The Algebraic Theory of Topological Quantum Information

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January 14, 2025

Abstract

This book aims to give a comprehensive account of the algebraic theory of topological quantum information. It is intended to be accessible both to mathematicians unfamiliar with quantum mechanics and theoretical physicists unfamiliar with category theory. Additionally, this text should make a good reference for working researchers in the field. A primary focus of this text is balancing powerful algebraic generalities with concrete examples, principles, and applications.

For my mentors

0 Preface

This book is a mathematical treatment of topological quantum information, with a focus on formal algebraic aspects and a special eye towards topological quantum computation. This manuscript began as an extended set of notes from a course on topological quantum field theory given by Zhenghan Wang in the winter of 2022 at UC Santa Barbara. Through his courses, his private tutoring, and his reccomendations, Zhenghan took me from a state of almost complete ignorance of mathematical physics to being a young researcher in the field. I am greatly emdebted to him for this, and it is certain that this book would not have existed without his guidance - he richly deserves of my apple.

This book would not have been possible without the tutelage of my esteemed mentors. Those who most directly contributed are the ones who used their time and energy to teach me topological quantum information - Dave Aasen, Mike Freedman, and Yuri Lensky. There are also those who took a chance on a young mathematician when they had every reason not to - Roald Dejean, Peter Bloomsburgh, Edward Frenkel, and Ken Ribet.

WORK: There's some other people to thank. Andrew Sylvester for letting me try out my arguments on him. Alexei Kitaev and Daniel Ranard now for advising me.

Great pains have been taken to make this book as pedagogical and accessable as possible. The hope is that it should be readable by both mathematicians unfamilar with quantum mechanics as well as theoretical physicists unfamiliar with category theory. A primary focus of this text is balancing powerful algebraic generalities with concrete examples, principles, and applications. The prerequisites for this book are a undergraduate-level understanding of topology, linear algebra, and group theory, as well as a popular-science level of familarity with quantum mechanics.

There are already many great references to learn aspects of the material covered in this book. An excellently written and relatively complete book on topological quantum information from the perspective of a physicist is Steven Simon's text [Sim23]. Simon's book is algebraic, but does *not* include any category theory. The main references for the relevant category theory are Bakalov-Kirillov [BK+01] and Etingof-Gelaki-Nikshych-Ostrik [EGNO16]. While both excellent texts, they suffer notable shortcomings for learning topological quantum information. Bakalov-Kirillov was written in 2001, making it outdated. Etingof-Gelaki-Nikshych-Ostrik is modern, but makes no connections to physics and does not use the language of string diagrams. The manuscript most similar to this one is Kong-Zhang's preprint [KZ22]. We distinguish ourself from Kong-Zhang by our rigorous mathematical treatment, our different choice of topics, and our extended scope. Other relevant books and review articles include Wang's monograph [Wan10] and Kauffman-Lomonaco's quantum topology themed review [KL09].

WORK: I will add a section detailing the structure of this book, and how it should be read. I have not written enough for this to be useful yet.

Contents

1 Category theory

1.1 Overview

1.1.1 Introduction

There is a lot of math in the world. The development of the subject has spanned thousands of years, and has enjoyed a large uptick in progress the last two hundred or so. This has given ample time for the most important ideas to rise to the top. Among these important concepts there is one which is the focus of chapter: **composition**.

Let A, B, C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. The *composition* of f and g is the function $g \circ f : A \to C$ defined by the formula $(g \circ f)(x) = g(f(x))$ for all $x \in A$. More generally, composition is the act of performing one process followed by performing a second process. Composition is distinguished in its importance for two reasons:

1. Composition is ubiquitous;
2. Many complicated structures can be described in terms of composition.

These two primary sources of importance lead to several emergent applications of composition:

1. It's a good organization principle - thinking in terms of composition gives a unifed approach to disperate subjects, which highlights the universality latent within mathematics;
2. It's a good compression technique - in a composition-first approach there's no need to remember details about objects or functions between them, only the way that those functions compose is used;
3. Sometimes composition rules are the only data we have about an object of study, making a composition-first technique the only approach possible.

This third point is the situation we find ourselves in with the algebraic theory of topological quantum information. We're trying to give a usable mathematical description of topologically ordered systems. The way we do this is by focusing on anyons (local quasiparticle excitations in topological order). In doing so we run into three important ponts:

1. Describing anyons exactly is hard. They are emergent phenomina, found within highly-entangled energy eigenstates of arbitrarily complicated gapped Hamiltonians;
2. Describing the ways anyons can transform is hard. This involves specifying intricate unitary operators on high-dimensional Hilbert spaces.
3. Describing how these transformations compose with one another is relatively simple. It can be done using explicit-to-describe rules, which are independent of the system size or choice of gapped Hamiltonian.

What to do in this situation is clear: we will take a composition-first approach to anyons. The mathematical structure which allows for an intelligent discussion of composition is known as a *category*. The composition-first approach to mathematics is known as *category theory*. Of course, to describe anyons we will need more than just the structure of composition. We will also need a way to encode what happens we we put anyons together, braid them, and fuse them. There structures are all completely compatible with the compostion-first approach, and correspond to adding extra structures onto the category. The type of category which fully describes anyons is known as a *modular tensor category*, and these categories will be the subject of much of this book. This chapter deals with introducing category theory, as well as some of the structures which will be important for discussing anyons and modular tensor categories.

1.1.2 Definition and important obervations

As discussed before, a category is the structure which allows for a composition-first approach to mathematics. Before going forward lets define what a category is:

Definition 1.1 (Category)**.** A category is the following data:

1. (Objects) A set $\mathcal{C}$;
2. (Morphisms) A set $\mathrm{Hom}(A, B)$ for all $A, B \in \mathcal{C}$;
3. (Composition) Functions

$$\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

   for all $A, B, C \in \mathcal{C}$;

Such that:

1. For all morphisms $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, and objects $A, B, C, D \in \mathcal{C}$,

$$(h \circ g) \circ f = h \circ (g \circ f).$$

2. (Identity) For all objects $A \in \mathcal{C}$ there exists a morphism $\mathrm{id}_A : A \to A$ such that for all $B \in \mathcal{C}$, $f \in \mathrm{Hom}(A, B)$, and $g \in \mathrm{Hom}(B, A)$,

$$f \circ \mathrm{id}_A = f, \qquad\qquad \mathrm{id}_A \circ g = g.$$

Remark 1.2. The structure of definition 1.1 is very typical of algebra. Roughly, algebra is defined to be the study of algebraic structures. An algebraic structure is some collection operations on some space, with rules outlining how these operations interact with each other. The general way of definding an algebraic structure is to first list its operations, and then list the axioms of how these operations inteact with each other. We will see many definitions of this sort throughout the rest of the book, so it is good to get used to it now.

Example 1.3. In this text we have already seen many examples of categories. We list some of them here:

- **Set**, the category of sets. The objects are sets and the morphisms are functions.
- **Top**, the category of topological spaces. The objects are topological spaces and the morphisms are continuous functions.
- $\mathbf{Vec}_k$, the category of finite dimensional vector spaces over a field k. The objects are finite dimensional vector spaces over k and the morphisms are linear operators.
- **Grp**, the category of groups. The objects are groups and the morphisms are group homomorphisms.
- **Hilb**, the category of quantum systems. The objects are finite dimensional Hilbert spaces and the morphisms are unitary operators.
- **Prob**, the category of probability spaces. The objects are finite dimensional real vector spaces with distinguished bases and the morphisms are operators which send normalized vectors to normalized vectors.
- $\mathbf{Ord}_M$, the category associated with ordered media with order space M. The objects are continuous maps $\phi : \mathbb{R}^2 \to M$ and the morphisms are continuous deformations.
- $\mathfrak{D}(G)$, the category associated with discrete gauge theory based on the finite group G. The objects are G-graded G-representations and the morphisms are linear maps which respect both the G-grading and the G-action.

Warning 1.4. One subtelty of the defintions in example 1.3 is that the collection of objects in some of these definitions are not sets. For instance, the collection of all sets does not itself form a set, due to logical paradoxes such as Bertrand's paradox. A more proper treatment of category theory would restrict to smaller categories (such as the category *finSet* of finite sets) where the collection of objects does really form a set. Alternatively, one can introduce the notation of a *class*, with is a collection of sets defined by some unambiguous property that all objects share. The collection of all objects in **Set** is not a set, but it is a class. In this framework, we call categories whose collection objects and whose hom spaces are all sets a *finite category*. The concerned reader can make all of our discussion correct by restricting to the case of small categories, and defining **Set** to be the space of all sets whose cardinality is at most the size of the real numbers, and in general restricting any definition of a category (such as **Grp**) to spaces whose underlying set whose cardinality has at most the size of the real numbers.

Remark 1.5. The objects and morphisms of a category do not have much complexity implicit to them. All of the interesting structure is encoded within the composition structure. This is despite the fact that when we listed our examples in example 1.3 we only described the objects and morphisms, and not the composition structure. The reason for this is that the composition structure between morphisms in all of our examples is clear. In all our examples the objects are sets with extra strcture, and the morphisms are maps of sets. The composition structure is inhereted from the composition structure on functions between sets. Going further, we remark that objects in abstract categories are *not* required to be sets and the morphisms are *not* required to be functions of sets. It is important to be aware of the fact that there are some categories for which there is no interpreation of morphisms as functions between sets [Fre70].

Remark 1.6. We will make several notational shorthands when dealing with categories. We will conflate a category and its set of objects whenever convenient. That is, we will label a category by its set of objects. Additionally, instead of writing "$\mathrm{Hom}_{\mathcal{C}}(A, B)$" we will write "$\mathrm{Hom}(A, B)$" when $\mathcal{C}$ is clear, and we will write "$f : A \to B$" to mean "$f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$". Additionally, we will use a single quantifier to instanciate multiple objects simultaneously. For example, we will use "$\forall f : A \to B$ in $\mathcal{C}$" to mean "$\forall A, B \in \mathcal{C},$ and $\forall f : A \to B$".

Remark 1.7. A category isn't just a space with a good notion of composition - it also has identity maps. These identity maps are important, and we include them in the definition purposefully. There are two primary reasons: firstly that all of the relevant examples of categories will have identity maps, and secondly that most interesting properties of categories only make sense because of the identity maps. Hence if we didn't require identity maps then we would find ourselves constantly requiring them as a condition, which is a waste of space.

It is important to take a closer look at what the identity map means, though. The identity map is trying to capture a very general phenominon about transformations: there is always the trivial transformation which results from doing nothing. This do-nothing map is the identity. In the category of sets, the identity maps on the set A is given by the formula $\mathrm{id}_A(x) = x$ for all $x \in A$ by lemma 1.8. The fact that these maps are the identities in the category of sets is the reason that the identity axiom for categories is defined like it is.

Lemma 1.8. *Let A be a set. For all sets B and for all $f : A \to B$, $g : B \to A$ we have*

$$f \circ \mathrm{id}_A = f, \qquad \mathrm{id}_A \circ g = g.$$

In particular, id_A *satisfies the axiom of an identity in the category of sets, and hence* **Set** *forms a category.*

Proof. The associativity axiom is satisfied because composition of set functions is associative, and for all $f : A \to B$, $g : B \to A$,

$$(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x),$$

$$(\mathrm{id}_A \circ g)(x) = \mathrm{id}_A(g(x)) = g(x),$$

so the identity axiom is satisfied. □

Definition 1.9 (Isomorphism)**.** Let $\mathcal{C}$ be a category, let $A, B \in C$ be objects, and let $f : A \to B$ be a morphism. We say that f is an *isomorphism* if there exists a morphism $f^{-1} : B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. We call f^{-1} the *inverse* of f. In this case, we say that A and B are *isomorphic objects.*

Lemma 1.10. *Let A, B be sets, and let $f : A \to B$ be a function. The map f is a bijection if and only if there exists a function $f^{-1} : B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. In particular, a function f in the category* **Set** *is an isomorphism if and only it is a bijection.*

Proof. Suppose that f is a bijection. Then, we can define a map $f^{-1} : B \to A$ which sends $b \in B$ to the unique element $f^{-1}(b)$ such that $f(f^{-1}(b)) = b$, which exists since f is surjective and is unique because f is injective. By definition of f^{-1}, $f \circ f^{-1} = \mathrm{id}_B$. To show that the composition the other direction is the identity, we observe that for all $a \in A$

$$f(f^{-1}(f(a)) = f(a),$$

so $f^{-1}(f(a)) = a$ by the injectivity of f. Thus, f has an inverse. Conversely, suppose that f has an inverse f^{-1}. Then, $f(a) = f(a')$ implies $a = f^{-1}(f(a)) = f^{-1}(f(a')) = a'$ so f is injective. Additionally, for all $b \in B$ we have $b = f(f^{-1}(b))$ so f is surjective. Thus, f is a bijection. We have proved both directions, so our proof is complete. □

Remark 1.11. Just like how the category-theoretic definitions of identity maps and isomorphisms are modeled after the abstract properties of identity maps and isomorphisms in the category of sets, many other definitions will be implicitely modeled after the abstract properties of the category of sets or vector spaces. Accompanied with most definitions, there is often an implicit lemma that the usual examples satisfy the axioms of the definition. Going forward, we will rarely remark on these implicit lemmas.

Proposition 1.12. *Let $\mathcal{C}$ be a category. Identities in $\mathcal{C}$ are unique. Explicitely, let $A \in \mathcal{C}$ be an object and let $\mathrm{id}_A, \tilde{\mathrm{id}}_A : A \to A$ be morphisms satisfying the identity axiom. We have that $\mathrm{id}_A = \tilde{\mathrm{id}}_A$.*

Proof. . Using the fact that $\mathrm{id}_A \circ f = f$ and $f \circ \tilde{\mathrm{id}}_A = f$ for any $f : A \to A$, we compute that

$$\mathrm{id}_A = \mathrm{id}_A \circ \tilde{\mathrm{id}}_A = \tilde{\mathrm{id}}_A$$

as desired. □

Proposition 1.13. *Let $\mathcal{C}$ be a category. Let A, B be objects and let $f : A \to B$ be an isomorphism. The inverse of f is unique. That is, let $f^{-1}, \tilde{f}^{-1}$ be morphisms satisfying the definition of the inverse of f. We have that $f^{-1} = \tilde{f}^{-1}$.*

Proof. Using the associativity axiom, we compute

$$f^{-1} = f^{-1} \circ \mathrm{id}_B = f^{-1} \circ (f \circ \tilde{f}^{-1}) = (f^{-1} \circ f) \circ \tilde{f}^{-1} = \mathrm{id}_A \circ \tilde{f}^{-1} = \tilde{f}^{-1}$$

as desired. □

Remark 1.14. Statements in category theory can be very broadly applied. This is in some sense obvious by the fact that there are so many different examples of categories, but it's good to state the observation explicitely. For instance, look at proposition 1.13. It applied equally well for showing that inverse elements in groups are unique and for showing that inverses of matrices are unique. Abstractly, proposition 1.13 demonstrates why the inverse of any reversible process is unique.

1.2 Structures in category theory

1.2.1 Products and universal properties

In this section we will work on defining important structures in category, with a focus on the broadly applicable principles behind the definitions. In this first subsection we focus on products, our first example of a definition via universal property.

Definition 1.15 (Product)**.** Let $A, B \in \mathcal{C}$ be objects in a category. A *product* of A and B is the following data:

1. An object $A \times B \in \mathcal{C}$;
2. A morphism $\pi_A : A \times B \to A$;
3. A morphism $\pi_B : A \times B \to B$;

such that for all other objects $C \in \mathcal{C}$ with morphisms $f_A : C \to A$, $f_B : C \to B$, there exists a unique morphism $f : C \to A \times B$ such that the diagram

$$\begin{tikzcd} & C \arrow[d, dashed, "f"] \arrow[ddl, "f_A"'] \arrow[ddr, "f_B"] & \\ & A \times B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A & & B \end{tikzcd}$$

commutes.

Remark 1.16. At first glance, the categorical definition of a product may look strange. For a first level of comfort, one should observe that the categorical notion of product agrees with the usual notion of Cartesian product in the category **Set**, by proposition 1.17. More generally, the same argument as in proposition 1.17 can be used to show that the Cartesian product endowed with the product topology is a product in the category **Top**, the direct sum of vector spaces is a product in the category $\mathbf{Vec}_k$ for all fields k, the Cartesian product endowed with component-wise multiplication is a product in the category **Grp**, and so on.

Proposition 1.17. *For all pairs of sets A, B, the triple $(A \times B, \pi_A, \pi_B)$ is a product of A, B in the category* **Set***, where $A \times B$ is the Cartesian product, π_A is the projection of $A \times B$ onto the A component, and π_B is the projection of $A \times B$ onto the B component.*

Proof. Consider a set C and functions $f_A : C \to A$, $f_B : C \to B$. We can define a function $f : C \to A \times B$ by $f(c) = (f_A(c), f_B(c))$. Clearly, this morphism f satisfies $f_A = \pi_A \circ f$ and $f_B = \pi_B \circ f$. Moreover, suppose $f : C \to A \times B$ is any function with $f_A = \pi_A \circ f$ and $f_B = \pi_B \circ f$. Then, the A component of $f(c)$ is $f_A(c)$ and the B component of $f(c)$ is $f_B(c)$. Thus, $f(c) = (f_A(c), f_B(c))$. Thus, we conclude that there is a unique map $f : C \to A \times B$ making the relevant diagram commute, and since C, f_A, f_B were chosen arbitrarily we conclude the result. $\square$

Remark 1.18. Even though the Cartesian product is a product in the category of sets, it is *not* true that every categorical product of two sets A, B in **Set** is equal to the Carteisan product. In particular, suppose that D is a set and $i : D \xrightarrow{\sim} A \times B$ is a bijection from D to $A \times B$. Define $g_A = \pi_A \circ i$ and $g_B = \pi_B \circ i$. Then, (D, g_A, g_B) is also a product of A and B. This fact can be seen as follows. Suppose C is set, and $f_A : C \to A$, $f_B : C \to B$ are functions. We can define $f : C \to D$ by $f(c) = i^{-1}((f_A(c), f_B(c)))$. This map satisfies $f_A = g_A \circ f$ and $f_B = g_B \circ f$ since

$$g_A \circ f = (\pi_A \circ i) \circ (i^{-1}(f_A(c), f_B(c))) = f_A(c).$$

This is, however, the only freedom we have for choosing products. Every product of A, B in **Set** will be obtained by starting with $(A \times B, \pi_A, \pi_B)$ and composing with a bijection. To summarize this situation, we say that categorical products are not unique but they are *unique up to isomorphism.* Moreover, given another product (D, g_A, g_B), there is a *unique* isomorphism $i : D \to A \times B$ such that $f_A = \pi_A \circ f$ and$g_B = \pi_B \circ i$. For this reason we say that products are *unique up to unique isomorphism.* The proof for the category of sets is no easier than the general case, which is given in proposition 1.19

Proposition 1.19. *Let $A, B \in \mathcal{C}$ be objects in a category. Let (C, f_A, f_B), (D, g_A, g_B) be products of A and B. There exists a unique isomorphism $i : C \to D$ such that $f_A = g_A \circ i$ and $f_B = g_B \circ i$.*

Proof. By the universal property of (D, g_A, g_B), there exists a unique morphism $i : C \to D$ making the relevant diagram commute ($f_A = g_A \circ i$ and $f_B = g_B \circ i$). Similarly, by the universal property of C, there exists a unique morphism $j : D \to C$ making the relevant diagram commute ($g_A = f_A \circ j$ and $g_B = f_B \circ j$). Composing, we find that $j \circ i : C \to C$ makes the relevant diagram commute ($f_A = f_A \circ (j \circ i)$ and $f_B = f_B \circ (j \circ i)$). The map $\mathrm{id}_C : C \to C$, however, also makes the relevant diagram commute ($f_A = f_A \circ \mathrm{id}_C$ and $f_B = f_B \circ \mathrm{id}_C$). The universal property of the product says that there is a *unique* map making the diagram commute. Thus, we must have $j \circ i = \mathrm{id}_C$. By an analagous argument using the universal property of D, we find that $i \circ j = \mathrm{id}_D$. Thus, $j = i^{-1}$ and i is an isomorphism as desired. □

Remark 1.20. Definition 1.15 is our first example of a definition by a *universal property.* The property that the triple $(A \times B, \pi_A, \pi_B)$ is asked to satisfy in the definition is the universal property. In words, we will sometimes say that the product of A, B is universal with respect to the property of having morphisms into A and B. In light of proposition 1.19, the categorical definition of product is unique up to isomorphism, and in light of proposition 1.17 this unique product is isomorphic to the usual Cartesian product. Thus, at least for the category of sets, definition 1.15 is a more-complicated and less-precise way of defining the Cartesian product. There are several general reasons why one might prefer definitions by universal property:

1. Universal properties are common throughout mathematics (and specifically in the study of topological quantum information). It is good to understand their general structure;

2. WORK: make this list better. There's a very well-written list on Wikipedia.

Definition 1.21 (Coproduct)**.** Let $A, B \in \mathcal{C}$ be objects in a category. A *coproduct* of A and B is the following data:

1. An object $A \sqcup B \in \mathcal{C}$;

2. A morphism $i_A : A \to A \sqcup B$;

3. A morphism $i_B : B \times A \sqcup B$;

such that for all other objects $C \in \mathcal{C}$ with morphisms $f_A : A \to A \sqcup B$, $f_B : B \to A \sqcup B$, there exists a unique morphism $f : A \sqcup B \to C$ such that the diagram

$$\begin{tikzcd}
& C & \\
& A \sqcup B \arrow[u, dashed, "f"] \arrow[dr, "i_B"] & \\
A \arrow[uur, "f_A"] \arrow[ur, "i_A"'] & & B \arrow[uul, "f_B"']
\end{tikzcd}$$

commutes.

Definition 1.22 (Opposite category)**.** Let $\mathcal{C}$ be a category. The *opposite category* of $\mathcal{C}$ is the category defined as follows $\mathcal{C}^{\mathrm{op}}$. The set of objects of $\mathcal{C}^{\mathrm{op}}$ is the same as the set of objects as $\mathcal{C}$, with the object $A \in \mathcal{C}$ corresponding to the object $A^{\mathrm{op}} \in \mathcal{C}^{\mathrm{op}}$. The hom-sets of $\mathcal{C}^{\mathrm{op}}$ are defined as

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A^{\mathrm{op}}, B^{\mathrm{op}}) = \mathrm{Hom}_{\mathcal{C}}(B, A).$$

The morphism in $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A^{\mathrm{op}}, B^{\mathrm{op}})$ corresponding to $f \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ is denoted f^{op}. Given any $A, B, C \in \mathcal{C}$, and functions $f : A \to B$, $g : B \to C$ we define

$$f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}.$$

Remark 1.23. The category $\mathcal{C}^{\mathrm{op}}$ is intuitively described as “reversing all the arrows in $\mathcal{C}$”. It is often a useful category to consider, especially in the context of mathematical physics were we see that the opposite category can be seen as the *time reversal* of $\mathcal{C}$. That is, the physical system analogous to $\mathcal{C}$ where the arrow of time has been reversed. This is also a mathematically fruitful perspective to take, because flipping source and target for arrows is the same as changing the direction of causation.

Remark 1.24. Definition 1.21 of the coproduct is another definition by universal property. The universal property, of course, is very similar to the universal property of the product. In a formal sense it is the same universal property but with all of the arrows reversed. In particular, proposition 1.25 shows that products and coproducts are formally dual in the sense that products (resp. coproducts) in a category $\mathcal{C}$ correspond to coproducts (resp. products) in the opposite category $\mathcal{C}^{\mathrm{op}}$. This is a common theme in category theory. Many notions have corresponding dual notions, and the general terminology for the dual notion is to add the prefix “co-”.

Proposition 1.25. *Let $A, B \in \mathcal{C}$ be objects in a category. Let $(A \times B, \pi_A, \pi_B)$ be a product of A, B in $\mathcal{C}$. The triple $((A \times B)^{\mathrm{op}}, \pi_A^{\mathrm{op}}, \pi_B^{\mathrm{op}})$ is a coproduct of $A^{\mathrm{op}}, B^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$. Similarly, if $(A \sqcup B, i_A, i_B)$ is a coproduct of A, B in $\mathcal{C}$ then $((A \sqcup B)^{\mathrm{op}}, i_A^{\mathrm{op}}, i_B^{\mathrm{op}})$ is a product in $A^{\mathrm{op}}, B^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$.*

Proof. We show that $((A \times B)^{\mathrm{op}}, \pi_A^{\mathrm{op}}, \pi_B^{\mathrm{op}})$ is a coproduct. Choose an arbitrary triple $C^{\mathrm{op}} \in \mathcal{C}^{\mathrm{op}}, f_A^{\mathrm{op}} : A^{\mathrm{op}} \to C^{\mathrm{op}}$, and $f_B^{\mathrm{op}} : B^{\mathrm{op}} \to C^{\mathrm{op}}$. Then considering the triple (C, f_A, f_B) and applying the universal property of the product, we conclude that there is a unique map $f : C \to A \times B$ making the relevant diagram for the product commute. The dual map $f^{\mathrm{op}} : (A \times B)^{\mathrm{op}} \to C^{\mathrm{op}}$ is thus the unique map making the relevant diagram for the coproduct commute, so $((A \times B)^{\mathrm{op}}, \pi_A^{\mathrm{op}}, \pi_B^{\mathrm{op}})$ is indeed a coproduct. The argument for why $((A \sqcup B)^{\mathrm{op}}, i_A^{\mathrm{op}}, i_B^{\mathrm{op}})$ is a product is analagous. $\square$

Corollary 1.26. *Let $A, B \in \mathcal{C}$ be objects in a category. The coproduct of A, B, if it exists, is unique up to unique isomorphism. That is, if (C, f_A, f_B), (D, g_A, g_B) are coproducts of A and B, there exists a unique isomorphism $i : C \to D$ such that $g_A = i \circ f_A$ and $g_B = i \circ f_B$.*

Proof. By proposition 1.25 we find that $(C^{\mathrm{op}}, f_A^{\mathrm{op}}, f_B^{\mathrm{op}})$, $(D^{\mathrm{op}}, g_A^{\mathrm{op}}, g_B^{\mathrm{op}})$ are products in $\mathcal{C}^{\mathrm{op}}$. By proposition 1.19, we conclude that there is a unique isomorphism between C^{op} and D^{op} making the relevant diagram commute. Taking the opposite of this isomorphism, we conclude that there is a unique isomorphism between C and D making the relevant diagram commute, as desired. $\square$

Remark 1.27. Unlike how the product looks roughly similar in all our basic examples of categories, the coproduct can change quite dramatically. For instance, in the category **Set** the coproduct is the disjoint union by proposition 1.28. However, the coproduct of two vector spaces V, W in $\mathbf{Vec}_k$ is the direct sum $V \oplus W$ with i_V, i_W given by $i_V(v) = (v, 0)$, $i_W(v) = (0, v)$, for all fields k. The coproduct in **Grp** is especially exotic. Given two groups G, H, their coproduct is the *free product*, defined to be the group whose elements are formal words where each letter is either an element of G or an element of H, and two words are considered equivalent if it is possible to go from one to another by adding/removing copies of the identity elements of G, H or by replacing a product $g_1 g_2$ of adjacent elements by their product (in G or H). For example, the free product of $\mathbb{Z}$ and $\mathbb{Z}$ is the free group on two generators. The definition of free product can sometimes feel cumbersome - it is an example of a case where the universal property can be helpful. Seeing as the free product of finite groups can be infinite, we conclude that the category **finGrp** does not have coproducts. Even though coproducts are unique up to unique isomorphism if they exist, they are not gauranteed to exist.

Proposition 1.28. *For all pairs of sets A, B, the triple $(A \sqcup B, i_A, i_B)$ is a coproduct of A, B in the category of* **Set***, where $A \sqcup B$ is the disjoint union of A and B and i_A (resp. B) is the inclusion of A (resp. B) into $A \sqcup B$.*

Proof. Suppose C is a set and $f_A : A \to C$, $f_B : B \to C$ are maps. Then, we can define a map $f : A \sqcup B \to C$ by the formula

$$f(x) = \begin{cases} f_A(x) & \text{if } x \in A \\ f_B(x) & \text{if } x \in B. \end{cases}$$

Moreover, suppose we had any other map $f : A \sqcup B \to C$ such that $f_A = f \circ i_A$ and $f_B = f \circ i_B$. The first condition says that $f(x) = f_A(x)$ for $x \in A$ and the second condition says that $f(x) = f_B(x)$ for $x \in B$, so f must be equal to the map defined above. Thus, we conclude that $(A \sqcup B, i_A, i_B)$ as a coproduct of A, B. $\square$

1.2.2 Functors and natural transformations

Category theory philosophy tells us to care about the relationships between things (or more precisely, how those relationships compose). Moreover, category theory philosophy tells us to care about categories (any self-respecting theory should care about its central object of study). Naively combining these two principles, category theory thus suggests we should care about relationships between different categories. These relationships are called *functors*. Going further, seeing as we care about functors, a naive application of the principles of category theory suggests we should care about the relationships between different functors. These relationships are known as *natural transformations*. Thankfully, this line of reasoning does not go on indefinitely. One could try to define a notation of a relationship between natural transformations, but the resulting notion is trivial. Thus, one is left with exactly two important and fundamental notions to define: functors and natural transformations.

