Clase 3:

Un homomorfismo es una función $\varphi: G \to H$ tq

$$\varphi(a \cdot b) = \varphi(a)\varphi(b)$$

El Kernel de φ:

$$\operatorname{Ker}\varphi = \{ g \in G / \varphi(g) = 1 \}$$

La imagen de φ:

$$\operatorname{im}\varphi = \{ f(g) / g \in G \}$$

Proposición: Sea $\varphi: G \to H$ un homomorfismo

1) $\varphi(1) = 1$
2) $\varphi(g^{-1}) = \varphi(g)^{-1}$
3) $\varphi(g^n) = \varphi(g)^n$
4) $\operatorname{Ker}\varphi \leq G$
5) $\operatorname{im}\varphi \leq H$

Ejemplos:

1) $\varphi: \mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_4$

$$(\bar{x}, \bar{y}) \mapsto \bar{x}$$

$$\operatorname{Ker}\varphi = \langle (\bar{0}, \bar{1}) \rangle$$

2) $\varphi: \overset{+}{\mathbb{Z}} \times \overset{+}{\mathbb{Z}} \to \overset{+}{\mathbb{Z}}$

$$(a, b) \mapsto a + b$$

$$\operatorname{Ker}\varphi = \langle (1, -1) \rangle$$

3) $\varphi : \overset{+}{\mathbb{R}} \to \overset{\cdot}{\mathbb{C}-\{0\}}$

$x \mapsto \cos x + i\,\mathrm{sen}\,x = e^{ix}$

- $\varphi(x+y) = e^{i(x+y)} = e^{ix} \cdot e^{iy} = \varphi(x)\,\varphi(y)$

- $\mathrm{Ker}\,\varphi = \langle 2\pi \rangle$

Definición: Un automorfismo es un isomorfismo

$$\varphi : G \to G$$

$$\mathrm{aut}(G) = \{ \varphi : G \to G \,/\, \varphi \text{ automorfismo} \}$$

Grupo cociente

G grupo ⇝ obtener subgrupos ↗ $a \in G,\ \langle a \rangle \leq G$; $S \subseteq G,\ \langle S \rangle \leq G$

↘ ¿ Cociente

En $\mathbb{Z}$ consideremos

$$a \sim b \iff n \mid a-b \iff \underline{a-b \in n\mathbb{Z}}$$

→ subgrupo

$[a] = \{ a + kn \,/\, k \in \mathbb{Z} \}$ ✓

$\Rightarrow \dfrac{\mathbb{Z}}{n\mathbb{Z}} = \{ [0], \ldots, [n-1] \}$ ✓ grupo ✓

↪ subgrupo

Ahora consideremos $H \leq G$

$$x \sim y \iff x^{-1}y \in H$$

$$[x] = \{y \in G \,/\, \underbrace{y \sim x}_{\substack{x^{-1}y = h \in H \\ y = xh}}\} = \{xh \,/\, h \in H\} := xH$$

$*$ Se puede def $a \sim b \Leftrightarrow ab^{-1} \in H \rightsquigarrow [a] = Ha$

$$¿\ \frac{G}{H} := \{xH : x \in G\} \text{ es un grupo ?}$$

$$(xH)(yH) = (xy)H \quad \text{esta bien definida}$$

Dado $\underline{y \in G}$ cualquiera

$$x \sim x',\ y \sim y' \Rightarrow \underline{x' = xh_1},\ h_1 \in H,\quad \underline{y' = yh_2},\ h_2 \in H$$

Queremos : $\underbrace{(xy)}_{\sim}H = \underbrace{(x'y')}_{\sim}H$

$$\Updownarrow$$

$$y^{-1}x^{-1}x'y' \in H$$

Reemp $\Rightarrow$ $y^{-1}x^{-1}(xh_1)y' = y^{-1}h_1\underbrace{y'}_{yh_2} = \underbrace{y^{-1}h_1 y}_{\in H}\overset{\in G}{\uparrow}\ \underset{\in H}{h_2} \in H$

