} ||x - s||$ represents the minimum distance between x and set $\mathcal{S}$ which is only positive when $x \notin \mathcal{S}$. The infinity norm for any essentially bounded function $g(t)$ is denoted by $||g(t)||_\infty := \mathrm{ess\,sup}_{t\in\mathbb{R}} ||g(t)||$. The parameterized superlevel set $\mathcal{C}_\epsilon := \{x \in \mathbb{R}^n : h(x) \geqslant -\epsilon\}$ where $h(x)$ is the associated ZCBF.

Remark 2 The second robust property in Theorem 1 indicates that the relaxed safety set is attracting for bounded disturbances. This is particularly useful in our task since it enables robots to safely move under bounded noise or unmodeled dynamics.

B. Quadratic Programming with ZCBF

One of the important role of ZCBF is to act as a hard constraint in the computation of the actual control signal. As mentioned above, the sole purpose of ZCBF is to certify for the forward invariance of safety set $\mathcal{C}$. However, a reasonable controller should always meet some performance requirement such as stabilization. Once a nominal stabilizing feedback controller, i.e. reference input $u = \kappa(x)$ is known, literatures always document the following Quadratic Programming (QP) problem as a "minimal invasive" solution to maintain the original control goal while keeping safe:

(Z)CBF-QP:

$$\begin{aligned} &\min_{u} \frac{1}{2} \|u-\kappa(x)\|^2 \\ &\text{s.t.} \quad L_f h(x) + L_g h(x) u + \alpha(h(x)) \geqslant 0 \end{aligned} \tag{5}$$

Notice that the term *zeroing(Z)* is usually omitted for sake of consistency with many other works provided that there is no ambiguity. Thus if not specified, the term *CBF* refers to *ZCBF* hereinafter. If the *reciprocal* CBF was used, an alternative form of constraint in (5) should be used, e.g. [13], [8]. Though intuitively clear, it is still necessary to show that the resulting u^* should be locally Lipschitz continuous in order to meet the requirement of Theorem 1. It has been shown in [7] that if $\kappa(x)=0$, the optimal solution to problem (5) is indeed locally Lipschitz. Motivated by this result, we present the following corollary:

Corollary 1 *Given any locally Lipschitz continuous reference controller $\kappa(x)$, the optimal solution to the CBF-QP problem (5) is locally Lipschitz.*

The proof simply follows the same argument in [7]:

Proof: Following [7], we assume $L_g h(x) \neq 0, \forall x \in \mathcal{D}$. The KKT condition implies the necessary and sufficient condition of optimality. For simplicity purpose, the variable x is omitted since there is no ambiguity:

$$\begin{cases} u^* = \kappa(x) + \lambda L_g h^T \\ L_f h + L_g h u^* + \alpha(h) \geqslant 0 \\ \lambda = 0 \quad \text{if} \quad L_f h + L_g h u^* + \alpha(h) > 0 \end{cases} \tag{6}$$

The closed form solution can be obtained:

$$u^* = \begin{cases} \kappa, \text{ if } L_f h + L_g h \kappa(x) + \alpha(h) > 0 \\ u_0^* + \kappa - \frac{L_g h \kappa L_g h^T}{L_g h L_g h^T} \end{cases} \tag{7}$$

where $u_0^* = -\frac{(L_f h + \alpha(h)) L_g h^T}{L_g h L_g h^T}$ is the optimal solution when $\kappa(x)=0$. Using the same argument in [7], if $\kappa(x)$ is locally Lipschitz (LL) and $L_g h(x) \neq 0$, the additional term due to nonzero reference $\kappa - \frac{L_g h \kappa L_g h^T}{L_g h L_g h^T}$ is also LL. Finally the solution is simply the sum of this term and the proved LL term u_0^*, which is apparently LL. ■

Remark 3 For the case of more than 1 constraints, sufficient conditions for u^* to be Lipschitz can be found [18]. We assume these conditions are satisfied in this work.

Therefore by the Theorem 1, the closed-loop system under $u^*(x)$ is not only forward invariant within the safety set $\mathcal{C}$ but also robust against disturbances within the domain $\mathcal{D}$.