Definition 1.29 (Functor)**.** A *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ is the following data:

1. A function of objects $F : \mathcal{C} \to \mathcal{D}$;
2. A function of morphisms $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ for all $A, B \in \mathcal{C}$;

such that for all $A, B, C \in \mathcal{C}$, $f : A \to B$, $g : B \to C$,

$$F(g \circ f) = F(g) \circ F(f).$$

Definition 1.30 (Natural transformation)**.** A *natural transformation* from a functor F to a functor G between categories $\mathcal{C}, \mathcal{D}$ is a family of morphisms $\eta_A : F(A) \to G(A)$ for all $A \in \mathcal{C}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

commutes for all $f : A \to B$ in $\mathcal{C}$.

Example 1.31. A important family of examples of functors are *forgetful functors*. A forgetful functor is a functor which acts on the level of objects by taking an algebraic structure and forgetting that it satisfies certain axioms or is equipped with certain operations so that it becomes a different algebraic structure, and it acts as the identity on the level of morphisms. For instance, there is a forgetful functor $F : \mathbf{Top} \to \mathbf{Set}$ which takes a topological space and assigns to it its underlying set. There is a forgetful functor $F : \mathbf{Vec} \to \mathbf{Grp}$ which takes a vector space and assigns to it its underlying additive group. There is also a forgetful functors $F : \mathbf{Field} \to \mathbf{Grp}$ from the category of fields to the category of groups which assigns to every fields its underlying additive group. Some functors which are almost like forgetful functors also get called forgetful functors, such as the functor $F : \mathbf{Field} \to \mathbf{Grp}$ which assigns to every field its multiplicative group of non-zero elements. The inclusion functors $F : \mathbf{finSet} \to \mathbf{Set}$ and $F : \mathbf{finGrp} \to \mathbf{Grp}$ from the categories of finite sets (resp. finite groups) to to the category of all sets (resp. groups) can also be viewed as forgetful functors, where we are forgetting the fact that the input objects are finite.

Example 1.32. For every $n \geq 1$, there is a functor $\mathrm{GL}_n : \mathbf{Field} \to \mathbf{Grp}$ which assigns to a field k to the general linear group of n by n matrices, $\mathrm{GL}_n(k)$. This map is em functorial (that is, it can be extended to a functor) since every field homomorphism $k \to k'$ induces a group homomorphism $\mathrm{GL}_n(k) \to \mathrm{GL}_n(k')$ by acting via the field homomorphism element-wise on a matrix. It is clear that this assignment is compatible with composition, and is thus a functor. There are many natural transformations between the functors GL_n. For instance, the inverse-transpose $((-)^{-1})^T$ is a natural transformation between the functor GL_n and itself. For every field k, there is a map $((-)^{-1})^T : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ which acts by first taking the inverse of a matrix and then taking its transpose. These maps form a natural transformation, because for any field homomorphism $k \to k'$ there is a commutative diagram

$$\begin{tikzcd} \mathrm{GL}_n(k) \arrow[r, "((-)^{-1})^T"] \arrow[d] & \mathrm{GL}_n(k) \arrow[d] \\ \mathrm{GL}_n(k') \arrow[r, "((-)^{-1})^T"'] & \mathrm{GL}_n(k'). \end{tikzcd}$$

WORK: the determinant is also a good example, and so is the map which pads a matrix with an extra row/collumn. Lots of nice linear algebra natural transformations. Not sure how relevant they are to this book - maybe I can do something better? Perhpas the additive group of matrices, with trace as an example. Want something that I could pull on later.

Definition 1.33. The way that functors relate categories and natural transformations relate functors is completely analagous to the way that morphisms relate objects in a category. In particular, we observe that there is a category **Cat** whose objects are categories, and whose morphisms are functors between categories. The identity morphisms in this category are the idenity functors $\mathrm{id}_{\mathcal{C}}$, which act as the identity both on objects and on hom spaces. For any two categories $\mathcal{C} \to \mathcal{D}$, we can define a catgegory $\mathbf{Hom}(\mathcal{C}, \mathcal{D})$ whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations. The identity morphisms in this category are the identity natural transformations id_F from a functor to itself which acts by the identity map on $F(A)$ for objects A.

Definition 1.34. Two categories are called *isomorphic* if they are isomorphic in the category of categories (denoted $\mathcal{C} \cong \mathcal{D}$), and two functors are called *naturally isomorphic* if they are isomorphic in the appropriate category of functors (denoted $F \cong G$). Two categories $\mathcal{C}$, $\mathcal{D}$ are called *equivalent* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ (denoted $\mathcal{C} \simeq \mathcal{D}$).

Remark 1.35. If two categories are isomorphic, then they are also equivalent. There are, however, equivalent categories which are not isomorphic. Let $\mathcal{C}$ be a non-empty category for which between any two objects are is always a unique morphism. Then, by proposition 1.36, $\mathcal{C}$ is equivalent to the category $\mathbf{1}$ which is defined to have a unique object labeled X and whose only morphism is id_X. We need not have $\mathcal{C} \cong \mathbf{1}$, however. In particular, if $\mathcal{C}$ has at least two objects than it cannot be isomorphic to $\mathbf{1}$ since $\mathbf{1}$ has a single objects and isomorphisms of categories must be bijections of sets of objects.

As an example, given objects $A, B \in \mathcal{D}$ in a category we define a category $\mathbf{Prod}(A, B)$ as follows. The objects of $\mathbf{Prod}(A, B)$ are products (C, f_A, f_B) of A, B. A morphism between (C, f_A, f_B) and (D, g_A, g_B) is a map $h : C \to D$ such that $f_A = g_A \circ h$ and $f_B = g_B \circ h$. That is, morphisms in $\mathbf{Prod}(A, B)$ are the morphisms in $\mathcal{C}$ which respect the additional structure of the triples. By the definition of a product, there is a unique morphism between any two objects in $\mathbf{Prod}(A, B)$ so by proposition 1.36 we have $\mathbf{Prod}(A, B) \simeq \mathbf{1}$ whenever the product of A, B exists. Philosophicaly, this language of equivalence says that even though products are not unique, the space of possible products between any two objects is equivalent to a point.

Proposition 1.36. *Let $\mathcal{C}$ be a non-empty category for there is a unique morphism between any two objects. Then, $\mathcal{C} \simeq \mathbf{1}$.*

Proof. Define a functor $F : \mathbf{1} \to \mathcal{C}$ by $F(X) = C$, $F(\mathrm{id}_X) = \mathrm{id}_C$ for some $C \in \mathcal{C}$. Define a functor $G : \mathcal{C} \to \mathbf{1}$ by $G(A) = X$, $G(f) = \mathrm{id}_X$ for all $A, B \in \mathcal{C}$, $f : A \to B$. It is clear that $G \circ F = \mathrm{id}_{\mathbf{1}}$. Conversely, we can show that $F \circ G \cong \mathrm{id}_{\mathcal{C}}$. To do this, we can define a natural transformation $\eta : (F \circ G) \to \mathrm{id}_{\mathcal{C}}$ which acts on each component by the unique morphism with the correct source and target. Namely, for all $A \in \mathcal{C}$, η is the unique morphism from $(F \circ G)(A) = F(X) = C$ to $\mathrm{id}_{\mathcal{C}}(A) = A$. The relevant diagram for showing that η is a natural transformation must commute, because there is a unique morphism between any two objects in $\mathcal{C}$ and thus every diagram in $\mathcal{C}$ commutes! Thus, η is indeed a natural transformation. We can see that η is invertible, it acts by invertible maps on all components (since all maps in $\mathcal{C}$ are invertible). Thus, η is a natural isomorphism as desired, so $\mathcal{C} \simeq \mathbf{1}$. ☐

Example 1.37. For all categories $\mathcal{C}$, the functor $i : (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{C}$ defined by $(A^{\mathrm{op}})^{\mathrm{op}} \mapsto A$ and $(f^{\mathrm{op}})^{\mathrm{op}} \mapsto f$ is an isomorphism of categories.

Remark 1.38. We can give an alternate definition of the opposite category via a universal property. Given two categories $\mathcal{C}, \mathcal{D}$, we define a *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ to be an assignment $F : \mathcal{C} \to \mathcal{D}$ of objects and $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ of morphisms for all $A, B \in \mathcal{C}$, such that for all $A, B, C \in \mathcal{C}$, $f : A \to B$, $g : B \to C$,

$$F(g \circ f) = F(f) \circ F(g).$$

Note that a contravariant functor is not a functor. We will sometimes call a standard functor a *covariant functor* to highlight that it is not contravariant. We observe that by the definition of the opposite category, every contravariant functor $F : \mathcal{C} \to \mathcal{D}$ naturally induces a covariant functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ by defining $G(A^{\mathrm{op}}) = F(A)$, $G(f^{\mathrm{op}}) = F(f)$. Moreover, this basic fact about the opposite category can be stated in terms of a universal property. This goes as follows. There is a canonical contravariant functor $i : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$, which takes $A \to A^{\mathrm{op}}$. This functor has the property that for any category $\mathcal{D}$ and for any contravariant functor $F : \mathcal{C} \to \mathcal{D}$, there exists a unique covaraint functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ as defined before, making the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[d, "F"'] & \mathcal{C}^{\mathrm{op}} \arrow[dl, "G"] \\ \mathcal{D} & \end{tikzcd}$$

commute. Moreover, for any other pair $(\mathcal{E}, i)$ with $\mathcal{E}$ a category $j : \mathcal{C} \to \mathcal{E}$ a contravariant with the same universal property then there would be a unique isomorphism of categories $I : \mathcal{E} \to \mathcal{C}^{\mathrm{op}}$ such that $I \circ j = i$. That is, the universal property that $\mathcal{C}^{\mathrm{op}}$ turns contravariant functors into covariant functors defines $\mathcal{C}^{\mathrm{op}}$ up to unique isomorphism.

Example 1.39. Let $\mathcal{C}$ be a category. For all $A \in \mathcal{C}$, there is a functor $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$ defined by $\mathrm{Hom}_{\mathcal{C}}(A, -)(B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$. This map is functorial because any morphism $f : B \to C$ induces a map $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ by postcomposition. Similarly, there is a contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, A)$. If a functor $F : \mathcal{C} \to \mathbf{Set}$ is of the form $\mathrm{Hom}_{\mathcal{C}}(A, -)$ or $\mathrm{Hom}_{\mathcal{C}}(-, A)$ for some $A \in \mathcal{C}$, we call it *representable*.

Definition 1.40. Let $\mathcal{C}$, $\mathcal{D}$ be categories. We define the *product category* $\mathcal{C} \times \mathcal{D}$ of $\mathcal{C}$ to $\mathcal{D}$ to be the category whose objects are pairs (A, B), $A \in \mathcal{C}$, $B \in \mathcal{D}$, whose morphisms between (A, A') and (B, B') are pairs (f, g), $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{D}}(A', B')$, and whose composition is defined component-wise. It is simple to check that this definition of the product of categories is an abstract product in the category $\mathbf{Cat}$.

Example 1.41. There is a nice rephrasing of the definition of product and coproduct in terms of representable functors. In particular, we claim that products of $A, B \in \mathcal{C}$ are in bijective correspondance with objects $A \times B \in \mathcal{C}$ paired with natural isomorphisms

$$\eta : \mathrm{Hom}(-, A \times B) \xrightarrow{\sim} \mathrm{Hom}(-, A) \times \mathrm{Hom}(-, B).$$

This can be seen as follows. Suppose η is a natural isomorphism as above. Then, we can define $(\pi_A, \pi_B) = \eta_{A\times B}(\mathrm{id}_{A\times B})$. We claim that $(A \times B, \pi_A, \pi_B)$ is a product in $\mathcal{C}$. Suppose that $C \in \mathcal{C}$, $f_A : C \to A$, $f_B : C \to B$ is another triple. Then, we can define $f = \eta_C^{-1}((f_A, f_B))$. By naturality the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(A \times B, A \times B) \arrow[r, "\eta_{A\times B}"] \arrow[d, "\mathrm{Hom}(f, A\times B)"'] & \mathrm{Hom}(A \times B, A) \times \mathrm{Hom}(A \times B, B) \arrow[d, "\mathrm{Hom}(f,A)\times\mathrm{Hom}(f,B)"] \\
\mathrm{Hom}(C, A \times B) \arrow[r, "\eta_C"] & \mathrm{Hom}(C, A) \times \mathrm{Hom}(C, B)
\end{tikzcd}$$

following $\mathrm{id}_{A\times B}$ through this square, we get that $f_A = \pi_A \circ f$ and $f_B = \pi_B \circ f$ as desired. Applying this same procedure in reverse allows one to define a natural transformation η given a product $(A \times B, \pi_A, \pi_B)$. Dually, we find that coproducts of $A, B \in \mathcal{C}$ are in bijective correspondance with objects $A \sqcup B \in \mathcal{C}$ paired with natural isomorphisms

$$\eta : \mathrm{Hom}(A \sqcup B, -) \xrightarrow{\sim} \mathrm{Hom}(A, -) \times \mathrm{Hom}(B, -).$$

1.2.3 Linear categories

WORK: I need to add the definition of *zero object*, so that proposition 2.17 makes sense.

To understand topological quantum information, we will need our categories to be in some real sense quantum mechanical. Quantum systems are always vector spaces over $\mathbb{C}$. Eventually, the morphisms in our categories will directly correspond to quantum states in certain quantum systems. Thus, we need the hom-spaces in our categories to be vector spaces over $\mathcal{C}$. This leads us to study the abstract properties of *linear categories*.

Definition 1.42 ($\mathbb{C}$-linear category)**.** A $\mathbb{C}$-linear category is the following data:

1. A category $\mathcal{C}$;
2. The structure of a $\mathbb{C}$-vector space on $\mathrm{Hom}(A, B)$ for all $A, B \in \mathcal{C}$;

such that the composition maps $\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ are bilinear maps of vector spaces for all $A, B, C \in \mathcal{C}$.

Definition 1.43 ($\mathbb{C}$-linear functor)**.** A $\mathbb{C}$-linear functor between $\mathbb{C}$-linear categories $\mathcal{C}, \mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a linear map of vector spaces for all $A, B \in \mathcal{C}$.

Remark 1.44. When we refer to a functor between two $\mathbb{C}$-linear categories, we will always assume that that functor is $\mathbb{C}$-linear unless otherwise stated. In general, whenever two categories are equipped with extra structure we will assume that functors between those categories respect that structure. There is no notion of a $\mathbb{C}$-linear natural transformation (or, if there were, it would have to be the same as the notion of usual natural transformation).

Example 1.45. The category $\mathbf{Vec}_{\mathbb{C}}$ of vector spaces over the complex numbers is $\mathbb{C}$-linear category, since the space of linear maps $\mathrm{Hom}_{\mathbf{Vec}_{\mathbb{C}}}(V, W)$ between any two vector spaces V, W is itself a vector space. Additionally, the product of two $\mathbb{C}$-linear categories is again a $\mathbb{C}$-linear category. So, the category $\mathbf{Vec}_{\mathbb{C}}^n$ of ordered n-tuples of vector spaces (equivalent to the product of n copies of $\mathbf{Vec}_{\mathbb{C}}^n$) is a $\mathbb{C}$-linear category.

Remark 1.46. A noteworthy feature of $\mathbb{C}$-linear categories is the existence of *zero morphisms.* Suppose that $A, B \in \mathcal{C}$ are objects in a $\mathbb{C}$-linear category. There is a distingished morphism $0 : A \to B$, called the *zero morphism*, corresponding to the zero element of $\mathrm{Hom}(A, B)$ as a vector space. The zero element is uniquely identified by the fact that for all $f : B \to C$ and for all $g : C \to A$,

$$f \circ 0 = 0, \qquad\qquad 0 \circ g = 0.$$

Note that the notation being used is ambiguous, since in the formulas above 0 refers to both the zero morphism $A \to B$ but also to the zero morphisms $A \to C$ and $C \to B$. The reason that the zero morphism satisfies these properties is that $\circ$ is a bilinear map, and bilinear maps evaluate to zero on any pair which has zero as one of its components.

Definition 1.47 (Biproduct)**.** Let $A, B \in \mathcal{C}$ be objects in a $\mathbb{C}$-linear category category. A *biproduct* (or *direct sum*) of A and B is the following data:

1. An object $A \oplus B \in \mathcal{C}$;
2. The structure of a product $(A \oplus B, \pi_A, \pi_B)$ on $A \oplus B$;
3. The structure of a coproduct $(A \oplus B, i_A, i_B)$ on $A \oplus B$;

such that:

1. $\pi_A \circ i_A = \mathrm{id}_A$ and $\pi_B \circ i_B = \mathrm{id}_B$;
2. $\pi_A \circ i_B = 0$ and $\pi_B \circ i_A = 0$.

Proposition 1.48. *For all vector spaces V, W, the direct sum $A \oplus B$ paired with its projections (π_A, π_B) onto its components and its inclusions $i_V(v) = (v, 0)$, $i_W(v) = (0, v)$ is a biproduct in $\mathbf{Vec}_{\mathbb{C}}$.*

Proof. We have observed in remarks 1.16 and 1.27 that $A \oplus B$ is individually a product and a coproduct. It is immediate that the structures are compatible in the sense of definition 1.47, and thus $A \oplus B$ is a biproduct. □

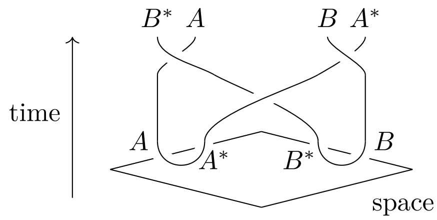
1.3 Monoidal categories

1.3.1 Motivation, definition, and string diagrams

The goal of this section is to introduce the languages of *monoidal categories* and *string diagrams*, which are neccecary for a proper algebraic discussion of anyons and modular tensor categories.

The structures in monoidal category theory and their interpretation in terms of string diagrams will allow us to discuss situations like the one below, where we create and braid quasiparticles:

Example 1.49. Suppose that we are braiding quasiparticles in a topological system. In spacetime, the trajectories of the anyons will look something like diagram 1.1.



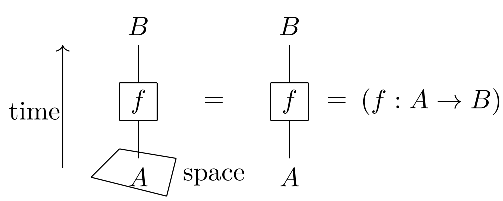
(1.1)

Using the formalism of monoidal categories and the language of string diagrams, we will be able to interpret the above diagram 1.1 as a certain morphism in a category. The exact category in which the morphism should live depends on the underlying topological system. The quasiparticle labels A, B, A^*, B^* represent objects in $\mathcal{C}$. The objects (A, A^*) form a particle/antiparticle pair, and the objects (B, B^*) form a particle/antiparticle pair. The diagram is broadly interpreted as follows. To begin, there are no particles. Then, we have creation maps create_{A,A^*} and $\text{create}_{B^*,B}$ which pairs of particles and their antiparticles. Then we have three different braiding operatations (braid_{A^*,B^*}, braid_{A,B^*}, and $\text{braid}_{A^*,B}$) which swap the positions of adjacent quasiparticles. The overall process is the composition of these sub-processes:

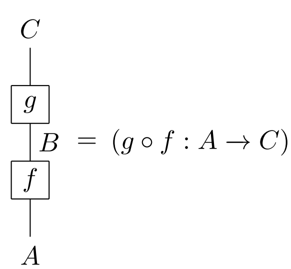
$$(\text{braid}_{A^*,B}) \circ (\text{braid}_{A,B^*}) \circ (\text{braid}_{A^*,B^*}) \circ (\text{create}_{B^*,B}) \circ (\text{create}_{A,A^*}).$$

It is thus clear why categories-with-structure are the right framwork for understanding diagrams like diagram 1.1. The additional structures on $\mathcal{C}$ will give the braiding and creation maps meaning, and the composition structure on $\mathcal{C}$ will tell us how these components fit together to make larger processes.

We now begin to introduce the language of string diagrams. The most basic principle os string diagrams is that morphisms are represented as follows:



The direction of time going from bottom to up and the space being two-dimensional slices is the same in every diagram, and hence is left implicit from here on out. Composition is expressed cleanly in this langauge as stacking. That is, for all $f : A \to B$, $g : B \to C$, we define



Accordingly, the identity map has a simple implementation:

$$= (\mathrm{id}_A : A \to A)$$

We now give our first major example of adding structure, and how that structure can be interpreted in terms of string diagrams. This structure is that of a *monoidal category*. For technical reasons we only define *strict* monoidal categories for now - we will come back to the general definition later. Monoidal categories give a way to put objects together. For instance, in diagram 1.1 we had four particles all together. We need to way to discuss composite-particle systems. In quantum mechanics, forming a composite system is done by taking the tensor product. Hence, we will use the notation $\otimes$ for joining particles in our current setting. We will even use the term "tensor product" to discuss it. In general, joining two systems is one way of going from pairs of systems to individual systems:

$$\begin{aligned} (\text{systems}) \times (\text{systems}) &\to (\text{systems}). \\ (\text{system } 1, \text{system } 2) &\mapsto (\text{system } 1) \otimes (\text{system } 2) \end{aligned}$$

In the world of category theory, we only require some basic properties of this joining. Namely, it should be functorial and satisfy some simple conditions:

Definition 1.50 (Strict monoidal category)**.** A strict monoidal category is the following data:

1. A category $\mathcal{C}$;
2. (Tensor product) A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
3. (Unit) A distinguished element $\mathbf{1} \in \mathcal{C}$;

Such that:

1. (Unit axiom) Let $A, A' \in \mathcal{C}$ be objects and let $f : A \to A'$ be a morphism. We have

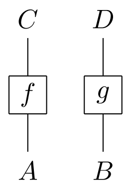
$$A \otimes \mathbf{1} = \mathbf{1} \otimes A = A, \qquad\qquad f \otimes \mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes f = f.$$

2. (Associativity) Let $A, B, C, A', B', C' \in \mathcal{C}$ be objects, and let $f : A \to A'$, $g : B \to B'$, $h : C \to C'$ be morphisms. We have

$$(A \otimes B) \otimes C = A \otimes (B \otimes C), \qquad\qquad (f \otimes g) \otimes h = f \otimes (g \otimes h).$$

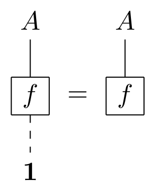
Remark 1.51. The object $\mathbf{1} \in \mathcal{C}$ is important. Just like how groups of symmetries always include the "do-nothing" symmetry, strict monoidal categories should always include the unit. In this case, $\mathbf{1} \in \mathcal{C}$ represents the empty particle - no particle at all. In every particle theory there should be the possibility of not having any particles. Joining the empty particle with any other particle should obviously do nothing, hence the axiom $\mathbf{1} \otimes A = A \otimes \mathbf{1} = A$.

We can now work strict monoidal categories into our graphical language. The tensor product of two objects is represented by putting two lines adjacent to one another. For instance, let $\mathcal{C}$ be a strict monoidal category, let $A, B, C, D \in \mathcal{C}$ be four objects, and let $f : A \to C$, $g : B \to D$ be morphisms. We have



$$= \quad (f \otimes g : A \otimes B \to C \otimes D)$$

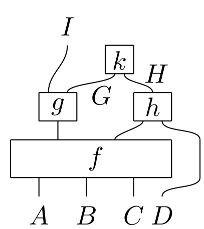
The monoidal unit $\mathbf{1}$ is distinguished in monoidal categories, and hence is represented with a special line. We will either use a dotted line, or no line at all:



$$= (f : \mathbf{1} \to A)$$

Remark 1.52. We *do not* require that the lines drawn in string diagrams be straight. They can curve any amount so long as it is clear that they are directly connecting an output to an input. The lines cannot cross each other or double back. Additionally, when it is clear from context, we *do not* require ourselves to include every label.

Example 1.53. Diagram 1.2 is valid in all strict monoidal categories $\mathcal{C}$, where $A,B,C,D,E,F,G,H \in \mathcal{C}$ are objects, and $f : A \otimes B \otimes C \to E \otimes F$, $g : E \to I \otimes G$, $h : F \otimes D \to H$, and $k : G \otimes H \to \mathbf{1}$ are morphisms:



(1.2)

1.3.2 Braided monoidal categories

We continue our definitions of structures on monoidal categories, and their expression in the language of string diagrams. Our next definition is that of a strict braided monoidal category:

Definition 1.54 (Strict braided monoidal category)**.** A strict braided monoidal category is the following data:

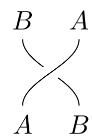
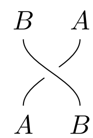
1. A strict monoidal category $\mathcal{C}$;
2. (Braiding) Isomorphisms $\beta_{A,B} : A \otimes B \to B \otimes A$ for all $A, B \in \mathcal{C}$ which form a natural isomorphism between the functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by $(A, B) \mapsto A \otimes B$ and $(A, B) \mapsto B \otimes A$.

Such that for all $A, B, C \in \mathcal{C}$, the diagrams

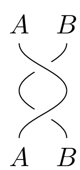
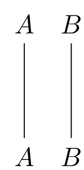
$$\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "\beta_{A,B} \otimes \mathrm{id}_C"] \arrow[dd, "\beta_{A,B\otimes C}"'] & & B \otimes A \otimes C \arrow[ddll, "\mathrm{id}_B \otimes \beta_{A,C}"] \\
& & \\
B \otimes C \otimes A & &
\end{tikzcd}, \qquad
\begin{tikzcd}
A \otimes B \otimes C \arrow[rr, "\beta^{-1}_{B,A} \otimes \mathrm{id}_C"] \arrow[dd, "\beta^{-1}_{B\otimes C,A}"'] & & B \otimes A \otimes C \arrow[ddll, "\mathrm{id}_B \otimes \beta^{-1}_{C,A}"] \\
& & \\
B \otimes C \otimes A & &
\end{tikzcd}$$

commute.

The idea for how to implement braided monoidal categories in the language of string diagrams is to introduce a special symbol for the braiding map $\beta_{A,B}$. Namely, we graphically define overcrossing and undercrossing as follows:

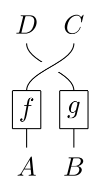
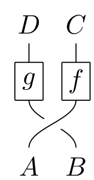

$= \beta_{A,B},$

$= \beta^{-1}_{B,A}.$

Remark 1.55. The fact that overcrossing and undercrossing are related by an inverse encodes the following topological fact:


$= \beta^{-1}_{A,B} \circ \beta_{A,B} = \mathrm{id}_{A\otimes B} =$


We can now describe the conditions on a strict braided monoidal category in a graphical way. The fact that β is a natural transformation can be reinterpreted as follows:

Lemma 1.56. *Let $\mathcal{C}$ be a strict braided monoidal category. For all $A, B, C, D \in \mathcal{C}$ and $f : A \to C$, $g : B \to D$, we have the following equality of string diagrams:*


$=$


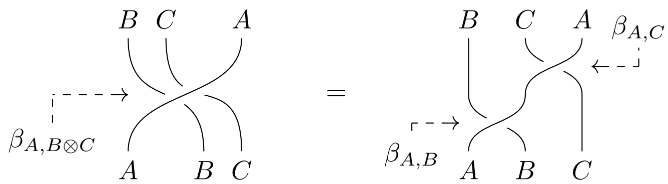
The same formula holds replacing overcrossing with undercrossing on both sides.

Proof. Consider the morphism $(f, g) : (A, B) \to (C, D)$ in $\mathcal{C} \times \mathcal{C}$. The naturality of β implies the following commutative square:

$$\begin{tikzcd}
A \otimes B \arrow[r, "f\otimes g"] \arrow[d, "\beta_{A,B}"] & C \otimes D \arrow[d, "\beta_{C,D}"] \\
B \otimes A \arrow[r, "g\otimes f"] & D \otimes C
\end{tikzcd}$$

exanding this square in diagramatic language gives the first part of the proposition. Reversing the direction of the arrows by taking inverses gives the second part. $\square$

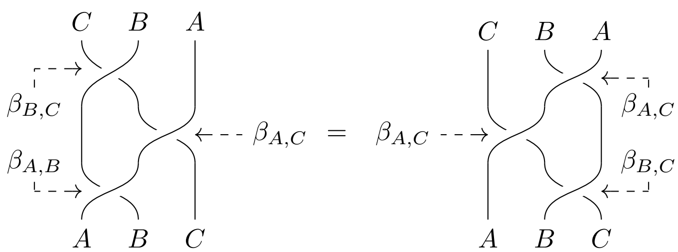
Remark 1.57. The first coherence axiom can be stated diagrammatically as follows,



and the second coherence axiom can be stated similarly replacing overcrossing with undercrossing. The importance of this axiom is that it means that our graphical langauge can express braid diagrams without other ambiguity. We can safely deform strings behind braids and not need to worry about whether we are applying $\beta_{A,B\otimes C}$ or $(\mathrm{id}_B \otimes \beta_{A,C}) \circ (\beta_{A,B} \otimes \mathrm{id}_C)$.

