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Asymptotic distributions for Random Median Quicksort

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Abstract

The first complete running time analysis of a stochastic divide and conquer algorithm was given for Quicksort, a sorting algorithm invented 1961 by Hoare. We analyse here the variant Random Median Quicksort. The analysis includes the expectation, the asymptotic distribution, the moments and exponential moments. The asymptotic distribution is characterized by a stochastic fixed point equation. The basic technic will be generating functions and the contraction method.

Keywords: Quicksort; Sorting algorithm; Analysis of algorithm; Recursive equation; Divide-and-conquer algorithm; Stochastic fixed point equation; Contraction method; Euler differential equation

1. Introduction

Quicksort was invented by Hoare [4,5] in 1961. Quicksort is one of the most widely used sorting algorithms. It is for instance the standard sorting procedure in Unix systems (see also [6,12–15]).

We consider here a variant of Quicksort, the Random Median Quicksort (RMQ) [7]. As pivot element take the median of $2k+1$ elements, where k itself is a random variable, drawn for every recall of the algorithm. In more detail: Let $K \in \mathbb{N}_0$ be fixed and $p = (p_0, \dots, p_K)$ be a probability vector on $0, 1, \dots, K$.

- Choose a k with probability p_k.
- Draw $2k+1$ random numbers from the list.
- Find the median of these.
- Form the lists of numbers strictly smaller than, equal to and strictly larger than the median.
- Arrange the lists in this order.
- Recall the algorithm (including a new choice of k) for each list with at least $2K+1$ elements.
- Continue as long as possible.

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- Sort all remaining lists with some reasonable sorting program.
- Exit.

This algorithm terminates because in every step we decrease the list sizes. After finitely many steps only lists remain of list size strictly smaller than $2K+1$. These are ordered in finitely many steps. The outcome is an ordered list.

RMQ is a generalization of the $(2k+1)$-median version of Quicksort with fixed k, [14]. This corresponds to RMQ with $p_k = 1$. Notice that our results therefore include the results for the $2k+1$ median version as special cases.

RMQ is a random divide-and-conquer algorithm with an internal randomness [11]. We are interested in the running time of the RMQ algorithm. An (complete) analysis includes

- the worst (and best) case
- the average performance
- the asymptotic distribution
- the tail behavior of the asymptotic distribution.

In our setting we consider the running time proportional (depending on the implementation and the computer) to the number of comparisons. Let X_S be the number of comparisons in order to sort a set S of n different numbers with RMQ. The distribution of X_S is the same for sets S of the same size. (Notice RMQ is a random divide-and-conquer algorithm with an internal randomness [11].) Therefore we are allowed to use X_n instead of X_S with $|S| = n$ in distributional equations.

The performance time analysis of random divide-and-conquer algorithms corresponds to the mathematical analysis of recurrence equations. The rvs $X_n, n \in \mathbb{N}_0$, satisfy the recursive structure

$$X_n \stackrel{\mathrm{D}}{=} X_{Z_n^{(A)}-1} + \overline{X}_{n-Z_n^{(A)}} + C_n^{(A)} \tag{1}$$

for $n \geqslant n_0 := 2K+2$. The rvs $(C_n^{(k)}, Z_n^{(k)})$, A, X_i, $\overline{X}_i$, $0 \leqslant k \leqslant K$, $0 \leqslant i \leqslant n$, are independent for every fixed $n \geqslant n_0$. The rv A has the distribution p. $Z_n^{(k)}$ denotes the final position of the pivot element, the median of $2k+1$ elements, conditioned to $A = k$. $C_n^{(A)}$ denotes the costs for this round of RMQ.

From the recurrence relation we obtain a recursion for the expectation $a_n = E(X_n),\ n \geqslant n_0$

$$a_n = E\big(C_n^{(A)}\big) + \sum_{m=1}^{n} P\big(Z_n^{(A)} = m\big)(a_{m-1} + a_{n-m}).$$

The first asymptotic analysis for the average number of comparisons for the $2k+1$ version ($p_k = 1$) was given by van Emden [16]. He showed

$$\frac{EX_n}{n \ln n} \mathop{\to}_{n} e_k := \frac{1}{-\int (t \ln t + (1-t)\ln(1-t)) g_k(t)\,\mathrm{d}t}$$

where g_k is the density of the $(k+1)$th order statistic on $2k+1$ independent random variables with uniform distribution. (This is a $beta(k+1, k+1)$ distribution on the unit interval.) He did not obtain the second leading term of $E(X_n)$.

The best available results for the expectation of the $2k+1$ median version are in [1]. She gives an asymptotic expansion of the average EX_n derived via generating functions. A general method deriving asymptotics for the expectation was developed in [3] and [10]. The results apply and provide the first and second asymptotic term in the expansion

$$E(X_n) = e_k n \ln n + f_k n + \mathrm{o}(n)$$

for some real f_k. In this paper we provide the same asymptotics for the RMQ. This precision is required in order to apply the contraction method.

Similar as in Quicksort or the $2k+1$ median version, the normalized rv for RMQ

$$Y_n := \frac{X_n - a_n}{n}$$

converges to a random variable, called Y. The Y satisfies the stochastic fixed point equation

$$Y \stackrel{\mathrm{D}}{=} YU_A + \overline{Y}(1-U_A) + C(U_A)$$

where $Y, \overline{Y}, A, U_k, 0 \leqslant k \leqslant K$ are independent, A has distribution p, U_k has a *beta*$(k+1, k+1)$ distribution and Y is distributed as $\overline{Y}$. C is the function

$$C(x) = \mu^{(K)}\bigl(x\ln x + (1-x)\ln(1-x)\bigr) + 1,$$

$x \in (0,1)$. The constant $\mu^{(K)}$ is determined by $E(C(U_A)) = 0$.

The basic technic for this result is the contraction method [8,11].

The fixed point equation provides for example higher moments, like the variance,

$$\operatorname{Var}(Y) = E\bigl(Y^2\bigr) = \frac{E(C^2(A, U_A))}{2E(U_A) - E(U_A^2)}.$$

Moreover from the fixed point equation we obtain finite Laplace transforms of the fixed point Y. As a consequence every moment or exponential moment of Y_n converges to the one of Y. This in turn provides via the Markov inequality tail estimates for the running time, for bad behavior of the running time and so on.

In order to avoid any complications in the discussion of the selection rule we have preferred the randomized version of Quicksort as presented.