The essential difference between CLF-CBF-QP and CBF-QP is that while the former aiming to find a safe min-norm controller, the latter problem minimizes the deviation from $\kappa(x)$ which is not necessary to be optimal in terms of the minimum norm. On the other hand, since the feedback controller $\kappa(x)$ always stabilizes the system, that is:

$$L_f V_\kappa + L_g V_\kappa \kappa(x) + \gamma(V_\kappa) \leqslant 0, \forall x \in \mathcal{D}, t \in \mathcal{I}(0,\infty) \tag{8}$$

Then an obvious connection between two QPs can be established. Denote the deviation by $\Delta u = u - \kappa(x)$, (8) becomes:

$$L_f V_\kappa + L_g V_\kappa u + \gamma(V_\kappa) \leqslant L_g V_\kappa \Delta u \tag{9}$$

Since the objective of CBF-QP is to minimize $||\Delta u||$, the right hand side term $L_g V_\kappa \Delta u$ is equivalent to the relaxation variable δ in CLF-CBF-QP. The natural interpretation of this observation is that the CBF-QP tries to find the minimum relaxation on the time derivative of the Lyapunov function, i.e. minimal impact on the closed-loop stability under the safety constraint.

C. Dynamic Feedback Linearization of Wheeled Robots

Apart from few full-actuated omnidirectional robots, the type of robots considered here is the typical wheeled robots including diff-drive and car-like robots which can be described as *nonholonomic* systems. These systems are subject to a set of nonholonomic constraints induced by the pure rolling and nonslipping conditions of the wheels. It has been shown that these systems cannot be stabilized by any smooth time-invariant feedback controller[19]. Nonetheless there are many time-variant discontinuous controller available that guarantee stability, e.g. [19], [20]. However, applying these strategies to multi-agents control can be very difficult.

Fortunately, it is possible for these robots to track *almost* arbitrary trajectories except for some singular states by using the *dynamic feedback linearization* [21], [16]. Since diff-drive and car-like robots can be simplified into the unicycle model as shown in [22], WRs discussed hereinafter are modeled as unicycles. The *dynamic feedback linearization* for unicycle models is:

$$u_r = \begin{bmatrix} a_r \\ \omega_r \end{bmatrix} = \begin{bmatrix} cos(\phi_r) & sin(\phi_r) \\ -v_r^{-1} sin(\phi_r) & v_r^{-1} cos(\phi_r) \end{bmatrix} \begin{bmatrix} \ddot{x}_r \\ \ddot{y}_r \end{bmatrix} \tag{10}$$

where ϕ_r, x_r, y_r are the robots' heading angle and Cartesian coordinate of the geometry center (the mid-point between wheels or rear wheels for diff-drive and car-like robots respectively); $\ddot{x}_r, \ddot{y}_r$ are the Cartesian acceleration of the robots; $u = [a_r, \omega_r]^T$ is the control input consisting of the longitudinal acceleration and steering angular velocity respectively. This dynamic linearizing controller (10) has a singularity at $v_r = 0$, which must be taken into account when designing the entire control law. Denote the state vector for robot by $\mathbf{x}_r = [x_r, y_r, v_r, \phi_r]^T$ then the dynamic model of the robot is:

$$\dot{\mathbf{x}}_r = f(\mathbf{x}) + g(\mathbf{x}) u_r = \begin{bmatrix} v_r \cos(\phi_r) \\ v_r \sin(\phi_r) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_r \\ \omega_r \end{bmatrix} \tag{11}$$

This model, however, doesn't take the rotation dynamic into account, which is the immediate result of *dynamic feedback linearization*. In this case, steering is assumed to be handled by ideal local feedback controller. Since physical controllers has finite gain and actuation saturation, the real dynamic can deviate from the ideal case, which will be modeled as disturbances. For sake of simplicity, the robot is assumed to be symmetric, otherwise the centrifugal and Coriolis forces should be considered. It is worth noting that *dynamic feedback linearization* is also valid for many other types of WRs, which is elaborated in [16].

