Senior Seminar Paper

Special Relativity

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"Space by itself, and time by itself, are doomed to fade away into mere shadows; and only a kind of union of the two will preserve an independent reality"

- H. Minkowski

Abstract

This seminar paper provides an overview of the mathematical foundations of Minkowski's space-time geometry and Lorentz transformations. We begin by reviewing the historical context and motivations that led Hermann Minkowski to introduce his novel geometric approach to space and time in the early 20th century, as well as Nobel Prize winner Hendrik Lorentz's infamous "Lorentz Transformations". We explore geometrically what it means to visualize and interpret space-time, which form the basis of special relativity. Finally, we discuss some of the key applications of Lorentz transformations which is relevant to our understanding of the structure of space-time and how it is used practically.

1 Introduction

1.1 Motivation

This paper aims to help readers gain a better understanding of the mathematical foundations behind space-time dimensions in Special Relativity. Minkowski space-time and Lorentz equations are essential in modern physics and are critical to our understanding of the structure of space and time. One reason why they are important is that they provide a mathematical framework for understanding the fundamental nature of the universe, including the behavior of matter and energy at the most basic level.

The special theory of relativity, which is based on Minkowski space-time and Lorentz transformations, has been confirmed experimentally to a high degree of accuracy and is considered one of the cornerstones of modern physics. The theory has important implications for the nature of space and time, such as time dilation and length contraction, and is the basis for the famous equation $E = mc^2$, which relates mass and energy.

Furthermore, Minkowski space-time and Lorentz transformations are critical to understanding phenomena such as black holes, gravitational waves, and the behavior of particles at high energies. They are also essential in the study of particle physics, where they are used to describe the behavior of subatomic particles and their interactions.

1.2 History and Background

1.2.1 Hermann Minkowski

Hermann Minkowski was a German mathematician born in Russia in 1864 to German parents. He showed an early talent for mathematics while studying at the Gymnasium in Königsberg, where he read the works of Dedekind, Dirichlet, and Gauss. He studied at the University of Königsberg and spent three semesters at the University of Berlin, where he became close friends with fellow mathematician Hilbert. Minkowski received his doctorate from Königsberg in 1885 for a thesis on quadratic forms. [4]

Minkowski's mathematical interests were originally in pure mathematics, and he spent much of his time investigating quadratic forms and continued fractions. However, his most original achievement was his "geometry of numbers," which he initiated in 1890. He published Geometrie der Zahlen in 1910, which was a major work on the geometry of numbers and was later reprinted multiple times. Minkowski also worked on convex bodies and packing problems. [4]

Minkowski developed a new view of space and time, which laid the mathematical foundation of the theory of relativity. By 1907, he realized that the work of Lorentz and Einstein could be best understood in a non-Euclidean space. He considered space and time, which were formerly thought to be independent, to be coupled together in a four-dimensional "space-time continuum." Minkowski worked out a four-dimensional treatment of electrodynamics. [4]

Unfortunately, Minkowski died suddenly at the young age of 44 from a ruptured appendix. Despite his short life, his contributions to mathematics and physics have been significant and long-lasting. [4]

1.2.2 Hendrik Lorentz

Hendrik Lorentz was a Dutch physicist and mathematician, born on July 18, 1853, in Arnhem, Gelderland. He was the eldest of three children of his parents Gerrit Frederik Lorentz and Geertruida van Ginkel. Hendrik's father was a shopkeeper, and after the death of Geertruida in 1861, he remarried Luberta Hupkes. Hendrik's early education was in Arnhem, where he attended the newly established high school from 1866 to 1869. In 1870 he passed the classical languages exams required for admission to University, and he enrolled in physics and mathematics at the University of Leiden. [3]

At the University of Leiden, Hendrik was influenced by the teaching of astronomy professor Frederik Kaiser, who was responsible for his decision to become a physicist. After earning a bachelor's degree, Hendrik returned to Arnhem in 1872 to teach high school classes in mathematics, but he continued his studies in Leiden alongside his teaching position. In 1875 Lorentz earned a doctoral degree under Pieter Rijke on a thesis entitled "On the theory of reflection and refraction of light," in which he refined the electromagnetic theory of James Clerk Maxwell. [3]

In 1878, at the age of only 24, Lorentz was appointed to the newly established chair in theoretical physics at the University of Leiden. He delivered his inaugural lecture on "The molecular theories in physics" on January 25, 1878. During the first twenty years in Leiden, Lorentz primarily focused on the theory of electromagnetism to explain the relationship of electricity, magnetism, and light. After that, he extended his research to a much wider area while still focusing on theoretical physics. From his publications, it appears that Lorentz made significant contributions to mechanics, thermodynamics, hydrodynamics, kinetic theories, solid-state theory, light, and propagation. His most important contributions were in the area of electromagnetism, the electron theory, and relativity. [3]

Lorentz theorized that atoms might consist of charged particles and suggested that the oscillations of these charged particles were the source of light. This was experimentally proven in 1896 by Pieter Zeeman, a colleague and former student of Lorentz. His name is now associated with the Lorentz-Lorenz formula, the Lorentz force, the Lorentzian distribution, and the Lorentz transformation.

In 1895, in an attempt to explain the Michelson-Morley experiment, Lorentz proposed that moving bodies contract in the direction of motion (see length contraction; George FitzGerald had already arrived at this conclusion, see FitzGerald-Lorentz Contraction). He introduced the term local time, which expresses the relativity of simultaneity between reference frames in relative motion. Henri Poincaré in 1900 called Lorentz's local time a "wonderful invention" and showed how it arose when clocks in moving frames are synchronized by exchanging light signals, which are assumed to travel with the same speed against and with the motion of the frame. [3]

In 1899 and again in 1904, Lorentz added time dilation to his transformations and published what Poincaré in 1905 named the Lorentz transformations. It was apparently unknown to Lorentz that Joseph Larmor had predicted time dilation, at least for orbiting electrons, and published the identical transformations in 1897. Larmor's and Lorentz's equations look unfamiliar, but they are algebraically equivalent to those presented by Poincaré and Einstein in 1905. These mathematical formulas describe basic effects of the theory. [3]

1.3 Theoretical Framework and Results

The crux of this paper pertains to the preservation of Lorentz Transformations within the framework of the Minkowski space-time which can be seen by through the condition $\Lambda^T \eta \Lambda = \eta$.

To comprehend the meaning and implications of Lorentz Transformations, it is crucial to establish a comprehensive understanding and foundation of Minkowski Space-Time diagrams, which can be represented by the $\mathbb{R}^4$ which is the matrix of the Minkowskian inner product given by:

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

One way to think of this geometry is that in everyday life, we are used to thinking about space and time as separate and distinct things. We measure distances in space using units like meters, and we measure time using units like seconds. However, in the context of special relativity, space and time are not completely separate, but are instead part of a single four-dimensional entity called space-time.

Minkowski space-time geometry provides a way to visualize this four-dimensional space-time by representing it using a set of coordinates that include both space and time. Instead of thinking of space and time as separate things, we can think of them as part of a single entity, where the distance between two points in space-time is calculated using a special formula called the space-time interval 'η'.

This four-dimensional model of space-time allows us to describe the motion of objects in a way that takes into account the effects of relativity. For example, if an object is moving very fast relative to an observer, its motion through space-time will be different than if it were moving more slowly. This is because the faster an object moves through space, the slower it appears to move through time, according to the theory of special relativity.