2. Recursive equation of comparisons

Let $K \in \mathbb{N}_0$ be a fixed positive number and $n_0 := 2K+2$. Let $p = (p_0, \ldots, p_K)$ be a probability vector on $\{0, 1, \ldots, K\}$. Let $\nu_n^{(k)}$, $n_0 \leqslant n \in \mathbb{N}, k \in \{0, \ldots, K\}$ be probability measures on $\mathbb{R} \times \{0, \ldots, n\}$. We denote with $(C_n^{(k)}, Z_n^{(k)})$, random variables with distribution $\nu_n^{(k)}$. The distribution of $Z_n^{(k)}$ is given by

$$P\bigl(Z_n^{(k)} = m\bigr) = \frac{\binom{m-1}{k}\binom{n-m}{k}}{\binom{n}{2k+1}}, \quad m = k+1, \ldots, n-k. \tag{2}$$

The distribution of $C_n^{(k)} - n$ depends not on n.

Definition 2.1. Let $\nu = (\nu_n^{(k)})_{n,k}$ be given as above and let $\mu_0, \mu_1, \ldots, \mu_{n_0-1}$ be probability measures on the reals. Then define the sequence $\mu_n, n \geqslant n_0$ of distributions on the reals recursively by

$$X_n \stackrel{\mathrm{D}}{=} X_{Z_n^{(A)}-1} + \overline{X}_{n-Z_n^{(A)}} + C_n^{(A)} \tag{3}$$

for $n \geqslant n_0$. The random variables $(C_n^{(k)}, Z_n^{(k)})$, A, X_i, and $\overline{X}_i$, $0 \leqslant k \leqslant K$, $0 \leqslant i \leqslant n$, are independent for every fixed $n \geqslant n_0$. The random variable X_i and $\overline{X}_i$ have the distribution μ_i and the probability distribution corresponding to the random variable A is $p_k = P(A = k)$, $0 \leqslant k \leqslant K$. The distribution of $(C_n^{(k)}, Z_n^{(k)})$ is $\nu_n^{(k)}$.

(To be precise, we use $X_{Z_n^{(A)}}(\omega) = X_{Z_n^{(A(\omega))}(\omega)}(\omega)$.) Eq. (3) is the short form writing of the equation

$$X_n \stackrel{\mathrm{D}}{=} \sum_{k=0}^{K} 1_{\{A=k\}}\bigl(X_{Z_n^{(k)}-1} + \overline{X}_{n-Z_n^{(k)}} + C_n^{(k)}\bigr).$$

Remark. Notice that X_n has the interpretation as the (random) number of comparisons of Random Median Quicksort in order to sort a list of length n. The random variable A corresponds to the randomly drawn index k, the $Z_n^{(k)}$ is the final position of the pivot element, the median of $2k+1$ random elements, (conditioned to $A = k$) and the cost $C_n^{(k)}$ is payed for that round.

Notice that the distribution of the random variable X_n does not depend on the actual order of the input list, but only of the length n of it.

In the sequel we use X, C, Z, and A as introduced here.

3. Expected number of comparisons

In this section we provide the asymptotic order of the expectation $a_n = E(X_n)$ of X_n. We obtain from the key equation (3)

$$a_n = Ea_{Z_n^{(A)}-1} + Ea_{n-Z_n^{(A)}} + EC_n^{(A)}. \tag{4}$$

for $n \geqslant n_0$ and $a_n = \int x\mu_n(\mathrm{d}x)$ for $n < n_0$.

Proposition 3.1. *The sequence $(a_n)_{n\geqslant 0}$ satisfies*

$$a_n = 2Ea_{Z_n^{(A)}-1} + EC_n^{(A)} \tag{5}$$

for $n \geqslant n_0$.

Proof. Notice the symmetry $P(Z_n^{(k)} = m) = P(Z_n^{(k)} = n - m + 1)$. Eq. (4) provides

$$\begin{aligned} a_n &= Ea_{Z_n^{(A)}-1} + Ea_{n-Z_n^{(A)}} + EC_n^{(A)} = \sum_{k=0}^{K} p_k \sum_{m=1}^{n} P\big(Z_n^{(k)} = m\big)(a_{m-1} + a_{n-m}) + EC_n^{(A)} \\ &= 2\sum_{k=0}^{K} p_k \sum_{m=k+1}^{n-k} P\big(Z_n^{(k)} = m\big)a_{m-1} + EC_n^{(A)} = 2Ea_{Z_n^{(A)}-1} + EC_n^{(A)} \end{aligned}$$

for $n \geqslant n_0$. $\square$

Define the generating function (formal power series)

$$a(z) = \sum_{n\geqslant 0} a_n z^n.$$

We will use the symbol

$$\binom{z}{k} := \frac{1}{k!} z(z-1)(z-2)\ldots(z-k+1)$$

$k \in \mathbb{N}$ for the above polynomial in z. Especially it follows $\binom{n}{k} = 0$ for $n < k$.

The number of comparison $C_n^{(k)}$ for given k and $n \geqslant n_0$ is the sum of the necessary comparisons in order to find the median out of $2k+1$ random elements and the necessary comparisons $n-2k-1$ in order to build the set of smaller and of larger elements than the median. The method in order to find the median is assumed to depend only on k and not on n. Taking the expectation we can write $EC_n^{(A)},\ n \geqslant n_0$ in the form

$$E\big(C_n^{(A)}\big) = \sum_{k=0}^{K} p_k EC_n^{(k)} = n + 1 + L_K. \tag{6}$$

The constant L_K does not depend on n. If necessary take $E(C_n^{(A)})$ as above also for $n < n_0$.

D denotes the derivative operator of functions f. We use Df instead of $D(f)$ and if necessary or appropriate also D_x for the derivative with respect to the variable x.

Lemma 3.2. *The generating function a satisfies the Euler differential equation*

$$\begin{aligned} &(1-z)^{2K+1} D^{2K+1} a(z) - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{k}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \frac{(2K+1)!}{(2K-k-j)!} (1-z)^{2K-k-j} D^{2K-k-j} a(z) \\ &\quad = \frac{L_K(2K+1)!}{1-z} + \frac{(2K+2)!}{(1-z)^2}. \end{aligned} \tag{7}$$

Proof. We will use the formal power series

$$c(z)=\sum_{n\geqslant 0} EC_n^{(A)} z^n$$

in z, where we shall use Eq. (6). Since we consider only the D^{2K+1} derivative of c to z it does not matter to sum for c over $n\geqslant 0$ or $n\geqslant n_0$.