D. Problem Statement

We are seeking for a robust and efficient control strategy to perform the safe formation control for a swarm of WRs in 2-D plane. In particular, we consider the relative state specification (RSS) formation control problem where the formation is defined by relative displacement between some pairs of agents. Formally [1], let $\Xi = \{\xi_1, \ldots, \xi_N\}, \xi_i \in \mathbb{R}^2, i = 1, \ldots, N$ be a set of agents and consider a spanning subgraph $\mathcal{D}$. Define the relative state vector $z(t) = (D(\mathcal{D})^T \otimes I_2)x(t)$ where $\otimes$ is the Kronecker product and $D(\mathcal{G})$ calculates for the incident matrix of a digraph $\mathcal{G}$. Let z_{ref} be the desired RSS formation, the goal is to drive the WR-type agents such that $z(t) \to z_{ref}$ while avoiding collision between agents and obstacles and respecting the predefined state and control constraints. Moreover, the controller should be robust in the presence of disturbances or unmodeled robots dynamics. In the rest of this paper, we make the following assumptions:

1) There is not position constraint, i.e. the virtual map is infinitely large;
2) Robots are identical and specifications will not change during the process;
3) The maximum velocity (state constraint) and maximum actuation input is finite.
4) Unless specified, all states are accessible and noiseless;
5) The formation graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_f)$ is always the subgraph of communication graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$.

III. Barrier Function Formulation

Designing CBF is highly task-dependent. For a category of safe swarm control problems, the safety criteria is to keep agents away from each other during the whole coordination process. Previous works [9], [10], [11] have constructed the following CBF to represent the invariant set $\mathcal{C}$ for mass point, viz. double integrator:

$$h_{ij}^I = \frac{\Delta p_{ij}^T}{\|\Delta p_{ij}\|}\Delta v_{ij} + \sqrt{2\Delta a_{max}(\|\Delta p_{ij}\| - D_s)} \quad (12)$$

where $\Delta p_{ij} = p_i - p_j$, $\Delta v_{ij} = v_i - v_j$ is the relative position and velocity between agent i and j respectively as illustrated in Fig. 1. The maximum relative deceleration

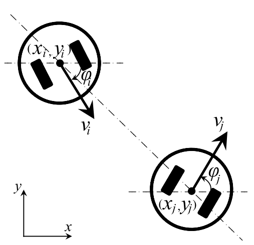

Fig. 1: Illustration of Wheeled Robots

is calculated by $\Delta a_{max} = a_{i,max} + a_{j,max}$. The predefined minimal distance between agents is denoted by D_s. The interpretation of (12) is that the current distance between agents must accommodate D_s plus the minimal braking distance when both agents brake as quickly as possible. Although effective, this CBF makes strong assumption that the collision is avoided by taking the maximum deceleration a_{max}. Since the control input is also subject to a_{max}, i.e. actuation constraint which might conflict with CBF constraint and results in infeasible QP. Ideally, the problem is always feasible if the initial state is feasible, which is guaranteed by forward invariance. In the presence of disturbance, even being slight, the actuation dependent CBF-QP might be infeasible during the evolution. This issue was addressed in [11] where the authors proposed a workaround to disable the CBF-QP and manually brake robots and hope the QP would be feasible later. However, this hand-crafted approach breaks the integrity of CBF-QP and doesn't actually leverage the robust property of ZCBF as mentioned before. The effectiveness of this method was also not validated for the case of swarm with more than two robots.

In effect, a simple and conservative ZCBF adopted in the Adaptive Cruise Control (ACC) problem[8], [7] can ameliorate the feasibility condition. This ZCBF is based on idea of the "headway" distance τv where τ is simply the looking ahead time. Apparently if τ is long enough that satisfies:

$$\frac{v(t)^2}{2a_{max}} \leqslant v(t)\tau, \forall t > 0 \implies \tau \geqslant \frac{v_{max}}{2a_{max}} \quad (13)$$

where v_{max} is the maximum velocity of robots. Then the headway is always longer than braking distance. Consequently, we propose a ZCBF based on headway time:

$$h_{ij}^{II} = \|\Delta p_{ij}\| - D_s - \tau \underbrace{\frac{(p_j - p_i)^T}{\|\Delta p_{ij}\|}\Delta v_{ij}}_{\overline{\Delta v_{ij}}} \quad (14)$$