In this work, we will introduce the Minkowski geometry of space-time diagrams, where η is expressed, initially, in $\mathbb{R}^2$, and then generalize to $\mathbb{R}^4$. We will establish the necessary foundations for understanding the geometry of Minkowski, including the notions of "inner-product" and "The hyperbolic Pythagorean theorem". This process is crucial for comprehending the mathematical relationships between events in space-time and their occurrences. [2]

Finally, once we have an understanding of the observation of events in space-time, we will move on to derivation of Lorentz Transformations. Lorentz transformations are a set of equations that describe how space and time appear to change for an observer who is moving relative to another observer. This means that if two observers are moving relative to each other, they will measure different lengths and times for the same events.

Lorentz transformations provide a way to convert measurements made by one observer into measurements that would be made by another observer who is moving relative to the first. These equations take into account the fact that time and space appear to be different for the two observers, and allow us to calculate the differences between the two sets of measurements.

The most common form of the transformation are the Lorentz "Boosts" which are parameterized by the real constant v, representing a velocity confined to the x-direction. In Special Relativity, we are more concerned with non rotational transformations, which are labelled as Lorentz Boost and can be expressed as:

$$t' = \gamma(t - \frac{vx}{c^2})$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

Where $S = (t, x, y, z)$ and $S' = (t', x', y', z')$ are coordinates of an event in two frames with the origins coinciding at $t = t' = 0$, where the S' frame is seen from the S as moving with speed v along with the x-axis, where c is the speed of light, and $\gamma = (\sqrt{1 - \frac{v^2}{c^2}})^{-1}$ is the Lorentz factor. [14]

To understand Lorentz transformations, it's important to realize that they involve a different way of thinking about space and time than we're used to in everyday life. But they are essential to understanding the behavior of light and other fundamental particles in the universe, and have been confirmed by countless experiments over the past century.

2 Establishing the Foundation

The purpose of this section is to establish a solid foundation for the main ideas that will be presented in this paper. To achieve this, we will begin by defining key terms that are relevant to our topic. We will then move on to outlining the theorems, propositions, and lemmas that are fundamental to our argument.

By presenting this preliminary information, we aim to ensure that our readers have a clear understanding of the concepts that will be discussed throughout the paper. This will enable them to engage with the material more effectively and critically evaluate the arguments that we put forward.

2.1 Preliminaries: Linear Algebra and Geometry

Definition 2.1. *We denote by V to be an arbitrary vector space of dimension $n \geqslant 1$ over the real numbers. A bilinear form on V is a map $g : V \times V \to \mathbb{R}$ that is linear in each variable, such that*

$g(a_1v_1 + a_2v_2, w) = a_1g(v_1, w) + a_2g(v_2, w)$ and $g(v, a_1w_1 + a_2w_2) = a_1g(v, w_1) + a_2g(v, w_2)$ whenever $a \in \mathbb{R}$ and $v, w \in V$.

(i) g is symmetric if $g(w, v) = g(v, w)$ for all v, w.

(ii) g is nondegenerate if $g(v, w) = 0$ for all $w \in V \implies (v = 0)$.

*A nondegenerate, symmetric, bilinear form g is generally called an **inner product** and the image of (v, w) under g and its convention is '$v \cdot w$' or '$\langle v, w\rangle$' instead of g(v, w).* [2]

Example (Euclidean Space): Let us suppose an inner product on $\mathbb{R}^n$ by defining, for $x, y \in \mathbb{R}^n$:

$$\langle x, y\rangle = x^T y$$

In order to verify that it is indeed an inner product, we need to show that all four properties hold. We shall only be verifying two properties:

$$\langle x, x\rangle = x^T x = x_1^2 + x_2^2 + \cdots + x_n^2 \geqslant 0,$$

with equality if and only if $x_i = 0$ for all i, that is, $x = 0$.

$$\langle x + y, z\rangle = (x + y)^T z = (x^T + y^T)z = x^T z + y^T z = \langle x, z\rangle + \langle y, z\rangle,$$

, where we have used that matrix multiplication distributes over addition. [9]

Definition 2.2. *if g is an inner product on V, then two vectors v and w for which $g(v, w) = 0$ are said to be **g-orthogonal**, or simply **orthogonal** if there is no ambiguity as to which inner product is intended.* [2]

Definition 2.3. *If W is a subspace of V, then the orthogonal complement $W^\perp$ of W in V is defined by the following set:*

$$W^\perp = \{v \in V : g(v, w) = 0, \quad \forall w \in W\}$$

Thus, by definition we can say that $W^\perp$ is a subspace of V. [2]

Definition 2.4. *The quadratic form associated with the inner product g on V is the map $Q : V \to \mathbb{R}$ defined by $Q(v) = g(v, v) = v \cdot v$ (often denoted v^2).* [2]

Theorem 2.1. *Let V be an n-dimensional real vector space on which is defined a nondegenerate, symmetric, bilinear form $g : V \times V \to \mathbb{R}$.*

Then there exists a basis $\{e_1, ..., e_n\}$ for V such that $g(e_i, e_j) = 0$ if $i \neq j$ and $Q(e_i) = \pm 1$ for each $i = 1, ..., n$. Moreover, the number of basis vectors e_i for which $Q(e_i) = -1$ is the same for any such basis. [2]

Proof. First, let us make the following observations:

(i) Since g is nondegenerate there exists a pair of vectors (v, w) for which $g(v, w) \neq 0$.

(ii) We are claiming that, in fact, there must be a single vector $u \in V$ such that $Q(u) \neq 0$.

(iii) Keep in mind that if one of $Q(v)$ or $Q(w)$ is nonzero we are done. On the other hand, if $Q(v) = Q(w) = 0$, then $Q(v + w) = Q(v) + 2g(v, w) + Q(w) = 2g(v, w) \neq 0$ so we may take $u = v + w$.

Let us suppose that $n > 1$ and that every inner product on a vector space of dimension less than n has a basis of the required type. Let the dimension of V be n.

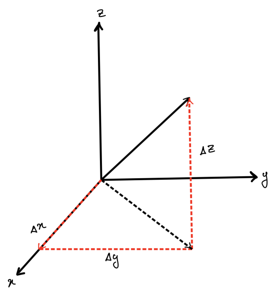
Now taking a $u \in V$ such that $Q(u) \neq 0$ and letting $e_n = (|Q(u)|)^{\frac{-1}{2}} \cdot u$ so that $Q(e_n) = \pm 1$. By definition, W is a subspace of V and since e_n is not in W, $\dim W < n$.

The restriction of $W \times W$ is an inner product on W so the induction hypothesis assures us the existence of a basis $\{e_1, ..., e_n\}$, $m = dimW$, for W such that $g(e_i, e_j) = 0$ if $i \neq j$ and $Q(e_i) = \pm 1$ for $i = 1, ..., m$. This we finally get that $m = n - 1$ $\{e_1, ..., e_m, e_n\}$ is a basis for V. $\square$

Theorem 2.2 (Pythagoras's Theorem)**.** *The Pythagorean Theorem, also known as Pythagoras' Theorem, is a fundamental principle in Euclidean geometry that establishes a relationship between the lengths of the three sides of a right triangle. Specifically, it states that the sum of the squares of the lengths of the two shorter sides of the triangle is equal to the square of the length of the longest side, which is also known as the hypotenuse. This relationship can be expressed mathematically as the Pythagorean equation: $a^2 + b^2 = c^2$, where a, b, and c represent the lengths of the sides of the right triangle.* [12]

Discussion: The Pythagorean theorem applies to infinitesimal triangles seen in differential geometry. In three dimensional space, the distance between two infinitesimally separated points satisfies the equation and can be visually represented as follows:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$



Let us note that this equation can be generalized to n dimensions.