La cond. que falta es :

$$y^{-1}h_1 y \in H,\quad \forall y \in G,\ h_1 \in H$$

$$\boxed{yHy^{-1} \subseteq H,\ \forall y \in G}\ ✓$$

Proposición: Sea $H \leq G$ tq $\forall g \in G$: $\underline{gHg^{-1} \subseteq H}$

$$\Rightarrow G/H = \{ gH : g \in G \} \text{ es un grupo con}$$

$$(g_1 H)(g_2 H) = (g_1 g_2) H$$

Demostración: resta ver está bien definida

Si $g_1 H = g_1' H \wedge g_2 H = g_2' H$

$$\Rightarrow \underline{g_1^{-1} g_1' \in H}, \quad \underline{g_2^{-1} g_2' \in H} \qquad \boxed{\underline{g_1 g_2} H = \underline{g_1' g_2'} H}$$

$$\Rightarrow g_2^{-1} \underbrace{g_1^{-1} g_1'}_{\in H} g_2' = g_2^{-1} h g_2' = \underbrace{g_2^{-1} h g_2}_{\in g_2^{-1} H g_2 \subseteq H} g_2^{-1} g_2'$$

$$= h' \underbrace{g_2^{-1} g_2'}_{\in H} \in H$$

$$\Rightarrow \underline{(g_1' g_2') H = (g_1 g_2) H}$$

$\blacksquare$

Definición: $H \leq G$ es normal si

$$\underline{gHg^{-1} \subseteq H}, \forall g \in G$$

Se denota $H \trianglelefteq G$

Si $H \trianglelefteq G \Rightarrow (G/H, \cdot)$ grupo cociente de G por H

Ejercicio: Sea $H \subseteq G$, se define

normalizador $N_G(H) = \{g \in G \,/\, gHg^{-1} = H\}$

- Pruebe que $N_G(H) \leq G$
- Si $H \leq G$ pruebe que H es normal $\Leftrightarrow N_G(H) = G$

Proposición: Sea $N \leq G$, son equivalentes:

1) $N \trianglelefteq G$
2) $gNg^{-1} = N, \forall g \in G$
3) $N_G(N) = G$
4) $gN = Ng, \forall g \in G$

Demostración:

1) $\Rightarrow$ 2) Como $N \trianglelefteq G \Rightarrow gNg^{-1} \subseteq N, \forall g \in G$

Sea $g \in G$, $x \in N$

$$\Rightarrow x = g\underbrace{g^{-1}xg}_{\in N}g^{-1} \in gNg^{-1}$$

$$\Rightarrow N \subseteq gNg^{-1}$$

2) $\Rightarrow$ 3) $gNg^{-1} = N, \ \forall g \in G$

$$N_G(N) = \{g \in G \,/\, gNg^{-1} = N\} = G$$

3) $\Rightarrow$ 4) $N_G(N) = G \Rightarrow gN = Ng, \forall g$

Sea $g \in G \Rightarrow gNg^{-1} = N \Rightarrow gN = Ng$ ✓

4) $\Rightarrow$ 1) Sea $g \in G$, $n \in N \rightsquigarrow gng^{-1} \in N$

Como $\underbrace{gn}_{\in gN = Ng} g^{-1} = n' \underbrace{g g^{-1}}_{1} = n' \in N \Rightarrow N \trianglelefteq G$ $\blacksquare$

Ejemplos:

1) Si G es abeliano $\Rightarrow (N \leq G) \Rightarrow (N \trianglelefteq G)$

2) $\{1\} \trianglelefteq G$, $G \trianglelefteq G$

3) $\varphi: G \to H$ homomorfismo $\Rightarrow \operatorname{Ker}\varphi \trianglelefteq G$

$$\underbrace{\varphi(gng^{-1})}_{\in \operatorname{Ker}\varphi} = \varphi(g) \underbrace{\varphi(n)}_{1} \varphi(g)^{-1} = \varphi(\underbrace{gg^{-1}}_{1}) = 1$$

Ejemplos:

1) $\varphi: \mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_4$ Sobreyectivo