And the resulting invariant set $\mathcal{C}$ is given by (2). Cautions must be taken that $\overline{\Delta v_{ij}}$ in (14) stands for the *approaching* speed. The CBF can be easily tailored to handle the obstacle avoidance as the latter can be seen as static agent without control input. In that case, Δv_{ij} is replaced by v_i. Next we present the fundamental property of (14):

Proposition 1 *If the headway time τ satisfies (13), then $h_{ij}^{II} \geqslant 0 \implies h_{ij}^I \geqslant 0$.*

Proof. First notice that the signs of $\overline{\Delta v_{ij}}$ in both expressions are minus, meaning if agents are moving away from each other,i.e. $\overline{\Delta v_{ij}} < 0$, then both CBF must be positive. Therefore we only consider the case where $\overline{\Delta v_{ij}} > 0$. Denote $\|\Delta p_{ij}\| - D_s$ by Δs and keep the definition of the approaching speed $\overline{\Delta v_{ij}}$. Then (12) and (14) can be written in a more compact form:

$$h_{ij}^I = \sqrt{2\Delta a_{max}\Delta s} - \overline{\Delta v_{ij}}$$
$$h_{ij}^{II} = \Delta s - \tau\overline{\Delta v_{ij}}$$

Then $h_{ij}^{II} \geqslant 0 \implies \Delta s \geqslant \tau\overline{\Delta v_{ij}}$. Substitute Δs into h_{ij}^I and since $\tau \geqslant v_{max}/(2a_{max})$, the square root term $\sqrt{2\Delta a_{max}\Delta s} \geqslant \sqrt{2v_{max}\overline{\Delta v_{ij}}}$. Consider the expression $\sqrt{2v_{max}\overline{\Delta v_{ij}}} - \overline{\Delta v_{ij}}$ and let $\sqrt{\overline{\Delta v_{ij}}}$ be the variable, it is trivial to confirm that the expression takes positive value iff. $0 \leqslant \sqrt{\overline{\Delta v_{ij}}} \leqslant \sqrt{2v_{max}}$, which is obviously true since the approaching speed is bounded by $2v_{max}$. Therefore $h^I_{ij} = \sqrt{2\Delta a_{max}\Delta s} - \overline{\Delta v_{ij}} \geqslant \sqrt{2v_{max}\overline{\Delta v_{ij}}} - \overline{\Delta v_{ij}} \geqslant 0$. In addition, the equality holds iff. $\tau = v_{max}/(2a_{max})$ and both agents move rightly towards each other with maximum speed. ∎

Interpretation of Proposition 1 is quite intuitive: if two agents come close with constant speed without reaching D_s, taking brake apparently guarantees the collision-free condition. In other word, (14) is more conservative than (12). By using the proposed CBF (14), the resulting CBF-QP would be more conservative and hence the feasible set is larger. More importantly, one can take advantages of the robustness feature of ZCBF introduced in Section II.A. To demonstrate the robust feature of ZCBF, before introducing the underlying controller, we present in Fig.2 an extreme example where two wheeled robots start moving from very up close position $\|\Delta p_{ij}\| \ll D_s$ (of course we suppose that the physical dimension of robots permits). Clearly this is not possible for CBF h^1 since the function is not defined when $\|\Delta p_{ij}\| < D_s$. The plot shows that the robots tend to move away from each other at the beginning even though the command is to the opposite direction. This can be also affirmed by Fig. 2 (b): during the first 3.5s, the distance between robots increases from the initial unsafe value to D_s and the barrier function value rises from the initial value -0.4 to 0.