Definition 2.5. *Hyperbolic geometry is a geometric system that is confined entirely within the open unit disc in the plane $\mathbb{R}^2$. This means that all geometric objects and relations in hyperbolic geometry are defined within this specific region.*

*The unit circle, which is the boundary of the open unit disc, is not considered as part of the hyperbolic geometry world. Instead, it is referred to as the **infinity circle**. This means that any point on the unit circle is considered to be infinitely far away from the interior of the open unit disc, and hence, it is not included in the hyperbolic geometry system.* [10]

Definition 2.6. *The length of a path in a disc is $\gamma(t) = (x(t), y(t))$ for $t_0 \leqslant t \leqslant t_1$, where the length of γ in hyperbolic geometry is given by the intergral $\int_{t_0}^{t_1} \frac{||\gamma'(t)||}{1-||\gamma'(t)||^2} dt$* [10]

Definition 2.7. *A path between points A and B is **Geodesics** if it has length no greater than the length of any other path between A and B. Geodesics in hyperbolic geometry are the analogue of straight lines in euclidean geometry. If there were light in the hyperbolic plane, it would travel along geodesics.* [10]

Lemma 2.1. *Any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies the following equation: (Proof in* [10]*)*

$$\sin A = \frac{\sinh a}{\sinh c} \quad \& \quad \frac{\tanh a}{\tanh c}$$

Definition 2.8. *Space-group symmetry is a combination of the translational symmetry of a lattice together with other symmetry elements such as rotation and/or screw axes.* [6]

Definition 2.9. *A translation symmetry (also called a slide) involves moving a figure in a specific direction for a specific distance. A vector (a line segment with an arrow on one end) can be used to describe a translation, because the vector communicates both a distance (the length of the segment) and a direction (the direction the arrow points).* [5]

We are going to require one definitions that pertain to physics, in order to derive the transformation of the Lorentz Boosts.

Definition 2.10. *Principle of Equivalence(2) is a frame that is linearly accelerated relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field.* [8]

3 Minkowski Space-Time Geometry

Before delving into the geometrical approach to Minkowski Space-time geometry, it is important to establish some formal definitions, theorems, and lemmas. Familiarizing oneself with these concepts will be helpful in comprehending the geometry of Minkowski Space-time.

3.1 Foundations for Minkowski Space-time Geometry

Definition 3.1. *Minkowski Space-time is a 4-dimensional real vector space M on which is defined a nondegenerate, symmetric, bilinear form g of index* 1*. The elements of M will be called events and g is referred to as a Lorentz inner product on M.*

Thus, there exists a basis $\{e_1, e_2, e_3, e_4\}$ for M with the property that if $v = v^a e_a$ and $w = w^a e_a$, then we have the following equation: [2]

$$g(v, w) = v^1 w^1 + v^2 w^2 + v^3 w^3 - v^4 w^4$$

Discussion: As mentioned in the definition "The elements of M will be called events", should be thought of as actual or physically possible point-events. An orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for M is to be identified with a "frame of reference". Let us then introduce the 4×4 matrix η defined by:

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Whose entries will be denoted either η_{ab} or η^{ab}, the choice in any particular situation being dictated by the requirements of the summation convention. Thus, $\eta_{ab} = \eta^{ab} = 1$ if $a = b = 1, 2, 3$, and $\eta_{ab} = \eta^{ab} = -1$ if $a = b = 0, 4$. [2]

In many physics textbooks, different sign convention is adopted to differentiate between space and time, where they will use either $-, -, -, +$ or $+, +, +, -$. Throughout this paper and for the maintenance of uniformity we will be using the sign convention of $+,+,+,-$ to describe our Space-time.

As consequence to the bases of the matrix above, how would one construct a *null basis* for M (i.e., a set of four linearly independent null vectors)?.

Such a null basis cannot consist of mutually orthogonal vectors! However, we shall follow the following theorem below:

Theorem 3.1. *Two nonzero null vectors v and w in M are orthogonal if and only if they are parallel, i.e., iff there is a t in $\mathbb{R}$ such that $v = tw$.* [2]

Proof. The *Schwartz Inequality* for $\mathbb{R}^3$ asserts that if $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$, then:

$$(x^1y^1 + x^2y^2 + x^4y^3) \leqslant ((x^1)^2 + (x^2)^2 + (x^3)^2)((y^1)^2 + (y^2)^2 + (y^3)^2$$

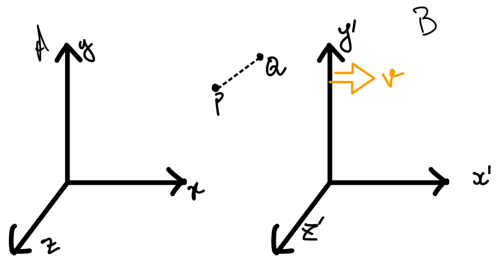
and that equality holds if and only if x and y are linearly dependent. □

3.2 Geometric Construction of Minkowski's Diagram

Now that we have provided some rigorous definitions and theorems for Minkowski Space-time, let us take a more geometrical approach to understand Hyperbolic Geometry of Minkowski space-time in $\mathbb{R}^2$, which can be generalized to $\mathbb{R}^4$.

This process involves a rigorous comparison and contrast of pertinent information from Euclidean Geometry to provide a comprehensive understanding of Minkowski Hyperbolic Geometry to the readers. For this construction and analysis, we will be referencing Professor Dibyajyoti Das lecture on Hyperbolic Geometry. [1]

Let us consider the following graphs in a Euclidean Space:



From this graph, we can make the following observations:

$$\Delta x^2 + \Delta y^2 + \Delta z^2 \neq \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

$$\Delta t \neq \Delta t'$$

Now the two observers $'A'$ and $'B'$ from the graph are such that B is in motion with velocity v relative to A who is at rest. So the following properties of this event must be true:

(i) Space-time Interval must remain **Invariant** under a Lorentz transformation (i.e., in other words transformations of the graphs from A to B must be preserved and must have an inverse)

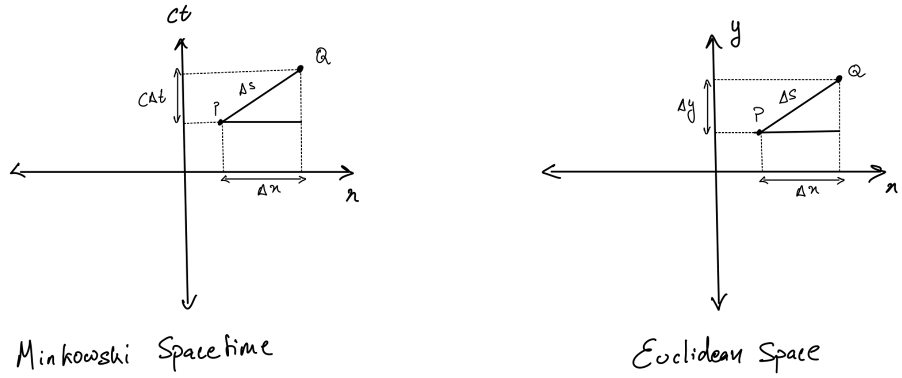
(ii) Speed of light is constant for all inertial observers.

These properties can be seen from the following equation:

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2\Delta t'^2$$

Now let us reduce total dimensionality to $\mathbb{R}^2$, a consequence to reducing the spacial dimension to 1 is that $\Delta y^2 = \Delta z^2 = 0$. So we can now use the following Space-time interval: $\Delta x^2 - c^2 \Delta t^2 = \Delta x'^2 - c^2 \Delta t'^2$.