$(\bar{x}, \bar{y}) \mapsto \bar{x}$

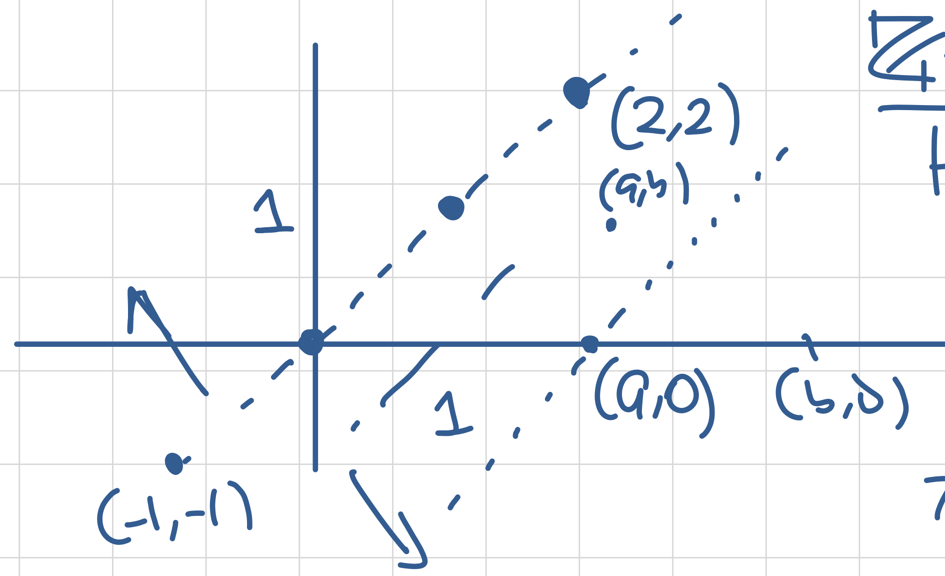
$\operatorname{Ker}\varphi = H = \langle(\bar{0},\bar{1})\rangle \trianglelefteq G = \mathbb{Z}_4 \times \mathbb{Z}_6$

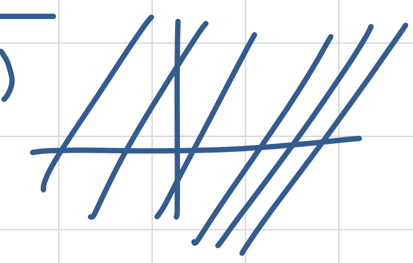
$\rightsquigarrow \dfrac{\mathbb{Z}_4 \times \mathbb{Z}_6}{H} = \{ \underbrace{(0,0)+H}_{\downarrow \bar{0}}, \underset{\downarrow}{(1,0)+H}, (2,0)+H, (3,0)+H \}$

$\rightsquigarrow \dfrac{\mathbb{Z}_4 \times \mathbb{Z}_6}{H = \operatorname{Ker}\varphi} \cong \mathbb{Z}_4 = \operatorname{im}(\varphi)$

2) $H=\langle(1,1)\rangle \subseteq \mathbb{Z}\times\mathbb{Z}$

$\dfrac{\mathbb{Z}\times\mathbb{Z}}{H}$





$$\rightsquigarrow \frac{\mathbb{Z}\times\mathbb{Z}}{H} = \{(a,0)+H \;/\; a\in\mathbb{Z}\} = \langle (1,0)+H\rangle$$

cíclico

$$(a,b)\in(a-b,0)+H$$

$$\rightsquigarrow \frac{\mathbb{Z}\times\mathbb{Z}}{H=\ker\varphi}\cong \mathbb{Z} = \mathrm{im}(\varphi)$$

$$\varphi: \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$$
$$(a,b)\mapsto a-b$$

Teorema de Lagrange

$H\leq G$

$a^{-1}b\in H$

$\Updownarrow$

$a\sim b$

$$|\{xH : x\in G\}| = ?$$

Teorema de Lagrange:

Sea G un grupo finito y $H \leq G$, entonces:

- $|H| \,\big|\, |G|$
- $\underline{|\{xH : x \in G\}|} = \dfrac{|G|}{|H|}$

Demostración:

La idea es probar que $|H| = |gH|$ ✓

$$\varphi : H \to gH \text{ biyección}$$
$$h \mapsto gh$$

Como: $G = \biguplus_{g \in G} \underbrace{gH}_{\text{clases de equiv}}$

$$\Rightarrow |G| = \sum_{g \in G} |gH| = \underbrace{|\{gH : g \in G\}|}\,.\,|H|$$

$$\Rightarrow |H| \,\big|\, |G| \wedge |\{gH : g \in G\}| = \frac{|G|}{|H|}$$

$\blacksquare$

Notación: $[G:H] = |\{xH : x \in G\}|$, índice de H en G

Corolario: G finito y $x \in G \Rightarrow |x| \,\big|\, |G|$

$$\langle x \rangle \leq G \Rightarrow \underbrace{|\langle x \rangle|}_{|x|} \,\big|\, |G| \checkmark$$

$$\Rightarrow \underbrace{\overbrace{x^{|G|} = 1}} \text{ pues } x^{|G|} = x^{|x|.k} = 1$$

Corolario: Si $|G| = p$, p primo $\Rightarrow G$ es cíclico ($G \cong \mathbb{Z}_p$)

$$\underset{\neq 1}{x \in G} \rightsquigarrow \underbrace{|x|}_{>1} \mid |G| = p \rightsquigarrow |x| = p \rightsquigarrow G = \langle x \rangle \Rightarrow G \cong \mathbb{Z}_p$$

(Si $G = \{1\} \Rightarrow G = \langle 1 \rangle$)

$\blacksquare$

Definición: $H, K \leq G$

$$HK = \{hk \;/\; h \in H, k \in K\}$$

¿HK subgrupo?