- $c(z)=\frac{1}{(1-z)^2}+\frac{L_K}{1-z}$.

$$\begin{aligned} c(z)&=\sum_{n\geqslant 0}(n+1+L_K)z^n = D\sum_{n\geqslant 0} z^{n+1}+\frac{L_K}{1-z}\\ &=D\frac{z}{1-z}+\frac{L_K}{1-z}=\frac{1}{(1-z)^2}+\frac{L_K}{1-z}. \end{aligned}$$

- $D^{2K+1}c(z)=L_K(2K+1)!(1-z)^{-2K-2}+(2K+2)!(1-z)^{-2K-3}$.
  Easy by the previous claim.
- $$D^{2K+1}a(z)=D^{2K+1}c(z)+2\sum_{k=0}^{K} p_k\frac{(2k+1)!}{k!}\sum_{j=0}^{2K-2k}\binom{2K-2k}{j}\frac{(k+j)!}{k!}(1-z)^{-k-1-j}D^{2K-k-j}a(z).$$

  We will use the identities

$$\sum_{n\geqslant 0}\binom{n+k}{k} z^n = z^k(1-z)^{-k-1},$$

and

$$\sum_{m\geqslant k+1}\binom{m-1}{k} a_{m-1}z^{m-1}=\frac{z^k}{k!}D^k a(z).$$

Then for $B:=D^{2K+1}a(z)-D^{2K+1}c(z)$, using Proposition 3.1

$$a_n=2\sum_{k=0}^{K} p_k\sum_{m=k+1}^{n-k} a_{m-1}P(Z_n^{(k)}=m)+EC_n^{(A)},$$

we obtain

$$\begin{aligned} B&=D^{2K+1}\sum_{n\geqslant n_0} a_n z^n - D^{2K+1}c(z)\\ &=D^{2K+1}\sum_{n\geqslant n_0}\left(2\sum_{k=0}^{K} p_k\sum_{m=k+1}^{n-k}\frac{\binom{m-1}{k}\binom{n-m}{k}}{\binom{n}{2k+1}}a_{m-1}+EC_n^{(A)}\right)z^n - D^{2K+1}c(z)\\ &=2\sum_{k=0}^{K} p_k D^{2K-2k}\left(\sum_{n\geqslant n_0}\sum_{m=k+1}^{n-k}\frac{\binom{m-1}{k}\binom{n-m}{k}}{\binom{n}{2k+1}}a_{m-1}D^{2k+1}z^n\right)\\ &=2\sum_{k=0}^{K} p_k D^{2K-2k}\left(\sum_{n\geqslant 2k+1}\sum_{m=k+1}^{n-k}\frac{\binom{m-1}{k}\binom{n-m}{k}}{\binom{n}{2k+1}}a_{m-1}D^{2k+1}z^n\right)\\ &=2\sum_{k=0}^{K} p_k D^{2K-2k}\left(\sum_{n\geqslant 2k+1}\sum_{m=k+1}^{n-k}\binom{m-1}{k}\binom{n-m}{k}(2k+1)!a_{m-1}z^{n-2k-1}\right)\\ &=2\sum_{k=0}^{K} p_k(2k+1)!D^{2K-2k}\left(z^{-2k}\sum_{m\geqslant k+1}\binom{m-1}{k}a_{m-1}z^{m-1}\sum_{n\geqslant k}\binom{n+k}{k}z^{n+k}\right) \end{aligned}$$

$$= 2\sum_{k=0}^{K} p_k (2k+1)! D^{2K-2k}\left(z^{-2k}\frac{z^k}{k!}\big(D^k a(z)\big) z^k (1-z)^{-k-1}\right)$$
$$= 2\sum_{k=0}^{K} p_k \frac{(2k+1)!}{k!} D^{2K-2k}\big((1-z)^{-k-1}\big(D^k a(z)\big)\big)$$
$$= 2\sum_{k=0}^{K} p_k \frac{(2k+1)!}{k!} \sum_{j=0}^{2K-2k}\binom{2K-2k}{j}\big(D^j (1-z)^{-k-1}\big)\big(D^{2K-2k-j} D^k a(z)\big)$$
$$= 2\sum_{k=0}^{K} p_k \frac{(2k+1)!}{k!} \sum_{j=0}^{2K-2k}\binom{2K-2k}{j}\frac{(k+j)!}{k!}(1-z)^{-k-1-j} D^{2K-k-j} a(z)$$
$$= 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k}\frac{\binom{k+j}{k}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}}\frac{(2K+1)!}{(2K-k-j)!}(1-z)^{-k-j-1} D^{2K-k-j} a(z).$$

This proves the last partial claim.

Multiplying both sides in the previous claim by $(1-z)^{2K+1}$ provides the Euler differential equation

$$\begin{aligned}&(1-z)^{2K+1} D^{2K+1} a(z) - (1-z)^{2K+1} D^{2K+1} c(z)\\ &\quad = 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k}\frac{\binom{k+j}{k}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}}\frac{(2K+1)!}{(2K-k-j)!}(1-z)^{2K-k-j} D^{2K-k-j} a(z). \qquad \square\end{aligned} \tag{8}$$

An Euler differential equation of order n is equivalent to a linear differential equation of order n with constant coefficients. In order to see this we use a change of variables $z = 1 - e^{-x}$. Let $y(x) = a(z) = a(1 - e^{-x})$. Then

Proposition 3.3. *Under the change of variables $z = 1 - e^{-x}$ holds*

$$D^j a(z) = e^{jx} D(D+1)\ldots(D+j-1)y(x) = j!e^{jx}\binom{D+j-1}{j} y(x)$$

for $j \in \mathbb{N}$.