In order to attain clarity on the above expression, Let us suppose two events P and Q takes place in both Minkowski space and Euclidean space such that the spatial change in distance for both is measured as Δs. Let observe what this looks like graphically:



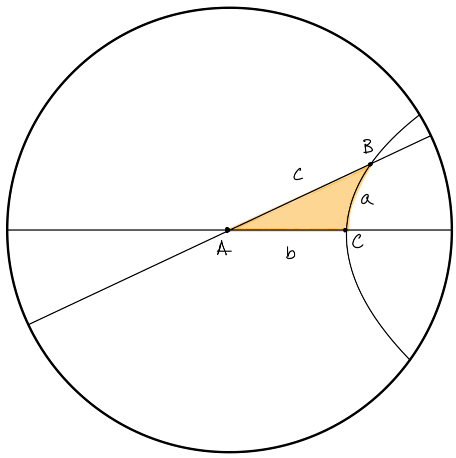
Notice that the distance for Euclidean space is $\Delta s^2 = \Delta x^2 + \Delta y^2$, by **Theorem 2.2** and the distance measured for Minkowski space is $\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$. We observe that there is a difference of a sign change in Minkowski Space. Well by **Definition 3.1**, we know that the negative sign for time dimension is convention so as to separate the temporal components from the spatial components.

Since this is a geometrical approach, we observe that **Theorem 2.2** does not valid on Minkowski-space time.

In fact the reason why the distance $\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$ has a minus sign is because of the **Hyperbolic Pythagorean Theorem**.

Theorem 3.2 (Hyperbolic Pythagorean Theorem)**.** *If a right triangle in the hyperbolic plane has sides of length a and b and a hypotenuse of length c, then* $\cosh a \cosh b = \cosh c$*.* [10]

Proof. Observe the picture below of a hyperbolic plane, where the edges of the triangle are geodesics, with the sides of length a and b adjacent to the right angle and the hypotenuse (of length c) is the edge across from the right angle. The function $\cosh x = \frac{e^x + e^{-x}}{2}$ is a hyperbolic cosine.



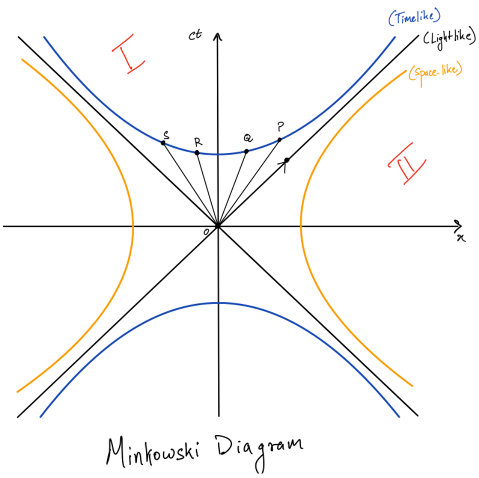
Let $\triangle$ ABC be a right triangle in the hyperbolic plane with C the right angle. Without loss of generality, we may assume that the vertex A is the origin and that two of the edges, one of which is the hypotenuse, are portions of diameters, as in our picture. Let $d(AB)$ be the hyperbolic distance from point A to point B and $\overline{AB}$ the euclidean distance.

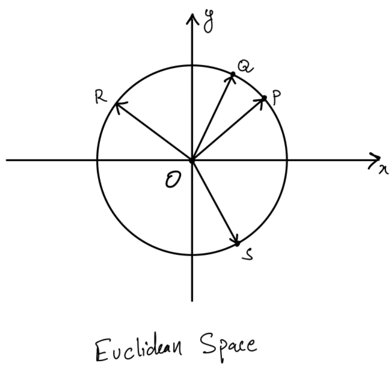
Since we have already defined the hyperbolic cosine function cosh, sinh is defined similarly: $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and the hyperbolic tangent function is simply $\tanh x = \frac{\sinh x}{\cosh x}$.

By applying **Lemma 2.3** the Hyperbolic Pythagorean Theorem can be derived from the equality $\sin A^2 + \cos A^2 = 1$. Thus by applying identities for sinh and cosh analogous to the identities involving sin and cos, finally get the following expression: $\cosh a \cosh b = \cosh c$ $\square$

Bringing it back to the construction of our hyperbolic space, we want to conceptualize a better understanding of this space. Thus let us look at the look at the line segments or vectors drawn from the origin. We will also compare the hyperbolic space with Euclidean space such that mathematically we are looking at the following equations: $x^2 - c^2t^2 = S^2$ and $x^2 + y^2 = S^2$, where $S \in \mathbb{R}$.

Observe the diagram below:





Observing the Euclidean space, The locus of all points that are equidistant from the origin can be described as a circle. Now, what is the locus of all points which are equidistant from the origin?

These points can be found using the following equation $x^2 - c^2t^2 = S^2$, which can be graphically represented as a hyperbola as seen in the diagram above. Thus the line segments $\{\overline{OP}, \overline{OQ}, \overline{OR}, \overline{OS}\}$ have equidistant lengths in **Space-time** from the origin. In fact the locus of all the points which are equidistant from the origin in Minkowski Space-time is represented by the blue hyperbola above the origin.

Finally, we can discuss what null vectors are and how can they be represented visually. By **Theorem 3.1**, we know that two nonzero null vectors v and w in M are orthogonal if and only if they are parallel. What does this mean graphically?

By definition, a null vector is a vector such that the n-dimensional vector 0 is of length 0. i.e., the vector with n components, each of which is 0. In other words it is a vector at the origin, such that its magnitude is equal to 0.

In regards to Minkowski Space-time diagram, which is represented by $x^2 - c^2t^2 = S^2$. Since this equation has a negative sign, S^2 can be equal to 0 even if x and t themselves are not equal to zero. So in the diagram above, the null vectors can be drawn on a line which is at 45 degrees with respect to the time and x-axis in both directions.

Let us mathematically describe this phenomenon:

$$x^2 - c^2t^2 = S^2 = 0$$

$$\implies x^2 = c^2t^2$$

$$\implies c^2 = \frac{x^2}{t^2}$$

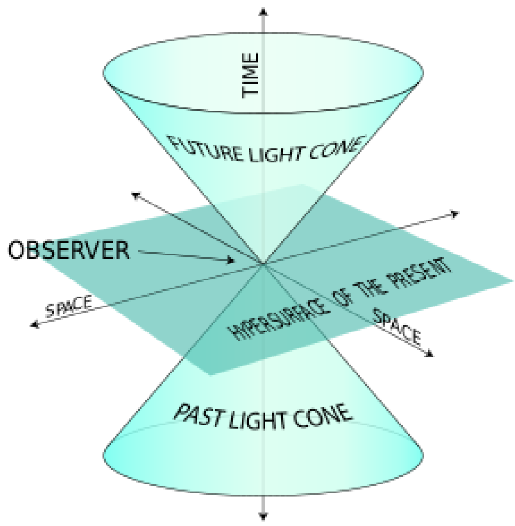
$$\boxed{\therefore c = \frac{x}{t}}$$

Using the equation above, the null vector can be described where the spatial displacement and the temporal displacement are such that it is equal to the speed of light. So when the magnitude of the vector comes out to be zero, that means any vector or line segment drawn at the 45 degrees with respect to the time and x-axis in both directions is a representation of light-speed.

Also, this is only possible for mass-less particles like light photons. By convention, this is called the **world line** or the trajectory of a light photon.