Ejercicio: $HK \leq G \Leftrightarrow HK = KH$

Proposición: $H, K \leq G$

no es grupo

$$\Rightarrow |HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$

Demostración: Se tiene

$$HK = \bigcup_{h \in H} hK \quad h'k$$

Se sabe que $|hK| = |K|$

$$\underbrace{h}_{\in H}K = \underbrace{h'}_{\in H}K \Leftrightarrow h^{-1}h' \in K \cap H \Leftrightarrow \underline{h(K \cap H) = h'(K \cap H)}$$

$$\Rightarrow \# \; hK \text{ dist.} = \# \underbrace{\overset{\in H}{h}(K \cap H)}_{\text{distintos}} = [H : K \cap H] = \frac{|H|}{|K \cap H|}$$

grupo

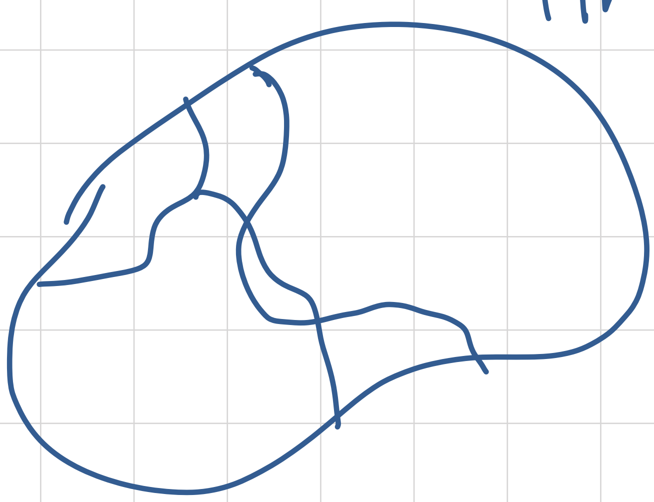
$K \cap H \leq H$

Como $HK = \bigcup_{h\in H} hK \Rightarrow |HK| = |K| \underbrace{\left(\frac{|H|}{|K\cap H|}\right)}$

$\Downarrow$

$\#\ hK$ distintos $= \dfrac{|H|}{|K\cap H|}$

HK



$\underbrace{h_1K}_{|K|} \quad \underbrace{h_2K}_{|K|} \quad \underbrace{h_3K}_{|K|}$

∎

Teoremas del isomorfismo

G, H cuando $G \cong H$

$\xrightarrow{\text{método}}$

Teorema (1er teorema) Si $\varphi : G \to H$ es un homom.

$$\Rightarrow \frac{G}{\operatorname{Ker}\varphi} \cong \operatorname{im}(\varphi)$$

Demostración:

$$\psi : G/\operatorname{Ker}\varphi \to \operatorname{im}\varphi$$
$$g\operatorname{Ker}\varphi \mapsto \varphi(g)$$

* ψ está bien definido

$g_1 \in g\operatorname{Ker}\varphi \Rightarrow \psi(g_1\operatorname{Ker}\varphi) = \psi(g\operatorname{Ker}\varphi)$

$g_1 = gk,\ k\in\operatorname{Ker}\varphi$

$\Rightarrow \varphi(g_1) = \varphi(g)\overbrace{\varphi(k)}^{1}$

Probar que φ es biyección y homom.

Ejemplos:

1) $\dfrac{\mathbb{Z}\times\mathbb{Z}}{\langle(1,1)\rangle} \cong \mathbb{Z}$, $\varphi: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ sobrey.