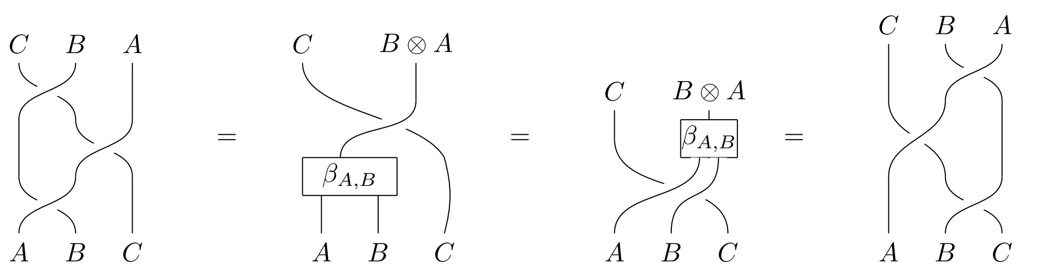
Remark 1.58. When discussing the theory of braiding in subsection **??**, we discusses braiding operations. It was asserted that two braiding operations are topologically equivalent if and only if they can be manipulated from one to another via manipulations like the one in equation 1.3. Thus, proposition 1.59 proves that any two topologically equivalent braids will correspond to the same morphism in a braided monoidal category.

Proposition 1.59 (Yang-Baxter equation)**.** *Let $\mathcal{C}$ be a strict braided monoidal category. Let $A, B, C \in \mathcal{C}$ be objects. We have*



(1.3)

Proof. We offer a graphical proof, using first the coherence condition and then naturality:



□

Corollary 1.60. *Let $\mathcal{C}$ be a strict braided monoidal category. Let $A \in \mathcal{C}$ be an object. The map*

$$\begin{aligned} B_n &\to \mathrm{Aut}(A^{\otimes n}) \\ \sigma_i &\mapsto \mathrm{id}_{A^{\otimes i-1}} \otimes \beta_{A,A} \otimes \mathrm{id}_{A^{n-i-1}} \end{aligned}$$

is a homomorphism of groups.

Proof. By definition B_n, the only relations we need to verify are $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ for $1 \leq i \leq n-1$. These conditions are satisfied by the map by proposition 1.59. $\square$

1.3.3 Examples, equivalences, and MacLane's coherence theorem

Warning 1.61. This section is not neccecary for a conceptual understanding of the subject matter. It is material of technical importance, and thus of interest to those who want a correct formal understanding of the mathematics at play.

In this section we will give concrete examples of monoidal categories and braided monoidal categories. What we will find, however, is that these examples will all demonstrate the same subtle problem. For example, here is the first category which we would want to give as an example of a monoidal category:

$$\mathcal{C} = \mathbf{Set}, \ \otimes = \text{Cartesian product}.$$

The Cartesian product is certainly functorial. Namely, given morphisms $f : A \to C$ and $g : B \to D$ we get a morphism

$$\begin{aligned}(f \times g) : A \times B &\to C \times D. \\ (a, b) &\mapsto (f(a), g(b))\end{aligned}$$

However we get a key issue $(A \times B) \times C \neq A \times (B \times C)$. We have an isomorphism

$$\begin{aligned}\alpha : (A \times B) \times C &\to A \times (B \times C), \\ ((a, b), c) &\mapsto (a, (b, c))\end{aligned}$$

but this isomorphism is *not* an equality. This means that **Set** does not satisfy the definiton of a strict monoidal category! In general, all the examples we would want to give of monoidal categories fail to be strict monoidal categories because the tensor product is not litereally associative. In this section we discuss a method for loosening the definition of monoidal category so that **Set** and other examples can be included in the definition.

Remark 1.62. The most naive way of loosening the definition of monoidal category is to only enforce the condition $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ instead of equality. However, this leads to a problem. The associativity axiom on morphisms $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ no longer makes sense because there is no way of comparing morphisms on $(A \otimes B) \otimes C$ and $A \otimes (B \otimes)C$. It is for this reason that we need to require specific isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ and require that those isomorphisms satisfy certain coherence conditions. In general, when looseing equalities to isomorphisms in category theory it is good to posit the existence of specific isomorphisms instead of only forcing that an isomorphism exists.

Definition 1.63 (Monoidal category)**.** A monoidal category is the following data:

1. A category $\mathcal{C}$.
2. (Tensor product) A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

3. (Unit) A distinguished element $\mathbf{1} \in \mathcal{C}$.

4. (Associator) A natural isomorphism

$$\alpha : - \otimes (- \otimes -) \xrightarrow{\sim} (- \otimes -) \otimes -,$$

   where $-\otimes(-\otimes-)$ denotes the functor $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ sending (A, B, C) to $A \otimes (B \otimes C)$, and similarly for $(- \otimes -) \otimes -$.

5. (Left unitor) A natural isomorphism $\lambda : \mathbf{1} \otimes - \xrightarrow{\sim} -$, where $\mathbf{1} \otimes -$ denotes the functor $\mathcal{C} \to \mathcal{C}$ sending A to $\mathbf{1} \otimes A$, and $-$ denotes the identity.

6. (Right unitor) A natural isomorphism $\rho : - \otimes \mathbf{1} \xrightarrow{\sim} -$, where $- \otimes \mathbf{1}$ is the functor $\mathcal{C} \to \mathcal{C}$ sending A to $A \otimes \mathbf{1}$.

Additionally, a monoidal category is required to satisfy the following properties:

1. (Triangle identity) The diagram

$$\begin{tikzcd}
(A \otimes \mathbf{1}) \otimes B \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (\mathbf{1} \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

   commutes for all $A, B \in \mathcal{C}$.

2. (Pentagon identity) The diagram

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{id}_D"'] & & A \otimes (B \otimes (C \otimes D)) \arrow[ul, "\alpha_{A,B,C\otimes D}"'] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{id}_A \otimes_{B,C,D}"']
\end{tikzcd}$$

   commutes for all $A, B, C, D \in \mathcal{C}$.

Example 1.64. The following collections of data form monoidal categories

- The category $\mathcal{C} = \mathbf{Set}$, with tensor product $\otimes$ = Cartesian product, monoidal unit $\mathbf{1} = \{*\}$, associator

$$\alpha_{A,B,C} : A \times (B \times C) \xrightarrow{\sim} (A \times B) \times C,$$
$$(a, (b, c)) \mapsto ((a, b), c)$$

  and unitors

$$\begin{aligned} \lambda : \mathbf{1} \otimes A &\to A \\ (*, a) &\mapsto a \end{aligned} \qquad\qquad \begin{aligned} \rho : A \otimes \mathbf{1} &\to A. \\ (a, *) &\mapsto a \end{aligned}$$

- The plain category $\mathcal{C} = \mathbf{Vec}_{\mathbb{C}}$, with its standard tensor product, monoidal unit $\mathbf{1} = \mathbb{C}$, associator

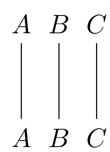
$$\begin{aligned} \alpha_{A,B,C} : A \times (B \times C) &\xrightarrow{\sim} (A \times B) \times C, \\ a \otimes (b \otimes c) &\mapsto (a \otimes b) \otimes c \end{aligned}$$

and unitors

$$\begin{aligned} \lambda : \mathbf{1} \otimes A &\to A \\ 1 \otimes a &\mapsto a \end{aligned} \qquad\qquad \begin{aligned} \rho : A \otimes \mathbf{1} &\to A. \\ a \otimes 1 &\mapsto a \end{aligned}$$

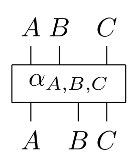
- The category $\mathcal{C} = \mathbf{Set}$ with tensor product $\otimes =$ Disjoint union and $\mathbf{1} = \{\}$, with a standard choice of associators and unitors;
- The category $\mathcal{C} = \mathbf{Vec}_{\mathbb{C}}$ with tensor product $\otimes =$ Direct sum, and $\mathbf{1} = 0$, with a standard choice of assoicators and unitors.

Remark 1.65. In expanding our definition from strict monoidal category to monoidal category we have introduced a subtle problem. The diagram



$= \mathrm{id}_{A \otimes B \otimes C}$

no longer makes sense! The map $\mathrm{id}_{A \otimes B \otimes C}$ no longer exists, because $A \otimes B \otimes C$ no longer exists. One must make a choice of $(A \otimes B) \otimes C$ or $A \otimes (B \otimes C)$. These maps may be isomorphic, but they have no need to be equal! The correct diagram would be



In general, string diagrams for non-strict monoidal categories need α maps thrown in at key points to make a well-defined language. This is quite complicated, and has issues that need to be adressed. Hence, we maintain that our graphical langauge only applies to strict monoidal categories.

Remark 1.66. In light of remark 1.65, we seem to have made very little progress. We defined the notion of a non-strict monoidal category so that we could include our favorite examples, but then we observed that string diagrams fail to describe those examples! This seemingly bad situation is rectified by theorem 1.72, which we first state informally.

- MacLane's cohrence theorem: *every monoidal category is equvialent to a strict monoidal category.*

This gives us a workflow for the book. We will frame our discussion so that it applies to arbitrary monoidal categories. That way, all our usual examples are included. Then, when we want to use string diagrams, we use MacLane's coherence theorem to pass to an equivalent strict category, in which our diagrams make sense. Then, when we are done using the diagram, we pass the conclusion of the argument through the equivalence! We will be using this subtle technique repeatedly throughout the book. To save time and energy, we won't explicitely mention it. We will implicitely pass to an equivalent strict category without making any special note. Sometimes we will want to pass to a strict monoidal category even before string diagrams come into play. Alternatively one can adopt the following simpler policy, with the price of making the usual examples only heuristic:

$$\boxed{\text{We assume monoidal categories are strict whenever it is convenient.}}$$

Example 1.67. To illustrate the workflow proposed in 1.66 we take a closer look at corollary 1.60, where we proved that every strict braided monoidal category $\mathcal{C}$ comes paired with a group homomorphism

$$B_n \to \mathrm{Aut}\left(A^{\otimes n}\right)$$

for all $A \in \mathcal{C}$, $n \geq 1$. Once we generalize strict braided monoidal categories to possibly non-strict braided monoidal categories, this proposition will become false. The object $A^{\otimes n}$ does not exist - a choice of parenthesization needs to be made. Every time that an element of the braid group acts on $A^{\otimes n}$, the parentheses need to be re-arranged using associators, then the braiding map β can be applied, and then the parentheses need to be re-arranged back into their original position using associators again. It is possible to formalize corollary 1.60 for non-strict categories, but it is space-consuming and makes the key insights less clear.

Remark 1.68. To state MacLane's coherence theorem, we need a notion of *equivalence* of monoidal categories. Our notion of equivalence is modeled after the more general notation of equivalence of categories - a pair of functors whose compositions are both naturally isomorphic to the identity. To translate to the present setting, we need a good notion of monoidal functor and monoidal natural transformation so that equivalence can preserve information about monoidal structure.

Definition 1.69 (Monoidal functor)**.** A monoidal functor between monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \alpha_{\mathcal{C}}, \lambda_{\mathcal{C}}, \rho_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \alpha_{\mathcal{D}}, \lambda_{\mathcal{D}}, \rho_{\mathcal{D}}, 1_{\mathcal{D}})$ is the following data:

1. A functor $F : \mathcal{C} \to \mathcal{D}$.
2. A morphism $\epsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$.
3. A natural isomorphism $\mu : F(-) \otimes_{\mathcal{D}} F(-) \xrightarrow{\sim} F(- \otimes_{\mathcal{C}} -)$.

Additionally, a monoidal functor is required to satisfy the following properties:

1. (Associativity) The diagram

$$\begin{tikzcd}
(F(A) \otimes_{\mathcal{D}} F(B)) \otimes_{\mathcal{D}} F(C) \arrow[rr, dashed, "\alpha_{\mathcal{D};F(A),F(B),F(C)}"] \arrow[d, "\mu_{A,B} \otimes \mathrm{id}_{F(C)}"] & & F(A) \otimes_{\mathcal{D}} (F(B) \otimes_{\mathcal{D}} F(C)) \arrow[d, "\mathrm{id}_{F(A)} \otimes \mu_{B,C}"] \\
F(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{D}} F(C) \arrow[d, "\mu_{A \otimes_{\mathcal{C}} B, C}"] & & F(A) \otimes_{\mathcal{D}} F(B \otimes_{\mathcal{C}} C) \arrow[d, "\mu_{A, B \otimes_{\mathcal{C}} C}"] \\
F((A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C) \arrow[rr, "F(\alpha_{\mathcal{C};A,B,C})"] & & F(A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C))
\end{tikzcd}$$

commutes for all $A, B, C \in \mathcal{C}$.

2. (Unitality) The diagrams

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes_{\mathcal{D}} F(A) \arrow[r, "\epsilon \otimes \mathrm{id}_{F(A)}"'] \arrow[d, "\lambda_{\mathcal{C};F(A)}"] & F(1_{\mathcal{C}}) \otimes F(A) \arrow[d, "\mu_{1_{\mathcal{C}},A}"'] \\
F(A) & F(1_{\mathcal{C}} \otimes A) \arrow[l, "F(\lambda_{\mathcal{C};A})"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(A) \otimes_{\mathcal{D}} 1_{\mathcal{D}} \arrow[r, "\mathrm{id}_{F(A)} \otimes \epsilon"'] \arrow[d, "\rho_{\mathcal{C};F(A)}"] & F(A) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \arrow[d, "\mu_{A,1_{\mathcal{C}}}"'] \\
F(A) & F(1_{\mathcal{C}} \otimes A) \arrow[l, "F(\rho_{\mathcal{C};A})"']
\end{tikzcd}$$

commute for all $A \in \mathcal{C}$.

Definition 1.70 (Monoidal natural transformation)**.** A monoidal natural transformation between two monoidal functors (F_0, μ_0, ϵ_0) and (F_1, μ_1, ϵ_1) between monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ is a natural transformation η between the underlying functors F_0, F_1. Additionally, a monoidal natural transformation is required to satisfy the following properties:

1. (Compatibility with tensor product) For all objects $A, B \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
F_0(A) \otimes_{\mathcal{D}} F_1(B) \arrow[r, "\eta_A \otimes \eta_B"] \arrow[d, "\mu_{0;A,B}"] & F_1(A) \otimes_{\mathcal{D}} F_1(B) \arrow[d, "\mu_{1;A,B}"] \\
F_0(A \otimes_{\mathcal{C}} B) \arrow[r, "\eta_{A \otimes B}"] & F_1(A \otimes_{\mathcal{C}} B)
\end{tikzcd}$$

commutes.

2. (Compatibility with unit) The diagram

$$\begin{tikzcd}
& 1_{\mathcal{D}} \arrow[dl, "\epsilon_0"'] \arrow[dr, "\epsilon_1"] & \\
F_0(1_{\mathcal{C}}) \arrow[rr, "\eta_{1_{\mathcal{C}}}"] & & F_1(1_{\mathcal{C}})
\end{tikzcd}$$

commutes.

Definition 1.71. A *monoidal equivalence* between two monoidal categories $\mathcal{C}, \mathcal{D}$ is a pair of monoidal functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ is monoidally naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is monoidally naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$. We say two categories are *monoidally equivalent* if there is a monoidal equivalence between them.

Theorem 1.72 (MacLane's coherence theorem)**.** *Every monoidal category is monoidally equivalent to a strict monoidal category.*

Remark 1.73. MacLane's coherence theorem allows us to enjoy a managable string diagram workflow for monoidal categories. However, as we add more structure onto our categories, it will be a non-trivial task to verify that we can still apply MacLane's coherence theorem. In particular, we will need to strengthen our notion of equivalence to make sure it is strong enough to pass through information about our additional structures. We can see this in the case of braidings already - if we have a braided monoidal category whose associativity is non-strict, will it be equivalent to a braided monoidal category whose associativity is strict? The answer is yes, by proposition 1.77.

Definition 1.74 (Braided monoidal category)**.** A braided monoidal category is the following data:

1. A monoidal category $(\mathcal{C}, \otimes, \alpha, \mathbf{1})$.
2. (Braiding) A natural isomorphism β between the functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ sending (A, B) to $A \otimes B$, and the functor sending (A, B) to $B \otimes A$.

Additionally, a braided monoidal category is required to satisfy the following properties:

1. (Hexagon identities) The diagrams

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathrm{id}_A \otimes \beta_{B,C}"'] & (A \otimes B) \otimes C \arrow[r, "\beta_{A \otimes B, C}"] & C \otimes (A \otimes B) \arrow[d, "\alpha_{B,C,A}"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[r, "\beta_{A,C} \otimes \mathrm{id}_B"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha^{-1}_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathrm{id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\beta_{A, B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha^{-1}_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha^{-1}_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "\mathrm{id}_B \otimes \beta_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

commute for all $A, B, C \in \mathcal{C}$.

Definition 1.75 (Braided monoidal functor)**.** A braided monoidal functor between braided monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \beta_{\mathcal{C}})$, $(\mathcal{D}, \otimes_{\mathcal{D}}, \beta_{\mathcal{D}})$ is a monoidal functor $(F, \mu) : \mathcal{C} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd}
F(A) \otimes_{\mathcal{D}} F(B) \arrow[r, "\beta_{\mathcal{D};F(A),F(B)}"'] \arrow[d, "\mu_{A,B}"] & F(B) \otimes_{\mathcal{D}} F(A) \arrow[d, "\mu_{B,A}"'] \\
F(A \otimes_{\mathcal{C}} B) \arrow[r, "F(\beta_{\mathcal{C};F(A),F(B)})"] & F(B \otimes_{\mathcal{C}} A)
\end{tikzcd}$$

commutes for all $A, B \in \mathcal{C}$.

Remark 1.76. There is no notion of braided monoidal natural transformation - any monoidal natural transformation will automatically repsect the braiding. Hence, we can define two braided monoidal categories to be equivalent if there are braided monoidal functors between them which have compositions which are naturally isomorphic to the identity.

Proposition 1.77 (Braided MacLane coherence theorem)**.** *Every braided monoidal equivalent is equivalent as a braided monoidal category to a strict braided monoidal category.*

Proof. Let $\mathcal{C}$ be a braided monoidal category. Let $\mathcal{C}'$ be a strict monoidal category equivalent to $\mathcal{C}$ (which exists by theorem 1.72), and let $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}$ be monoidal functors inducing the equivalence. Let $\eta : F \circ G \cong \mathrm{id}_{\mathcal{C}}$ be a monoidal natural isomorphism. We define a natural transformation β' on the A, B component by the followiing composition:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\eta^{-1}_{A\otimes B}"] & F(G(A \otimes B)) \arrow[r, "F(\mu^{-1}_{G;A,B})"] & F(G(A) \otimes G(B)) \arrow[d, "F(\beta_{G(A),G(B)})"] \\
B \otimes A & F(G(B \otimes A)) \arrow[l, "\eta_{B\otimes A}"'] & F(G(B) \otimes G(A)) \arrow[l, "F(\mu_{G;B,A})"']
\end{tikzcd}$$

It is a straightforward to show that the axioms of a braided monoidal category on $\mathcal{C}$, the axioms of a monoidal functor on F, G and the axioms of a natural transformation on η imply that β' is the structure of a braiding on $\mathcal{C}'$. Moreover, the monoidal equivalence of categories between $\mathcal{C}$, $\mathcal{C}'$ is a braided monoidal equivalence. $\square$

Remark 1.78. As we go through this text, we will define increasingly more structure on monoidal categories. We will be implicitely assuming theorems which assert that every structured monoidal categories is equivalent to a structure monoidal category whose underlying monoidal category is strict. Importantly, we will assume that this equivalence respects the relevant structure. We will not state these theorems as we go along the way, but they are true and neccecary for our discussion.

1.3.4 Pivotal monoidal categories

So far we have defined a language for putting particles together and braiding them. The next frontier is to introduce a langauge for creating and fusing particles/antiparticles. Categories with a mechanism for creating and fusing particles/antiparticles is known as a *pivotal monoidal category.*

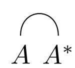
Remark 1.79. In the relevant physical systems, every particle has a dual *antiparticle.* Particle/antiparticle pairs can always spontaneously be created from the vaccuum. Often, particles/antiparticles can annhilate each other to go back to the vaccum. This process of annhilation is subtle however, because a particle/antiparticle pair could also fuse to make a particle which is not the vacuum. We delay the subtleties of fusion to our chapter on modular tensor categories, and focus instead on the abstract underpinnings of antiparticles (which we call *duals*), pair-creation (which we call *coevaluation*) and fusion (which we all *evaluation*).

Definition 1.80 (Right-rigid monoidal category)**.** A right-rigid monoidal category is the following data:

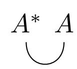
1. A monoidal category $\mathcal{C}$.
2. Objects A^* for all $A \in \mathcal{C}$.
3. Morphisms $\mathrm{ev}_A : A \otimes A^* \to 1$, and $\mathrm{coev}_A : 1 \to A^* \otimes A$ for all $A \in \mathcal{C}$.

Such that $(\mathrm{ev}_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \mathrm{coev}_A) = \mathrm{id}_A$ and $(\mathrm{id}_{A^*} \otimes \mathrm{ev}_A) \circ (\mathrm{coev}_A \otimes \mathrm{id}_{A^*}) = \mathrm{id}_{A^*}$ for all $A \in \mathcal{C}$.

We implement right-rigid monoidal categories in string diagrams as follows. We denote evaluation and coevaluation as follows:

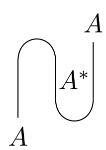


$= \mathrm{ev}_A,$

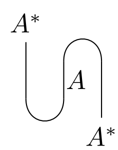


$= \mathrm{coev}_A\,.$

The compatibility conditions are stated graphically as follows:



$= \mathrm{id}_A,$



$= \mathrm{id}_{A^*}$

Definition 1.81 (Left-rigid monoidal category)**.** A left-rigid monoidal category is the following data:

1. A monoidal category $\mathcal{C}$.
2. Objects A^* for all $A \in \mathcal{C}$.
3. Morphisms $\mathrm{ev}_A : A^* \otimes A \to 1$, and $\mathrm{coev}_A : 1 \to A \otimes A^*$ for all $A \in \mathcal{C}$.

Additionally, a rigid category is required to satisfy the property that $(\mathrm{id}_A \otimes \mathrm{ev}_A) \circ (\mathrm{coev}_A \otimes \mathrm{id}_A) = \mathrm{id}_A$ and $(\mathrm{ev}_A \otimes \mathrm{id}_{A^*}) \circ (\mathrm{id}_{A^*} \otimes \mathrm{coev}_A) = \mathrm{id}_{A^*}$ for all $A \in \mathcal{C}$.

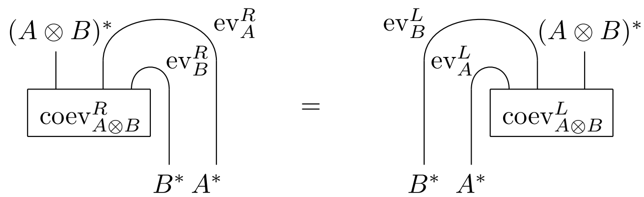
Remark 1.82. We want to discuss categories which have a full theory of particles/antiparticles. This means that they should be able to create particle/antiparticle pairs on both sides, leading to a left-rigid and right-rigid structure on $\mathcal{C}$. As per usual, there should be some compatibility conditions between these two rigid structures.

Definition 1.83 (Pivotal monoidal category)**.** A pivotal monoidal category is the following data:

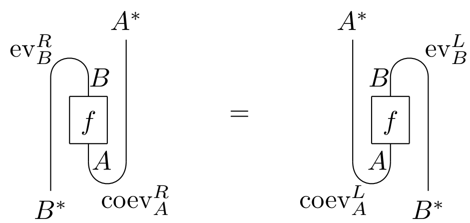
1. A monoidal category $\mathcal{C}$;
2. A right-rigid structure $(\mathrm{ev}^R, \mathrm{coev}^R)$ on $\mathcal{C}$;
3. A left-rigid structure $(\mathrm{ev}^L, \mathrm{coev}^L)$ on $\mathcal{C}$.

Such that:

1. The right-duals and left-duals of all objects are equal;
2. For all $A, B \in \mathcal{C}$, we have an equality of morphisms $B^* \otimes A^* \to (A \otimes B)^*$,



3. For all $A, B \in \mathcal{C}$ and $f : A \to B$,

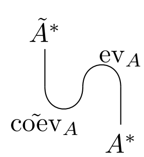


Remark 1.84. The first thing to observe is that even though there is a lot of structure involved in the definition of a rigid monoidal category, most of it is in a real sense innessential. Proposition 1.85 tells us we could have chosen different duals and the result would have been essentially the same, or in other words, duals are unique up to unique isomorphism.

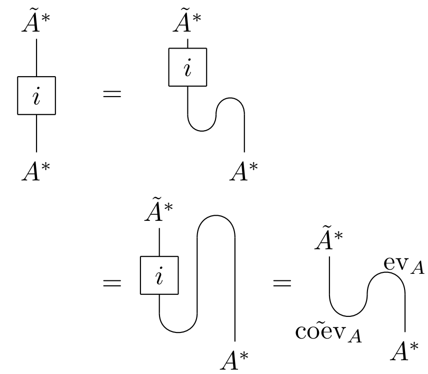
Proposition 1.85. *Let $\mathcal{C}$ be right (resp. left) rigid monoidal category. Let $A \in \mathcal{C}$ be an object, and let $(\tilde{A}^*, \tilde{\mathrm{ev}}_A, \tilde{\mathrm{coev}}_A)$ be another triple satisfying the axioms of rigidity. There is a unique morphism $i : A^* \to \tilde{A}^*$ making the diagram*

$$\begin{tikzcd}
& A^* \otimes A \arrow[dd, "\sim"] \\
1 \arrow[ur, "\mathrm{coev}_A"] \arrow[dr, "\tilde{\mathrm{coev}}_A"'] & \\
& A \otimes \tilde{A}^*
\end{tikzcd}$$

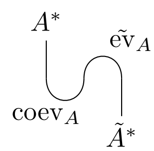
commute (resp. reverse order of tensor factors). This unique morphism is an isomorphism, and it is given by

$i =$ 

Proof. By the computation



we find that i is unique, and it has the desired formula. To prove that i is an isomorphism we observe that the map



serves as an inverse. This gives a proof of the result. $\square$

Remark 1.86. Seeing as we will be working with rigid and pivotal categories, it behooves us to make sure that we have the correct notion of functors and natural transformations between these categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between pivotal categories. Given an object $A \in \mathcal{C}$, the evaluation and coevaluation maps naturally extend through the functor to endow $F(A^*)$ with the structure of a dual for A. Thus, by proposition 1.85, we have a canonical isomorphism $F(A^*) \cong F(A)^*$. This isomorphism exists without needing to add any extra conditions on F. In this way, the correct notion of functor between right/left rigid categories is just functor! There is, however, extra an compatibility condition needed for pivotal category. Both the left-rigid *and* right-rigid structures induce isomorphisms $F(A^*) \cong F(A)^*$. These induced isomorphisms should be the same. A functor between pivotal categories with this property is known as a *pivotal functor*.

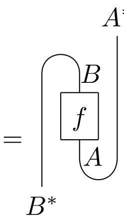
Definition 1.87. Define a monoidal category $\overline{\mathcal{C}}$ as follows. The underlying category on $\overline{\mathcal{C}}$ is the opposite category for $\mathcal{C}$. The tensor product is given by $A \overline{\otimes} B = B \otimes A$, and the monoidal unit is $\mathbf{1} \in \mathcal{C}$. WORK: I'm not sure if I like this definition, notation, or location.

Remark 1.88. An important feature of rigid categories is that duality is automatically *functorial*, by proposition 1.89. This perspective can be used to motivate the axioms of a pivotal category. By proposition 1.89 both the right and left rigid structures in a pivotal category induce functors $\mathcal{C} \to \overline{\mathcal{C}}$. The coherence condition is that these two functors should be equal.

Proposition 1.89. *Let $\mathcal{C}$ be right (resp. left) rigid monoidal category.*

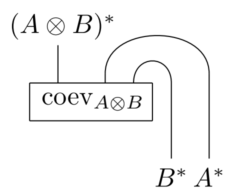
(i) The right (resp.left) rigid structure on $\mathcal{C}$ induces a left (resp. right) rigid structure on $\overline{\mathcal{C}}$;

(ii) Given any morphism $f : A \to B$ in $\mathcal{C}$, define

$f^* =$ 

to be the dual for f (resp. same diagram using left rigidity). The assignment $A \mapsto A^*$, $f \mapsto f^*$ induces a functor from $\mathcal{C}$ to $\overline{\mathcal{C}}$ which we denote $(-)^*$.