Discussion: In general discussions of Minkowski Space-time we look at the curves that connect the positions and time periods in space-time where they are known as world lines. So the world line of a light photon that is traveling in the $+x$ direction and the world line of a light photon that is traveling in the $-x$ direction divide the entire space-time diagram into two quadrants as seen in the diagram.

The combination of these two quadrants of world lines of a light photon end up creating what is known as a **Light Cone** in $\mathbb{R}^3$. So right now we have one space and one time dimension, but if we create a two space and 1 time dimension and we observe all possible scenarios of how a light photon can travel from the origin outwards, then all those points can be traced into some kind of a cone, which is known as a light Cone. We can visually observe this discussion in the image below:



.

The figure above can be be generalized to $\mathbb{R}^4$ in a similar way. Now we have observed and discussed what happens when the magnitude $S^2 = 0$, by convention is labelled as **Lightlike**. Now we want to define what **Timelike** and **Spacelike** intervals represent.

Timelike intervals are such that the magnitude $S^2 < 0$ or $x^2 < c^2t^2$, this means that the distance that light photons can travel in that time period is greater than the spacial distance between two events.

Spacelike intervals are such that the magnitude $S^2 > 0$ or $x^2 > c^2t^2$, this means that the distance that light photons can travel in that time period is greater than the spacial distance distance between two events.

Finally, we have built all the necessary foundations needed to understand Minkowski Hyperbolic Space-time Geometry, with a solid visual representation. Now we can come to the crux of this paper, which is Lorentz Transformations.

In the next section, we will address Lorentz Transformations as well as deriving the equation and by the end of it, gain a similar visual understanding of how it works.

4 Lorentz Transformations

Before we proceed with the geometrical approach, let us first state some formal definitions, Theorems and lemmas from the discussion in the previous section that will be beneficial to understand and derive Lorentz Transformations.

Definition 4.1. *The null cone (or light cone) $C_N(x_0)$ at x_0 in M is given by:* [2]

$$C_N(x_0) = \{x \in M : Q(x - x_0) = 0\}$$

Definition 4.2. *The null world line (or light ray) $R_{x_0,x}$ containing x_0 and x is given by:* [2]

$$R_{x_0,x} = \{x_0 + t(x - x_0) : t \in \mathbb{R}\}$$

Theorem 4.1. *Let x_0 and x be two distinct events with $Q(x - x_0)$, then:* [2]

$$R_{x_0,x} = C_N(x_0) \cap C_N(x)$$

Proof. First let $z = x_0 + t(x - x_0)$ be an element of $R_{x_0,x}$. Then $z - x_0 = t(x - x_0)$ so $Q(z - x_0) = t^2Q(x - x_0) = 0$ so z is in $C_N(x_0)$. Since the latter is true, we can then say that z is in $C_N(x)$ and so $R_{x_0,x} \subseteq C_N(x0) \cap C_N(x)$.

In order to prove the reverse containment we assume that z is in $C_N(x_0) \cap C_N(x)$. Then each of the vectors $z - x, z - x_0$ and $x_0 - x$ is null. But $z - z_0 = -2g(z - z, z_0 - x)$ so we observe the following computations:

$$0 = Q(z - z_0) - Q(z - x) - 2g(z - x, x_0 - x) + Q(x_0 - x) = -2g(z - x, x_0 - x).$$

$$\therefore \quad g = (z - x, x_0 - x) = 0$$

If $z = x$ we are done. If $z \neq x$, then $x \neq x_0$, so when we apply **Theorem 3.1** to the orthogonal null vectors $z - x$ and $x_0 - x$ to obtain a 't' in $\mathbb{R}$ such that $z - x = t(x_0 - x)$ and it follows that z is in $R_{x_0,x}$ as required. ☐

Lemma 4.1. *Let $L : M \to M$ be a linear transformation. Then the following are equivalent:*
(a) L is an orthogonal transformation.
(b) L preserves the quadratic form M, i.e., $Q(Lx) = Q(x)$ for all $x \in M$.
(c) L carries any othonormal basis for M onto another orthonormal basis for M.

Proof. Observe an abbreviated proof in [2] □

Definition 4.3. *We define the matrix $\Lambda = [\Lambda^a_b]_{a,b=1,2,3,4}$ associated with the orthogonal transformation L and the orthonormal basis e_a by the following:* [2]

$$\Lambda = \begin{bmatrix} \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 & \Lambda^1_4 \\ \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 & \Lambda^2_4 \\ \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 & \Lambda^3_4 \\ \Lambda^4_1 & \Lambda^4_2 & \Lambda^4_3 & \Lambda^4_4 \end{bmatrix}$$

Discussion: We regard matrix Λ associated with L and e_a as a coordinate transformation matrix in the usual way. Specifically, if the event x in M has coordinates $x = x^1e_1 + x^2e_2 + x^3e_3 + x^4e_4$ relative to e_a, then its coordinates relative to $\{\hat{e_a}\} = \{Le_a\}$ are $x = \hat{x}^1\hat{e_1} + \hat{x}^2\hat{e_2} + \hat{x}^3\hat{e_3} + \hat{x}^4\hat{e_4}$, where: [2]

$$\begin{aligned} \hat{x}^1 &= \Lambda^1_1 x^1 + \Lambda^1_2 x^2 + \Lambda^1_3 x^3 + \Lambda^1_4 x^4 \\ \hat{x}^2 &= \Lambda^2_1 x^1 + \Lambda^2_2 x^2 + \Lambda^2_3 x^3 + \Lambda^2_4 x^4 \\ \hat{x}^3 &= \Lambda^3_1 x^1 + \Lambda^3_2 x^2 + \Lambda^3_3 x^3 + \Lambda^3_4 x^4 \\ \hat{x}^4 &= \Lambda^4_1 x^1 + \Lambda^4_2 x^2 + \Lambda^4_3 x^3 + \Lambda^4_4 x^4 \end{aligned}$$

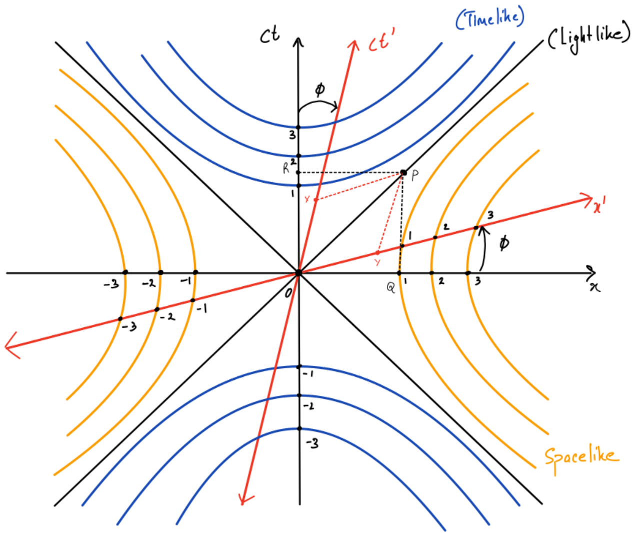
which we generally write more concisely as

$$\hat{x}^a = \Lambda^a_b x^b \quad a = 1, 2, 3, 4.$$

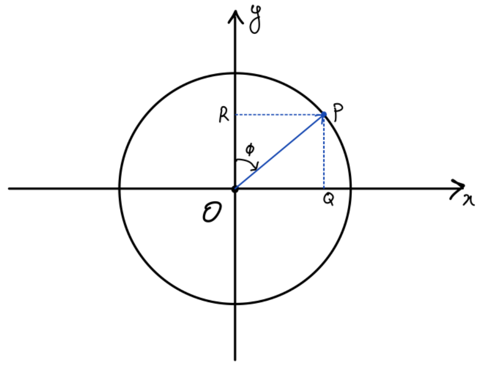
4.1 Geometric Construction of Lorentz Transformation