Proof. The claim is true for $j = 1$ since $Da(z) = e^x Dy(x)$. Notice the derivative on the left side is with respect to z, on the right side with respect to x, in more detail $D(a)(z) = e^x D(y)(x)$. The induction step from j to $j+1$ is

$$\begin{aligned} D_z^{j+1} a(z) &= e^x D_x\big(e^{jx} D_x(D_x+1)\ldots(D_x+j-1)\big)y(x)\\ &= e^x e^{jx} j D_x(D_x+1)\ldots(D_x+j-1)y(x) + e^x e^{jx} D_x\big(D_x(D_x+1)\ldots(D_x+j-1)\big)y(x)\\ &= e^{x+jx} D_x(D_x+1)\ldots(D_x+j-1)(D_x+j)y(x). \qquad \square\end{aligned}$$

Define the polynomial P in the variable λ by

$$P(\lambda) = \binom{\lambda+2K}{2K+1} - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k}\frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}}\binom{\lambda+2K-k-j-1}{2K-k-j}.$$

Replacing z by $1 - e^{-x}$ and dividing by $(2K+1)!$ in Eq. (8), yields

$$P(D)y(x) = L_K e^x + (2K+2)e^{2x}. \tag{9}$$

This is inhomogeneous linear differential equation of order n. The homogeneous differential equation can be written as follows:

$$P(D)y(x) = 0. \tag{10}$$

$P = P(\lambda)$ is the characteristic polynomial of the differential equation (9). Our strategy is now to establish a specific solution. Then every solution of the inhomogeneous differential equation (9) is a convex combination of the special and a solution of the homogeneous differential equation.

3.1. Inhomogeneous differential equation

Proposition 3.4. *Let P be the characteristic polynomial in λ as above. Then*

$$P(1) = -1, \tag{11}$$

$$P(2) = 0, \tag{12}$$

$$DP(2) = (2+2K)H_{2K+2} - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}}(2K-k-j+1)H_{2K-k-j+1} > 0. \tag{13}$$

Proof. For the first statement we use the identity

$$\sum_{j=0}^{r} \binom{m+j}{j}\binom{n+r-j}{r-j} = \binom{m+n+r+1}{r}.$$

Substituting $r = 2K - 2k$, $m = k$, $n = k + 1$ we obtain

$$\sum_{j=0}^{2K-2k} \binom{k+j}{j}\binom{2K-k+1-j}{k+1} = \binom{2K+2}{2k+2}. \tag{14}$$

Then

$$\begin{aligned}
P(1) &= \binom{2K+1}{2K+1} - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \binom{2K-k-j}{2K-k-j} \\
&= 1 - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \\
&= 1 - 2\sum_{k=0}^{K} p_k \frac{\binom{2K+1}{2k+1}}{\binom{2K+1}{2k+1}} = 1 - 2 = -1.
\end{aligned}$$

For the second

$$\begin{aligned}
P(2) &= \binom{2K+2}{2K+1} - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \binom{2K-k-j+1}{2K-k-j} \\
&= 2K + 2 - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}}(2K-k-j+1).
\end{aligned}$$

We will use the identities

$$\binom{m}{l}(m+1) = \binom{m+1}{l+1}(l+1)$$

and (14). Therefore

$$P(2) = 2K + 2 - 2\sum_{k=0}^{K} p_k (k+1) \frac{1}{\binom{2K+1}{2k+1}} \sum_{j=0}^{2K-2k} \binom{k+j}{j}\binom{2K-k-j+1}{k+1}$$

$$= 2K + 2 - 2\sum_{k=0}^{K} p_k (k+1) \frac{\binom{2K+2}{2k+2}}{\binom{2K+1}{2k+1}}$$
$$= 2K + 2 - 2\sum_{k=0}^{K} p_k (K+1) = 0.$$

For the third statement we notice that first the formal derivative of a polynomial in z is as follows:

$$D_z \binom{z+l}{k} = \sum_{0 \leqslant i \leqslant k-1} \frac{1}{z+l-i} \binom{z+l}{k}.$$

Therefore

$$\begin{aligned}
DP(2) &= \sum_{0 \leqslant i \leqslant 2K} \frac{1}{2+2K-i} \binom{2+2K}{2K+1} \\
&\quad - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \sum_{0 \leqslant i \leqslant 2K-k-j-1} \frac{1}{2+2K-k-j-1-i} \binom{2K-k-j+1}{2K-k-j} \\
&= (2+2K) \sum_{0 \leqslant i \leqslant 2K} \frac{1}{2+2K-i} \\
&\quad - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} \sum_{0 \leqslant i \leqslant 2K-k-j-1} \frac{1}{2K-k-j+1-i} (2K-k-j+1) \\
&= (2+2K)(H_{2K+2} - 1) - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} (2K-k-j+1)(H_{2K-k-j+1} - 1) \\
&= (2+2K)H_{2K+2} - 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} (2K-k-j+1)H_{2K-k-j+1} - P(2).
\end{aligned}$$

Continue

$$\begin{aligned}
&2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} (2K-k-j+1) H_{2K-k-j+1} \\
&\quad < 2\sum_{k=0}^{K} p_k \sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j}{k}}{\binom{2K+1}{2k+1}} (2K-k-j+1) H_{2K+2} \\
&\quad = 2H_{2K+2} \sum_{k=0}^{K} p_k \frac{k+1}{\binom{2K+1}{2k+1}} \sum_{j=0}^{2K-2k} \binom{k+j}{j} \binom{2K-k-j+1}{k+1} \\
&\quad = 2H_{2K+2} \sum_{k=0}^{K} p_k (k+1) \frac{\binom{2K+2}{2k+2}}{\binom{2K+1}{2k+1}} \\
&\quad = 2H_{2K+2} \sum_{k=0}^{K} p_k (K+1) = (2K+2)H_{2K+2}.
\end{aligned}$$

This proves $DP(2) > 0$. □

Lemma 3.5. *The function* $y : \mathbb{R} \to \mathbb{R}$, $y(x) = \frac{e^x}{P(1)}$ *solves the inhomogeneous differential equation*

$$P(D)y(x) = e^x.$$

The function $y:\mathbb{R}\to\mathbb{R}$, $y(x)=\frac{xe^{2x}}{P'(2)}$ *solves the inhomogeneous differential equation*

$$P(D)y(x)=e^{2x}.$$

The function $y:\mathbb{R}\to\mathbb{R}$, $y(x)=-L_K e^x+\frac{2K+2}{P'(2)}xe^{2x}$ *solves the inhomogeneous differential equation* (9)

$$P(D)y(x)=L_K e^x+(2K+2)e^{2x}.$$

Proof. These solutions are known in the literature. For simplicity and completeness we provide the short argument. Consider the function $(x,\lambda)\mapsto e^{x\lambda}$. Then

$$P(D_x)e^{\lambda x}=e^{\lambda x}P(\lambda).$$

Notice $P(1)\neq 0$. For the second statement, $\lambda=2$, argue

$$\begin{aligned}P(D_x)\big(xe^{x\lambda}\big)&=P(D_x)\big(D_\lambda e^{x\lambda}\big)=D_\lambda P(D_x)\big(e^{x\lambda}\big)\\&=D_\lambda\big(e^{x\lambda}P(\lambda)\big)=xe^{x\lambda}P(\lambda)+e^{x\lambda}DP(\lambda).\end{aligned}$$

We obtain for $\lambda=2$

$$P(D_x)\big(xe^{2x}\big)=xe^{2x}P(2)+e^{2x}DP(2)=e^{2x}DP(2).$$

Notice $DP(2)\neq 0$. □

The last claim is easy.

