Homotopy (Type) Theory take-home exam

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For full credit solve *at least 51 points* worth of problems. As you are training to become a researcher, you are free to refer to constructions and proofs in existing literature, namely peer-reviewed papers and monographs. References to blog posts and other non-standard sources are allowed, but in those cases you need to verify the veracity of the claims yourself. It is probably a good idea to verify your sources even when they are of a reputable origin. In the end, you are responsible for your solutions.

Part I: homotopy theory

Problem 1 (7 points)**.** Let $n \geqslant 1$, $1 \leqslant k \leqslant n-1$, and let $G_k(\mathbb{R}^n)$ denote the set of k-planes in $\mathbb{R}^n$. Also, let $V_k(\mathbb{R}^n)$ denote the set of (ordered) k-tuples of orthonormal vectors in $\mathbb{R}^n$. Topologize the latter by viewing it as a subset of $\mathbb{R}^{n\times k}$ in the obvious way.

(a) Topologize $G_k(\mathbb{R}^n)$ as a quotient space of $V_k(\mathbb{R}^n)$.

(b) Show that $E_k^n = \{(\Lambda, x) \mid \Lambda \in G_k(\mathbb{R}^n),\ x \in \Lambda\} \subset G_k(\mathbb{R}^n) \times \mathbb{R}^n$, together with the obvious projection map, is a vector bundle of rank k over $G_k(\mathbb{R}^n)$.

(c) Let S^2 be the 2-sphere and let $f\colon S^2 \to G_2(\mathbb{R}^3)$ assign to $\zeta \in S^2$ the plane perpendicular to ζ. Show that f is continuous and identify the pullback bundle $f^*(E_2^3)$. You may want to consult Davis-Kirk [1] for the latter.

Problem 2 (7 points)**.** Suppose given $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ in the category of topological spaces over B. A map $f\colon E_0 \to E_1$ over B is called a *fibre homotopy equivalence* if there exist a map $g\colon E_1 \to E_0$ over B and homotopies $gf \simeq \mathrm{id}_{E_0}$ and $fg \simeq \mathrm{id}_{E_1}$ over B. Here, $E_i \times [0,1]$ is a space over B by virtue of $P_i = p_i \circ \mathrm{pr}_{E_i}$. Let $p\colon E \to B$ be a fibration and let $h\colon X \times [0,1] \to B$ be a homotopy from h_0 to h_1. Using a lifting function for p, construct an explicit fibre homotopy equivalence of pullbacks $h_0^*(E)$ and $h_1^*(E)$ as spaces over X.

Problem 3 (7 points)**.** Look up the definition of a *diagram* in $\mathcal{C}$ with a given *shape* $\mathcal{D}$ and its colimit in Dwyer-Spalinski [2].

(a) Make sense of the colimit functor $\operatorname{colim} \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ for a finite (small) category $\mathcal{C}$ with finite (small) colimits and a finite (small) shape $\mathcal{D}$. (Define it and prove that it is a functor.)

(b) Consider the diagrams $\mathcal{D}$ and $\mathcal{E}$,

$$\mathcal{D}: \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \mathcal{E}: \begin{tikzcd} \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dl] \arrow[dd] \\ & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

Employing the pushout, define a suitable map $\mathcal{C}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{E}}$ and study its properties.

Problem 4 (7 points)**.** Let $\mathcal{C}$ be a category. A bounded direct sequence in $\mathcal{C}$ is a diagram of objects and morphisms of $\mathcal{C}$ of the form

$$\dots \xrightarrow{\xi_{-2}} X_{-1} \xrightarrow{\xi_{-1}} X_0 \xrightarrow{\xi_0} X_1 \xrightarrow{\xi_1} X_2 \xrightarrow{\xi_2} X_3 \xrightarrow{\xi_3} \dots$$

where for all small enough $i \in \mathbb{Z}$, the ξ_i are identity morphisms. We denote such a direct sequence simply by $\{(X_i, \xi_i)\}$. A morphism $f\colon \{(X_i, \xi_i)\} \to \{(Y_i, \eta_i)\}$ is a collection of morphisms $f_i\colon X_i \to Y_i$ in $\mathcal{C}$ satisfying $f_{i+1}\xi_i = \eta_i f_i$ for all i, such that $f_i = f_{i-1}$ for all small enough i (i.e. for all $i \leqslant b$ where $b \in \mathbb{Z}$ depends on f). This defines a category of bounded direct sequences in $\mathcal{C}$, which we denote $\mathrm{bdSeq}(\mathcal{C})$.

Suppose that $\mathcal{C}$ is a model category. We call $f\colon \{(X_i, \xi_i)\} \to \{(Y_i, \eta_i)\}$ a weak equivalence (respectively a fibration) if all f_i are weak equivalences (respectively fibrations) in $\mathcal{C}$. Next, we call f a cofibration if for all i, the natural morphism $Y_i \sqcup_{X_i} X_{i+1} \xrightarrow{\eta_i + f_{i+1}} Y_{i+1}$ is a cofibration in $\mathcal{C}$, and, moreover, f_i is a cofibration in $\mathcal{C}$ for all small enough i.

(a) Prove that $\mathrm{bdSeq}(\mathcal{C})$ is a model category.

(b) Identify the fibrant and cofibrant objects in $\mathrm{bdSeq}(\mathcal{C})$.

(c) Suppose that $\mathcal{C}$ has small colimits. Define a colimit functor $\mathrm{colim}\, \mathrm{bdSeq}(\mathcal{C}) \to \mathcal{C}$ and prove that it preserves cofibrations and trivial cofibrations. **Hint.** Use adjoint functors.

Problem 5 (7 points)**.** Let $\mathcal{C}$ be a pointed model category. For a cofibrant X, we defined an association $[\Sigma X, Y] \to \pi_1^l(X, Y) = \pi_1^l(X, Y; 0, 0)$ which is a natural equivalence of functors $[\Sigma X, \_]$ and $\pi_1^l(X, \_)$ on the category $\mathcal{C}_f$ (the full subcategory of $\mathcal{C}$ of fibrant objects). See Theorem 2 of Quillen [3] for a proof. State the dual of the former and prove it. **Warning.** Mind the notation.

Part II: homotopy type theory

Problem 6 (5 points)**.** Prove that the coproducts have the expected universal property:

$$(A + B \to C) \simeq (A \to C) \times (B \to C).$$

Problem 7 (5 points)**.** Let A be a type and $a : A$ a point. Prove that $\Sigma(x : A) \,.\, a =_A x$ is contractible.

Problem 8 (5 points)**.** Prove that $\mathbb{N}$ is a set.

Problem 9 (5 points)**.** Show that $(2 \simeq 2) \simeq 2$.

Problem 10 (5 points)**.** Show that $S^1 \simeq \mathsf{Susp}(2)$, where S^1 is the circle and $\mathsf{Susp}(2)$ the suspension of 2.

Problem 11 (5 points)**.** Construct the *double cover* of the circle as a dependent type, i.e., a dependent type $D : S^1 \to \mathcal{U}$ such that $D(\mathsf{base}) \simeq 2$ and $(\Sigma(x : S^1) \,.\, D(x)) \simeq S^1$.

Problem 12 (5 points)**.** How would you define the *Möbius band* as a type?

References

[1] J. F. Davis, P. Kirk, *Lecture notes in algebraic topology.* Graduate Studies in Mathematics, 35. American Mathematical Society, Providence, RI, 2001.

[2] W. G. Dwyer, J. Spalinski, *Homotopy theories and model categories.* Handbook of algebraic topology, 73-126, North-Holland, Amsterdam, 1995.

[3] D. G. Quillen, *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.