We are initially interested in the calculations between two observers or two events "A" and "B", such that A is at rest and B is moving at high speeds. Now we know that their calculations for the space-time interval are equal: $\Delta x^2 - c^2\Delta t^2 = \Delta x'^2 - c^2\Delta t'^2$. Moreover, we ascertain that they are also connected or related to each other by the Lorentz Transformations (Mathematically true). For this construction and analysis, we will be referencing Professor Dibyajyoti Das lecture on Lorentz Transformation. [1]

So in order to understand how these transformations are equal to each other and connected, we need to gain a visual understanding between two inertial observers in space-time itself. Below is a graph that represents represents two observers A which is in black B which is in red. Let us make the following observations based on the hyperbolic geometry below in comparison with the Euclidean space:



Lorentz Transformations on Minkowski Space-Time



Euclidean Space

Since A and B are related under Lorentz Transformations, the transformation gives us a relationship or a set of transformation equations of the coordinates of space and time between two observers or events. Thus, by usage of the Lorentz Transformations, we end up getting the invariant of this space-time interval.

In regards to the Euclidean Space, what kind of transformations of the coordinated axis (x, y) will preserve the length of a given vector? The magnitude of a position vector $\overrightarrow{OP}$ is invariant under a rotation of the coordinated axes.

Now what kind of transformation of the coordinate axes in Minkowski Space-time will keep the Space-time interval invariant? Notice that a rotation like in Euclidean space would not preserve the magnitude of our vector but a combination of rotation and stretching of space-time would be invariant.

Space-time geometry is different from Euclidean in a way such that an object in euclidean coordinate set of axis an object at rest will be associated with a point. But in Space-time, an object at rest will have a **world line**. In order words, for space-time the temporal coordinate keeps moving forward (time does not stop). Thus, observe "A" has world line ct or the coordinates axis (x, ct).

Similarly, observer B moves at constant speed with respect to A, has a world line such that its slope is lesser than that of light Cone with respect to the x axis. Thus in the figure above we see that B has the coordinate axis (x', ct'). The x' axis rotates with an angle φ from x such that the angle it makes with the world line must be the same as that of when ct rotates to ct' with an angle φ. This rotation is done to preserve the magnitude of the vector and also that the speed of light is a constant universal truth.

Observe the components of the Euclidean space (using Pythagorean Theorem):

$$OR = OP \cos \varphi$$

$$OQ = OP \sin \varphi$$

Let us compare this with the vector components of A in Space-time (Using Hyperbolic Pythagorean Theorem) and observe the following computations:

$$OR = OP \cosh \varphi$$
$$OQ = OP \sinh \varphi$$
$$\frac{OQ}{OR} = \frac{\sinh \varphi}{\cosh \varphi}$$
$$\frac{x}{ct} = \tanh \varphi$$
$$\boxed{\therefore \quad \frac{v}{c} = \tanh \varphi}$$

Now we have a relationship between two observers A, B and the visual representation of their axes, so if B is traveling with velocity v and A is at rest, then their time axes will have an inclination of an angle of φ such that $\frac{v}{c} = \tanh \varphi$, where v is the relative velocity between A and B.

Another observation we can make is that, the vector components for B is not at an angle 90 degrees. Instead we can use the inner product of two vectors and ensure that they are equal to zero in order to find their components.

4.2 Preservation of Lorentz Transformations

Following the discussion in this section we understand that an orthogonal transformation associated with orthonormal basis has the general form $\hat{x}^a = \Lambda^a_b x^b \quad a = 1, 2, 3, 4$. So the Linear transformation that is needed such that Λ preserves the quadratic form of M is $\Lambda^T \eta \Lambda = \eta$. In particular, if $x - x_0$ is the displacement vector between two events for which $Q(x - x_0) = 0$, then $\Delta \hat{x}^a = 0$. [2]

Definition 4.4. *Any 4×4 matrix Λ that satisfies $\Lambda^T \eta \Lambda = \eta$ is called a **general (homogeneous) Lorentz Transformation**. Since the orthogonal transformation of M are isomorphisms, the matrix Λ must be invertible, which we find to be $\Lambda^{-1} = \eta \Lambda^T \eta$.* [2]

To gain a deeper understanding of preservation, we narrow our focus to the two-dimensional case. Specifically, we examine the Minkowski plane, which is represented by $\mathbb{R}^2$ and has two coordinates, x and t. We also use the inner product η, which is defined by a particular matrix.

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The following theorem below, a detailed treatment of the four dimensional case very similar to this two dimension space can be found in [2].

Theorem 4.2. *let $\Lambda = \begin{bmatrix} \Lambda^1_1 & \Lambda^1_2 \\ \Lambda^2_1 & \Lambda^2_2 \end{bmatrix}$ be such that $\Lambda^T \eta \Lambda = \eta$ holds, then λ is given by $\Lambda = \pm \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{bmatrix}$ for a constant v such that $|v| < 1$. The inverse transformation $\Lambda^{-1} = \eta \Lambda^T \eta$ must also be true for velocity $-v$* [7]

Proof. Solving the left hand side of $\Lambda^T \eta \Lambda = \eta$, we observe the following computations:

$$\Lambda^T \eta \Lambda = \eta$$

$$\implies \begin{bmatrix} (\Lambda^1_1)^2 - (\Lambda^2_1)^2 & \Lambda^1_2\Lambda^1_1 - \Lambda^2_1\Lambda^2_2 \\ \Lambda^1_1\Lambda^1_2 - \Lambda^2_2\Lambda^2_1 & (\Lambda^1_2)^2 - (\Lambda^2_2)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, it follows that the components of Λ must fulfill the following conditions when equating it with η

$$\begin{cases} (\Lambda^1_1)^2 - (\Lambda^2_1)^2 = 1 \\ \Lambda^1_2\Lambda^1_1 - \Lambda^2_1\Lambda^2_2 = 0 \\ \Lambda^1_1\Lambda^1_2 - \Lambda^2_2\Lambda^2_1 = 0 \\ (\Lambda^1_2)^2 - (\Lambda^2_2)^2 = -1 \end{cases}$$

Solving these equations for $\{\Lambda^1_1, \Lambda^1_2, \Lambda^2_1, \Lambda^2_2\}$, we obtain the following:

$$\begin{cases} \Lambda^1_1 = \pm\frac{1}{\sqrt{1-v^2}} \\ \Lambda^1_2 = \pm\frac{v}{\sqrt{1-v^2}} \\ \Lambda^2_1 = \pm\frac{v}{\sqrt{1-v^2}} \\ \Lambda^2_2 = \pm\frac{1}{\sqrt{1-v^2}} \end{cases} = \pm \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{bmatrix}$$

for $|v| < 1^1$. This gives us Λ. Now for velocity being $-v$ we find that inverse $\Lambda^{-1} = \eta\Lambda^T\eta$ can be computed in a similar way:

$$\Lambda^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \Lambda^1_1 & -\Lambda^2_1 \\ -\Lambda^1_2 & \Lambda^2_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{bmatrix} = \eta\Lambda^T\eta$$

□

By generalizing Minkowski's space to $\mathbb{R}^4$, **Theorem 4.2** confirms to us that the Lorentz transformations preserves the quadratic form of Minkowski Space M. In order to derive Lorentz Boosts, we need to know have further definitions Lorentz Transformations to proceed.