For the Euler differential equation (8) we obtain the special solutions $a_s(z)$:

$$a_s(z)=-\frac{L_K}{1-z}-\frac{2K+2}{P'(2)}\frac{\ln(1-z)}{(1-z)^2}.$$

Proposition 3.6. *The identity*

$$\frac{\ln(1-z)}{(1-z)^2}=-\sum_{n\geqslant 0}\big((n+1)H_n-n\big)z^n$$

as formal power series in z for $|z|<1$ is true.

Proof. Since

$$\frac{1}{1-z}=\sum_{i\geqslant 0}z^i,$$

$$-\ln(1-z)=\sum_{j\geqslant 1}\frac{z^j}{j},$$

we obtain by changing the order of summation

$$-\frac{\ln(1-z)}{1-z}=\sum_{i\geqslant 1}H_i z^i,$$

$$-\frac{1}{1-z}\frac{\ln(1-z)}{1-z}=\sum_{n\geqslant 1}\sum_{i=1}^{n}H_i z^n=\sum_{n\geqslant 0}\big((n+1)H_n-n\big)z^n. \qquad \square$$

Corollary 3.7. *The special solution a_s is*

$$a_s(z)=\sum_{n\geqslant 0}\left(\frac{2K+2}{DP(2)}\big((n+1)H_n-n\big)-L_K\right)z^n. \tag{15}$$

Proof. Apply the last proposition to the special solution. □

3.2. Homogeneous differential equation

It remains to solve the homogeneous equation $P(D)y = 0$ respectively the corresponding homogeneous Euler equation.

Let $\lambda_1, \ldots, \lambda_l \in \mathcal{C}$ be the roots of the characteristic polynomial P in λ and $1 \leqslant r_1, \ldots, r_l$ be the multiplicities, $\sum_i r_i = n$. Then

$$P(\lambda) = \prod_{i=1}^{l} (\lambda - \lambda_i)^{r_i}.$$

Let $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real and imaginary part of a complex number λ. It is well known, that the homogeneous linear differential equation (9) of order n has the solutions

$$x^j e^{x\,\mathrm{Re}(\lambda_i)} \cos\bigl(x\,\mathrm{Im}(\lambda_i)\bigr),\ x^j e^{x\,\mathrm{Re}(\lambda_i)} \sin\bigl(x\,\mathrm{Im}(\lambda_i)\bigr),$$

$i = 1, \ldots, l,\ j = 0, \ldots, r_i - 1$.

The set of all above solutions forms a fundamental system. Every solution of the homogeneous differential equation (10) is a convex combination of these. For any given special solution of the inhomogeneous equation (9) is every solution of the inhomogeneous equation (8). The sum of the special solution and a solution of the homogeneous equation is the general solution of (9).

We obtain for the Euler differential equation (8) the fundamental system

$$\bigl(-\ln(1-z)\bigr)^j (1-z)^{-\mathrm{Re}(\lambda_i)} \cos\bigl(-\mathrm{Im}(\lambda_i)\ln(1-z)\bigr),$$
$$\bigl(-\ln(1-z)\bigr)^j (1-z)^{-\mathrm{Re}(\lambda_i)} \sin\bigl(-\mathrm{Im}(\lambda_i)\ln(1-z)\bigr),$$

$1 \leqslant i \leqslant l$, $0 \leqslant j < r_i$ of real valued solutions.

We will show in the following, that one of the eigenvalues is 2 with multiplicity 1 and all other eigenvalues contribute only a $o(n)$ term to a_n, where a_n is the nth coefficient of a solution corresponding to the eigenvalue.

Proposition 3.4 shows 2 is an eigenvalue of the characteristic polynomial P with multiplicity 1.

Lemma 3.8. *All eigenvalues $\lambda_i \neq 2$ have a real part strictly less than 2.*

The main observation in order to prove Lemma 3.8 is the following observation: Define $b_n = \frac{a_n}{n+1}$ where a_n are the coefficients of a solution to the homogeneous Euler equation.

Proposition 3.9. *Let the sequence $(b_n)_n$ satisfy the recursive equation*

$$b_n = E b_{\overline{Z}_n - 1}$$

for $n \geqslant n_0$, where the distribution of $\overline{Z}_n$ is given by

$$P(\overline{Z}_n = m) = \frac{2m P(Z_n^{(A)} = m)}{n+1},$$

$m \in \{1, \ldots, n\}$. *Then*

$$\sup_n |b_n| \leqslant \sup_{0 \leqslant j < n_0} |b_j|.$$

Proof. It remains only to show that $\overline{Z}_n$ is a well defined rv. This is equivalent to

$$\sum_{m=1}^{n} \frac{2m P(Z_n^{(A)} = m)}{n+1} = 1.$$

An easy calculation shows

$$\sum_{m=1}^{n} \frac{2mP(Z_n^{(A)}=m)}{n+1} = \sum_{k=0}^{K} p_k \sum_{m=1}^{n} \frac{2mP(Z_n^k=m)}{n+1} = \sum_{k=0}^{K} p_k = 1. \qquad \square$$

Proof of Lemma 3.8. We could continue to work with real valued solutions of the homogeneous Euler equation. We prefer (with equivalent arguments) to consider complex valued solutions a.

Let λ_0 be an eigenvalue of the characteristic polynomial with multiplicity $r > 0$. The corresponding solutions are $f_{j,\lambda_0}(z)$, $j = 0, 1, \ldots, r-1$, where

$$f_{j,\lambda}(z) := \ln^j(1-z)(1-z)^{-\lambda}$$

with $\lambda \in \mathbb{R}$.