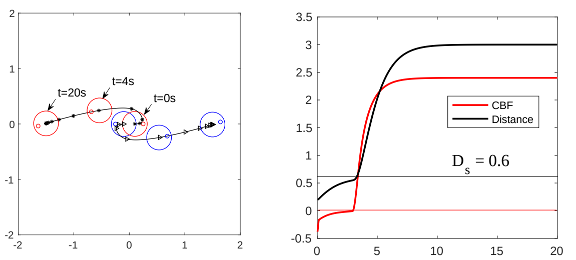

Fig. 2: Robustness of ZCBF. (a) Robots' movement path. The small circle inside the robot represents the ahead direction. (b) The CBF value and distance between robots during movement. Initial distance between two robots is 0.2 and D_s = 0.6. The destination of two robots are $[-1.5, 0]$ and $[1.5, 0]$ respectively.

IV. Cascaded CBF-QP Controller

Given a well-defined CBF, one can in principle formulate the CBF-QP (5) to safely control the system while keeping the nominal control goal with minimal modification. Indeed, there are many nominal formation controllers without safety guarantee. In [17], the unicycle-type agents are transformed into mass point using the *dynamic feedback linearization* and then perform a special relative state based formation control. This approach essentially decouples the formation control for point mass with unicycle tracking control therefore it is reasonable to expect that inserting a CBF-QP can bring safety feature to their control law. However, in the case where actuation saturation and model uncertainties are present, the real trajectory of unicycles might deviate from expectation and the safety might be no longer guaranteed. In [23], a backstepping based consensus control protocol was devised, which can be adapted to formation control task. Again, applying CBF-QP that take as reference the generated unicycle control input might introduce safety, which, however, might incur frequent deadlock. Motivated by these works, we propose a cascaded CBF-QP controller that respects both safety and state/input constraints without sacrificing the performance of nominal control law.

A. Main Result

The controller is composed of four cascaded layers. Each layer takes as input the output of previous layer:

1) Top level nominal controller (NC). The choice of the nominal controller is task-dependent. Any stabilizing and Lipschitz continuous controller can be used. For our task, the simplest relative state based PD controller [1] is sufficient:

$$\begin{aligned} &e(t) = z_{ref} - z(t) \quad \text{where} \quad z(t) = D(\mathcal{D})^T x(t) \\ &u_{NC} = kD(\mathcal{D})(e(t) + \dot{e}(t)), \quad k > 0 \end{aligned} \tag{15}$$

2) Mass point CBF-QP (QP1). The robots is modeled as mass points in this layer. The state of robots are first transformed into Cartesian position and velocity which are passed as parameters to a CBF-QP utilizing the ZCBF (14). The output of the top nominal controller serves as the reference for the CBF-QP. No real state and actuation constraints are considered in this layer. Concretely, the CBF-QP is written as:

$$\begin{aligned} u^*_{QP1} &= \sum_{i=1}^{N} \arg\min_{u_{QP1}} \left\| u^i_{QP1} - u^i_{NC} \right\|^2 \\ \text{s.t} \quad & \dot{h}_{ij}(x) + \alpha(h_{ij}(x)) \geqslant 0, \; \forall i \neq j \end{aligned} \tag{16}$$

where N is the number of robots. The optimization variable u_{QP1} is composed of the Cartesian control input for each robot u^i_{QP1}, $i = 1, \ldots, N$. Similarly, the reference generated by NC is denoted by u_{NC} consisting of reference for each robot u^i_{NC}, $i = 1, \ldots, N$. (16) basically does not differ from the one in previous works except for dropping actuation constraints.

3) Dynamic feedback linearization (FL). The output of previous layer is the Cartesian acceleration $[a_x, a_y]^T$ expressed in the world frame which has to be transformed into appropriate control input for robots using (10). This layer must also handle the singular state where a robot stands. The output of this layer is denoted by u_{FL}.