$(a,b) \mapsto a-b$

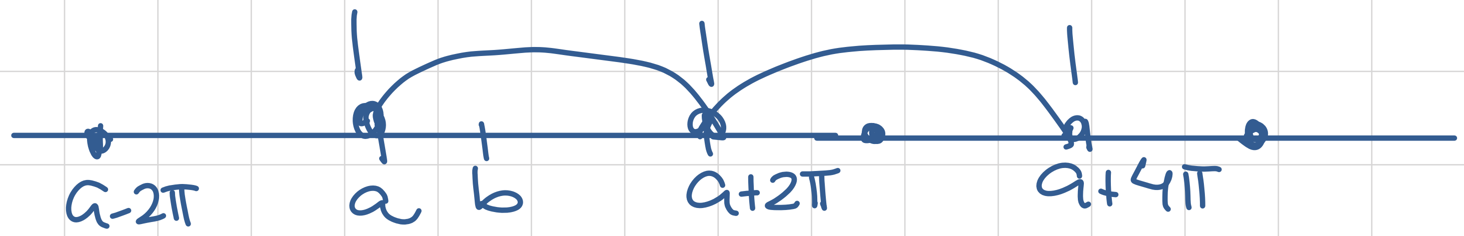
2) $\dfrac{\mathbb{R}}{\langle 2\pi\rangle} \cong \{z\in\mathbb{C} / |z|=1\}$

$\varphi: \mathbb{R} \to \mathbb{C}-\{0\}$

$x \mapsto e^{ix} = \cos x + i\operatorname{sen} x$

$\Rightarrow \dfrac{\mathbb{R}}{\langle 2\pi\rangle} \cong \varphi(\mathbb{R}) = \{z\in\mathbb{C} / |z|=1\}$

Clases $a + \langle 2\pi\rangle = \{a + 2\pi k, k\in\mathbb{Z}\}$



$\mathbb{R}/\langle 2\pi\rangle$

Teorema (2^{DO})

$A, B \leq G$, $B \trianglelefteq G$, entonces:

- $AB \leq G$ ✓
- $B \trianglelefteq AB$ ✓
- $A \cap B \trianglelefteq A$ ✓
- $\boxed{\dfrac{AB}{B} \cong \dfrac{A}{A \cap B}}$

$$\begin{tikzcd}
 & G \arrow[d, dash] & \\
 & AB \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] & & B \arrow[dl, dash] \\
 & A \cap B \arrow[d, dash] & \\
 & 1 &
\end{tikzcd}$$

Demostración:

$$\varphi : A \to AB/B \qquad a \in AB$$
$$a \mapsto aB$$

- $\operatorname{Ker} \varphi = A \cap B$

$$\varphi(a) = 1 \iff aB = B \iff a^{-1} \in B \iff a \in B \cap A$$

- φ es sobreyect. pues si $a \underbrace{b}_{\in B} B \in AB/B$

$$abB = aB = \varphi(a) \checkmark$$

- $\varphi(a_1 a_2) = (a_1 a_2)B = (a_1 B)(a_2 B) = \varphi(a_1)\varphi(a_2)$

$$\underset{\substack{1^{\circ}\text{Teo} \\ \text{isom}}}{\Longrightarrow} \quad \frac{A}{A \cap B} \cong \frac{AB}{B} \qquad \blacksquare$$

Teorema (3^er teo. del isomorfismo)

$H, K \trianglelefteq G$ con $\underbrace{H \leq K}_{H \triangleleft K} \Rightarrow \underbrace{K/H \triangleleft G/H}$ y :

$$\frac{G/H}{K/H} \cong G/K$$

Demostración:

Sea $gH \in G/H$ y $kH \in K/H$

$$(gH)(kH)(gH)^{-1} = \underbrace{g^{-1}kg}_{\in K} \overset{\in G}{H} \in \underline{K/H}$$

$$\Rightarrow K/H \trianglelefteq G/H$$

$$* \frac{G/H}{\underset{\sim}{K/H}} \cong G/K$$

$$\varphi : G/H \to G/K$$
$$gH \mapsto gK$$

$\underbrace{\varphi(gH)}_{gK} = K$, $gK = K$, $g \in K$

φ esta bien definida $\quad g'K = gK$

$g' \in gH \Rightarrow g' = gh \Rightarrow g^{-1}g' = g^{-1}gh = h \in H \leq K$

$$\Rightarrow \underbrace{gK}_{\varphi(gH)} = \underbrace{g'K}_{\varphi(g'H)} \quad \checkmark$$

φ es homom. sobreyectivo con $\operatorname{Ker}\varphi = K/H$ $\Rightarrow \dfrac{G/H}{K/H} \cong G/K$ $\blacksquare$

Proposición (4^{TO} teo. isomorfismo)