We treat firstly the case $\lambda_0 \notin -\mathbb{N}_0$ and $\lambda_0 \neq 2$.

The derivative of the solutions is

$$Df_{j,\lambda} = -jf_{j-1,\lambda+1} + \lambda f_{j,\lambda+1}.$$

Introduce the vector

$$v_\lambda = \begin{pmatrix} f_{0,\lambda} \\ \vdots \\ f_{r-1,\lambda} \end{pmatrix}$$

and the matrix

$$A_\lambda = \lambda I - M,$$

where I is the identity matrix and M has the entries $M_{i,j} = \mathbb{1}_{i-1=j}$, $1 \leqslant i, j \leqslant r-1$. Notice $M^{r-1} = 0$. Then

$$Dv_\lambda = A_\lambda v_{\lambda+1},$$

and

$$D^n v_\lambda = A_\lambda A_{\lambda+1} \ldots A_{\lambda+n-1} v_{\lambda+n}.$$

We obtain

$$\frac{D^n v_\lambda(0)}{n!} = \frac{\lambda(\lambda+1)\ldots(\lambda+n-1)}{n!} \left(\prod_{j=0}^{n-1} \left(I - \frac{1}{\lambda-j} M \right) \right) v_{\lambda+n}(0) = I \cdot II \cdot III.$$

The easiest of them is the third factor $III = v_{\lambda+n}(0)$, which is the unit vector $(1, 0, \ldots, 0)$. The second factor $II = \prod_{j=0}^{n-1}(I - \frac{1}{\lambda-j}M)$ is a product of matrices. The determinant is 1. Since $M^{r-1} = 0$ the operator norm of the matrix is of the order $\sum_{j=0}^{n} \frac{1}{|\lambda+j|} \approx \ln n$.

The absolute value of the first factor $I = \frac{\lambda(\lambda+1)\ldots(\lambda+n-1)}{n!}$ behaves like a power of n as $n \to \infty$, use $\lambda = s + it$,

$$\begin{aligned} |I|^2 &= \frac{(s^2+t^2)((s+1)^2+t^2)\ldots((s+n-1)^2+t^2)}{n!n!} \\ &= \frac{s^2(s+1)^2\ldots(s+n-1)^2}{n!n!} \prod_{j=0}^{n-1} \left(1 + \frac{t^2}{(s+j)^2}\right). \end{aligned}$$

The product $\prod_{j=0}^{n-1}(1 + \frac{t^2}{(s+j)^2})$ converges to some real number, since

$$1 \leqslant \prod_{j \geqslant j_0} \left(1 + \frac{t^2}{(s+j)^2}\right) \leqslant \prod_{j \geqslant j_0} \mathrm{e}^{\frac{t^2}{(s+j)^2}} = \mathrm{e}^{t^2 \sum_{j \geqslant j_0} \frac{1}{(s+j)^2}}$$

is finite and converges to 1 as $j_0 \to \infty$.

The factor $\frac{s^2(s+1)^2\dots(s+n-1)^2}{n!n!}$ behaves like a power of n. Since every coordinate of the $\frac{D^n v_\lambda(0)}{n!}$ are possible candidates of a_n for specific initial conditions $a_0,\dots,a_{n_0-1}$, we conclude by Proposition 3.9

$$\frac{1}{n^2}\frac{s^2(s+1)^2\dots(s+n-1)^2}{n!n!}$$

is bounded. Therefore we have $s \leqslant 2$.

In case $s<2$ the above converges to 0 as $n\to\infty$. Taking care of the matrix II we have to show

$$\frac{\ln n\ln n}{n^2}\frac{s^2(s+1)^2\dots(s+n-1)^2}{n!n!}$$

converges to 0 as $n\to\infty$. Therefore the corresponding a_n are of order $o(n)$.

If $s=2$ then $t\neq 0$ since we excluded $\lambda_0=2$. In a more detailed study we will exclude this value.

Consider the real solutions

$$f_j(z)=(1-z)^{-j}\cos\bigl(t\ln(1-z)\bigr)\quad\text{and}\quad g_j(z)=(1-z)^{-j}\sin\bigl(t\ln(1-z)\bigr)$$

for $j=2$ of the homogeneous Euler differential equation. The derivative is

$$Df_j=jf_{j+1}+tg_{j+1}, \tag{16}$$
$$Dg_j=-tf_{j+1}+jg_{j+1}. \tag{17}$$

Introduce the vector

$$v_j=\begin{pmatrix} f_j\\ g_j\end{pmatrix}$$

and the matrix

$$A_j=jI+t\mathcal{O},$$

where $\mathcal{O}$ is the orthogonal rotation

$$\mathcal{O}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Then

$$Dv_j=A_jv_{j+1},$$
$$D^nv_j=A_jA_{j+1}\dots A_{j+n-1}v_{j+n}.$$

We obtain

$$\frac{D^nv_s(0)}{n!}=\frac{s(s+1)\dots(s+n-1)}{n!}\left(\prod_{l=0}^{n-1}\left(I+\frac{t}{s+l}\mathcal{O}\right)\right)v_{s+n}(0).$$

The first factor is $n+1$. The third factor $v_{s+n}(0)$ is $(1,0)$.

We consider now the second factor, the matrix

$$M_n:=\prod_{l=0}^{n-1}\left(I+\frac{t}{s+l}\mathcal{O}\right).$$

The determinant

$$\det M_n=\prod_{l=0}^{n-1}\left(1+\frac{t^2}{(s+l)^2}\right)$$

converges as $n \to \infty$. The normalized matrix $\frac{1}{\sqrt{\det M_n}} M_n$ is an orthogonal rotation, since every component $I + x\mathcal{O}$ is an orthogonal rotation,

$$(I + x\mathcal{O})(I + x\mathcal{O})^* = I + x^2 I = \det(I + x\mathcal{O}) I.$$

Let the orthogonal rotation $\frac{1}{1+x^2}(I + x\mathcal{O})$ on $\mathbb{R}^2$ correspond to $\mathrm{e}^{-i\varphi_x}$ for the complex numbers, where $\sin\varphi_x = \frac{x}{\sqrt{1+x^2}}$ is continuous for small x and $\varphi_0 = 0$. Notice for small x

$$\left|\varphi_x - \frac{x}{\sqrt{1+x^2}}\right| \leqslant \mathrm{const}\left(\frac{x}{\sqrt{1+x^2}}\right)^3.$$

In our case with $x = \frac{t}{s+l}$ we obtain

$$\sum_l \left(\frac{t}{\sqrt{(s+l)^2 + t^2}}\right)^3 < \infty$$

and consequently

$$\left|\sum_{l=0}^{N} \varphi_{\frac{t}{s+l}} - \sum_{l=0}^{N} \frac{t}{\sqrt{t^2 + (s+l)^2}}\right| \leqslant \sum_{l=0}^{N} \left(\frac{t}{\sqrt{(s+l)^2 + t^2}}\right)^3 < \infty$$

uniformly in N.