(iii) Given any objects $A, B \in \mathcal{C}$, define the map



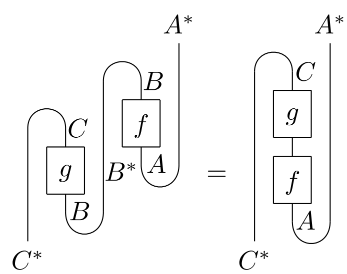
from $B^* \otimes A^*$ to $(A \otimes B)^*$ (resp. same diagram using left rigidity). These maps endows $(-)^*$ with the structure of a monoidal functor.

(iv) The functor $(-)^*$ is fully faithful. If $\mathcal{C}$ is a pivotal category, then then functor above is an equivalence of monoidal categories between $\mathcal{C}$ and $\overline{\mathcal{C}}$.

Proof. We do only the proofs for right-rigid categories. The left-rigid proof is identitical.

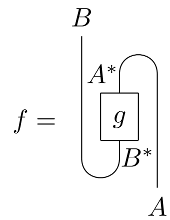
(i) This follows immediately from the definitions;

(ii) Functoriality is the condition that $(f \circ g)^* = g^* \circ f^*$. The follows from the following argument:

$g^* \circ f^* =$  $= (f \circ g)^*$

(iii) This is an unenlightening and straightforward computation;

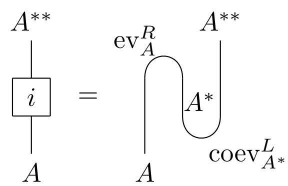
(iv) We first prove that $(-)^*$ is fully faithful. Given any objects $A, B \in \mathcal{C}$ and any morphism $g : B^* \to A^*$, the morphism



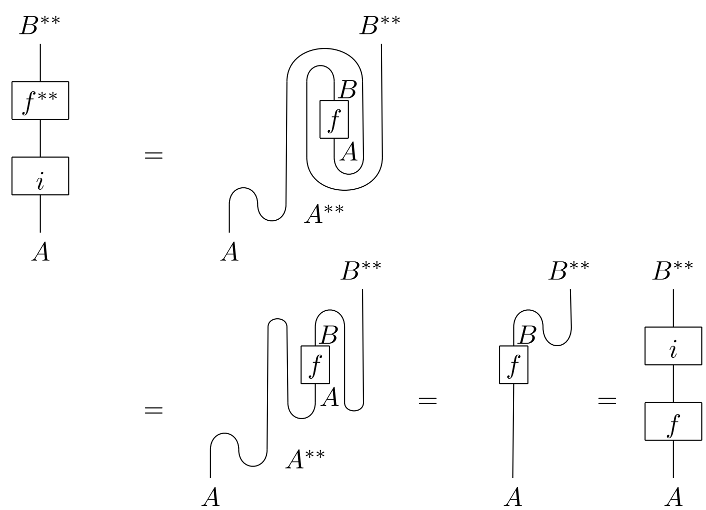
has the property that $f^* = g$. Hence, $(-)^*$ is bijective on hom-spaces as desired.

We now show that $(-)^*$ is an equivalence of categories with $\mathcal{C}$ is pivotal. By part (i), $\overline{\mathcal{C}}$ is a pivotal monoidal category. Hence duality once again induces a monoidal functor, this time $\overline{\mathcal{C}} \to \overline{\overline{\mathcal{C}}}$. Clearly, by our definition of $\overline{\mathcal{C}}$, $\overline{\overline{\mathcal{C}}} = \mathcal{C}$. Hence we have a pair of functors $\mathcal{C} \to \overline{\mathcal{C}}$ and $\overline{\mathcal{C}} \to \mathcal{C}$, each given by duality. Proving this proposition hence amounts to showing that the double dual map is monoidally naturally isomorphic to the identity.

To do this, we define a natural isomorphism explicitely by the isomorphisms $i : A \xrightarrow{\sim} A^{**}$



for all $A \in \mathcal{C}$. To show that these morphisms induce a natural transformation, we observe that for all $f : A \to B$

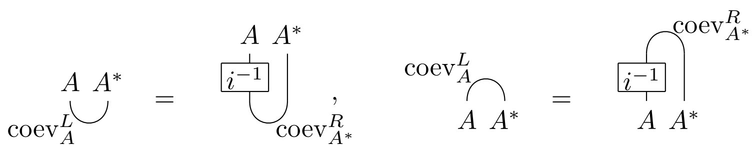


The fact that $\mathcal{C}$ is compatible with the tensor product is a straightforward computation, using the fact that computing the tensor product using right-rigidity and left-rigidity gives the same answer, and compatibility of $\mathcal{C}$ with the unit is immediate.

□

Remark 1.90. As a key part of proposition 1.89, we showed that every pivotal structure on a right-rigid monoidal category induces a natural isomorphism between the identity functor and the double dual functor. This gives an alternate description of pivotal categories as rigid categories equipped with isomorphisms between the identity functor and the double dual functor. This is stated precisely in corollary 1.91.

Corollary 1.91. *Let $\mathcal{C}$ be a right-rigid monoidal category. Let $i : \mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} (-)^{**}$ be a monoidal natural isomorphisms between the identity functor and the double dual functor. The maps*



induce a pivotal structure on $\mathcal{C}$. Moreover, this assignement induces a bijection between pivotal structures on $\mathcal{C}$ and monoidal natural isomorphisms $\mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} (\mathrm{id}_{\mathcal{C}})^{**}$.

Proof. Proving that the maps provided satisfy the axioms of a left-rigid structure is immediate. Proving that they satisfy the axioms of a pivotal structure comes from running the arguments in the proof of proposition 1.89 in reverse. The operations of inducing a monoidal natural isomorphism from a pivotal structure and inducing a pivotal structure from a monoidal natural isomorphism are inverses of one another. Hence, they induce a bijection between the two types of structures as desired. □

History and further reading:

WORK: an early reference for string diagrams is [Mou84]. Maybe I should cite it?
Category theory was first introduced and formalized by Saunders Mac Lane and Samuel Eilenberg in 1945 [EM45]. Of course, the ideas underlying category theory were present earlier and can be traced back arbitrarily far. In the subsequent decades the formalism of category theory spread far and wide, bringing with it the discovery of many deep theorems. The first major explicit appearance of category theory in physics was Vladimir Drinfeld's work on so-called *quantum groups* in the early 1980s [Dri86]. Quantum groups are certain kinds of mathematical objects righly related to content in this book. They were introduced as tools to help generate exactly-solvable models in condensed matter physics. Very quickly quantum groups were absorbed into the theory of the ideas of string theory of topological quantum field theory, which were both new at the time [BPZ84, Wit88]. The physics in this area has since become and remained extremely categorical in nature [Lur08, BDSPV15].

There are many excellent introductory texts to category theory. Some authors find it fruitful to reformulate all of quantum mechanics, and especially quantum information, in terms of category theory. A good source outlining this approach and introducing category theory through it is Coecke-Kissinger's textbook [CK18]. The Kong-Zhang textbook [KZ22] gives an introduction to category theory in the context of topological order. A good general-purpose textbook on category theory is Fong-Spivak [FS19], and a classical but slightly dated reference is [ML13].

Exercises:

1.1. WORK: If $\mathcal{C}$ is a category with products, then the product forms a monoidal structure (with a good $\mathbf{1}$ given of course), and same for coproducts.

1.2. WORK: Show that endofunctors form a *strict* monoidal category.

1.3. WORK: Add an exercise giving some compatibility conditions between monoidal/rigid structures and direct sums. Namely, they distribute nicely.

WORK: I'm not consistent about up/down orientation for my string diagrams. I need to go through and fix this.

2 Modular categories

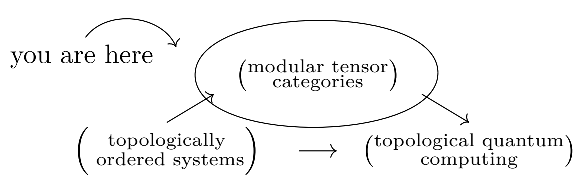
WORK: I am using nonstandard language. I am using the term “modular category” to refer to what most authors call “modular tensor categories”. I do this because modular category is shorter than modular tensor category and I want to be succinct. For many people (notably Zhenghan) modular category means potentially non-semsimple modular category. For instance, Turaev’s original definition did not include the simisimplicity axiom and he called them modular categories. The term tensor is used to highlight semisimplicity by many authors, so I can highlight semisimplicity with a footnote.

WORK: There’s an issue in this treatment. There is one piece of data beyond the scope of an modular categories - the chiral central charge. This is a remnant of the fact that the bulk-to-boundary correspondane is not exact because the boundary can have stacked E_8 phases, see [Bon21]. Of course this is not something to dive into now. However, I want to be maximally honest - point out that there is a unique piece of topologically invariant information beyond the scope of modular categories. Maybe include somewhere (as an exercise?) the treatment of chiral central charge mod 8? I think this would make for a good footnote. I’m realizing now that the fact that the chiral central charge is a root of unity is part of Vafa’s theorem. So, it makes the most sense for this to be in the number theory section.

2.1 Overview

2.1.1 Introduction

In this chapter we will be giving a detailed analysis of modular categories, the abstract algebraic structures used to describe anyons in topological order. We recall below how this fits into the general framework of this book:



Describing exactly what an anyon is and how it can transform in terms of states and unitary operators on a Hilbert space can be difficult. However, describing abstractly how these transformations compose with one another can be done realtively simply. Hence we take a composition-first category-theoretic approach to anyons. We will make heavy use of the diagramatic language of braided monoidal categories established in chapter 1. Concretely, we will think of a modular category as being the category with the following data:

$$\begin{pmatrix} \textbf{objects}\text{: finite collections of anyons} \\ \textbf{morphisms}\text{: motions/behaviors of anyons} \end{pmatrix}$$

WORK: In a sense the above picture is not correct. Suppose I give you the object $2A \oplus B$. What “finite collection of anyons” does this correspond to? None! It is more like a composite object which has two fusion channels to A and one fusion channel to B. You could also maybe think of it as a probability distribution over A and B.

We have the following general pictuture for our algebraic theory:

Physics-math dictionary 2.1. Topological phases of matter are algebraically described by unitary modular categories $\mathcal{C}$ called the *anyon theory* of the phase, and a choice of integer $c_- \in \mathbb{Z}$ called the *chiral central charge* of the phase.

Remark 2.2. Throughout these notes, there are some aspects of the general dictionary 2.1 that we have not emphasized. For instance, we have not emphasized that the modular categories describing topological phases are supposed to be *unitary* modular categories - we will discuss this later, but by analagy we can say that a unitary modular category is related to a modular category just like a Hilbert space is related to a vector space.

Another aspect we have not emphasized is the *chiral central charge*. This is a real invariant of topological phases which is beyond the description of modular categories. In particular, there are topological phases with no non-trivial anyons (and hence correspond to the trivial modular category) yet are non-trivial as topological phases. These are called *invertible* topological phases. Invertible topological phases are paramaterized by an integer invariant, their chiral central charge. Not every pair $(\mathcal{C}, c_-)$ describes a valid topological phase. In particular, $\mathcal{C}$ determines c_- modulo 8 but there is no other condition. The way that $\mathcal{C}$ determins c_- modulo 8 is known as the Gauss-Milgram formula and is discussed in subsection 2.5.6.

Remark 2.3. The major structures of a modular tensor category can be motivated by considering abstractly the possible motions and behaviors of anyons. The most basic thing anyons can do is move anyons around each other - this is known as *braiding*. If the anyons touch each other then they can congeal into a single anyon - this is known as *fusion*. Even if there are no anyons in a system, however, there is always something possible. Anyons can be spontaneously created, so long as every anyon which is created comes along with its corresponding antiparticle. This is known as *pair-creation*. These three operations are the fundamental structures which we will building into modular categories:

1. braiding;

2. fusion;

3. pair-creation.

Remark 2.4. Another potentially useful way of thinking about modular categories comes from analogy with classical physics. We saw in chapter **??** that topological classical systems have an algebraic description in terms of finite groups. Namely, quasiparticles in the system of ordered media with order space M is algebraically characterized by the fundamental group $\pi_1(M, m)$ of M relative to some basepoint $m \in M$. Seeing as topological order is a vast quantum generalization of classical ordered media, we can think of modular categories as being a vast quantum generalization of finite groups. Every finite group G induces a modular category $\mathfrak{D}(G)$, by first constructing the Kitaev quantum double model based on that finite group and then describing its anyons. Many modular categories do not come from the group-theoretical construction.

2.1.2 Using the final product

Before developping the theory of modular categories, it is good to get a feel for what using the final product is like. A modular category itself will be a big infinite structure, with infinitely many objects and infinitely many morphisms between those objects. However, all modular categories are in a real sense *finitely generated*. What we mean by this is that plugging in a finite number of objects and morphisms, the rest of the obejcts and morphisms can be recovered by the abstract rules encoded in the formalism. In this way, the axioms of modular categories are not only neccecary by the fact that they restrict which categories can be modular categories, but they are also vital in the fact that they allow us to generate a full description of anyons from a minimal collection of data. For practically-minded readers, this can be viewed as the main motivation for putting so much work into defining modular categories..

Example 2.5. To highlight how the formalism used to define an object impacts its finitely-generated description, we take an example from group theory. Consider the 3-strand braid group B_3. This group has ininitely many elements and the group operation $B_3 \times B_3 \to B_3$ naively takes an infinite amount of data to describe. However, the presentation

$$B_3 = \langle \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

gives completely finite description of B_3. It is important to note, however, that this presentation would *not* have been enough to recover B_3 if we had just been told that B_3 is a monoid. The fact that B_3 is a group implied the existence of elements σ_1^{-1}, σ_2^{-1}, and defined how they interacted with σ_1, σ_2. We see in this way that the axioms of a group not only serve as a restriction on what mathematical objects are allowed to be groups, but they also serve as a compression technique. They give the rules by which a minimal collection of data can be used to generate the rest.

An important step in going from modular categories to their description in terms of a finite set of data is in comming up with an efficient standard way of descrbing morphisms in a modular category. This is done using skeletonization, as discussed in section 2.6. A large table of these descriptions are found in appendix **??**. We now give a worked example of how this data is used to compute observable quantities.

Example 2.6. WORK: add toric code modular category data

Example 2.7. Or, for a more complicated example, we can consider the data for $G = S_3$: WORK: add $G = S_3$ modular category data

Example 2.8. WORK: add good example, computing some probability of annhilation of two anyons in $G = S_3$.

2.2 First properties

2.2.1 Definition

In this section we define modular categories, which are the main mathematical subject of this book. Seeing as lots of data is involved, we spread out the definition over a series of steps as to not overload the senses. These intermediate definitions are also important in their own right, because they will be used in other places in the algebraic theory of topological phases.

Definition 2.9 (Fusion category)**.** A fusion category is the following data:

1. A category $\mathcal{C}$;
2. The structure of a right-rigid monoidal category on $\mathcal{C}$;

3. The structure of a $\mathbb{C}$-linear category on $\mathcal{C}$.

Such that:

1. The tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces bilinear maps hom-spaces;
2. There is an equivalence $\mathcal{C} \cong \mathbf{Vec}^n_{\mathbb{C}}$ as $\mathbb{C}$-linear categories;
3. $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \cong \mathbb{C}$ as $\mathbb{C}$-vector spaces

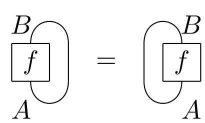
A fusion category is part of the way towards having all of the requisite structures of a modular category: it has a method for fusion inherited from the tensor product, and it has half of a method for pair-creation coming from right-rigidity. The $\mathbb{C}$-linearity allows us to think of hom-spaces as vector spaces, which allows us to treat hom-spaces as quantum systems. The condition (1) is a comptability between the $\mathbb{C}$-linear structure and the monoidal structure. The conditions (2)-(3) are strong niceness and finiteness conditions - we will explain them in detail later.

Definition 2.10 (Spherical fusion category)**.** A spherical fusion category is the following data:

1. A fusion category $\mathcal{C}$;
2. A left-rigid structure on $\mathcal{C}$.

Such that:

1. The left-rigid and right-rigid structures on $\mathcal{C}$ satisfy the axioms of a pivotal structure on $\mathcal{C}$;
2. For every object $A \in \mathcal{C}$ and for every morphism $f : A \to A$, we have



A spherical fusion category now has a structure for fusion, and a full structure for pair-creation. The 2nd condition is known as the *spherical axiom.* We will explain this axiom in more detail later.

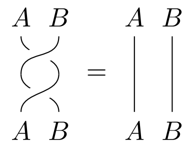
Definition 2.11 (Pre-modular category)**.** A pre-modular category is the following data:

1. A spherical fusion category $\mathcal{C}$;
2. A braided structure on $\mathcal{C}$.

No extra compatibility conditions are required.

A pre-modular category has all of the structure we require of a modular category: fusion, pair-creation, and braiding. The final axiom it is missing is a non-degeneracy condition. The non-degeneracy condition is subtle in its interpretation, and we will explain it several different ways throughout this chapter.

Definition 2.12 (Modular category)**.** A modular category is a pre-modular category satisfying the following condition. Let $A \in \mathcal{C}$ be an object. If



for all $B \in \mathcal{C}$, then $A \cong \mathbf{1}$.

2.2.2 Anyons in modular categories

Modular categories are supposed to be theories of anyons in topological order. So, now that we have the definition of modular category, it is natural to ask: what do anyons mathematically correspond to, in modular categories? The answer lies within the condition in a fusion category $\mathcal{C}$ that there is an equivalence $\mathcal{C} \cong \mathbf{Vec}_{\mathbb{C}}^n$ as $\mathbb{C}$-linear categories. We explore the importance of this condition.

Remark 2.13. Suppose we are given an object $V = (V_1, V_2 ... V_n) \in \mathbf{Vec}_{\mathbb{C}}^n$. For all $1 \leq i \leq n$, let $\mathbb{C}_i \in \mathbf{Vec}_{\mathbb{C}}^n$ denote the object which has dimension zero in every index $j \neq i$ and is equal to $\mathbb{C}$ in index i. We observe the isomorphism

$$\begin{aligned} V &\cong \bigoplus_{i=1}^{n} (0 ... V_i ... 0) \\ &\cong \bigoplus_{i=1}^{n} (0 ... \mathbb{C}^{\dim(V_i)} .. 0) \\ &\cong \bigoplus_{i=1}^{n} \dim(V_i) \cdot \mathbb{C}_i \end{aligned}$$

where $\dim(V_i) \cdot (\mathbb{C}_i) = \mathbb{C}_i \oplus \mathbb{C}_i ... \oplus \mathbb{C}_i$, $\dim(V_i)$ many times. This computation shows that any object in $\mathbf{Vec}_{\mathbb{C}}^n$ can be decomposed into irriducible components $\mathbb{C}_i$. These objects $\mathbb{C}_i$ are in a real sense the building blocks of $\mathbf{Vec}_{\mathbb{C}}^n$. In the language of definition 2.14, the objects $\mathcal{C}_i$ are the simple objects of $\mathbf{Vec}_{\mathbb{C}}^n$.

Definition 2.14. A *simple object* A in a fusion category $\mathcal{C}$ is an object which has no direct sum decomposition into smaller objects. That is, $A \ncong B \oplus C$ for any non-zero objects $B, C \in \mathcal{C}$ where $\oplus$ denotes the biproduct in $\mathcal{C}$.

Physics-math dictionary 2.15. Isomorphism classes of simple objects in $\mathcal{C}$ correspond to anyon types in a topological phase described by the MTC $\mathcal{C}$.

Definition 2.16. For all fusion categories $\mathcal{C}$, we define $\mathcal{L}(\mathcal{C})$ (or simple $\mathcal{L}$ when $\mathcal{C}$ is clear from context) to the the set of isomorphism classes of simple objects. By entry 2.15 of the physics-math direction, elements of $\mathcal{L}(\mathcal{C})$ correspond bijectively to anyon types in any topological phase described by $\mathcal{C}$.

Proposition 2.17. *Let $\mathcal{C}$ be a fusion category. The biproduct of any two elements in $\mathcal{C}$ exists, and $\mathcal{C}$ has a zero object. The set $\mathcal{L} = \mathcal{L}(\mathcal{C})$ is finite. Choose an object $X \in \mathcal{C}$. There exist unique nonnegative integers $c_{[A]}$, $[A] \in \mathcal{L}$ such that*

$$X \cong \bigoplus_{[A]\in\mathcal{L}} N_{[A]} \cdot A.$$

Proof. We first show that $\mathcal{C}$ has biproducts. Let $F : \mathcal{C} \to \mathbf{Vec}_{\mathbb{C}}^n$, $G : \mathbf{Vec}_{\mathbb{C}}^n \to \mathcal{C}$ be a pair of functors which induces an equivalence of categories, for some $n \geq 1$ Let $A, B \in \mathcal{C}$ be objects. Since G and F are fully faithfull, the universal property of the direct sum $F(A) \oplus F(B)$ guarantees that $G(F(A) \oplus F(B))$ will be a direct sum of $G(F(A))$ and $G(F(B))$. Since $G(F(A)) \cong A$ and $G(F(B)) \cong B$, we conclude that $\mathcal{C}$ has biproducts. The object $G(0)$ is a zero object for $\mathcal{C}$.

We now prove that $\mathcal{C}$ has finitely many isomorphism classes of simple objects. It is clear that an object $A \in \mathcal{C}$ is simple if and only fi $F(A) \in \mathbf{Vec}_{\mathbb{C}}^n$ is simple. Thus, since G serves as an inverse, F establishes a bijection between isomorphism classes of simple obejcts in $\mathcal{C}$ and isomorphism classes of simple objects in $\mathbf{Vec}_{\mathbb{C}}^n$. Every simple obejct in $\mathbf{Vec}_{\mathbb{C}}^n$ will be isomorphic to $\mathbb{C}_i$ for some $1 \leq i \leq n$. Hence, there are n simple objects in $\mathbf{Vec}_{\mathbb{C}}^n$. Hence, there are n simple objects in $\mathcal{C}$, which is finite. The unique direct sum decomposition is clearly true in $\mathbf{Vec}_{\mathbb{C}}^n$. It is immediate that it passes to a unique direct sum decomposition in $\mathcal{C}$. $\square$

Proposition 2.18 (Schur's Lemma)**.** *Let $\mathcal{C}$ be a fusion category. An object $A \in \mathcal{C}$ is simple if and only if its endomorphism ring* $\mathrm{End}(A)$ *is one-dimensional. Additionally, if $A, B \in \mathcal{C}$ are nonisomorphic simple objects then* $\mathrm{Hom}(A, B) = 0$.

Proof. Let $F : \mathcal{C} \to \mathbf{Vec}_{\mathbb{C}}^n$, $G : \mathbf{Vec}_{\mathbb{C}}^n \to \mathcal{C}$ be a pair of $\mathbb{C}$-linear functors which establishes an equivalence between $\mathcal{C}$ and $\mathbf{Vec}_{\mathbb{C}}^n$ as $\mathbb{C}$-linear categories. The simple objects in $\mathbf{Vec}_{\mathbb{C}}^n$ are all isomorphic to $\mathbb{C}_i$ for some $1 \leq i \leq n$. We compute that

$$\dim\left(\mathrm{Hom}_{\mathbf{Vec}_{\mathbb{C}}^n}\left(\bigoplus_{i=1}^n n_i\mathbb{C}_i, \bigoplus_{i=1}^n m_i\mathbb{C}_i\right)\right) = \sum_{i=1}^n n_i m_i.$$

As a corrolary of this formula, we find that if $A = \bigoplus_{i=1}^n n_i\mathbb{C}_i$ then $\dim(\mathrm{End}_{\mathbf{Vec}_{\mathbb{C}}^n}(A)) = \sum_{i=1}^n n_i^2$. Clearly, this dimension is equal to one if $A = \mathbb{C}_i$ for some $1 \leq i \leq n$, and is greater than one otherwise. As a second corrolary, we compute that $\mathrm{Hom}(\mathbb{C}_i, \mathbb{C}_j) = 0$ whenever $i \neq j$. The functor G induces a bijection between isomorphism classes of simple objects in $\mathbf{Vec}_{\mathbb{C}}^n$ and isomorphism classes of simple objects in $\mathbf{Vec}_{\mathbb{C}}^n$, and it induces vector space isomorphisms on hom spaces. This means that the results for $\mathbf{Vec}_{\mathbb{C}}^n$ translate to the desired result on $\mathcal{C}$. $\square$

Remark 2.19. Schur's lemma is a first verification that simple objects are a good choice of mathematical characterization of anyons. If A, B are distinct anyon types, then there should not be any physical process which goes from one to another. There is no physical mechanism for locally turning one anyon type into another. This is captured by the formula $\mathrm{Hom}(A, B) = 0$. Similarly, given an anyon A, there is no nontrivial action that can be locally performed on A. This comes from the fact that information is topologically protected, and thus cannot be changed by acting on a single particle - topological information processing requires global braiding between multiple particles. This is encoded in the fact that $\mathrm{Hom}(A, A) \cong \mathbb{C}$ is one dimensional and hence consists only of trivial phase gates.

Remark 2.20. As an application of Schur's lemma, we can observe that the monoidal unit $\mathbf{1}$ is a simple object in every fusion category. By our physics-math dictionary, this means that $\mathbf{1}$ corresponds to an anyon type. This anyon $\mathbf{1}$ is the trivial "no-anyon" type, corresponding to a quasiparticle which happens to be the same as the homogenous bulk.

Physics-math dictionary 2.21. The monodial unit $\mathbf{1}$ corresponds to the trivial anyon type, which is the same as the homogenous bulk (sometimes called the *vaccuum* anyon type).

Remark 2.22. Continuing in our expansion of the physics-math dictionary, we can assert that dual objects (defined using the right-rigid structure on a fusion category) correspond to antiparticles (entry 2.23). This is consistent with the fact that anyon types correspond to simple objects, by proposition 2.24.

Physics-math dictionary 2.23. For every simple object A in a modular category $\mathcal{C}$, the dual object A^* corresponds to the *antiparticle* of A. Expanding our physics-math dictionary, we say that for every anyon A its *antiparticle* is the dual A^* which comes from right-rigidity. This gives a valid anyon type by proposition 2.24.

Proposition 2.24. *Let $\mathcal{C}$ be a fusion category. If $A \in \mathcal{C}$ is a simple object, then so is A^*.*

Proof. By proposition 1.89 duality induces a bijection on hom-spaces. Since composition is bilinear, duality is thus an isomorphism of vector spaces on all hom-spaces. Hence, for all $A \in \mathcal{C}$ there is an isomorphism $\mathrm{Hom}(A, A) \cong \mathrm{Hom}(A^*, A^*)$. So, $\dim \mathrm{Hom}(A, A) = 1$ if and only if $\dim \mathrm{Hom}(A^*, A^*) = 1$, and thus the result follows from Schur's lemma. ☐

Physics-math dictionary 2.25. The tensor product $\otimes$ physically corresponds to joining anyons, forming a composite anyon configuration. That is, the object $A \otimes B$ corresponds to the configuration with one A-type anyon and one B-type anyon.

Remark 2.26. By entry 2.25 of the physics-math dictionary and proposition 2.17, it behooves us to take a moment to interpret the direct sum decomposition

$$A \otimes B \cong \bigoplus_{[C] \in \mathcal{L}} N_C^{A,B} \cdot C, \tag{2.1}$$

where $A, B \in \mathcal{C}$ are objects in a fusion category. We call the non-negative integers $N_C^{A,B}$ the *fusion coefficients* of $\mathcal{C}$. We observe that

$$\dim \mathrm{Hom}(A \otimes B, C) = N_C^{A,B}. \tag{2.2}$$

Morphisms in $\mathcal{C}$ correspond to physical processes. So, equation 2.2 can be interpreted as saying that whenever $N_C^{A,B} > 0$, there is a physical process $A \otimes B \to C$. That is, there is a physical process which goes from the composite system of one A-type anyon and one B-type anyon and ouputs one C-type anyon. Neccecarily, such a process can be interpreted as *fusion* of A and B to C. Moreover, equation 2.2 tells us that the space of possible physical processes which fuse A and B to C (known as *fusion channels* $A \otimes B \to C$) is $N_C^{A,B}$-dimensional.

2.2.3 States in modular categories and unitarity

WORK: This whole section only discusses *pure* states. For a good discussion of mixed states and entropy, see [BKP17]

It is now worth reflecting on what exactly states correspond to in modular categories. Certainly, objects in modular categories are *not* quantum systems. They don't have vector space structure. Objects correspond to anyon configurations, which are classical observables. The hom-spaces are what have vector space structure, by $\mathbb{C}$-linearity. However, they are not yet Hilbert spaces. It is exactly for this reason we need to define *unitary modular categories*. These are modular categories in which hom-spaces have Hilbert space structure.

Definition 2.27 (Unitary fusion category)**.** A unitary fusion category is the following data:

1. An fusion category $\mathcal{C}$.
2. (Conjugation) A linear map $\dagger : \mathrm{Hom}(A, B) \to \mathrm{Hom}(B, A)$ for all $A, B \in \mathcal{C}$.