Finally, we have reached a point in our knowledge where we have built all the necessary knowledge required to derive the transformations known as Lorentz Boosts.

4.3 The Derivation of Lorentz Boosts

A Lorentz transformation is a mathematical technique used in special relativity to transform coordinates between different reference frames. While it is possible for a Lorentz transformation to include a rotation of space, special relativity is mainly concerned with a particular type of transformation known as a "boost," which does not involve any spatial rotation.

Boosts can be thought of as analogous to rotations in Euclidean space, but they occur in planes that are defined by one timelike and one spacelike direction, such as the xt-plane. In other words, boosts involve changing the relative motion of two objects, but without any change in the orientation of the space through which they are moving.

Definition 4.5. *The set of all general (Homogeneous) Lorentz transformations forms a group under matrix multiplication, i.e., that is closed under the formation of products and inverses. This group is called the* ***general (homogeneous) Lorentz group****, which is denoted as* $\mathfrak{L}_{GH}$ [2]

For this derivation we will be referencing Professor Victor Yakovenko paper on "Derivation of the Lorentz Transformation" [13].

The fundamental principles of equivalence between all inertial reference frames and the symmetries of space and time can be used to derive the most general transformation of space and time coordinates, the Lorentz Boosts. This transformation is characterized by one free parameter that has the dimensions of speed and can be identified as the speed of light (c).

The derivation of this transformation relies on the group property of the Lorentz transformations, which by **Definition 4.5** means that a combination of two Lorentz transformations also belongs to the class of Lorentz transformations. Thus, Let us Observe the following computations:

Let's think about two inertial reference frames S and S'. The reference frame S' is in motion relative to S with a velocity v along the x-axis. We are aware that the coordinates y and z, which are perpendicular to the velocity, are identical in both reference frames, denoted as $y = y'$ and $z = z'$.

Hence, we only need to consider the transformation of the coordinates x and t from the reference frame S to $x' = f_x(x,t)$ and $t' = f_t(x,t)$ in the reference frame S'. Here we have the transformation function defined to be $f_{(x,t)} : \Lambda(x,t) \to \Lambda(x',t')$.

By **definition 2.8** the symmetry of space and time tells us that the functions $f_x(x,t)$ and $f_t(x,t)$ must be linear functions. Moreover by **Definition 2.9**, we find that the relative distances between two events in one reference frame must depend only on the relative distances in another frame. Thus, we observe the following equations:

$$x'_1 - x'_2 = f_x(x_1 - x_2, t_1 - t_2), \quad t'_1 - t'_2 = f_t(x_1 - x_2, t_1 - t_2) \tag{1}$$

Because (1) must be valid for any two events, the function $f_x(x,t)$ and $f_t(x,t)$ must be linear function. Thus, we find that:

$$x' = Ax + Bt, \tag{2}$$
$$t' = Cx + Dt, \tag{3}$$

Where $\{A, B, C, D\}$ are some coefficients that depend on v. The origin of the frame of reference S' has the coordinate $x' = 0$ and moves with velocity v relative to the reference frame S, so that $x = vt$. Substituting these values into (2) we find that $B = -vA$. Substituting in (2), we observe the following:

$$x' = A(x - vt), \tag{4}$$

The only unknowns we need to find now are functions A, C, D of v. The origin of reference frame S has the coordinate $x = 0$ and moves with velocity $-v$ relative to the reference frame S', so that $x' = -vt'$. Substituting these values in (4) and (3), we observe the following computation:

$$t' = Cx + At = A(Fx + t), \tag{5}$$

Where $F = \frac{C}{A}$. Now let us use convention notation and replace $A = \gamma$. Then (4) and (5) has the form:

$$x' = \gamma(x - vt), \tag{6}$$

$$t' = \gamma(Fx + t), \tag{7}$$

Which in matrix form looks like

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -v \\ F & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}, \tag{8}$$

Now we narrowed down our unknowns to the functions γ_v and F_v of v. By **Definition 4.5**, we know that a combination of two Lorentz Transformations must also be a Lorentz Transformations. Let us consider a reference frame S' moving relative to S with velocity v_1 and a reference frame S'' moving relative to S' with velocity v_2. Then we observe the following:

$$x'' = \gamma_{v_2}(x' - v_2 t'), \quad x' = \gamma_{v_1}(x - v_1 t), \tag{9}$$
$$t'' = \gamma_{v_2}(F_{v_2} x' + t'), \quad t' = \gamma_{v_1}(F_{v_1} x + t), \tag{10}$$

Substituting x' and t' from (10) into (9), we get the following computation:

$$x'' = \gamma_{v_2}\gamma_{v_1}\left[(1 - F_{v_1} v_2)x - (v_1 + v_2)t\right], \tag{11}$$
$$t'' = \gamma_{v_2}\gamma_{v_1}\left[(F_{v_1} + F_{v_2})x - (1 - F_{v_2} v_1)t\right], \tag{12}$$

For a general Lorentz transformation, the coefficients in front of x in (6) and (7) are equal, i.e. the diagonal matrix elements in (8) are equal. Moreover, (10) and (11) must also satisfy the following requirements:

$$1 - F_{v_1} v_2 = 1 - F_{v_2} v_1 \implies \frac{v_2}{F_{v_2}} = \frac{v_1}{F_{v_1}}, \tag{13}$$

In the second part of (13), the left hand side depends only on v_2, and the right hand side only on v_1. This equation can be satisfied only if the ratio of $\frac{v}{F_v}$ is a constant a independent of velocity v, which gives us:

$$f_v = \frac{v}{a} \tag{14}$$

Now substituting (14) into (6), (7), and in (9), we observe the following changes:

$$x' = \gamma_v(x - vt), \quad t' = \gamma_v(\frac{xv}{a} + t), \tag{15}$$

Which in matrix form looks like

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma_v \begin{bmatrix} 1 & -v \\ \frac{v}{a} & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}, \tag{16}$$

Now we need to find only one unknown function γ_v, whereas the coefficient a is a fundamental constant independent of v.

Let us make the Lorentz transformation from the reference frame S to S' and then from S' back to S. The first transformation is performed with the velocity v, whereas the second transformation with velocity $-v$. The equations are similar to (9) and (10). So we then observe the following computation:

$$x = \gamma_{-v}(x' + vt'), \quad x' = \gamma_v(x - vt), \tag{17}$$
$$t = \gamma_{-v}(\frac{-x'v}{a} + t'), \quad t' = \gamma_v(\frac{xv}{a} + t), \tag{18}$$

Let us substitute x' and t' from the second parts of equation (17) and (17) into the first parts, we arrive at the following equation:

$$x = \gamma_{-v}\gamma_v(1 + \frac{v^2}{x})x, \quad t = \gamma_{-v}\gamma_v(1 + \frac{v^2}{a})t, \tag{19}$$

Now from (19), we ascertain that it must be valid for any x and t, so

$$\gamma_{-v}\gamma_v = \frac{1}{1 + \frac{v^2}{a}} \tag{20}$$

By **definition 2.8** of space symmetry, the function γ_v must depend only on the absolute value of velocity v, but not on its direction, so $\gamma_{-v} = \gamma_v$. Thus we find:

$$\gamma_v = \frac{1}{\sqrt{1 + \frac{v^2}{a}}} \tag{21}$$

Now substituting (21) into (15), we find the final expression for the most general transformation:

$$x' = \frac{x - vt}{\sqrt{1 + \frac{v^2}{a}}}, \quad t' = \frac{\frac{-xv}{a^2} + t}{\sqrt{1 + \frac{v^2}{a}}} \tag{22}$$

Equations (22) have one fundamental parameter a, which has the dimensionality of velocity squared. If $a < 0$, we can write it as:

$$a = -c^2 \tag{23}$$

So then our equation (22) become the standard Lorentz transform which is as follows:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{\frac{-xv}{c^2} + t}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{24}$$

Thus substituting in the Lorentz factor into equation (24), $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$, we finally get the Lorentz boosts and formally write them as:

$$\begin{aligned} t' &= \gamma(t - \frac{vx}{c^2}) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

We can easily verify the boosts, that when a particle is in motion with a velocity of c in one particular reference frame, it will also be moving at the same velocity c in any other reference frame. This means that if we define the particle's position as $x = ct$ in one reference frame, then its position in any other reference frame can be represented as $x' = ct'$.