The normalized matrix $\frac{1}{\sqrt{\det M_n}} M_n$ corresponds to $\prod_{l=0}^{n-1} \mathrm{e}^{-\varphi_{\frac{t}{s+l}}}$ and behaves asymptotic like

$$\sim \prod_{l=0}^{n-1} \mathrm{e}^{-\frac{t}{\sqrt{(s+l)^2+t^2}}} \sim \mathrm{e}^{-it(\ln n + c)}$$

for some constant c. (Asymptotic $\sim$ in the sense of the quotient converges to 1.)

The consequence for our sequence $b_n = \frac{a_n}{n+1}$ is a periodic solution

$$b_n \sim \mathrm{e}^{-it(\ln n + c)}. \tag{18}$$

Then from Proposition 3.9

$$b_n = E b_{\overline{Z}_n - 1}.$$

Neglecting the constant in (18), yields $b_n \sim \cos(t \ln n)$ is an asymptotic solution.

Next observe, that $\frac{\overline{Z}_n}{n}$ converges in distribution to a limiting random variable $\overline{Z}$. The distribution is concentrated on $(0, 1)$ and has a Lebesgue density. Therefore we should have

$$\cos(t \ln n) \approx E \cos\left(t \ln n + t \ln\left(\frac{\overline{Z}_n - 1}{n}\right)\right). \tag{19}$$

Choose a subsequence n_i such that $\cos(t \ln n_i)$ converges to 0. We obtain a contradiction by

$$1 \approx E \cos\left(t \ln n_i + t \ln\left(\frac{\overline{Z}_{n_i} - 1}{n_i}\right)\right) \underset{i}{\to} E \cos(t \ln \overline{Z}) < 1.$$

Next we treat the remaining case, $\lambda_0 \in -\mathbb{N}_0$. The argument runs the same lines besides the fact, that some derivatives disappear. The notation we used takes care of that, since in that case we multiply the (pseudo) derivative with a factor, which is 0. □

Corollary 3.10. *The general solution a of the homogenous Euler equation (10) has coefficients a_n of the form*

$$a_n = c_1 n + \mathrm{o}(n).$$

Proof. The eigenvalue $\lambda = 2$ with multiplicity 1 contributes a multiple of $n+1$ to a_n. The other eigenvalues contribute only $o(n)$ terms. □

Lemma 3.11. *The solution of the Euler differential equation (8) is given by*

$$a_n = \mu^{(K)} n \ln n + f^{(K)} n + o(n), \tag{20}$$

where

$$\mu^{(K)} := \frac{2(K+1)}{DP(2)} \tag{21}$$

and the constant $DP(2)$ is given in Proposition 3.4 and $f^{(K)}$ denotes some constant.

Proof. From Corollaries 3.7 and 3.10 we get (20). □

4. The limit law

In this section we show the Mallows metric l_2 convergence via the contraction method of the normalized random variable

$$X_n^* := \frac{X_n - E(X_n)}{n} \tag{22}$$

to some limit X^* solving the stochastic fixed point equation

$$X^* \stackrel{\mathrm{D}}{=} X^* U_A + \overline{X}^*(1 - U_A) + C(U_A). \tag{23}$$

Here X^*, $\overline{X}^*$, A, U_k, $0 \leqslant k \leqslant K$ are independent rvs. The rvs X^* and $\overline{X}^*$ have the same distribution, A has the distribution vector $(p_0, \ldots, p_K)$ and the U_k have a *beta*$(k+1, k+1)$ distribution. The function $C : [0,1] \to \mathbb{R}$ is given by

$$C(x) := \mu^{(K)}\big(x \ln x + (1-x)\ln(1-x)\big) + 1, \tag{24}$$

where

$$\mu^{(K)} := \frac{2(K+1)}{DP(2)}$$

and the constant $DP(2)$ is given in Proposition 3.4

$$DP(2) = (2+2K)H_{2K+2} - 2\sum_{k=0}^{K} p_k(k+1)\sum_{j=0}^{2K-2k} \frac{\binom{k+j}{j}\binom{2K-k-j+1}{k+1}}{\binom{2K+1}{2k+1}} H_{2K-k-j+1} > 0. \tag{25}$$

For the Mallows metric l_2 on the space of measures see [2].

Theorem 4.1. *Let X_n be given by (3) and X_n^* (22) be the normalized form. Then X_n^* converges in l_2-Mallows metric to a solution X^* of the fixed point equation (23). X^* is unique within the class of centered fixed points with finite variance.*

Proof. The normalized rvs X_n^* satisfy by (3) the recursive relation

$$X_n^* \stackrel{\mathrm{D}}{=} \frac{Z_n^{(A)} - 1}{n} X^*_{Z_n^{(A)}-1} + \frac{n - Z_n^{(A)}}{n} \overline{X}^*_{n - Z_n^{(A)}} + C_n, \tag{26}$$

$$C_n := \frac{1}{n}\big(a_{Z_n^{(A)}-1} + a_{n-Z_n^{(A)}} - a_n\big) + \frac{C_n^{(A)}}{n}. \tag{27}$$

Plugging in $a_n = \mu^{(K)} n \ln n + f^{(K)} n + g(n)$, where $g(n)$ is of the order $o(n)$, we obtain

$$C_n = \frac{\mu^{(K)}}{n}\big(\big(Z_n^{(A)} - 1\big)\ln\big(Z_n^{(A)} - 1\big) + \big(n - Z_n^{(A)}\big)\ln\big(n - Z_n^{(A)}\big) - n\ln n\big) + \frac{1}{n}\big(g\big(Z_n^{(A)} - 1\big) + g\big(n - Z_n^{(A)}\big) - g(n)\big) - \frac{f^{(K)}}{n} + \frac{C_n^{(A)}}{n}.$$

The above theorem is now a consequence of Theorem 3 [10]. We have to verify the assumptions, first for the existence of a fixed point of (23).