Such that:

1. (Unitarity) Given $f : A \to A$ an endomorphism of $A \in \mathcal{C}$, define

$$\mathrm{tr}(f) = \mathrm{ev}_A \circ (\mathrm{id}_{A^*} \otimes f) \circ (\mathrm{ev}_A)^\dagger .$$

The map $\langle \cdot | \cdot \rangle : \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, B) \to \mathbb{C}$ defined by $\langle f|g\rangle = \mathrm{tr}(f^\dagger \circ g)$ is an inner product, endowing $\mathrm{Hom}(A, B)$ with the structure of a Hilbert space.

2. $\left(f^\dagger\right)^\dagger = f$ for all $f \in \mathrm{Hom}(A, B)$, $A, B \in \mathcal{C}$.
3. $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$ for all $f \in \mathrm{Hom}(B, C)$,$g \in \mathrm{Hom}(A, B)$, $A, B, C \in \mathcal{C}$.
4. $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for all $f \in \mathrm{Hom}(A, B)$,$g \in \mathrm{Hom}(C, D)$, $A, B, C, D \in \mathcal{C}$.
5. $(\mathrm{coev}_A)^\dagger \circ (f \otimes \mathrm{id}_{A^*}) \circ \mathrm{coev}_A = \mathrm{tr}(f)$ for all $A \in \mathcal{C}$

Remark 2.28. Unitary fusion categories make for a pleasant object of study because by proposition 2.29 the distinguished maps $(\mathrm{ev}_A)^\dagger : 1 \to A^* \otimes A$ and $(\mathrm{coev}_A)^\dagger : A \otimes A^* \to 1$ induce a pivotal structure.

Proposition 2.29. *Let $\mathcal{C}$ be a unitary fusion category. The maps $\mathrm{ev}^L_A = (\mathrm{coev}_A)^\dagger$ and $\mathrm{coev}^L_A = (\mathrm{ev}_A)^\dagger$ give a left-rigid structure on $\mathcal{C}$. This left-rigidity endows $\mathcal{C}$ with the structure of a spherical fusion category.*

Proof. WORK: This is actually subtler than I expected. Either give a proof here, or postpone it to the unitarity section ☐

Definition 2.30 (Unitary pre-modular category)**.** A unitary modular category is the following data:

1. A modular category $\mathcal{C}$;
2. (Conjugation) A linear map $\dagger : \mathrm{Hom}(A, B) \to \mathrm{Hom}(B, A)$ for all $A, B \in \mathcal{C}$.

Such that:

1. Forgetting the left-rigid structure and braiding, $(\mathcal{C}, \dagger)$ forms a unitary fusion category.
2. $\left(\mathrm{ev}^R_A\right)^\dagger = \mathrm{coev}^L_A$;
3. $\left(\mathrm{coev}^R_A\right)^\dagger = \mathrm{ev}^L_A$;
4. $(\beta_{A,B})^\dagger = \beta^{-1}_{B,A}$.

Definition 2.31 (Unitary modular category)**.** A unitary modular category is a unitary pre-modular category which satisfies the non-degeneracy axiom of a modular category.

Remark 2.32. The compatibility conditions for the twist are chosen so that the definition of trace as a modular category and the definition of trace as a unitary fusion category coincide.

Physics-math dictionary 2.33. For any phase whose anyons as described by a unitary modular category $\mathcal{C}$.

$$\begin{pmatrix} \text{states of topological order } \mathcal{C} \\ \text{on the sphere } S^2 \\ \text{with anyon configuraiton } A_1, A_2...A_n \end{pmatrix} = \begin{pmatrix} \text{normalized vectors in the Hilbert space} \\ \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A_1 \otimes A_2 ... \otimes A_n) \end{pmatrix} \tag{2.3}$$

where by "anyon configuration $A_1, A_2...A_n$" we mean that the state has anyons present in n sites, arranged left to right on a line segment in the sphere, with corresponding anyon type $A_1, A_2...A_n$.

Remark 2.34. It is not immediately clear where in equation 2.3 we chose the sphere as the physical space. As one source of motivation, we can observe the following corollary of equation 2.3:

$$\dim \begin{pmatrix} \text{Hilbert space of topological order } \mathcal{C} \\ \text{on the sphere } S^2 \\ \text{with exactly one anyon of type } A \end{pmatrix} = \dim \mathrm{Hom}(\mathbf{1}, A) = \begin{cases} 1 & A = \mathbf{1} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, equation 2.3 tells us that if the sphere has exactly one anyon on it then that anyon type must be trivial. Moreover, there is unique state on the sphere with no anyons. This is consistent with our general principles about topological order on the sphere.

WORK: sketch nice argument for why there is a unique ground state on the sphere. What I'm struggling with here is why an anyon type in a region must neccecarily be detectable by its surrounding region.

Physics-math dictionary 2.35. For any modular category $\mathcal{C}$,:

$$\begin{pmatrix} \text{states of topological order } \mathcal{C} \\ \text{on the infinite flat plane } \mathbb{R}^2 \\ \text{with anyon configuraiton } A_1, A_2...A_n \end{pmatrix} = \begin{pmatrix} \text{normalized vectors in the Hilbert space} \\ \mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{[B] \in \mathcal{L}} B, A_1 \otimes A_2 ... \otimes A_n\right) \end{pmatrix} \tag{2.4}$$

Remark 2.36. Replacing $\mathbf{1}$ with $\bigoplus_{[B]\in\mathcal{L}} B$ reflects the differences between the sphere and the infinite flat plane.

WORK: sketch nice argument for why states on infinite flat plane are determined by their overall charge. The subtelty here is exactly the same as the one for the sphere. Think about it then put it down.

WORK: What happens for higher genus surfaces? I should add a few words about them. Zhenghan says all of this is contained in Turaev's book about the Reshetikhin-Turaev construction.

Remark 2.37. The anyon configurations in equations 2.3 and 2.4 are always assumed to be linear. The main reason to do this is because it makes the mathematics much simpler. If we kept track of the positions of each of the anyons in two dimensional space it would add more pieces of data and structures to keep track of. Seeing as every anyon configuration can be pushed onto a one-dimensional space, only working with a one-dimensional configuration does not affect the generality of the answers and hence it is prefered.

Remark 2.38. The formula $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A_1 \otimes A_2 ... \otimes A_n)$ encodes the fact that states can be specified by their history. A good first question to ask when seeing the Hilbert space $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A_1 \otimes A_2...A_n)$ is *why* this should describe a state with anyon configuration A_1...

A_n. The answer is that states can be described their history. WORK: give good example of making a state by specifying its history; argue why it has to be this way in general.

Remark 2.39. In the definition of an modular category hom-spaces are vector spaces and not Hilbert spaces, so this choice of physics-math correspondance is incorrect as literally written. To make this definition work, all of the hom-spaces of the modular category $\mathcal{C}$ should be equipped with Hilbert space structures. Furthermore, the natural operators we wish to perform like braiding should all be unitary with respect to these inner products. This amounts to adding a large number of compatibility conditions on the Hilbert space structures. A modular category with this choice of structure is known as a *unitary* modular category.

For this reason, the correct algebraic structure to underlie the theory of topological order is not a modular category, but a unitary modular category. We have chosen to not emphasize this before because the difference between unitary modular categories and non-unitary modular categories is very small. WORK: talk about uniqueness + positive q.d. criterion this will make more sense once we write the actual section about unitarity. A good thing to emphaize is that unitary modular categories don't let you use less data in your definition, and you can still do essentially everything you want to do. It's just way more cumbersome. They're all equivalent but you still have to choose, c.f. the fact that the category of vector spaces and Hilbert spaces with linear maps as morphisms are equivallent.

2.2.4 Topological charge measurement

When two anyons are fused together, they will form a superposition of other anyon types. Measuring the result of the fusion will collapse the answer into a specific anyon type. The outcome of this measurment is an observable quantity, which allows for the measurement of topological quantum information. In many cases this is the *only* local observable quanitity. We give the formalism behind computing these probabilities now.

WORK: do this right - I don't know it well but it shouln't be hard to learn. Don't introduce anything too general, like trace or whatnot. Just quantum dimension, which should already have been introduced in previous chapter.

WORK: The correct reference for this subsection is [Bon21]. The paper [CCW17] claims to introduce the term topological charge measurement and gives a nice formal treatmenet. Clarifying the situation seems important.

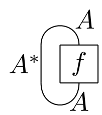
2.3 The modular category toolkit

In this section, we will introduce and prove the basic facts about the most important structures in the theory of modular categories. These facts and structures are the tools used for solving problems about the algebraic theory of anyons.

WORK: I don't have a section on fusion coefficients yet. I guess this isn't a problem, because there isn't that much to say. I would like to have the associativity of fusion cooficients and the fact that braiding $\implies$ commutative said somewhere explicitely, though. Find a place?

2.3.1 Trace

The first structure to define in the theory of modular categories is the *trace*. Let $\mathcal{C}$ be a spherical fusion category. Given any object $A \in \mathcal{C}$ and any endomorphism $f : A \to A$, we define the *trace of f* by the following formula:

$\mathrm{tr}(f) =$ 

Initially, the trace is a morphism, $\mathrm{tr}(f) : \mathbf{1} \to \mathbf{1}$. However, we will choose to think of the trace of a morphism as a *complex number*, $\mathrm{tr}(f) \in \mathbb{C}$. This can be done because the definition of a fusion category $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$. This isomorphism can be made canonical by identifying an endomorphism $g \in \mathrm{End}(\mathbf{1})$ with the unique $\lambda \in \mathbb{C}$ such that $g = \lambda \cdot \mathrm{id}_{\mathbf{1}}$.

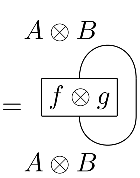
The trace is used mainly as a tool for linearization. Morphisms and objects are hard to describe, but the trace is a complex number.

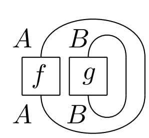
Proposition 2.40. *Let $\mathcal{C}$ be a spherical fusion category. For all $A, B \in \mathcal{C}$, $f \in \mathrm{End}(A)$ the following claims are all true:*

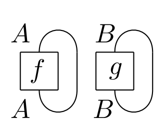
1. $\mathrm{tr} : \mathrm{End}(A) \to \mathbb{C}$ *is a linear map of vector spaces,*
2. $\mathrm{tr}(f^*) = \mathrm{tr}(f)$*,*
3. $\mathrm{tr}(f \oplus g) = \mathrm{tr}(f) + \mathrm{tr}(g)$ *for all* $g \in \mathrm{End}(B)$*,*
4. $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \cdot \mathrm{tr}(g)$ *for all* $g \in \mathrm{End}(B)$*,*
5. $\mathrm{tr}(h \circ g) = \mathrm{tr}(g \circ h)$ *for all* $g : A \to B$*,* $h : B \to A$*.*
6. *Trace is preserved by functors. That is, let $\mathcal{C}, \mathcal{D}$ be spherical categories with traces $\mathrm{tr}_{\mathcal{C}}, \mathrm{tr}_{\mathcal{D}}$ respectively. Let $F : \mathcal{C} \to \mathcal{D}$ be a pivotal functor. We have that $\mathrm{tr}_{\mathcal{C}}(f) = \mathrm{tr}_{\mathcal{D}}(F(f))$;*

Proof. We prove the claims one by one.

1. This follows immediately from the bilinearity of composition.
2. This is a straightforward computation.
3. WORK: do proof. This uses facts about the direct sum we haven't established yet.
4. Using Proposition [ref] we compute

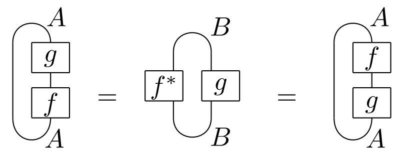
$\mathrm{tr}(f \otimes g) =$ 

$=$ 

$=$ 

$= \mathrm{tr}(f) \cdot \mathrm{tr}(g).$

5. Using Proposition [ref] we find that

$$\mathrm{tr}(f \circ g) =$$


$$= \mathrm{tr}(g \circ f).$$

6. WORK: do the proof. It's not very hard, but involves a diagram and uses pivotality of the functor.

This completes the proof. ☐

With these properties in hand, we can explicitely compute the trace using a straightforward procedure:

Corollary 2.41. *Let $f : A \to A$ be an endomorphism in a fusion category $\mathcal{C}$. Fix a decomposition $A \cong \bigoplus_{i\in I} A_i$ of A into simple objects A_i. Moreover, we take the decomposition such that if $A_i \cong A_j$ then $A_i = A_j$. We can decompose*

$$\mathrm{Hom}(A, A) \cong \mathrm{Hom}(\bigoplus_{i\in I} A_i, \bigoplus_{i\in I} A_i) = \bigoplus_{i\in I, j\in I} \mathrm{Hom}(A_i, A_j).$$

Let M be the matrix whose collums and rows are labeled by I, and whose (i, j) entry is 0 if $A_i \not\cong A_j$ and $\lambda \cdot d_{A_i}$ if $A_i = A_j$, where $\lambda \in \mathbb{C}$ is the unique value such that the $\mathrm{Hom}(A_i, A_j)$ component of f is $\lambda \cdot \mathrm{id}_{A_i}$. We have that

$$\mathrm{tr}_{\mathcal{C}}(f) = \mathrm{tr}_{\mathbf{Vec}}(M).$$

Proof. Suppose that $A = A_0 \oplus A_1$ is the direct sum of two objects, not neccecarily simple. By proposition [ref] we have a canonical decomposition

$$(A_0 \oplus A_1) \otimes (A_0 \oplus A_1)^* \cong (A_0 \otimes A_0^*) \oplus (A_0 \otimes A_1^*) \oplus (A_1 \otimes A_0^*) \oplus (A_1 \otimes A_1^*).$$

Suppose that $h : A \to A$ is an endomorphism. We can decompose $h = h_{A_0,A_0} + h_{A_0,A_1} + h_{A_1,A_0} + h_{A_0,A_1}$ as a sum of morphisms which restrict to maps $A_i \to A_j$. We find that $\mathrm{coev}_{A\oplus B}$ restricts to a map whose codomain is $(A_0 \otimes A_0^*) \oplus (A_1 \otimes A_0)^*$ and similarly ev restricts to a map whose domain is $(A_0 \otimes A_0^*) \oplus (A_1 \otimes A_0)^*$, since $\mathrm{coev}_{A\oplus B} = \mathrm{coev}_A \oplus \mathrm{coev}_B$ and $\mathrm{ev}_{A\oplus B} = \mathrm{ev}_A \oplus \mathrm{ev}_B$.

Hence, in the definition of trace, we find that the cross terms $h_{A_0,A_1} + h_{A_1,A_0}$ act by zero since they send the codomain of $\mathrm{coev}_{A\oplus B}$ to elements with no effect on the map $\mathrm{ev}_{A\oplus B}$. Moreover, we compute in this way that $\mathrm{tr}(h) = \mathrm{tr}(h_{A_0,A_0}) + \mathrm{tr}(h_{A_1,A_1})$.

In this way, the trace splits over direct sums and only picks out diagonal elements. Applying this result inductively reduces the proof to the case that A is a simple object. This follows directly from the definition of quantum dimension. ☐

2.3.2 Duality

Duality is baken into our definition of modular categories as a fundamental part of the structure. It acts in a very controlled way on fusion cooefficients:

Proposition 2.42. *Let $\mathcal{C}$ be a fusion category and let $A, B, C \in \mathcal{C}$ be simple objects. We have the following:*

(i) (Anti-involution) $N_C^{A,B} = N_{C^*}^{B^*,A^*}$;

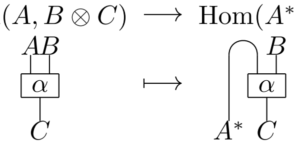
(ii) (Frobenius reciprocity) $N_C^{A,B} = N_B^{A^*,C} = N_A^{C,B^*}$.

Proof. Part (i) follows from the fact that the duality functor is fully faithful and monoidal from propositon [ref], so

$$N_C^{A,B} = \dim \operatorname{Hom}(C, A \otimes B) = \dim \operatorname{Hom}(A^* \otimes B^*, C^*) = N_{C^*}^{B^*,A^*}.$$

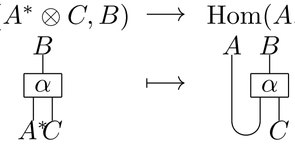
Part (ii) follows from the following computation. Consider the map

$$i : \operatorname{Hom}(A, B \otimes C) \longrightarrow \operatorname{Hom}(A^* \otimes C, B)$$



Since composition is bilinear, i is a linear map. The map

$$\operatorname{Hom}(A^* \otimes C, B) \longrightarrow \operatorname{Hom}(A, B \otimes C)$$



serves as an inverse for i by rigidity. Hence, we conclude that

$$N_C^{A,B} = \dim \operatorname{Hom}(C, A \otimes B) = \dim \operatorname{Hom}(A^* \otimes C, B) = N_B^{A^*,C}.$$

The third equality in Frobenius reciprocity follows from an identical argument, and hence we conclude the proof. $\square$

In particular, we can describe the fusion rules of a simple object with its dual:

Corollary 2.43. *Let $\mathcal{C}$ be a fusion category. Let $A, B \in \mathcal{C}$ be simple objects. We find that*

$$N_{\mathbf{1}}^{A,B} = N_{\mathbf{1}}^{B,A} = \begin{cases} 1 & B \cong A^* \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows from Frobenius reciprocity and Schur's lemma:

$$N_{\mathbf{1}}^{A,B} = N_A^{A^*,\mathbf{1}} = \dim(\operatorname{Hom}(A, A^*)) = \begin{cases} 1 & B \cong A^* \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Specializing even more, we get the following corollary:

Corollary 2.44. *If $\mathcal{C}$ is a fusion category, then $A \cong A^{**}$ for all $A \in \mathcal{C}$.*

Proof. Since $N_{\mathbf{1}}^{A^*,A^{**}} > 0$, we conclude that $A^{**} \cong A$ by the $(iii) \implies (i)$ implication in proposition [ref]. ☐

However, despite this corollary, we *cannot* conclude that every fusion category admits a pivotal structure. The isomorphism $A \cong A^{**}$ may fail to form a monoidal natural transformation. It is an open problem whether or not every fusion category admits a pivotal structure, and it is furthermore an open problem whether every pivotal fusion category admits a spherical structure [ENO05].

2.3.3 Quantum dimension and Frobenius-Perron dimension

WORK: include something about global quantum dimension $\mathcal{D}$.

Our next tool to discuss is the *quantum dimension*. Given any spherical fusion category $\mathcal{C}$ and any object $A \in \mathcal{C}$, we define its quantum dimension using the following formula:

$$d_A = A \bigcirc A^*$$

As usual, we identitfy d_A with a complex number via the canonical isomorphism $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$. The quantum dimension is clearly equal to the trace of the identity map on A, $d_A = \mathrm{tr}(\mathrm{id}_A)$. The first properties of quantum dimension follow from our general analysis of trace:

Proposition 2.45. *For every spherical fusion category $\mathcal{C}$ and any objects $A, B \in \mathcal{C}$, we have the following formulas:*

(i) *If $A \cong B$, then $d_A = d_B$;*

(ii) $d_{A\oplus B} = d_A + d_B$;

(iii) $d_{A\otimes B} = d_A \cdot d_B$;

(iv) $d_{A^*} = d_A$.

(v) $d_A \neq 0$.

Proof.

(i) Let $f : A \cong B$ be an isomorphism. Using Proposition [ref] we find

$$d_A = \mathrm{tr}(\mathrm{id}_A) = \mathrm{tr}(f^{-1} \circ f) = \mathrm{tr}(f \circ f^{-1}) = \mathrm{tr}(\mathrm{id}_B) = d_B.$$

(ii) This follows from proposition [ref].

(iii) This follows from proposition [ref].

(iv) This follows from proposition [ref].

(v) From proposition [ref], we know that $A \otimes A^* \cong \mathbf{1} \oplus X$ for some $X \in \mathcal{C}$ which does not have any factors of $\mathbf{1}$ in its direct sum decomposition. The map $\mathrm{coev}_A^R : \mathbf{1} \to A \otimes A^*$ is thus a non-zero scalar times the inclusion $\mathbf{1} \hookrightarrow \mathbf{1} \oplus X$, and the map $\mathrm{ev}_A^L : A \otimes A^* \to \mathbf{1}$ is a non-zer scalar times the projection $\mathbf{1} \oplus X \to \mathbf{1}$. Since inclusion composed with projection is the identity, we find that $\mathrm{ev}_A^L \circ \mathrm{coev}_A^R$ is a non-zero scalar times the identity, as desired.

$\square$

The above propositions tell us that the values d_A as A ranges over isomorphism classes of simple objects determs all the other values of d_A. Moreover, proposition [ref] tells us that the quantum dimensions of simple objects determines the trace of *every* endomorphism! Hence, computing d_A for each isomorphism class $[A] \in \mathcal{L}$ is an important step in analysing an modular category. The following formula and its linear-algebraic reformulaion are the primary insight in performing the computation:

Proposition 2.46. *Let $\mathcal{C}$ be a spherical fusion category.*

(i) Let $A, B \in \mathcal{C}$ be simple objects. We have that

$$d_A d_B = \sum_{[C] \in \mathcal{L}} N_C^{A,B} d_C.$$

(ii) Let $A \in \mathcal{C}$ be a simple object. Define an operator

$$\begin{aligned} N^A : \mathbb{C}[\mathcal{L}] &\to \mathbb{C}[\mathcal{L}]. \\ |[B]\rangle &\mapsto \sum_{[C] \in \mathcal{L}} N_C^{A,B} |[C]\rangle \end{aligned}$$

Define $\mathbf{d} = \sum_{[B] \in \mathcal{L}} d_B |[B]\rangle \in \mathbb{C}[\mathcal{L}]$. We have that

$$N^A \mathbf{d} = d_A \mathbf{d}.$$

Proof. From proposition [ref], we have an isomorphism

$$A \otimes B \cong \bigoplus_{[C] \in \mathcal{L}} N_C^{A,B} \cdot C$$

and thus

$$\operatorname{tr}(\mathrm{id}_{A \otimes B}) = \operatorname{tr}(\mathrm{id}_{\bigoplus_{[C] \in \mathcal{L}} N_C^{A,B} \cdot C}).$$

Expanding using the rules in proposition [ref] gives part (i). Part (ii) follows from expanding the definition of the linear operator and applying part (i). $\square$

We now make commentary about the above proposition. It tells us that d_A is an eigenvalue of N^A. Since N^A is an operator with integer coefficients, this immediately tells us that d_A is the root of polynomial with integer coeffients. Namely, the characteristic polynomial of N^A. We can even be more precise about the nature of d_A:

WORK: On the MathOverflow question "Modular Tensor Categories: Reasoning behind the axioms", a commentator said "You also have to assume that the categorical dimensions arising from the pivotal structure are all real. This is called the spherical axiom". Why is this the same thing as the spherical axiom? What is the motivation? I should include this as a remark on the below proposition.

Corollary 2.47. *Let $\mathcal{C}$ be a spherical fusion category. The quantum dimensions of all simple objects in $\mathcal{C}$ are real numbers.*

Proof. WORK: do proof. □

The question is whether or not the quantum dimensions are *positive* real numbers. We recall that we defined a unitarizable spherical fusion category to be one in which the quantum dimensions are all positive. It is at this point that this becomes relevant. In particular, if $\mathcal{C}$ is unitarizable then its quantum dimensions are eigenvalues of N^A, and their corresponding eigenvector $\mathbf{d}$ has positive real entries. There is a theorem about eigenalues of non-negative matrices with positive eigenvectors:

Theorem 2.48 (Frobenius-Perron theorem, [EGNO16])**.** *Let B be a square matrix with nonnegative real entries.*

(i) B has a non-negative real eigenvalue. The largest non-negative real eigenvalue $\lambda(B)$ of B dominates the absolute values of all other eigenvalues μ of B: $|\mu| \leq \lambda(B)$. Moreover, there is an eigenvector of B with non-negative entries and eigenvalue $\lambda(B)$.

(ii) If B has strictly positive entries then $\lambda(B)$ is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries. Moreover, $|\mu| < \lambda(B)$ for any other eigenvalue μ of B.

(iii) If a matrix B with non-negative entries has an eigenvector v with strictly positive entries, then the corresponding eigenvalue is $\lambda(B)$.

We call the largest positive real eigenvalue of a matrix its *Frobenius-Perron eigenvalue*. The Frobenius-Perron theorem tells us the following:

Corollary 2.49. *Let $\mathcal{C}$ be a unitarizable spherical fusion category. Let $A \in \mathcal{C}$ be a simple object. The quantum dimension d_A is equal to the Frobenius-Perron eigenvalue of N^A.*

Proof. Since $\mathcal{C}$ is unitarizable, the vector $\mathbf{d} = \sum_{[B]\in\mathcal{L}} d_B\, |[B]\rangle \in \mathbb{C}[\mathcal{L}]$ has positive entries and has eigenvalue d_A. Hence, d_A is the Frobenius-Perron eigenvalue of N^A as desired. □

In this chapter we will mostly work with generic spherical fusion categories with no conditions on unitarizability. Hence, it is useful to make the following definition. Let $A \in \mathcal{C}$ be a simple object in a spherical fusion category. We define

$$\mathrm{FPdim}(A) = (\text{Frobenius-Perron eigenvalue of } N^A).$$

When $\mathcal{C}$ is unitarizable, $\mathrm{FPdim}(A) = d_A$. Many formulas about quantum dimension in the unitary world apply to the Frobenius-Perron dimension in the non-unitary world. An interesting observation is that the definition of quantum dimension strongly uses the spherical structure on $\mathcal{C}$. However, the Frobenius-Perron dimension only uses the fusion coeffients, and those are well-defined in any fusion category. Hence, the Frobenius-Perron dimension also derives utility from being applicable in a broader set of situations than the quantum dimension.

We now give an alternate interpretation of the Frobenius-Perron dimension in terms of growth in tensor powers. This sort of alternate perspective of dimension applies to several types of objects outside the scope of tensor category theory [COT24].

Proposition 2.50. *Let $\mathcal{C}$ be a fusion category, and let $A \in \mathcal{C}$ be a simple object.*

(i) $\mathrm{FPdim}(A) = \lim_{n\to\infty} \dim(\mathrm{Hom}(A^{\otimes n}, A^{\otimes n}))^{1/(2n)}$

(ii) $\mathrm{FPdim}(A) = \lim_{n\to\infty} \dim(\mathrm{Hom}(\mathbf{1}, A^{\otimes n}))^{1/n}$

(iii)

$$\mathrm{FPdim}(A) = \lim_{n\to\infty} (\#\ \textit{of simple objects in the direct sum decomposition of } A^{\otimes n})^{1/n}.$$

Proof. WORK: I can do a good part of this when $\mathcal{C}$ is unitarizable, so that its largest eigenvalue is strictly larger than all of the others. When there are multiple large eigenvalues all of the same size then the proofs go wrong. Is there something about the structure of N^A I can exploit? Are these theorems true for fusion categories, or do I need to pass to unitarizable fusion categories? ☐

This proposition can be interpreted as saying that the simple object A has $\mathrm{FPdim}(A)$ internal degrees of freedom "on average". Elements of the vector space $\mathrm{Hom}(\mathbf{1}, A^{\otimes n})$ correspond to states in the system with n anyons of type A arranged in a line. If the internal configuration space of each anyon was $\mathrm{FPdim}(A)$-dimensional, then the overall dimension would be $\mathrm{FPdim}(A)^n$. By Proposition [ref], $\mathrm{FPdim}(A)^n$ is approximately $\mathrm{Hom}(\mathbf{1}, A^{\otimes n})$ for large n. Hence, each anyon has approximately $\mathrm{FPdim}(A)$ internal degrees of freedom. Of course, $\mathrm{FPdim}(A)$ has no reason to be an integer! In the Fibonacci theory $\mathrm{FPdim}(\tau) = \phi = 1.61...$. Frobenius-Perron dimension just gives an average amount for large values.

WORK: re-do this explanation way better + add diagram for it.

2.3.4 Twist

In this section we will discuss *twists*. The twist is a subtle concept, which we have not explicitely mentioned up to now. The idea is that anyons can *rotate in place*. Since the space of endomorphisms of an anyon is one dimensional, this rotation must act by a phase. This phase is physically relevant, and can be measured in experiment.

For example, consider the Y-type on the toric code. It consists of the fusion of an X-type anyon and a Z-type anyon, as shown below:

WORK: add figure of Y as a thick X and Z together; could be hard to draw these nice

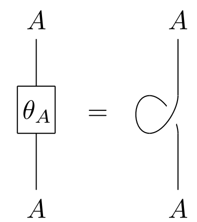
Twisting Y in place will correspond to twisting X and Z around each other. This twsiting thus results in a phase of -1. In general, we can imagine anyons as having some thickness to them. Anyons are not localized at points - they are localized at small regions. Twisting this region all the way around can be viewed visually as

WORK: twisted anyon.