4) Low level CBF-QP (QP2). This layer directly deals with safety as well as state/actuation constraints for robots. It takes as reference the transformed control input from previous layer and incorporate all physical constraints when synthesizing CBF-QP which is based on the same CBF (14). When calculating the time derivative of the CBF, the dynamic (11) is used. Concretely, The CBF-QP

is written as:

$$
\begin{aligned}
u^*_{QP2} = & \sum_{i=1}^{N} \arg\min_{u_{QP2}} (u^i_{QP2} - u^i_{FL})^T H (u^i_{QP2} - u^i_{FL}) \\
\text{s.t} \quad & \dot{h}_{ij}(\mathbf{x}_r) + \alpha(h_{ij}(\mathbf{x}_r)) \geqslant 0, \forall i \neq j \\
& |u^i_{QP2.a_r}| \leqslant a_{r,max} \\
& |u^i_{QP2.a\omega_r}| \leqslant \omega_{r,max} \\
& |u^i_{QP2}\Delta t + v^i_r| \leqslant v_{r,max}
\end{aligned} \tag{17}
$$

where the notations are analogous to that defined in (QP1). Δt is the discretization time. The essential differences consist in using robots' state vector $\mathbf{x}_r$ and involving state/actuation constraints of the robots.

The idea behind the cascaded controller is to let wheeled robots safely track moving paths which lead to collision-free formation for mass points. It should be highlighted that the reason for not directly utilizing CBF method for WRs is its vulnerability to *deadlock*, which will be explained in *Deadlock Consideration* part. Having presented the basic structure of the controller, we bring forward the following proposition:

Proposition 2 *The invariant safety set $\mathcal{C}$ associated with the cascaded controller is asymptotically stable. Additionally, if the disturbances acting on dynamic (11) is essentially bounded, there exists a set $\mathcal{C}_{\gamma(||g_2||_\infty)} \subseteq \mathcal{D}$ that is asymptotically stable.*

Proof: Suppose that the (FL) layer can transform u_{QP1} in a way that whenever singularity arises, the output u_{FL} is Lipschitz. Under this assumption, by Corollary 1, the continuity can propagate through layers as long as the output of (NC) is Lipschitz which is apparently true for linear controller. Therefore by the Theorem 1/2, this robustness property is a direct result of (QP2). ■

Remark 4 The essential stage in the cascaded controller is to appropriately handle singularity as implied in the proof. Since singularity occurs only when $v_r = 0$, it is always possible to switch to another control law without losing continuity.

Remark 5 The same CBF is shared between (QP1) and (QP2) while the latter has stricter constraints. In the absence of disturbance, once the system starts within the invariant set $\mathcal{C}$, i.e. $h_{ij} > 0, \forall i \neq j$, the state will never leave the set. If the dynamic model (11) is subject to bounded additive disturbances, i.e. $\dot{\mathbf{x}}_r = f(\mathbf{x}_r) + g(\mathbf{x}_r)u_r + w(\mathbf{x}_r)$ or starts outside $\mathcal{C}$, the robustness property will enforce the state back to a larger invariant set.

B. Singularity Handling

As opposed to the double stages control law without safety features proposed by [17] in which the singularity will provably happen at most once, there is in general no guarantee for safe coordination to be sufficiently smooth. In fact, the simple CBF method is a sort of "passive" control that allows the state to evolve on the manifold. Even for ideal mass point model, experiment observation has manifested the possible nonsmoothness of the trajectories especially when the number of agents is large. On the other hand, as mentioned in Remark 4, it is possible to find a smooth controller that is continuous at singularity. Specifically, for unicycle-like robots which can rotate in place, a P controller that aligns ϕ_r with the direction of the reference acceleration $[\ddot{x}_r, \ddot{y}_r]$ can substitute for the second row of singular control law (10):

$$a_{FL}(t) = 0 \tag{18}$$

$$\omega_{FL}(t) = k_p(\phi^*(t) \ominus \phi_r(t)) \tag{19}$$

$$\text{where} \quad \phi^*(t) = \operatorname{atan2}(\ddot{x}_r(t), \ddot{y}_r(t))$$

weher $\ominus$ is the minimum angle difference operator. Theoretically it needs infinite time to reach $\phi^*(t)$. However in practice, the convergence is assumed when the error is within small tolerance. Once alignment is achieved at t_0, the ω_r row in (10) becomes 0/0-type expression which is defined as 0. With the extra definition at singular state, any Lipschitz u_{QP1} will be transformed into Lipschitz u_{FL}.

If the robots cannot turn in place, e.g. car-like robots, one possible way is to wiggle and steer within the safe radius $D_s/2$ as long as the robot is *Short Time Local Controllable (STLC)* [22].