- $EC(U_A) = 0$.
  We show this in a moment.
- $E(C(U_A))^2 < \infty$.
  Easy since the function $[0,1] \ni x \mapsto x\ln x$ is bounded.
- $E(U_A)^2 + E(1-U_A)^2 < 1$.
  Argue $U_A \in (0,1)$ a.e. and therefore $E((U_A)^2 + (1-U_A)^2) < E(U_A + (1-U_A)) = 1$.
  Now for the convergence $X_n^* \to X^*$ in l_2 metric.
- $EC_n = 0$.
  Straight forward by definition.
- For any $n_1 \in N$ holds

$$E\left(\mathbb{1}_{Z_n^{(A)} \leqslant n_1}\left(\frac{Z_n^{(A)} - 1}{n}\right)^2\right) + E\left(\mathbb{1}_{n - Z_n^{(A)} \leqslant n_1}\left(\frac{n - Z_n^{(A)}}{n}\right)^2\right) \underset{n}{\to} 0.$$

The terms $(\frac{Z_n^{(A)}-1}{n})^2$ and $(\frac{n-Z_n^{(A)}}{n})^2$ are bounded by 1 and the probability of the sets $Z_n^{(A)} \leqslant n_1$ and $n - Z_n^{(A)} \leqslant n_1$ converge to 0.

- $l_2((C_n^{(A)}, \frac{Z_n^{(A)}-1}{n}, \frac{n-Z_n^{(A)}}{n}), (C(U_A), U_A, 1-U_A)) \to_n 0$.
  Conditioning on A it suffices to show this for every $0 \leqslant k \leqslant K$.

Since $\frac{Z_n^{(k)}}{n}$ converges in distribution to U_k and is bounded, we have also l_2 convergence. Now choose on some probability space for all $0 \leqslant k \leqslant K$ a version of U_k such that $E(Z_n^{(k)} - U_k)^2 \to_n 0$. It suffices to show $E((C_n - C(U_A))^2 \mid A = k) \to_n 0$. This is now straight forward on the set $A = k$ since

$$\begin{aligned} C_n - C(U_A) &= \mu^{(K)}\left(\frac{Z_n^{(A)} - 1}{n}\ln\frac{Z_n^{(A)} - 1}{n} - U_A \ln U_A\right) \\ &\quad + \mu^{(K)}\left(\frac{n - Z_n^{(A)}}{n}\ln\frac{n - Z_n^{(A)}}{n} - (1-U_A)\ln(1-U_A)\right) \\ &\quad - \mu^{(K)}\frac{\ln n}{n} - \frac{f_K}{n} + \frac{1}{n}\big(g\big(Z_n^{(A)} - 1\big) + g\big(n - Z_n^{(A)}\big) - g(n)\big) + \frac{C_n^{(A)}}{n} - 1 \underset{n}{\overset{l_2}{\to}} 0. \end{aligned}$$

- $E(C(U_A)) = 0$.
  Follows by $E(C_n) = 0$ and $E(C_n - C(U_A))^2 \to_n 0$. □

5. Laplace transforms

The techniques developed by [8,9] to obtain results on the existence and convergence of Laplace transforms for the scaled running time of the Quicksort algorithm can be applied to Random Median Quicksort.

Lemma 5.1. $\forall L > 0\ \exists K_L > 0\ \forall n \in N\ \forall \lambda \in [-L, L]$:

$$E\exp(\lambda X_n^*) \leqslant \exp\big(\lambda^2 K_L\big). \tag{28}$$

Proof. In place of the random variable U_n in Lemma 4.1 [8], we use

$$V_n = \left(\frac{Z_n^{(A)} - 1}{n}\right)^2 + \left(\frac{n - Z_n^{(A)}}{n}\right)^2 - 1.$$

Then with C_n given by (26) it holds:

(a) $\forall n \in \mathbb{N}$: $-1 \leqslant V_n < 0$;
(b) $\sup_{n\in\mathbb{N}} EV_n < 0$;
(c) $\sup_{n\in\mathbb{N}} \|C_n\|_\infty < \infty$.

For the proof of (b) notice $V_n < 0$ a.e. and the L_1-convergence

$$V_n \xrightarrow{L_1}_n U_A^2 + (1-U_A)^2 - 1.$$

Now, using (a)–(c) we can conclude as in Lemma 4.1 and 4.2 in [8] which leads to our assertion. □

Theorem 5.2 *(Convergence of Laplace transforms). The normalized sequence* (X_n^*) *given in* (26) *and the fixed-point* X^* *of* Theorem 4.1 *satisfy for all* $\lambda \in \mathbb{R}$

$$E\exp(\lambda X_n^*) \underset{n\to\infty}{\rightarrow} E\exp(\lambda X^*) < \infty. \tag{29}$$

Proof. The exponential bound in (28) implies uniform integrability of $\exp(\lambda X_n^*)$ which by Theorem 4.1 yields (29). □

Finally using the expansion of the mean EX_n in (20) one obtains as in Corollary 4.3 of [8] the following bounds for (large) deviations.

Corollary 5.3. *For any fixed* $\lambda, \epsilon > 0$ *exists a constant* c *such that for all* $n \in \mathbb{N}$

$$P\big(\big|X_n - E(X_n)\big| \geqslant \epsilon E(X_n)\big) \leqslant cn^{-2\epsilon\lambda\mu^{(K)}}.$$

This implies

$$P\big(|X_n - EX_n| \geqslant \epsilon EX_n\big) = \mathrm{O}\big(n^{-l}\big)$$

for all $l \in \mathbb{N}$.

Proof. Using expansion (20) the Markov inequality provides for $\lambda := \frac{k}{n\epsilon\mu^{(K)}}$ and $\mu^{(K)}$ defined in (21). We derive

$$P\big(|X_n - EX_n| \geqslant \epsilon EX_n\big) = P\left(\exp\big(\lambda|X_n^*|\big) \geqslant \exp\left(\epsilon\lambda\frac{a_n}{n}\right)\right) \leqslant \frac{E\exp(\lambda|X_n^*|)}{\exp(\lambda\epsilon(\mu^{(K)}\ln n + d_K + \frac{g(n)}{n}))}$$
$$\leqslant \mathrm{const}(\lambda,\epsilon) n^{-\lambda\epsilon\mu^{(K)}}. \qquad \square$$

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