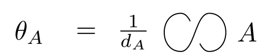
This is the twist. One way of working with the twist is to work with thickened diagrams, where strings are replaced with ribbons. While popular in some parts of the literature, we will continue to work with string diagrams for simplicity. The key observation is that the twist can be constructed using string diagramatic structures we already have as follows:

WORK: twist as a swirl diagram, compared with ribbon.

Hence, letting $\mathcal{C}$ be a pre-modular fusion category, we *define* the twist θ_A of an object $A \in \mathcal{C}$ to be

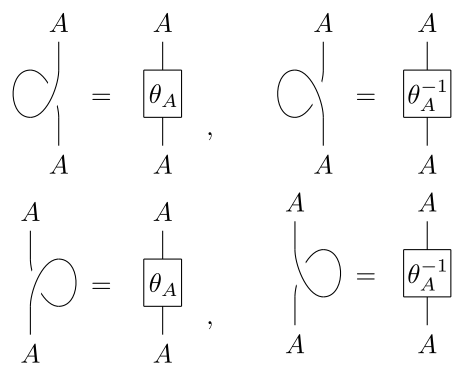


For every simple object $A \in \mathcal{C}$, the map $\theta_A \in \mathrm{End}(A)$ can be identified with the unique complex number λ such that $\theta_A = \lambda \cdot \mathrm{id}_A$. Equivilantly, we can identify θ_A with the complex number $\lambda = \mathrm{tr}(\theta_A)/d_A$ which gives the graphical formula

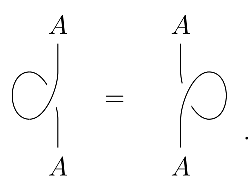


We can reinterpret all other twist-like maps in terms of θ:

Lemma 2.51. *Let $\mathcal{C}$ be a pre-modular fusion category. We have that*



Proof. To begin we show that



When A is simple, this follows from the spherical axiom. Taking the trace of both sides gives the same formula for θ_A as a figure-eight. Additionally, pushing through duals it is clear that both sides in the above proposed equality are natural isomorphisms. Natural isomorphisms are determined by their action on simple objects because they commute with direct sums. Hence, we conclude that the sides are equal for all objects.

To get that the two reversed formulas are equal to θ_A^{-1}, it suffices to compose with θ_A and use string-diagram manipulations to show that it results in the identity. This is a simple exercise and is left as an exercise to the reader. $\square$

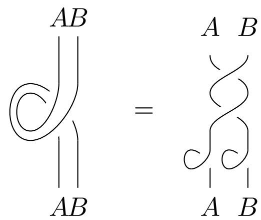
We now characterize the key properties of the twist:

Proposition 2.52. *Let $\mathcal{C}$ be a pre-modular fusion category. The twists θ induce a monoidal natural isomorphism $\mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$. Additionally, θ satisfies the identity*

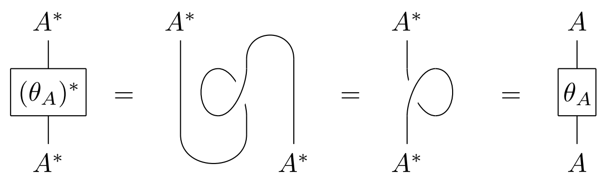
$$\theta_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\theta_A \otimes \theta_B)$$

for all $A, B \in \mathcal{C}$*, and* $\theta_{A^*} = (\theta_A)^*$*.*

Proof. Naturality of θ follows from pushing through duals. The formula $\theta_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\theta_A \otimes \theta_B)$ comes from manipulating string diagrams to get the equation

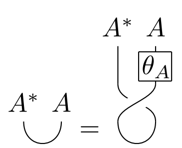


Finally, $\theta_{A^*} = (\theta_A)^*$ comes from the string-diagram manipulation and proposition [ref]:

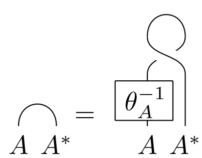


as desired. ∎

The naive reason to care about twists is that they descrbe a physically relevant quantity and hence should be studied. The more subtle reason to care about twists is that they are the most efficient way of encoding the spherical structure on $\mathcal{C}$. A spherical structure is first and foremost a pivotal structure, meaning that it has a right and left rigid structure which are compatible. Given a spherical structure one can always obtain twists. Conversely, given a right-rigid structure and twists one can recover the left-rigid structure via the formulas



and

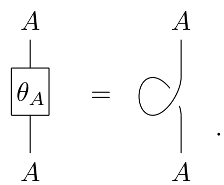


In this way, giving a spherical structure on a right-rigid monoidal category is the *same* as giving a twist structure. This is codified in the following lemma:

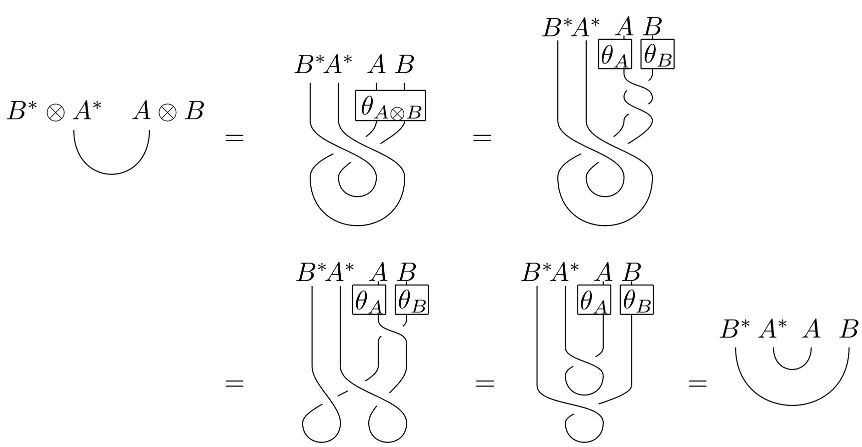
Proposition 2.53 (Deligne's twisting lemma, [Yet92])**.** *Let* $\mathcal{C}$ *be a right-rigid braided monoidal category. Every pivotal structure on* $\mathcal{C}$ *naturally gives a twist natural transformation* $\theta : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$*. This assignment induces a canonical bijection between the set of pivotal structures on* $\mathcal{C}$ *and the set of natural isomorphism* $\theta : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$ *satisfying* $\theta_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\theta_A \otimes \theta_B)$ *for all* $A, B \in \mathcal{C}$*.*

Moreover, restricting the assignment to the space of spherical structures on $\mathcal{C}$ *induces a canonical bijection between the set of spherical structures on* $\mathcal{C}$ *and the set of isomorphisms* $\theta : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$ *satisfying* $\theta_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\theta_A \otimes \theta_B)$ *for all* $A, B \in \mathcal{C}$ *and* $\theta_{A^*} = (\theta_A)^*$*.*

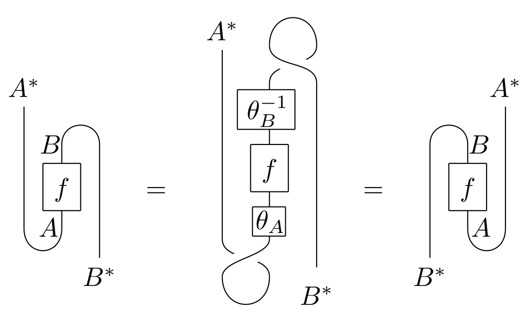
Proof. We already showed in proposition [ref] that every spherical category gives a twist natural transformation satisfying the desired axioms. Restriciting the proof to only a possibly non-spherical pivotal category still gives a twist natural transformation satisfying $\theta_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\theta_A \otimes \theta_B)$ for all $A, B \in \mathcal{C}$. The heart of the proof is showing that the formulas [ref] induce pivotal and spherical structures with the twist satisfies the right axioms. The process of inducing a pivotal structure and inducing a twist are inverses to one another because



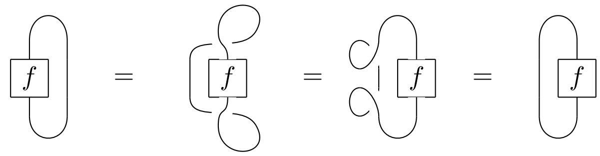
To begin, we assume that $\theta_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\theta_A \otimes \theta_B)$ and we seek to prove that the corresponding ev^L, coev^L maps induce a pivotal structure. We first axiom of pivotality follows from use of the axiom $\theta_{A\otimes B} = \beta_{B,A} \circ \beta_{A,B} \circ (\theta_A \otimes \theta_B)$:



The second axiom of pivotality follows from the use of the naturality of θ:



Finally, we assume that $(\theta_A)^* = \theta_{A^*}$ and we seek to prove the spherical axiom. Taking the dual of theta we can get all of the equalities in Lemma [ref]. Applying them we get that



as desired. $\square$

2.3.5 Functors, natural transformations, and equivalence

In this section, we will talk about functors, natural transformations, and equivalences between fusion, spherical, pre-modular, and modular categories. Given a topological order, there is *not* a unique modular category describing it. There is a unique modular category *up to equivalence*. Hence, the notion of equivalence of categories is baked into our physics-math correspondance so it is important that we state it explicitely.

Functors which do not induce equivalences of categories are also physically relevant. In certain contexts, a functor $F : \mathcal{C} \to \mathcal{D}$ is used to model a *phase transition* from $\mathcal{C}$ to $\mathcal{D}$. We will see a lot more functors and natural transformations between modular categories throughout the book, especially in chapter [ref].

Even though structures in categories require a lot of compatibility conditions, the conditions on the functors do not. This means that we have the following:

- The correct notion of functor between fusion categories is $\mathbb{C}$-linear monoidal functor. There is no compatibility condition required between the $\mathbb{C}$-linear structure and the monoidal structure. The correct notion of natural transformation between $\mathbb{C}$-linear monoidal functors is a monoidal natural transformation.

- The correct notion of functor between spherical fusion categories is $\mathbb{C}$-linear pivotal monoidal functor. There is no compatibility condition required between the $\mathbb{C}$-linear structure and the pivotal monoidal structure. The correct notion of natural trasnformation is monoidal natural transformation.

- The correct notion of functor between pre-modular categories is $\mathbb{C}$-linear pivotal braided monoidal functor. There is no compatibility condition required between the $\mathbb{C}$-linear structure, pivotal monoidal structure, or braided monoidal structure. The correct notion of natural transformation is monoidal natural transformation.

- The correct notions of functors/natural transformations for modular categories are the same as for pre-modular categories.

WORK: : this section is very short. I don't have much to say, actually. Should this be moved? Maybe I keep a very short section? I don't know.

2.3.6 Deligne tensor product

In the theory of any class of mathematical object, an important consideration is the ways in which examples can be put together to give new examples. In the case of fusion categories, this basic operation is known as the *Deligne tensor product*. Given any fusion categories $\mathcal{C}$, $\mathcal{D}$, their Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ is a new fusion category. The Deligne tensor product of spherical fusion categories will be equipped with the structure of a spherical fusion category, and the Deligne tensor product of (pre-)modular categories will be equipped with the structure of a (pre-)modular category.

Physically, the Deligne tensor product corresponds to *stacking*. Consider two sheets of material. We choose two modular categories $\mathcal{C}$, $\mathcal{D}$. We enodow the top sheet with the structure of a topologically ordered quantum system described by $\mathcal{C}$ and we endow the bottome with the structure of a topologically ordered quantum system described by $\mathcal{D}$. The algebraic description of this bilayer system is $\mathcal{C} \boxtimes \mathcal{D}$. This can be viewed as the physical definition of $\mathcal{C} \boxtimes \mathcal{D}$.

WORK: add bilayer system diagram

We now mathematically define the Deligne tensor product.

Definition 2.54. Let $\mathcal{C}, \mathcal{D}$ be a $\mathbb{C}$-linear categories, isomorphic as a $\mathbb{C}$-linear categories to $\mathbf{Vec}_{\mathbb{C}}^n$, $\mathbf{Vec}_{\mathbb{C}}^m$ respectively. We define a Deligne tensor product of $\mathcal{C}$ and $\mathcal{D}$ to be be the following data:

1. A $\mathbb{C}$-linear category $\mathcal{C} \boxtimes \mathcal{D}$;
2. A $\mathbb{C}$-linear functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$.

Such that:

1. Every object $X \in \mathcal{C} \boxtimes \mathcal{D}$ has a direct sum decomposition
$$X \cong \bigoplus_{i=1}^{n} A_i \boxtimes B_i$$
for some $n \geq 1$, $A_i \in \mathcal{C}$, $B_i \in \mathcal{D}$.
2. There is an equality of vector spaces
$$\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(A \boxtimes B, A' \boxtimes B') = \mathrm{Hom}_{\mathcal{C}}(A, A') \otimes \mathrm{Hom}_{\mathcal{D}}(B, B').$$
3. Given any $A, A', A'' \in \mathcal{C}$, $B, B', B'' \in \mathcal{D}$, $f : A \to A'$, $f' : A' \to A''$, $g : B \to B'$, $g' : B' \to B''$, the diagram
$$\begin{tikzcd} A \boxtimes B \arrow[r, "f\boxtimes g"] \arrow[rr, bend right, "(f'\circ f)\boxtimes(g'\circ f)"'] & A' \boxtimes B' \arrow[r, "f'\boxtimes g'"] & A'' \boxtimes B'' \end{tikzcd}$$
commutes.

We know state the main existence/uniqueness result about the Deligne tensor product:

Proposition 2.55. *Let $\mathcal{C}, \mathcal{D}$ be $\mathbb{C}$-linear categories isomorphic as $\mathbb{C}$-linear categories to $\mathbf{Vec}_{\mathbb{C}}^n$ and $\mathbf{Vec}_{\mathbb{C}}^m$ respectively. There exists a Deligne product $\mathcal{C} \boxtimes \mathcal{D}$ for $\mathcal{C}$ and $\mathcal{D}$. Moreover, given any other deligne tensor product $\mathcal{C} \boxtimes' \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ there exists a unique functor $F : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C} \boxtimes' \mathcal{D}$ making the diagram*
$$\begin{tikzcd} \mathcal{C} \times \mathcal{D} \arrow[r] \arrow[dr] & \mathcal{C} \boxtimes \mathcal{D} \arrow[d, "F"] \\ & \mathcal{C} \boxtimes' \mathcal{D} \end{tikzcd}$$
commute. This functor is an equivalence of categories.

Proof. It is clear that $\mathbf{Vec}_{\mathbb{C}}^{n} \boxtimes \mathbf{Vec}_{\mathbb{C}}^{m} = \mathbf{Vec}_{\mathbb{C}}^{nm}$. Every equivalence of categories $\mathcal{C} \to \mathcal{C}'$ induces an equivalence of categories $\mathcal{C} \boxtimes \mathcal{D} \to \mathcal{C}' \boxtimes \mathcal{D}$. Hence, since $\mathcal{D}$ and $\mathcal{D}$ are equivalent to $\mathbf{Vec}_{\mathbb{C}}^{n}$ and $\mathbf{Vec}_{\mathbb{C}}^{m}$ respectively, their Deligne tensor product exists and is equivalent to $\mathbf{Vec}_{\mathbb{C}}^{nm}$.

Any functor making the diagram commute must send $A \boxtimes B$ to $A \boxtimes' B$. The definition of Deligne tensor produce tells us this is enough to conclude that the map is an equivalence of categories, since axiom 3 this map is always a functor, axiom 2 implies it is fully faithful, and axiom 1 implies it is essentially surjective, and hence we can apply proposition [ref]. □

Now that we have defined the Deligne tensor product of $\mathbb{C}$-linear categories equivalent to $\mathbf{Vec}_{\mathbb{C}}^{n}$, we move on to defining the Deligne tensor product of fusion categories, spherical fusion categories, pre-modular categories, and modular categories.

Proposition 2.56. *The following claims are all true.*

(i) Let $\mathcal{C}$, $\mathcal{D}$ be fusion categories. On the level of objects, define a monoidal structure $\mathcal{C} \boxtimes \mathcal{D}$ by the formula

$$(A \boxtimes B) \otimes (A' \boxtimes B') = (A \otimes A') \boxtimes (B \otimes B').$$

Along with a natural choice of action of the tensor product on morphisms, unit $\mathbf{1}_{\mathcal{C}\boxtimes\mathcal{D}} = \mathbf{1}_{\mathcal{C}} \boxtimes \mathbf{1}_{\mathcal{D}}$, and a natural choice of associator and unitors, this induces the structure of a monoidal category on $\mathcal{C}$.

Define a right-rigid structure on $\mathcal{C} \boxtimes \mathcal{D}$ as follows. The dual of an object $A \boxtimes B$ is $A^* \boxtimes B^*$. Define $\mathrm{ev}_{A\boxtimes B} = \mathrm{ev}_A \boxtimes \mathrm{ev}_B$, $\mathrm{coev}_{A\boxtimes B} = \mathrm{coev}_A \boxtimes \mathrm{coev}_B$. This induces a well-defined right-rigid structure on $\mathcal{C} \boxtimes \mathcal{D}$.

The above definitions induce the structure of a fusion category on $\mathcal{C} \boxtimes \mathcal{D}$.

(ii) Let $\mathcal{C}, \mathcal{D}$ be spherical fusion categories. The evaluation and coevaluation maps $\mathrm{ev}^L_{A\boxtimes B} = \mathrm{ev}^L_A \boxtimes \mathrm{ev}^L_B$ and $\mathrm{coev}^L_{A\boxtimes B} = \mathrm{coev}^L_A \boxtimes \mathrm{coev}^L_B$ induce a left-rigid structure on $\mathcal{C}\boxtimes\mathcal{D}$. Along with the canonial structure of a fusion category on $\mathcal{C}\boxtimes\mathcal{D}$, this induces the structure of a spherical fusion category on $\mathcal{C} \boxtimes \mathcal{D}$.

(iii) Let $\mathcal{C}, \mathcal{D}$ be pre-modular categories. The braiding map $\beta_{\mathcal{C}\boxtimes\mathcal{D}} = \beta_{\mathcal{C}} \boxtimes \beta_{\mathcal{D}}$ induces the structure of a pre-modular category on $\mathcal{C} \boxtimes \mathcal{D}$. The product $\mathcal{C} \boxtimes \mathcal{D}$ is modular if and only if $\mathcal{C}, \mathcal{D}$ are both modular.

Proof. Given any of the above structures, all of the axioms on $\mathcal{C}\boxtimes\mathcal{D}$ immediately follow from their respective axioms on $\mathcal{C}$ and $\mathcal{D}$. Hence, the proof is an exercise is recalling definitons which we omit. □

2.4 The category of G-graded G-representations

2.4.1 Overview

We've talked about a lot of general theory of modular categories. It's time for us to focus on our main family of *examples*. Namely, the categories $\mathfrak{D}(G)$ of G-graded G-representations. These categories describe discrete gauge theory based on the finite group G.

Before we can prove that $\mathfrak{D}(G)$ is a modular category, we need to endow $\mathfrak{D}(G)$ with the neccecary structures. In particular, we will endow $\mathfrak{D}(G)$ with $\mathbb{C}$-linear, monoidal, braided, right-rigid, and left-rigid structures. We will need to show that all of these structures are comptabile with each other in the neccecary ways, and that $\mathfrak{D}(G)$ satisfies the non-degeneracy axiom. We will use this as an oppurtunity to introduce tools of general use for proving that categories satisfy the axioms of a modular category.

Additionally, we will also study two categories similar to $\mathfrak{D}(G)$ which will serve as extra examples to get our grip on definitions. These categories will also appear later as relevant in and of themselves. The first is $\mathbf{Vec}_G$, the category of G-graded vector spaces. It is defined as follows:

WORK: define $\mathbf{Vec}_G$ in terms of objects and composition.

Our second structure of interest is $\mathrm{Rep}(G)$, the category of G-representations. It is defined as follows:

WORK: define $\mathrm{Rep}(G)$ in terms of objects and composition.

We will show that both $\mathbf{Vec}_G$ and $\mathrm{Rep}(G)$ can be naturally equipped with the structures of spherical fusion categories. We then show that $\mathrm{Rep}(G)$ admits a braiding which turns it into a pre-modular category. This braiding is symmetric in the sense that $\beta_{B,A} \circ \beta_{A,B} = \mathrm{id}_A \otimes \mathrm{id}_B$ for all $A, B \in \mathcal{C}$, and hence $\mathrm{Rep}(G)$ is not a modular category. The category $\mathbf{Vec}_G$ is shown to not admit a braiding whenever G is non-abelian.

2.4.2 Higher linear algebra

WORK: In this section we define the $\mathbb{C}$-linear structures on $\mathbf{Vec}_G$, $\mathrm{Rep}(G)$, and $\mathfrak{D}(G)$. Our goal is to show that they are all equivalent to $\mathbf{Vec}_{\mathbb{C}}^n$ for some $n \geq 1$.

WORK: It seems like the best approach is through higher linear algebra. Namely, we show that if $\mathcal{C}$ is abelian, $\mathbb{C}$-linear, semisimple, and has finitely many isomorphism classes of simple obejcts then it must be isomorphism to $\mathbf{Vec}_{\mathbb{C}}^n$. Its a good time to wax philosophical about higher linear algebra and 2-vector spaces. However, its not clear that this approach actually helps at all. It might be easier to immediately note that everybody is the direct sum of irriducibles, prove a Schur's lemma, and call it a day. Of course these approaches are all equivalent but its not clear what's best.

WORK: Here's a lemma I would like to use. Since $\times$ and $\boxtimes$ look too similar, I should use $\boxplus$ to denote the Cartesian product and call it the direct sum. I should then prove this:

Lemma 2.57. *Consider the category **2Vec** whose objects are $\mathbb{C}$-linear categories equivalent to $\mathbf{Vec}_{\mathbb{C}}^n$ for sme $n \geq 0$, and whose morphisms are $\mathbb{C}$-linear functors. The direct sum $\boxplus$ is a biproduct in **2Vec***

Proof. WORK: do proof ∎

2.4.3 Spherical fusion structures

WORK: show that the categories have duals and monoidal structure. This should be pretty easy and painless. Pentagon identity should follow from the pentangon identity on $\mathbf{Vec}_{\mathbb{C}}$.

2.4.4 Braiding and modularity

WORK: Introduce braidings. Show that $\mathrm{Rep}(G)$ is symmetric. Show that $\mathbf{Vec}_G$ does not admit a braiding if G is not abelian and does admit a symmetric braiding if G is abelian. Show that $\mathfrak{D}(G)$ admits a non-degenerate braiding.

2.5 The modular representation

2.5.1 defn

In this chapter we are going to talk about the *modular representations* of modular categories. Here's the point. Let $\mathcal{C}$ be a modular category. Let $\mathcal{L}$ be the set of isomorphism classes of simple objects of $\mathcal{C}$. We will define a group homomorphism

$$\rho_{\mathcal{C}} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[\mathcal{L}])$$

assocociated to $\mathcal{C}$, where $\mathrm{SL}_2(\mathbb{Z})$ is the group of 2-by-2 matrices with integer coefficients and unit determinant. The group $\mathrm{SL}_2(\mathbb{Z})$ is sometimes known as the *modular group*, due to its connection with moduli spaces of elliptic curves. Hence, $\rho_{\mathcal{C}}$ is known as the *modular representation* of $\mathcal{C}$.

The goal of this chapter is to introduce $\rho_{\mathcal{C}}$, show it is well defined, and then prove a series of theorems related to $\rho_{\mathcal{C}}$.

Before defining $\rho_{\mathcal{C}}$, we recall the basic group theory of $\mathrm{SL}_2(\mathbb{Z})$. It is generated by the matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

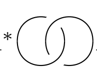
These two matrices satisfy the relations $s^2 = -1$ and $(st)^3 = -1$, where 1 is used the represent the identity matrix. These relations generate $\mathrm{SL}_2(\mathbb{Z})$, in the sense that we have the following presentation:

Proposition 2.58. *The following presentation is valid:*

$$\mathrm{SL}_2(\mathbb{Z}) = \left\langle s, t \,\middle|\, s^4 = 1,\ (st)^3 = s^2 \right\rangle.$$

Proof. This is a standard fact about $\mathrm{SL}_2(\mathbb{Z})$. Se for instance [ref] □

Hence, to define a homomorphism $\rho_{\mathcal{C}} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[\mathcal{L}])$ it suffices to choose automorphisms $\rho_{\mathcal{C}}(s)$, $\rho_{\mathcal{C}}(t)$ of $\mathbb{C}[\mathcal{L}]$, and show that they satisfy the relations $\rho_{\mathcal{C}}(s)^4 = 1$ and $(\rho_{\mathcal{C}}(s)\rho_{\mathcal{C}}(t))^3 = \rho_{\mathcal{C}}(s)^2$. Since $\mathbb{C}[\mathcal{L}]$ has a canonical basis, we can think of its automorphisms as being matrices with rows and collumns labeled by $\mathcal{L}$. We define an operator $S : \mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$ via the matrix coeffients

$S_{A,B}=$ A^*  B^*

We next define the matrix $T : \mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$ to be the diagonal matrix with $([A],[A])$-entry θ_A, for all $[A] \in \mathcal{L}$.

As currently stated, the S and T matrices defined do not satify $S^4 = 1$ and $(ST)^3 = S^2$. They only satisfy these formula up to phases in $\mathbb{C}$. They still need to be normalized before we can defined $\rho_{\mathbb{C}}$. The normalization factors come in terms of the *Gauss sums*,

$$p_{\mathcal{C}}^{\pm} = \sum_{[A] \in \mathcal{L}} \theta_A^{\pm 1} d_A^2.$$

We can now state the main theorem of this chapter:

Theorem 2.59. *Let $\mathcal{C}$ be a modular category. The values $p_{\mathcal{C}}^{+}$ and $p_{\mathcal{C}}^{-}$ are nonzero, and the map*

$$\begin{aligned}\rho_{\mathcal{C}} : \mathrm{SL}_2(\mathbb{Z}) &\to \mathrm{Aut}(\mathbb{C}[\mathcal{L}]) \\ s &\mapsto \frac{1}{\mathcal{D}} \cdot S \\ t &\mapsto \left(p_{\mathcal{C}}^{-}/p_{\mathcal{C}}^{+}\right)^{1/6} \cdot T\end{aligned}$$

is a group homomorphism.

We will prove this theorem and motivate why it should be true over the course of this chapter. We will also prove key facts about the image and kernel of this representation, as well as other formulas of interest relating to twists, S-matrix entries, and Gauss sums.

2.5.2 Torus perspective

It's good to reflect on why MCs have $\mathrm{SL}_2(\mathbb{Z})$ representations assocated with them in the first place. Not only does the representation exist, but it is so fundamental to the modular category that it is chosen as the namesake. This begs the question. What's going on?

The answer has to do with topological phases on the torus.

WORK: add torus

Every modular category $\mathcal{C}$ is supposed to describe a topological order. Up to now we have only considered what happens when this topological order is applied to an infnitely large flat sheet. We have not examined what happens when this topological order is put on a space with nontrivial topology. For instance, the torus. Suppose we analyse the system of $\mathcal{C}$ applied to the torus. This amounts to breaking up the torus into some microscopic lattice and applying some Hamiltonian. This Hamiltonian will have group states $V_{\text{g.s.}}^{T^2}$, which are independent of the choice of microscopic realization of $\mathcal{C}$.

Suppose we start with a torus, cut it across, twist one of its legs, then glue it back together, as shown below:

WORK: add Dehn twist picture.

If the initial torus has some state $|\psi\rangle \in V_{\text{g.s.}}^{T^2}$ on it, then applying this procedure would give back another group state, though possibly a different one. The key phenominon is that continues transformations on physical space correspond to linear transformations on state space:

WORK: add schematic.

We can make this more formal as follows. We define the *mapping class group* of a topological space X as follows:

$$\mathrm{MCG}(X) = (\text{homeomorphisms } X \to X)/(\text{continuous deformations}).$$

If two homeomorphisms can be continuously deformed from one another then they will act the same on the ground states $V_{\text{g.s.}}^{T^2}$. This is because ground states are topologically protected and hence slowly changing the diffeomorphism cannot affect them. Hence, we get a well-defined group homomorphism

$$\rho_{\mathcal{C}}^{T^2} : \mathrm{MCG}(T^2) \to \mathrm{Aut}\left(V_{\text{g.s.}}^{T^2}\right).$$

This homomorphism connects back to our modular representation as follows:

- **Claim 1:** $\mathrm{MCG}(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$;
- **Claim 2:** $V_{\mathrm{g.s.}}^{T^2} \cong \mathbb{C}[\mathcal{L}]$;
- **Claim 3:** $\rho_{\mathcal{C}}^{T^2} \cong \rho_{\mathcal{C}}$, passing through the identifications in claims 1 and 2.

In general, we see that associated to every modular category $\mathcal{C}$ there should not only be a modular representation, but also a representation of $\mathrm{MCG}(\Sigma)$ for many other choices of topological space Σ. For instance, if $\Sigma = \Sigma_g$ is the g-holed torus then putting $\mathcal{C}$ on Σ_g we get a map

$$\rho_{\mathcal{C}}^{\Sigma_g} : \mathrm{MCG}(\Sigma_g) \to \mathrm{Aut}(V_{\mathrm{g.s.}}^{\Sigma_g}).$$

WORK: : maybe say a few words about these representations. I'm sure they must have an explicit description in terms of generators and relations. A good reference (though a bit early) is this one: [Lyu95]

We now examine and motivate claims 1-3.