C. Deadlock Consideration

A potential risk of the naive CBF approach is deadlock [14], [15], [24]. The deadlock pattern of robots depends on the nominal controller. For simple PD controller, deadlock occurs when perfect symmetric constellation appears [15]. Nevertheless, there are many possible resolution strategy including coordinated perturbance[11] and position swapping[15]. As suggested in the beginning of this section, bypassing the mass point modeling might cause regular deadlock. We can show by construction that the deadlock configuration for directly applying CBF-QP method to WRs control is not zero measure while the one for mass point control is indeed zero measure[15].

Consider the configuration depicted in Fig. 3. If such

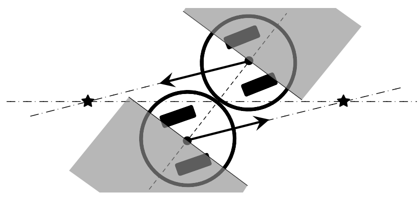

Fig. 3: Robots configuration causing unicycle CBF-QP to deadlock. Thick arrows stand for "pulling" force as termed in [15] and stars represent the destination. The shaded region is the admissible Cartesian force induced by CBF constraints.

situation happens to mass points with Cartesian driving force, the CBF constraints still allow the acceleration orthogonal to the relative position between two robots so that they will not collide. On the contrary, if the QP is to minimize the 2-norm error expressed in $[a_r, \omega_r]$, the longitudinal must be zero in this configuration otherwise collision can happen. Meanwhile, since rotation makes no contribution to collision, the error for ω_r can always be zero. For any reasonable nonholonomic controller e.g. [19], [20] that stabilizes the robot to the origin, if the heading angle is to the tangent direction of the expected path, the angular velocity will converge to zero at some t_1. The final result is $u^* = 0, \forall t > t_1$. As opposed to the mass point case where the deadlock set is nonempty but zero measure, the deadlock set $\{\mathcal{S} \mid u_{ref} \neq 0, u^* = 0\}$ is open and dense. Therefore in our scheme, the point mass model is included to leverage avoidance maneuver with minimal risk of deadlock.

V. Simulation

In this section, we valid our controller by simulation. The control objective is to drive 6 wheeled robots to form a billiard-like pattern while avoiding any collisions and respecting robots' physical constraints. The simulation result is shown in Fig. 4. The plot clearly testified the validity of our controller.

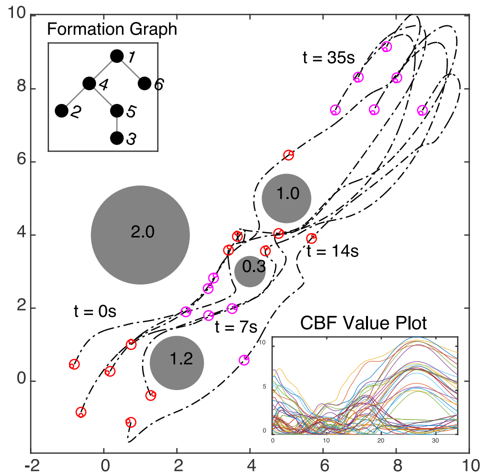

Fig. 4: Safe Formation Control Simulation. For each robot, the safety parameters are τ=0.8s,D_s=0.3, state/actuation constraints are $v_{max} = 1, a_{r,max} = 1, \omega_{r.max} = 2\pi$. The discretization time $T_s = 0.02$. In each time interval, ode45 solver is used to solve the dynamic. Obstacles are shown in the plot. The formation graph used is illustrated in the left-top. The function plot of all pair-wise CBFs is shown in the right-bottom. It it clear that all CBFs never go negative.

VI. Conclusions

In this paper, we proposed a cascaded CBF-QP controller for safe swarm control of wheeled robots. The proposed headway time based ZCBF not only enlarges the feasible set but also leverages the robust feature which makes our controller robust against disturbances. Furthermore, We demonstrated the mass point modeling using dynamic feedback linearization would minimize the risk of deadlock. Even though, the control algorithm is still operating in a centralized manner which addresses future work on decentralized controller.

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