G, $N \trianglelefteq G$ entonces

$$\left\{ \begin{array}{l} \text{subgrupos } A \leq G \\ \text{tq' } N \subseteq A \quad N \trianglelefteq A \end{array} \right\} \overset{\text{biy}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{subgrupos de } \bar{A} \\ \text{de } G/N \end{array} \right\}$$

$$\varphi : \{ \quad \} \to \{ \quad \}$$

$$\underline{A} \mapsto A/N = B = \pi(A)$$

Sobrey: Si $\underline{B \leq G/N} \Rightarrow \exists A \leq G$ con $N \subseteq A$ / $\underline{\frac{A}{N} = B}$ $\boxed{\text{Z}}$

$$\pi : G \to G/N \quad \text{proyección canónica}$$
$$g \mapsto gN$$

homom. Sobrey.

$T(G) = \{ g \in G / |g| < \infty \}$ $|g| = o(g)$

grupo $\hookrightarrow I$ $\hookrightarrow$ orden

$G = GL_2(\mathbb{R})$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq I$

Si $\underline{G \text{ es abeliano}} \Rightarrow T(G) \leq G$ ✓

De un ejemplo de $\underline{T(G) \not\leq G}$ ✓

$a, b \in T(G)$ pero $ab \notin T(G)$

$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix}$ $(ab)^n = \begin{pmatrix} & \\ & \end{pmatrix} \neq \underline{I}$

$\underline{|a| = |b| = 2}$ $|ab| = \infty$

$G = GL_{\cancel{n}\,2}(\mathbb{R})$

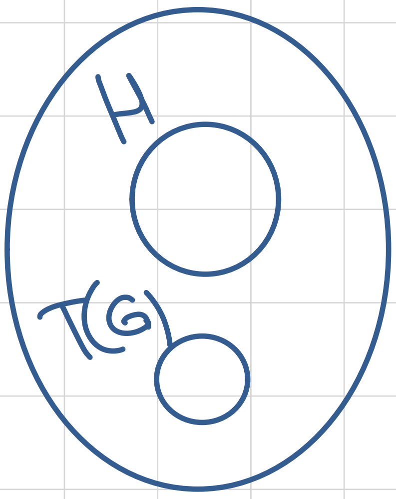
$T(G) = \{A \in G \,/\, |A| < \infty\} \leq G$

$\boxed{a,b \in T(G) \,/\, ab \notin T(G)}$

$H \leq G \iff \forall a,b \in H : ab^{-1} \in H$

$ab \in H$

$H \not\leq G$



G $\quad H = \left\{ \begin{pmatrix} a & d \\ c & b \end{pmatrix} \,/\, a \neq b \right\} \subseteq G$ no es grupo

$\not\leq$ $\quad //$

$\underline{T(G)}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin H$ pero tiene orden $< \infty$

$T(G) = \underbrace{\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,/\, \text{———} \right.}\left.\right\} \subseteq G$

$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,/\, ac \neq 0 \right\}$ $\quad N \trianglelefteq G$

$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,/\, b \in \mathbb{R} \right\}$ $\quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

$a \neq 1$

$c \neq 1$

$ac \neq 0$

$\frac{G}{N} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} N$

$\hookrightarrow \notin N$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} N \cdot \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} N \qquad \begin{matrix} a \neq 1 & a' \neq 1 \\ c \neq 1 & c' \neq 1 \end{matrix}$$

$$\begin{pmatrix} \underline{aa'} & ab' + bc' \\ 0 & \underline{cc'} \end{pmatrix} N \qquad T$$

$$=$$

$$\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} \underline{aa'} & a'b + b'c \\ 0 & \underline{cc'} \end{pmatrix} N \quad \perp$$

$$\underbrace{\begin{pmatrix} aa' & a'b + b'c \\ 0 & cc' \end{pmatrix}^{-1} \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}} \in N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{\cancel{aa'}cc'} \underbrace{\begin{pmatrix} cc' & -a'b - b'c \\ 0 & aa' \end{pmatrix} \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}}$$

$$\begin{pmatrix} \cancel{aa' \cdot cc'} & b' \\ 0 & \cancel{aa'}cc' \end{pmatrix} \in N$$

$$T(G) = \{ g \in G \mid |g| < \infty \}$$

$$\varphi : A \times G \to A$$

$$\underbrace{ka}_{a^k} = 0 \;, \quad \underline{k \text{ no div de cero}}$$

(S)