Claim 1: $\mathrm{MCG}(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$. This claim is best seen by thinking of the torus as a a gluing diagram,

WORK: add gluing diagram

WORK: add diagram with s acting by rotating by 90 degrees. Clearly, $s^4 = 1$.

WORK: add diagram with t as a shift. $(st)^3 = s^2$ can be left as an exercise.

WORK: writing presentation for $MCG(T^2)$, note that it is the same as $\mathrm{SL}_2(\mathbb{Z})$.

Claim 2: $V_{\mathrm{g.s.}}^{T^2} \cong \mathbb{C}[\mathcal{L}]$.

WORK: explain this. Cut into cylinder, label by charge on boundary

Claim 3: $\rho_{\mathcal{C}}^{T^2} \cong \rho_{\mathcal{C}}$.

WORK: Showing that the Dehn twist acts diagonally by θ_A is obvious. θ_A and Dehn twist are both defined as a 2π twist. For S we need another argument, more subtle but not too hard. I think Simon has it.

WORK: Finish by saying this is something like TQFTs. TQFT = bundled collection of mapping class group representations. Link this to TQFT appendix.

WORK: I wrote a little extra about this for the Kapustin final, might be useful

To demonstrate the TQFT perspective in action, we focus on the case of the genus one surface - the torus. As part of the data of every TQFT $(V_g, \rho_g, Z_{g,g'})$ there is a map

$$\rho_1 : \mathrm{MCG}(\Sigma_1) \to \mathrm{Aut}(V_1).$$

We seek to understand this map in full detail, and as such gain some level of enlightenment about the behavior of topological order on the torus. On physical groups, the space V_1 is easy to describe. We argued before it should be $\mathbb{C}[\mathcal{L}]$. The mapping class group $\mathrm{MCG}(\Sigma_1) = \mathrm{MCG}(T^2)$ is also straightforward to describe. We state a classical theorem about $\mathrm{MCG}(T^2)$ below:

Theorem 2.60 ([FM11], Theorem 2.5)**.** *There is an isomorphism* $\mathrm{MCG}(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$*, which sends the element of* $\mathrm{MCG}(T^2)$ *represented by* $\pi/2$*-rotation to* s*,*

WORK: add picture

and which sends the element of $\mathrm{MCG}(T^2)$ *represented by a Dehn twist around a handle of the torus to* t*.*

Remark 2.61. For simplicity of notation, we will from now on identify $\mathrm{MCG}(T^2)$ with $\mathrm{SL}_2(\mathbb{Z})$, using the letters s, t to represent both the elements of $\mathrm{SL}_2(\mathbb{Z})$ and their corresponding preimage in $\mathrm{MCG}(T^2)$.

A more elementary way of stating this theorem is that every self-homeomorphism of the torus can be deformed to a composition of Dehn twists and rotations, and that the rotation and Dehn twist satisfy the $\mathrm{SL}_2(\mathbb{Z})$ relations $s^4 = 1$ and $(st)^3 = s^2$.

The Dehn twist is described as follows. WORK: describe Dehn twist in terms of theta-symbols θ_A.

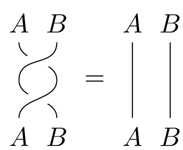
The rotation is described as follows.WORK: describe the rotation in terms of the S-matrix.

Now that we have described $\mathrm{MCG}(T^2) = \mathrm{SL}_2(\mathbb{Z})$ and $V_1 = \mathbb{C}[\mathcal{L}]$, we are tasked with understanding $\rho_1 : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}[\mathcal{L}]$. That is, we must describe the maps $\rho_1(s), \rho_1(t) \in \mathrm{Aut}(\mathbb{C}[\mathcal{L}])$.

]

2.5.3 Bruguieres's modularity theorem and the Verlinde formula

In this section we prove Bruguieres's modularity theorem. This theorem asserts that, given a pre-modular category $\mathcal{C}$, the S-matrix S is invertible if and only if $\mathcal{C}$ is modular. Historically, this theorem is backwards. The original definition of modular category included that the S-matrix should be invertible. This was the only definition of modular category, until Bruguieres proved in [ref] that the invertability of the S-matrix is equivalent to $\mathcal{C}$ having the non-degenerate braiding property that if



for all $B \in \mathcal{C}$ then $A \cong \mathbf{1}$. We are thus stating a historically incorrect definition of modular category, and Bruguieres's modularity theorem tells us that this is equivalent to the original definition. The proof of the modularity theorem relies on the *Verlinde algebra* of $\mathcal{C}$. This algebra will be of use for us in proving other theorems in the future, in particular the Verlinde formula in section [ref].

We define an *algebra* over $\mathbb{C}$ to a vector space V paired with a bilinear map $\cdot : V \times V \to V$ called multiplication, such that multiplication is associative and has a unit. An algebra is called *commutative* if its multiplication is commutative.

We define the Verlinde algebra $K_{\mathbb{C}}(\mathcal{C})$ of $\mathcal{C}$ as follows:

$$K_{\mathbb{C}}(\mathcal{C}) = \left\{ \mathbb{C}[\mathcal{L}] \text{ with algebra structure } \ |[A]\rangle \cdot |[B]\rangle = \sum_{[C]\in\mathcal{L}} N_C^{A,B}\, |[C]\rangle \right\}.$$

We additionally define the function algebra

$$\mathbb{C}[\mathcal{L}]^{\text{func}} = \left\{ \mathbb{C}[\mathcal{L}] \text{ with algebra structure } \left(\sum_{[A]\in\mathcal{L}} c_A\, |[A]\rangle\right) \cdot \left(\sum_{[A]\in\mathcal{L}} c'_A\, |[A]\rangle\right) = \sum_{[A]\in\mathcal{L}} c_A c'_A\, |[A]\rangle \right\}.$$

Lemma 2.62. *Both $K_{\mathbb{C}}(\mathcal{C})$ and $\mathbb{C}[\mathcal{L}]^{func}$ are commutative algebras.*

Proof. The fact that $K_{\mathbb{C}}(\mathcal{C})$ is associative follows from the associativity of the tensor product. Its unit is $|[\mathbf{1}]\rangle$. It is commutative because $\mathcal{C}$ is braided. The fact that $\mathbb{C}[\mathcal{L}]^{\text{func}}$ is a commutative algebra is a standard exercise in algebra. Its unit is $\sum_{[A]\in\mathcal{L}} |[A]\rangle$. □

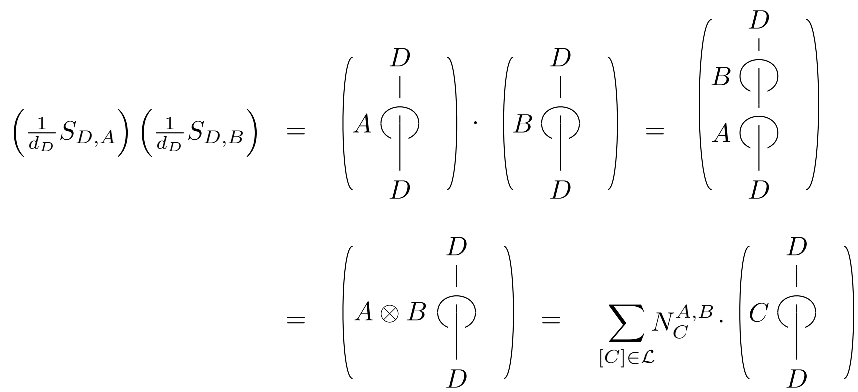
We now state and prove the core theorem which underlies the core properties of the S matrix:

Proposition 2.63. *The map*

$$\begin{aligned}\mathcal{S} : K_{\mathbb{C}}(\mathcal{C}) &\to \mathbb{C}[\mathcal{L}]^{func}\\ |[A]\rangle &\mapsto \sum_{[B]\in\mathcal{L}} \frac{1}{d_B} S_{B,A}\, |[B]\rangle\end{aligned}$$

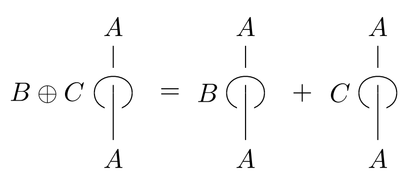
is a morphism of algebras.

Proof. Since it was defined on a basis, $\mathcal{S}$ is clearly a linear map. We now verify that $\mathcal{S}$ preserves multiplication. In the below computation, we identity endomorphisms of simple objects with the unique scalar they are times the identity. We let A, B, D be simple objects.



$$= \sum_{[C]\in\mathcal{L}} N_C^{A,B} \left(\tfrac{1}{d_D} S_{D,C}\right).$$

Note our key use of the fact that



which follows from the facts that $\text{id}_{B\oplus C}$ can be decomposed as projection onto B plus projection onto C, and composition is bilinear. We now conclude that

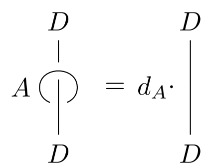
$$
\begin{aligned}
\mathcal{S}\left(|[A]\rangle\right)\cdot\mathcal{S}\left(|[B]\rangle\right) &= \sum_{[D]\in\mathcal{L}}\left(\frac{1}{d_D}S_{D,A}\right)\left(\frac{1}{d_D}S_{D,B}\right)|[D]\rangle \\
&= \sum_{[D]\in\mathcal{L}}\left(\sum_{[C]\in\mathcal{L}} N_C^{A,B}\left(\frac{1}{d_D}S_{D,C}\right)\right)|[D]\rangle \\
&= \mathcal{S}\left(|[A]\rangle\cdot|[B]\rangle\right)
\end{aligned}
$$

as desired. $\square$

By Proposition [ref], we have constructed a map of algebras $\mathcal{S} : K_{\mathbb{C}}(\mathcal{C}) \to \mathbb{C}[\mathcal{L}]^{\text{func}}$. As a map of vector spacces, $\mathcal{S}$ is equal to the S-matrix up to a rescaling of rows by nonzero factors. Hence, it is clear that the S-matrix is invertible if and only if the algebra map $\mathcal{S}$ is invertible. We now use special properties of the map $\mathcal{S}$ to prove the main theorem of the section:

Theorem 2.64 (Bruguieres's modularity theorem)**.** *Let $\mathcal{C}$ be a pre-modular category. The braiding on $\mathcal{C}$ satisfies the non-degenerate braiding axiom if and only if the S-matrix is invertible.*

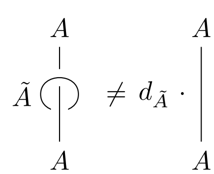
Proof. We observe that $\mathcal{C}$ has a degenerate braiding if and only if there exists some $A \ncong \mathbf{1}$ such that



for all $D \in \mathcal{D}$. If such an element A exists, then clearly $\mathcal{S}(|[A]\rangle) = d_A\mathcal{S}(|[1]\rangle)$. Hence, two linearly independent vectors map to linearly dependent vectors and thus $\mathcal{S}$ is not invertible. Thus, the invertibilty of the $\mathcal{S}$-matrix implies that the braiding is non-degenerate.

We now prove the converse, and hence we suppose that $\mathcal{C}$ has nondegenerate braiding. The proof is in two main steps. First, we prove that $|[\mathbf{1}]\rangle$ is in the image of $\mathcal{S}$. Then, we use the fact that $|[\mathbf{1}]\rangle$ is in the image of $\mathcal{S}$ to construct the rest of the image, which proves that $\mathcal{S}$ is surjective hence invertible.

Part 1: $|[\mathbf{1}]\rangle$ is in the image of $\mathcal{S}$. Since $\mathcal{C}$ has nondegenerate braiding, for all simple objects $A \ncong \mathbf{1}$ there exists some simple object $\tilde{A}$ such that



Thus, the vector $\mathcal{S}(|[\tilde{A}]\rangle) - \frac{S_{A,\tilde{A}}}{d_A}\mathcal{S}(|[\mathbf{1}]\rangle)$ has a cooficient zero of for $|[A]\rangle$ but a non-zero cooficient for $|[\mathbf{1}]\rangle$. Thus, using the product structure on $\mathbb{C}[\mathcal{L}]^{\text{func}}$, we find that the vector

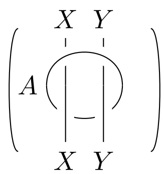
$$
\prod_{\substack{[A]\in\mathcal{L}\\ A\ncong\mathbf{1}}}\left(\mathcal{S}(|[\tilde{A}]\rangle) - \frac{S_{A,\tilde{A}}}{d_A}\mathcal{S}(|[\mathbf{1}]\rangle)\right)
$$

has a cooficient of zero for all $|[A]\rangle$, $A \ncong \mathbf{1}$, but a non-zero cooficient of $|[\mathbf{1}]\rangle$. Hence, it is a scalar multiple of $|[\mathbf{1}]\rangle$. Since $\mathcal{S}$ is a morphism of algebras, it is in the image of $\mathcal{S}$. Hence, $|[\mathbf{1}]\rangle$ is in the image of $\mathcal{S}$.

This completes the first part of the proof. We now use the fact that $|[\mathbf{1}]\rangle$ is in the image of $\mathcal{S}$ to construct the rest of the vectors.

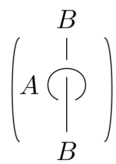
Part 2: $\mathcal{S}$ is surjective. Let $\omega = \sum_{[A]\in\mathcal{L}} \omega_A \,|[A]\rangle \in K_{\mathbb{C}}(\mathcal{C})$ be a vector such that $\mathcal{S}(\omega) = |[\mathbf{1}]\rangle$, which exists by part 1 of the proof. We now compute the quantity

$$h_{X,Y} = \sum_{[A]\in\mathcal{L}} \omega_A \cdot \ \mathrm{tr}$$

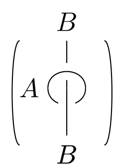


two ways, for all simple objects $X, Y \in \mathcal{C}$. The first way folllows by expanding $X \otimes Y$ as a direct sum and using the fact that $\mathcal{S}(\omega) = |[\mathbf{1}]\rangle$:

$$h_{X,Y} = \sum_{[A]\in\mathcal{L}} \sum_{[B]\in\mathcal{L}} \omega_A N_B^{X,Y} d_B \cdot$$



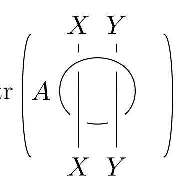
$$= \sum_{[B]\in\mathcal{L}} N_B^{X,Y} d_B \sum_{[A]\in\mathcal{L}} \omega_A \cdot$$



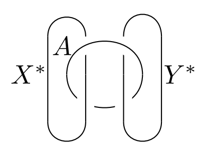
$$= N_{\mathbf{1}}^{X,Y} d_1 = \begin{cases} 1, & X \cong Y^* \\ 0, & \text{otherwise.} \end{cases}$$

In our second way of computing $h_{X,Y}$, we relate the string diagram trace to S-matrix values:

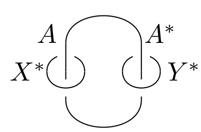
$$\mathrm{tr}$$



$$=$$



$$=$$



$$= \frac{1}{d_A} S_{X,A} S_{Y,A}$$

and thus, combining our two computations, we find

$$h_{X,Y} = \sum_{[A]\in\mathcal{L}} \frac{\omega_A}{d_A} S_{X,A} S_{Y,A} = \begin{cases} 1 & X \cong Y^* \\ 0 & \text{otherwise.} \end{cases}$$

We now define the vector

$$\omega^{(X)} = \sum_{[A]\in\mathcal{L}} \frac{\omega_A}{d_A} S_{A,X^*} \left|[A]\right\rangle$$

for all simple objects $X \in \mathcal{C}$. We compute

$$\begin{aligned}
\mathcal{S}(\omega^{(X)}) &= \sum_{[A]\in\mathcal{L}} \frac{\omega_A}{d_A} S_{A,X^*} \left(\sum_{[Y]\in\mathcal{L}} \frac{1}{d_Y} S_{A,Y} \left|[Y]\right\rangle \right) \\
&= \sum_{[Y]\in\mathcal{L}} \frac{1}{d_Y} \left(\sum_{[A]\in\mathcal{L}} \frac{\omega_A}{d_A} S_{A,X^*} S_{A,Y} \right) \left|[Y]\right\rangle \\
&= \sum_{[Y]\in\mathcal{L}} \frac{h_{X^*,Y}}{d_Y} \left|[Y]\right\rangle = \frac{1}{d_X} \left|[X]\right\rangle .
\end{aligned}$$

Hence $\left|[X]\right\rangle$ is in the image of $\mathcal{S}$ for $[X] \in \mathcal{L}$, as desired.

□

2.5.4 Verlinde formula

In this section, we prove the *Verlinde formula*. This formula was first conjectured by Verlinde [Ver88], and proven the following year by Moore-Seiberg [MS89]. There are now many Verlinde-type formulas. Most imporantly, there is one for vertex operator algebras [Hua08] and one in algebraic geometry [Fal94]. With proposition [ref] in hand, the proof is very quick:

Theorem 2.65 (Verlinde formula)**.** *Let $\mathcal{C}$ be a modular category.*

(i) For all simple objects $A, B, C \in \mathcal{C}$,

$$N_C^{A,B} = \sum_{[E]\in\mathcal{L}} \frac{S_{A,E} S_{B,E} \left(S^{-1}\right)_{C,E}}{d_E}$$

where $(S^{-1})_{C,E}$ denotes the (C, E)-coeffiiceint of the inverse of the S matrix.

(ii) For all simple objects $A \in \mathcal{C}$, the matrix

$$D^A = S N^A S^{-1}$$

is diagonal with $([B], [B])$-entry $S_{A,B}/d_B$, where $N^A = (N_C^{A,B})_{([B],[C])\in\mathcal{L}^2}$ is the fusion matrix of A.

Proof. We begin by proving part (*ii*). The main observation of the proof is that that the operator $N^A : \mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$ is exactly left multiplication by A in $K_{\mathbb{C}}(\mathcal{C})$. Proposition [ref] says that $\mathcal{S}$ is a morphism of algebras, and hence we can commute N^A past $\mathcal{S}$, and turn it into multiplication by A in $\mathbb{C}[\mathcal{L}]^{\text{func}}$. Hence, using the appropriate multiplication in the appropriate algebra, we find

$$
\begin{aligned}
(\mathcal{S}N^A\mathcal{S}^{-1})\,|[B]\rangle &= \mathcal{S}\left(|[A]\rangle \cdot_{K_{\mathbb{C}}(\mathcal{C})} \mathcal{S}^{-1}(|[B]\rangle)\right) \\
&= \mathcal{S}(|A\rangle) \cdot_{\mathbb{C}[\mathcal{L}]^{\text{func}}} |[B]\rangle \\
&= \frac{S_{A,B}}{d_B}\,|[B]\rangle\,.
\end{aligned}
$$

Thus, $\mathcal{S}N^A\mathcal{S}^{-1}$ is diagonal with $([B],[B])$ entry $S_{A,B}/d_B$. Scaling rows of $\mathcal{S}$ does not change the effect of diagonalizition. Hence, we conclude that SN^AS^{-1} is diagonal as well, with the same entries, and thus our proof of (ii) is complete.

We now move on to proving part (i). Expanding the formula $N^A = S^{-1}D^AS$, we find

$$
\begin{aligned}
N^A\,|[B]\rangle &= S^{-1}D^AS\,|[B]\rangle \\
&= S^{-1}\left(\sum_{[E]\in\mathcal{L}} \frac{S_{A,E}S_{B,E}}{d_E}\,|[E]\rangle\right) \\
&= \sum_{[E]\in\mathcal{L}} \frac{S_{A,E}S_{B,E}}{d_E}\left(\sum_{[C]\in\mathcal{L}} (S^{-1})_{C,E}\,|[C]\rangle\right).
\end{aligned}
$$

Comparing coeffients with the definition of N^A, we conclude the result.

□

2.5.5 Proof of modularity

In this section we prove that the S-matrix and T-matrix indeed give a representation of the modular group. That is, we will prove Theorem [ref]. At its heart, the fact that the modular representation of modular category is a homomorphisms comes down to proving a series of relations between the coeffients of the S-matrix and the coeffients of the T-matrix. That is, we are proving a series of relations between braiding and twisting. The general method is to take traces of certain diagrams, and then compute those traces in two ways. One way will involve more twists and the other will involve more braiding. This will give some algebraic relation, and choosing the right diagrams we will get enough algebraic relations to deduce Theorem [ref].

We begin with the most fundamental relationship between S-matrix and T-matrix entries:

Lemma 2.66. *Let $\mathcal{C}$ be a pre-modular category. We have that*

$$
S_{A,B} = \theta_A^{-1}\theta_B^{-1}\sum_{[C]\in\mathcal{L}} N_C^{A,B}\theta_C d_C
$$

and

$$
S_{A^*,B} = \theta_A\theta_B\sum_{[C]\in\mathcal{L}} N_C^{A,B}\theta_C^{-1} d_C.
$$

Proof. By Proposition [ref], we have $\beta_{B,A} \circ \beta_{A,B} = (\theta_A^{-1} \otimes \theta_B^{-1}) \circ \theta_{A \otimes B}$. Taking the trace of this formula we get

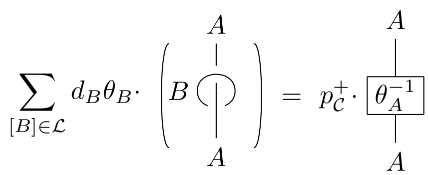
$$S_{A,B} = \theta_A^{-1} \theta_B^{-1} \operatorname{tr}(\theta_{A \otimes B}).$$

Now, since θ is a natural transformation it splits over direct sums. Traces split over direct sums as well by proposition [ref] and hence $\operatorname{tr}(\theta_{A\otimes B}) = \sum_{[C]\in\mathcal{L}} N_C^{A,B} \theta_C d_C$. This concludes the proof of the first formula.

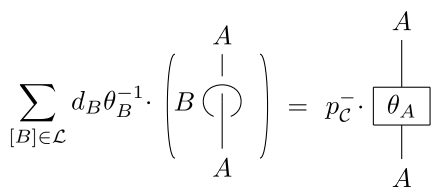
For the second formula, we observe that replacing overcrossings with undercrossings in the definition of $S_{A,B}$ has the effect of taking the dual of one of elements, replacing it with $S_{A^*,B}$. Hence $\operatorname{tr}(\beta_{A,B}^{-1} \circ \beta_{B,A}^{-1}) = S_{A^*,B}$. Thus, taking the trace of the formula $\beta_{A,B}^{-1} \circ \beta_{B,A}^{-1} = \theta_{A\otimes B}^{-1} \circ (\theta_A \otimes \theta_B)$ yields the desired result. □

Before continuing to our proof of theorem [ref] we observe a key lemma:

Lemma 2.67. *Let $\mathcal{C}$ be a pre-modular category. Let $A \in \mathcal{C}$ be a (possibly non-simple) object. We have that*



and



Proof. We only prove the first formula - the second follows by a formally dual argument. We restrict to the case that A is simple. Seeing as both sides are linear with respect to direct sums, the case that A is simple will immedaitely imply the general case. Since A is simple, it suffices to prove that the traces of both sides are equal. The trace on the left hand side has the effect of replacing the diagram with $S_{A,B}$. Hence, we compute as follows using Lemma [ref] and the fact that $\sum_{[C]\in\mathcal{L}} N_C^{A,B} d_B = d_A d_C$ from proposition [ref]:

$$\begin{aligned}
\sum_{[B]\in\mathcal{L}} d_B \theta_B S_{A,B} &= \sum_{[B]\in\mathcal{L}} d_B \theta_B \left(\theta_A^{-1} \theta_B^{-1} \sum_{[C]\in\mathcal{L}} N_C^{A,B} \theta_C d_C \right) \\
&= \theta_A^{-1} \sum_{[C]\in\mathcal{L}} \theta_C d_C \left(\sum_{[B]\in\mathcal{L}} N_C^{A,B} d_B \right) \\
&= \theta_A^{-1} d_A \sum_{[C]\in\mathcal{L}} \theta_C d_C^2 = p_{\mathcal{C}}^{+} \theta_A^{-1} d_A.
\end{aligned}$$

This result is exact the trace of the right hand sides of the lemma. Hence, the proof is complete. □

We now give the heart of the proof of theorem [ref]:

Theorem 2.68. *Let $\mathcal{C}$ be a pre-modular category. Define the charge conjugation operator $\check{C} : \mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$ to the matrix with $([A],[A^*])$ coefficient 1 for all $[A] \in \mathcal{L}$, and all other coefficients zero.*

(i) $\check{C}S = S\check{C}$, $\check{C}T = T\check{C}$, *and* $\check{C}^2 = 1$*;*

(ii) $(ST)^3 = p^+_{\mathcal{C}} S^2$*;*

(iii) $(ST^{-1})^3 = p^-_{\mathcal{C}} S^2 \check{C}$.

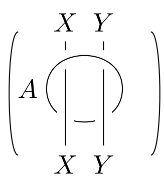
If $\mathcal{C}$ is modular, then

(iv) $S^2 = p^+_{\mathcal{C}} p^-_{\mathcal{C}} \check{C}$.

Proof. Part (i) follows from the fact that $S_{A^*,B^*} = S_{A,B}$, $\theta_{A^*} = \theta_A$, and $A^{**} \cong A$. Parts (ii) and (iii) have formally dual proofs, which arise from replacing θ with θ^{-1} at every opportunity. Part (iv) follows algebraically from combining formulas (ii) and (iii) whenever S is invertible, which is always the case when $\mathcal{C}$ is modular by theorem [ref].

Hence, it suffices to prove part (ii). The proof comes from comuting the quantity

$$h_{X,Y} = \sum_{[A]\in\mathcal{L}} d_A \theta_A \cdot \ \mathrm{tr}$$



two ways.

In the first way of computing $h_{X,Y}$, we use lemma [ref]. We find the following:

$$h_{X,Y} = p^+_{\mathcal{C}} \cdot \ \mathrm{tr}\left(\boxed{\theta^{-1}_{X\otimes Y}} \right) = p^+_{\mathcal{C}} \theta_X \theta_Y S_{X^*,Y}$$

In our second way of computing $h_{X,Y}$, we use computation of the trace of two lines through a loop in the proof of Theorem [ref]. We find this way that

$$h_{X,Y} = \sum_{[A]\in\mathcal{L}} \theta_A S_{X,A} S_{Y,A}.$$

Thus, we find that $\sum_{[A]\in\mathcal{L}} \theta_A S_{X,A} S_{Y,A} = p^+_{\mathcal{C}} \theta_X \theta_Y S_{X^*,Y}$. Thinking of these quantites as the $([X],[Y])$ entries in operators $\mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$, we get the equation

$$STS = p^+_{\mathcal{C}} TST\check{C}.$$

WORK: This formula is WRONG. It should be

$$STS = p^+_{\mathcal{C}} T^{-1} S T^{-1}.$$

From this we get

$$(ST)^3 = p^+_{\mathcal{C}} S^2$$

as desired. I'm not sure where I went wrong, but something is off in here. ☐

We now know that the S and T matrices give a modular representation *up to phase*! We still need to work out the details of the phases. A key part is the following computation:

Corollary 2.69. *Let $\mathcal{C}$ be a modular category. The quantities $p_{\mathcal{C}}^{+}$ and $p_{\mathcal{C}}^{-}$ are nonzero, and*

$$p_{\mathcal{C}}^{+} p_{\mathcal{C}}^{-} = \mathcal{D}^2.$$

Proof. The values $p_{\mathcal{C}}^{+}$ and $p_{\mathcal{C}}^{-}$ must be nonzero because S is invertible and $S^2 = p_{\mathcal{C}}^{+} p_{\mathcal{C}}^{-} \check{C}$. The formula $S^2 = p_{\mathcal{C}}^{+} p_{\mathcal{C}}^{-} \check{C}$, when expanded, says that

$$\sum_{[C]\in\mathcal{L}} S_{C,A} S_{C,B} = \begin{cases} p_{\mathcal{C}}^{+} p_{\mathcal{C}}^{-} & A \cong B^* \\ 0 & \text{otherwise.} \end{cases}$$

Applying this formula to $A = B = \mathbf{1}$, we find

$$\sum_{[C]\in\mathcal{L}} S_{C,\mathbf{1}} S_{C,\mathbf{1}} = \sum_{[C]\in\mathcal{L}} d_C^2 = p_{\mathcal{C}}^{+} p_{\mathcal{C}}^{-}$$

as desired. $\square$

Now, it is clear that we can normalize the representation appropriately and conclude theorem [ref], as desired.