As a result, the value of the parameter c remains constant and is unaffected by changes in reference frame. This constant value of c is known as the invariant speed.

Moreover, It is straightforward to check that the Lorentz Boosts preserves the space-time interval:

$$(ct')^2 - (x')^2 = (ct)^2 - x^2$$

Thus the Lorentz Boosts have Minkowski metric.

In the next section, we shall take a look at a few examples at how these Boosts equation are used in order to cement the idea on our heads.

4.4 Applications of Lorentz Boosts

Example 1: (Using Lorentz Transformation for time) [11]

Spacecraft S' is at rest, eventually heading toward Alpha Centauri, when Spacecraft S passes it at relative speed $\frac{c}{2}$. The captain of S' sends a radio signal that lasts 1.2 s according to that ship's clock. Use the Lorentz transformation to find the time interval of the signal measured by the communications officer of spaceship S.

Solution: Let us first identify the known information, $\Delta t' = t'_2 - t'_1 = 1.2s$ and $\Delta x' = x'_2 - x'_1 = 0$. Now the unknown information is $\Delta t = t_2 - t_1$.

let us express the answer as an equation. The time signal starts at (x', t'_1) and stops at $(x',, t'_2)$. Note that the x' coordinate for both events is the same because the clock is at rest in S'. So writing out the first Lorentz transformation equation in terms of $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, and similarly for the primed coordinated, we get the following equation:

$$\Delta t = \frac{\Delta t' + \frac{v\Delta x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substituting $\Delta t' = 1.2s$, we get the following equation:

$$\boxed{\therefore \quad \Delta t = \frac{1.2s}{\sqrt{1 - (\frac{1}{2})^2}} = 1.4s}$$

Note that the transformation reproduces the time dialation.

Example 2: (Using Lorentz Transformation for length) [11]

A surveyor measures a street to be $L = 100m$ long in Earth frame S. Use the Lorentz transformation to obtain an expression for its length measured from a spaceship S', moving by at speed $0.20c$, assuming the x coordinates of the two frames coincide at time $t = 0$.

Solution: Let us first identify the known information, $L = 100m, v = 0.20c, \Delta t = 0$. Now the unknown information is L'.

let us express the answer as an equation. The surveyor in frame S has measured the two ends of the stick simultaneously, and found them at rest at x_2 and x_1 a distance $L = x_2 - x_1 = 100m$ apart. The spaceship crew measures the simultaneous location of the ends of the sticks in their frame. To relate the lengths recorded by observers in S' and S, respectively, write the second of the four Lorentz transformation equations as:

$$x_2 - x_1 = \frac{x_2' + vt}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1' + vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x_2 - x_1 = \frac{x_2' - x_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substituting $x_2 - x_1 = 100m$, the length of the moving stick is equal to:

$$L' = x_2' - x_1'$$

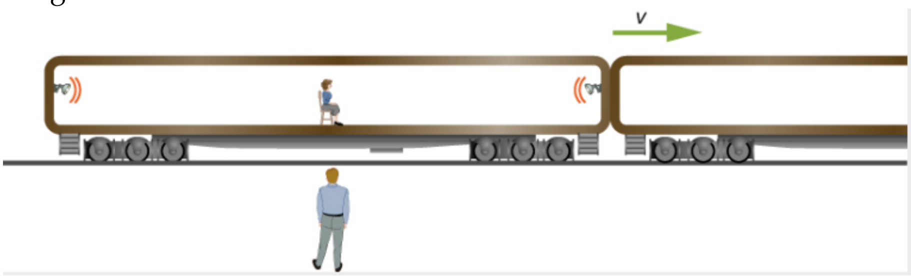
$$= (L)\sqrt{1 - \frac{v^2}{c^2}}$$

$$(100m)\sqrt{1 - (0.20)^2}$$

$$\boxed{\therefore \quad L' = 98.0m}$$

Example 3: (Lorentz Transformation and Simultaneity) [11]

The observer shown in Figure below standing by the railroad tracks sees the two bulbs flash simultaneously at both ends of the 26m long passenger car when the middle of the car passes him at a speed of $\frac{c}{2}$. Find the separation in time between when the bulbs flashed as seen by the train passenger seated in the middle of the car.



A person watching a train go by observes two bulbs flash simultaneously at opposite ends of a passenger car. There is another passenger inside of the car observing the same flashes but from a different perspective.

Solution: Let us first identify the known information, $\Delta t = 0$. Note that the spatial separation of the two events is between the two lamps, not the distance of the lamp to the passenger. Now the unknown information is $\Delta t' = t_2' - t_1'$. Again, note that the time interval is between the flashes of the lamps, not between arrival times for reaching the passenger. let us expression the answer as an equation and observe the computations below:

$$\Delta t = \frac{\Delta t' + \frac{v\Delta x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$0 = \frac{\Delta t' + \frac{\frac{c}{2}(26m)}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta t' = \frac{-26m/s}{2c} = -\frac{26m/s}{2(3.00 \times 10^8 m/s}$$

$$\boxed{\therefore \quad \Delta t' = -4.33 \times 10^{-8} s}$$

the significance of this problem is that the sign indicates that the event with the larger x_2', namely, the flash from the right, is seen to occur first in the S' frame, as found earlier for this example, so $t_2 < t_1$.

5 Conclusion

In conclusion, this seminar paper has provided an in-depth exploration of the mathematical foundations of Minkowski's space-time hyperbolic geometry, and Lorentz transformations. The paper started by reviewing the historical context and motivations that led Hermann Minkowski to introduce his novel geometric approach to space and time in the early 20th century, as well as Nobel Prize winner Hendrik Lorentz's history and motivation behind the infamous "Lorentz Transformations". The paper then delved into the fundamental mathematical concepts underlying Minkowski's space-time geometry, including concepts from Linear Algebra, Pythagoras's Theorem, and associated definitions.

The paper extensively explored the geometric interpretation of Lorentz transformations, which form the basis of special relativity, and derived the mathematical equations that describe these transformations. The visuals and examples presented in the paper can greatly aid in the reader's comprehension of the complex topic, and the building up of the foundation from the ground up ensured that the reader was well-prepared to understand the advanced concepts presented.

Finally, the paper discussed some of the key applications of Minkowski's geometry and Lorentz transformations, including their importance in modern physics and their relevance to our understanding of the structure of space and time. These topics are crucial to our understanding of the universe, and the paper has provided an excellent introduction to them.

Overall, this paper is an excellent resource for anyone interested in the intersection of mathematics and physics, and the ways in which mathematical concepts and tools have aided in the development of our modern understanding of the physical world.

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