2.5.6 Vafa's theorem, unitarity of S-matrix, and the Chiral central charge

In this section we discuss some finer points of the structure of the modular representation. In particular, we will prove that modular representation of every modular category is *unitary*. That is, the S and T matrices are both unitary operators on $\mathbb{C}[\mathcal{L}]$ when it is endowed with its canonical inner product coming from its basis

We begin with the matrix T. For a diagonal matrix to be unitary, it is neccecary and sufficient for its diagonal entries to have absolute value 1. We will prove something even stronger: that all of the entries are roots of unity! We recall that a number $z \in \mathcal{C}$ is called a root of unity if $z^n = 1$ for some integer $n \geq 1$. We begin with a key topological lemma which will underscore our proof:

Lemma 2.70 (Lantern identity)**.** *Let $\mathcal{C}$ be a pre-modular category. Let $A, B, C \in \mathcal{C}$ be objects. As maps $A \otimes B \otimes C \to A \otimes B \otimes C$, we have the identity*

$$\theta_{A\otimes B} \circ \theta_{A\otimes C} \circ \theta_{B\otimes C} = \theta_{A\otimes B\otimes C} \circ (\theta_A \otimes \theta_B \otimes \theta_C).$$

Proof. In the language of string diagrams, this formula becomes

WORK: add diagram.

It is a matter of elementary manipulations to convince one's self that these two diagrams are equal. An alternate algebraic approach is to expand the relation both sides using the formula $\theta_{X\otimes Y} = (\beta_{Y,X} \circ \beta_{X,Y}) \circ (\theta_X \otimes \theta_Y)$, cancel $\theta_A^2 \otimes \theta_B^2 \otimes \theta_C^2$, and compare the resulting braids using the hexagon and naturality. $\square$

We state one more linear-algebraic lemma neccecary for the proof:

Lemma 2.71. *Let $n \geq 1$ be an integer and let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator whose (a, b) entry is $M_{a,b}$. Suppose that all off diagonal entries of M are positive, and that $\sum_{b=1}^{n} M_{a,b} < 0$ for all $1 \leq a \leq n$. Then, M is invertible.*

Proof. WORK: This follows from the Gershgorin circle theorem, but that's way to general for our purposes. Find a nice proof which is appropriate for the situation. □

Theorem 2.72 (Vafa). *Let $\mathcal{C}$ be a pre-modular category. The values θ_A are roots of unity for all simple object $A \in \mathcal{C}$.*

Remark 2.73. This theorem is interesting in part due to its history. WORK: add history.

Proof. To begin, we choose a simple obejct A. We observe that every endomorphism of $A \otimes A^* \otimes A$ induces a linear map

$$\mathrm{Hom}(A, A \otimes A^* \otimes A) \to \mathrm{Hom}(A, A \otimes A^* \otimes A)$$

by postcomposition. We now consider the lantern identity (proposition [ref]) on the stands A, A^*, A,

$$\theta_{A\otimes A^*} \circ \theta_{A\otimes A} \circ \theta_{A^*\otimes A} = \theta_{A\otimes A^*\otimes A} \circ (\theta_A \otimes \theta_{A^*} \otimes \theta_A).$$

viewed as an equality of linear operators on $\mathrm{Hom}(A, A \otimes A^* \otimes A)$. We compute the determinant of both sides. We first compute the determinant of $\theta_{A\otimes A^*}$. The eigenvalues of the operator $\theta_{A\otimes A^*}$ are the twists θ_B. The dimension of the θ_B eigenspace is exactly the dimension of the subspace of $\mathrm{Hom}(A, A\otimes A^*\otimes A)$ in which $A\otimes A^*$ fuse to B. The dimension of this space is $N_B^{A,A^*} N_A^{B,A} = (N_B^{A,A^*})^2$. The determinant is equal to the product of the eigenvalues counted with multiplicity, and hence

$$\det \theta_{A\otimes A^*} = \prod_{[B]\in\mathcal{L}} \theta_B^{\left(N_B^{A,A^*}\right)^2}.$$

Continuing this way for all the other twists and plugging them into the lantern identity we get

$$\prod_B \theta_B^{2\left(N_B^{A,A^*}\right)^2 + \left(N_B^{A,A}\right)^2} = \theta_A^{4\cdot \dim \mathrm{Hom}(A, A\otimes A^*\otimes A)}.$$

We now define the coefficients

$$M_{A,B} = \begin{cases} 2\left(N_B^{A,A^*}\right)^2 + \left(N_B^{A,A}\right)^2 & A \not\cong B \\ 2\left(N_B^{A,A^*}\right)^2 + \left(N_B^{A,A}\right)^2 - 4\cdot \dim \mathrm{Hom}(A, A\otimes A^* \otimes A) & A \cong B. \end{cases}$$

We observe that all of the off-diagonal entries of $M_{A,B}$ are positive and

$$\begin{aligned} \sum_{[B]\in\mathcal{L}} M_{A,B} &= \sum_{[B]\in\mathcal{L}} 2\left(N_B^{A,A^*}\right)^2 + \left(N_B^{A,A}\right)^2 - 4\cdot \dim \mathrm{Hom}(A, A\otimes A^*\otimes A) \\ &= -\dim \mathrm{Hom}(A, A\otimes A^* \otimes A) < 0. \end{aligned}$$

Thus, by lemma [ref], we conclude that the matrix $M = (M_{A,B})_{([A],[B])\in\mathcal{L}^2} : \mathbb{C}[\mathcal{L}] \to \mathbb{C}[\mathcal{L}]$ is invertible. Let $\tilde{M}$ be the adjugate of M. That is, an integer valued matrix such that $M \cdot \tilde{M} = \tilde{M} \cdot M = n$ where $n = \det M$. We find for all simple objects A that

$$\theta_A^n = \prod_{[C]\in\mathcal{L}} \left(\prod_{[B]\in\mathcal{L}} \theta_B^{M_{C,B}} \right)^{\tilde{M}_{A,C}} = 1,$$

and hence θ_A is a root of unity as desired.

□

We immediately get several corollaries from this theorem. The first is that braiding enough times gets you back where you started:

Corollary 2.74. *Let $\mathcal{C}$ be a pre-modular category. There exists an integer $n \geq 1$ such that*

$$(\beta_{B,A} \circ \beta_{A,B})^n = \mathrm{id}_{A\otimes B}$$

for all $A, B \in \mathcal{C}$

Proof. Choose $n \geq 1$ so that $\theta_C^n = 1$ for all simple objects $C \in \mathcal{C}$, which exists by Vafa's theorem (theorem [ref]). Seeing that $\beta_{B,A} \circ \beta_{A,B} = \theta_{A\otimes B} \circ (\theta_A^{-1} \otimes \theta_B^{-1})$, in the direc sum decomposition $A \otimes B \cong \bigoplus_{[C]\in\mathcal{L}} N_C^{A,B} \, |[C]\rangle$ the transformation $\beta_{B,A} \circ \beta_{A,B}$ acts by the scalar $\theta_C/(\theta_A\theta_B)$ on every $|[C]\rangle$ summand. Thus, $(\beta_{B,A} \circ \beta_{A,B})^n$ acts by $(\theta_C/(\theta_A\theta_B))^n = 1$ and thus $(\beta_{B,A} \circ \beta_{A,B})^n$ is the identity as desired. □

Corollary 2.75. *Let $\mathcal{C}$ be a modular category. The quantity $p_{\mathcal{C}}^-/p_{\mathcal{C}}^+$ is a root of unity.*

Proof. We take determinants. From $S^2 = p_{\mathcal{C}}^+ p_{\mathcal{C}}^- \check{C}$ we find that $\det(S)^2 = \pm p_{\mathcal{C}}^+ p_{\mathcal{C}}^-$, since $\det \check{C} = \pm 1$. From the formula $(ST)^3 = p_{\mathcal{C}}^+ S^2$ we find that

$$\left(p_{\mathcal{C}}^+/\det(S)\right)^2 = \det(T)^6$$

so $p_{\mathcal{C}}^+/p_{\mathcal{C}}^- = \pm \det(T)^6$. By Vafa's theorem $\det(T)$ is a root of unity. Hence, we conclude that $p_{\mathcal{C}}^+/p_{\mathcal{C}}^-$ is a root of unity as desired. □

We take a moment to observe that the quantity $p_{\mathcal{C}}^-/p_{\mathcal{C}}^+$ is of great interest to physicists. It is related to the *chiral central charge* c_{top} of the theory by the formula

$$p_{\mathcal{C}}^-/p_{\mathcal{C}}^+ = e^{-\frac{2\pi i c_{\mathrm{top}}}{4}}.$$

WORK: probably say a bit more? Maybe there will be a better spot somewhere else.

We now move on to proving that the S matrix is unitary. The main technical ingredient is as follows:

Proposition 2.76. *Let $\mathcal{C}$ be a modular category. For all simple objects $A, B \in \mathcal{C}$ we have that $S_{A^*,B} = \overline{S_{A,B}}$.*

Proof. By the Verlinde formula [ref], we know that for all simple objects A there exists a vector $\mathbf{v}_B \in \mathbb{C}[\mathcal{L}]$ such that

$$N^A \mathbf{v}_B = \frac{S_{A,B}}{d_B} \mathbf{v}_B$$

for all simple objects A. Namely, $\mathbf{v}_B$ is the $[B]$-collumn of the S matrix. Let $\mathbf{v}_B^*$ be the row vector which is the Hermitian adjoint to $\mathbf{v}_B$. We have that

$$\mathbf{v}_B N^A \mathbf{v}_B = \frac{S_{A,B}}{d_B} |\mathbf{v}_B|^2.$$

Now, we observe that by Frobenius reciprocity (proposition [ref]) $(N^A)^\dagger = N^{A^*}$. Hence,

$$\begin{aligned} \mathbf{v}_B N^A \mathbf{v}_B &= \left(N^{A^*} \mathbf{v}_B\right)^* \mathbf{v_B} \\ &= \left(\frac{S_{A^*,B}}{d_B} \mathbf{v}_B\right)^* \mathbf{v_B} \\ &= \overline{\frac{S_{A^*,B}}{d_B}} |\mathbf{v}_B|^2. \end{aligned}$$

Comparing, we get the desired result.

□

We now get the following theorem:

Theorem 2.77 (Etingof-Nikshych-Ostrik)**.** *Every matrix in the image of the modular representation* $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[\mathcal{L}])$ *is a unitary operator on* $\mathbb{C}[\mathcal{L}]$.

Proof. We already know by theorem [ref] and corollary [ref] that the normalized T-matrix is unitary. It thus suffices to show that the modular S matrix is unitary. From theorem [ref] we have that $\left(\frac{1}{\mathcal{D}}S\right)^{-1} = (\frac{1}{\mathcal{D}}S) \cdot \check{C}$. By proposition [ref] $(\frac{1}{\mathcal{D}}S) \cdot \check{C} = (\frac{1}{\mathcal{D}}S)^\dagger$ and thus $\frac{1}{\mathcal{D}}S$ is unitary as desired. □

2.6 Skeletonization

2.6.1 Principle

WORK: lots of big choices need to be made here. Do I call this the skeletonization, or do I call it something else? Do I work with multiplicity-free categories, or do I allow multiplicity? I don't know what the correct statements are or what the proofs look like so this section might be a tough one. A good mathematical reference is [DHW13].

2.6.2 F-symbols

2.6.3 R-symbols

2.6.4 θ-symbols

2.6.5 Reconstruction theorem

2.7 Quantum double modular categories

2.7.1 The Drinfeld center

A quantum double modular category is a special type of modular category. They are particularly important because many of the constructions of topological order only deal with quantum double modular categories. For instance, there are constructions of modular categories/topological order coming from the theory of tensor networks [ref], subfactors [ref], vertex operator algebras [ref], WORK: add more sources. All of these constructions only give quantum double modular categories. Hence, understanding quantum doubles is key to understanding how topological order work in practice.

At the heart of quantum doubles is a construction known as the *Drinfeld center*. In its most basic form the Drinfeld center induces an assignment

$$\mathcal{Z} : (\text{monoidal categories}) \to (\text{braided monoidal categories}).$$

In our context, we care about a more structured version of the Drinfeld center. It is a theorem of Muger that the Drinfeld center induces an assignment as follows:

$$\mathcal{Z} : (\text{spherical fusion categories}) \to (\text{modular categories}).$$

This theorem is fantastic because it allows one to construct modular categories using much less data than would otherwise be neccecary. Without needing a braiding, non-degenerate or otherwise, the Drinfeld center allows one to construct a modular category. This makaes the Drinfeld center an abundant source of modular categories. We call an modular category $\mathcal{C}$ a quantum double if it is of the form $\mathcal{Z}(\mathcal{C}_0)$ for some spherical fusion category $\mathcal{C}_0$. A a major goal of this chapter is to set up and prove Muger's theorem

We now define the Drinfeld center. The Drinfeld center is a somewhat direct categorification of the usual notion of center for finite groups. If G is a finite group, its center is defined as follows:

$$Z(G) = \{g \in G | gh = hg \; \forall h \in G\}.$$

The first guess at $\mathcal{Z}(\mathcal{C})$ is thus

$$\mathcal{Z}(\mathcal{C}) = \{A \in \mathcal{C} | A \otimes B \cong B \otimes A \; \forall B \in \mathcal{C}\}.$$

This is almost correct, but not quite. The issue is that $\mathcal{Z}(\mathcal{C})$ is not quite a braided monoidal category yet. Even though $A \otimes B \cong B \otimes A$ for all $A, B \in \mathcal{Z}(\mathcal{C})$, we don't have a distinguished choice of isomorphism. A braided monoidal category requires a distinugished isomorphism $\beta_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$. Moreover, these distinguished isomorphisms are required to satisfy the hexagon equations. Hence, we make a new definition of center which keeps track of the choice of isomorphism and enforces the hexagon equation along the way:

Proposition 2.78. *The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a monoidal category $\mathcal{C}$ is a braided monoidal category defined as follows:*

- *(Objects) Pairs $(A, \beta_{A,-})$, where $A \in \mathcal{C}$, and $\beta_{A,-}$ is a natural isomorphism of monoidal natural isomorphism between the two functors $A \otimes -$ and $- \otimes A$ from $\mathcal{C}$ to $\mathcal{C}$, satisfying the additional condition that*

$$\beta_{A,B\otimes C} = (\mathrm{id}_B \otimes \beta_{A,C}) \circ (\beta_{A,B} \otimes \mathrm{id}_C).$$

- *(Morphisms) Given $(A, \beta_{A,-}), (B, \beta_{B,-}) \in \mathcal{Z}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((A, \beta_{A,-}), (B, \beta_{B,-}))$ is the subspace of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that for all $C \in \mathcal{C}$*

$$(\mathrm{id}_C \otimes f) \circ \beta_{A,C} = \beta_{B,C} \circ (f \otimes \mathrm{id}_C).$$

- *(Tensor product) Given* $(A, \beta_{A,-}), (B, \beta_{B,-}) \in \mathcal{Z}(\mathcal{C})$*, we define*

$$(A, \beta_{A,-}) \otimes (B, \beta_{B,-}) = (A \otimes B, (\beta_{A,-} \otimes \mathrm{id}_{\mathcal{C}}) \circ (\mathrm{id}_{\mathcal{C}} \otimes \beta_{B,-}))\,.$$

- *(Unit) The element* $(1, \rho \circ \lambda^{-1})$
- *(Braiding) We define the braiding between two elements* $(A, \beta_{A,-}), (B, \beta_{B,-}) \in \mathcal{Z}(\mathcal{C})$ *to be* $\beta_{A,B} = \beta_{A,B}$*.*

Inheriting associativity, unitors, and composition from $\mathcal{C}$*, this gives* $\mathcal{Z}(\mathcal{C})$ *the structure of a braided monoidal category.*

Proof. Since morphisms in $\mathcal{Z}(\mathcal{C})$ are a subspace of morphisms in $\mathcal{C}$, commutative diagrams don't change when going from $\mathcal{C}$ to $\mathcal{Z}(\mathcal{C})$. Hence, the triangle and pentagon axioms for $\mathcal{Z}(\mathcal{C})$ follow immediately from the triangle and pentagon axioms on $\mathcal{C}$. One thing to be checked is that evaluation/co-evaluaiton satisfy the compatibility condition required to a morphism in $\mathcal{Z}(\mathcal{C})$, but this is straightforward. We remark on the hexagon identities. The condition imposed on $\beta_{A,B\otimes C}$ given is technically incorrect. To make the parentheses work in the braiding one has to add associators, and impose the longer condition

$$\beta_{A,B\otimes C} = \alpha^{-1}_{C,A,B} \circ (\mathrm{id}_B \otimes \beta_{A,C}) \circ \alpha_{A,C,B} \circ (\beta_{A,B} \otimes \mathrm{id}_C) \circ \alpha^{-1}_{A,B,C}.$$

This condition makes the second hexagon identity tautological. Similarly, the definition of tensor product given is not strictly correct - one must add the correct associator terms, making the first hexagon identity immediate. Lastly one must verify the half-braidings defined on the tensor unit/tensor product are actually half braidings, i.e., that they satisfy the hexagon condition. These follow from straightforward computations, which we leave as exercises. This completes the proof.

□

2.7.2 Muger's theorem

WORK: through Mueger's theorem. The exposition will greatly differ based on what the proof looks like, which I haven't done before. The standard proof people use is completeley steeped in the language of module categories. The original proof uses the tube algebra, and is "elementary" in the sense that you don't need tube algebras but it is heineous, and I would very much like to avoid it. I'll figure out what to write hence once I've done the module category section and I've digested the proof. Maybe there's a way to de-module-ify it, but maybe not. Maybe I don't want to do that if it'll lose its essence.

2.7.3 Discrete gauge theory as a quantum double and Morita equivalence

We saw in chapter [ref] that $\mathbf{Vec}_G$ anad $\mathrm{Rep}(G)$ are both naturally spherical fusion categories. Thus, Muger's theorem tells us that $\mathcal{Z}(\mathbf{Vec}_G)$ and $\mathcal{Z}(\mathrm{Rep}(G))$ are both modular categories. Hence, given a finite group G we have three different modular categories we can associate to it: $\mathfrak{D}(G)$, $\mathcal{Z}(\mathbf{Vec}_G)$, $\mathcal{Z}(\mathrm{Rep}(G))$. The amazing fact is that these are all the same category:

Proposition 2.79. *Let* G *be a finite group. There are equivalences of modular categories* $\mathfrak{D}(G) \cong \mathcal{Z}(\mathbf{Vec}_G) \cong \mathcal{Z}(\mathrm{Rep}(G))$*.*

Proof. WORK: do proof. Shouldn't be too hard. ☐

We now make a few comments about this theorem. The first is that it proves that $\mathfrak{D}(G)$ is a quantum double modular category. Secondly, it gives a second proof that $\mathfrak{D}(G)$ has a non-degenearte braiding, using Muger's theorem. Thirdly, it demonstrates the concept of *Morita equivalence*.

WORK: introduce Morita equivalence. I know that there's some important basic facts to tell - I should include those.

2.7.4 Factorizability and time reversal symmetry

Given a modular category $\mathcal{C}$, we can forget the braiding on $\mathcal{C}$ and only remember its structure as a spherical fusion category. Hence, Muger's theorem tells us that $\mathcal{Z}(\mathcal{C})$ is canonically a modular category. It is a fantastic fact that in this case $\mathcal{Z}(\mathcal{C})$ can be explicitely computed in terms of $\mathcal{C}$. We describe this computation now.

WORK: Define the time-reversal conjugate $\overline{\mathcal{C}}$. Setup the map $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C})$

Proposition 2.80. *Let $\mathcal{C}$ be a pre-modular category. The canonical map $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{Z}(\mathcal{C})$ is an equivalence of categories if and only if $\mathcal{C}$ is modular.*

Proof. WORK: proof ☐

This theorem is fantastic because it not only computes $\mathcal{Z}(\mathcal{C})$ for every modular category $\mathcal{C}$, but also it gives an equivalent definition of modularity. This gives us our third definition of modularity. Namely a pre-modular category $\mathcal{C}$ is modular if and only if its braidings are all non-degenerate, or equivalently if its S-matrix is non-degenerate, or equivalently if it is factorizable in the above sense.

2.7.5 Levin-Wen model

WORK: : work though the Levin-Wen model. I think that this model is fantastic because it shows how all of the ideas of tensor category theory can manifest themselves extremely concretelely on the level of gapped Hamiltonians. Namely, the coherence relations on the category theory side correspond exactly the the formulas needed to make terms in a Hamiltonian commute with one another. It would be nice if I could give a motivation for which the category which describes the Levin-Wen model is the Drinfeld center, though I've never seen that before.

2.8 Unitarity

2.8.1 Characterization of unitarizability

WORK: An early reference about unitary MTCs is [KJ96]. I should read this.

As we've mentioned beofre, the algebraic theory of topological order is not modular category theory but *unitary* modular category theory because hom-spaces need Hilbert space structure to define valid quantum systems. We define unitary fusion categories and unitary modular categories in section [ref].

We've mostly ignored unitary fusion categories for the following reasons: their additional structure is a large semantic burden, and is in a way inessential. We mean this in the following sense:

1. Given a spherical fusion category, $\mathcal{C}$ every unitary structure on $\mathcal{C}$ is equivalent. That is, all the unitary fusion categories obtained by enriching $\mathcal{C}$ with unitary structure are equivalent to one another as unitary fusion categories;
2. All braidings on unitary fusion categories are automattically unitary so there is no subtelty in passing to unitary modular categories;
3. A spherical fusion category admits a unitary structure if and only if its quantum dimensions are all positive.

Still, it is worthwhile to study unitary fusion categories. They are the correct algebraic structure to describe topological order, so understanding *why* points (1-3) above are true gives insight into unitarity in practice. There is also a characterization of unitary fusion categories in the skeletized perspective which is of much utility in practice. Namely, a fusion category admits a unitary structure if and only if its F-matrices and R-matrices can all be made unitary. We discuss this more and offer a proof in section [ref].

We now prove the first main result of the section:

Proposition 2.81. *Let $\mathcal{C}$ be a spherical fusion category. There is a compatible unitary structure on $\mathcal{C}$ if and only if the quanutum dimensions d_A are positive for all simple objects $A \in \mathcal{C}$.*

Proof. Suppose first that $\mathcal{C}$ admits a unitary structure. Then, for all simple objects $A \in \mathcal{C}$,

$$d_A = \mathrm{tr}(\mathrm{id}_A) = \langle \mathrm{id}_A \,|\, \mathrm{id}_A \rangle > 0$$

because the inner product is positive definite. Conversely, suppose that all of the quantum dimensions of $\mathcal{C}$ are positive. We will suppose for simplicity that $\mathcal{C}$ is skeletal, which is possible by proposition [ref]. Suppose that $f : A \to B$ is a morphism between objects $A, B \in \mathcal{C}$. We know that there are direct sum decompositions $A = \bigoplus_i A_i$ and $B = \bigoplus_j B_j$. Writing f as an element of $\bigoplus_{i,j} \mathrm{Hom}(A_i, B_j)$, we find that f acts by some scalar $f_{i,j}$ on each homspace $\mathrm{Hom}(A_i, B_j)$, where $f_{i,j} = 0$ if $A \ncong B$. We define $f^\dagger : B \to A$ to be the map whose $\mathrm{Hom}(B_j, A_i)$ is $f^\dagger_{j,i} = \overline{f_{i,j}}$.

We now prove that this defines a unitary structure on $\mathcal{C}$. Given any two morphism $f : A \to B$, we compute

$$\langle f \,|\, f \rangle = \mathrm{tr}(f^\dagger \circ f) = \sum_{i,j} |f_{i,j}|^2 d_{A_i}.$$

Clearly, $\langle f \,|\, f \rangle = 0$ if and only if $f = 0$ because all of the quantum dimensions d_{A_i} are positive real numbers. Hence, $\langle \cdot \,|\, \cdot \rangle$ is positive definition.

It is clear that $(f^\dagger)^\dagger = f$, $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$, and $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for appropriate choices of f, g. It remains to show that $\left(\mathrm{ev}_A^R\right)^\dagger = \mathrm{coev}_A^L$ and $\left(\mathrm{coev}_A^R\right)^\dagger = \mathrm{ev}_A^L$.

WORK: I don't know how to do this part of the proof. It's something to come back to. ☐

2.8.2 Unitary braidings

WORK: Show that every braiding is automatically unitary. This is the content of [Gal14].

2.8.3 Uniqueness of unitary structure

WORK: Show that unitary structures are unique. This is the content of [Reu23b].

2.8.4 Skeletonization of unitarity

WORK: Show that a fusion category admits a unitary structure if and only if its F and R symbols can be made unitary in some basis. There are good references for this in the papers cited above.

2.9 Number theory in modular categories

2.9.1 Techniques and first results

WORK: A good general reference about this stuff is [GS19]. To-read, for sure. There is also interesting work in [MS12] which gives some counterexamples. Another reference is [DHW13].

2.9.2 Galois conjugation

WORK: An early reference is [CG93]. Some other papers not to ignore in this area are [PSYZ23, BR22].

2.9.3 Ocneanu rigidity

2.9.4 Rank-finiteness theorem

WORK: Of course there is the original paper on the topic. However, there is also the generaliation for G-crossed MTCs and fermionic MTCs - [JMNR21]. Should I bring this up now or later?

2.9.5 Schauenberg-Ng theorem

WORK: Go through Schauenberg-Ng's original paper and understand the proof. It seems on the face of it like it is a hard theorem. Certainly, it uses strongly the theory of the Drinfeld center as well as the modular representation. It is good to push this proof as far down as possible since it will use a lot of machinery. I think it can be boiled down to something elegant, though, if the machinery has been set up.

WORK: I would quite like to show that fusion categories (or at least modular categories) have finitely automorphisms of the identity functor. The proof I know follows from generalities about the universal grading group - [GN08]. I wonder if this proof can be done completely without the use of grading. Then, the fact that the universal grading group can be interpreted in terms of grading can be put as an exercise in the "further structure" section!

WORK: this section is going to host a lot more theorems

History and further reading:

Modular categories were born from conformal field theory in the late 1980s. In a series of papers, Moore and Seiberg analysed deeply the underlying content within conformal field theory to find what essential algebraic data lied within it [MS88, MS89]. They wrote out the axioms of this essential algebraic data in their subsequent notes on conformal field theory [MS90]. They used the name modular category to describe their data, as suggested by Igor Frenkel. This definition was then refined and re-introduced by Turaev [Tur92]. The first major application of modular categories was the Reshetikhin-Turaev construction [RT91, Tur10b]. Prior to this result nobody had succeed in constructing topological quantum field theories. In this way, modular categories and the Reshetikhin-Turaev construction completed Witten's programme of quantizing Chern-Simons theory.

By the early 2000s, the proposal of topological quantum computing was attracting a lot of interest in anyons and their algebraic properties. Seeing as topological order can be described by topological quantum field theory and topological quantum field theory is essentially equivalent to modular categories, it was understood that modular categories could be used to understand topological order. This latent description of anyons in terms of modular categories was made explicit in an appendix in the seminal 2006 paper of Kitaev [Kit06]. This approach to anyons in terms of modular categories was popularized by Wang's early monograph [Wan10]. This has since become the standard approach towards the algebraic theory of topological quantum information.

Exercises:

2.1. WORK: apply Verlinde formula to group-theoretical modular categories to recover classical theorem by Burnside

2.2. WORK: show that irriducible G-graded G-reps are equivalent to irriducible reps of centralizers of conjugacy classes

2.3. WORK: One of the pivotal axioms automatically holds in fusion categories. Namely, the one with the twists going the two different ways. The proof is very standard - reduce to the case A, B are both simple, reduce to case $A = B$, $f = \lambda \cdot \mathrm{id}_A$ and hence commutes with everything. This would be a nice exercise.

2.4. WORK: there's a notion of prime factorization of MTCs. If $\mathcal{C}$ is an MTC, then it can be decomposed as a tensor product

$$\mathcal{C} = \mathcal{C}_1 \boxtimes \mathcal{C}_2 ... \boxtimes \mathcal{C}_n,$$

where $\mathcal{C}_i \neq 0$ all have no proper non-degenerate braided fusion subcategories other than $\mathbf{Vec}_{\mathbb{C}}$. We call suchMTCs *prime MTCs*. Suppose that $\mathcal{C}$ has NO abelian anyons. Then, the decomposition is unique. However, if $\mathcal{C}$ has abelian anyons then the decomposition can fail to be unique. In particular, the color code is equivalent to both bilayer toric code and bilayer 3-fermion (c.f. [KPEB18]). The factorization results I asserted come from the (largely ignored?) paper by Muger: [Müg02]. This could make for a very nice exericse. First, prove Muger's double centralizer theorem. Then, prove the factorization into prime MTCs. Then, if $\mathcal{C}$ has no abelian anyon anyons, prove that the decomposition is unique. Then, prove that bilayer toric code is equivalent to bilayer 3-fermion (or maybe just leave that as a comment)!

[WORK:
List of things to add:

- A big "physics-math dictionary" which allows the translation of everything;
- A table of notation;

]

References

[AA81] Marcia Ascher and Robert Ascher. Code of the quipu. *Ann Arbor: University of Michigan Press*, pages 56–74, 1981.

[AA11] Dorit Aharonov and Itai Arad. The bqp-hardness of approximating the jones polynomial. *New Journal of Physics*, 13(3):035019, 2011.

[Aar13] Scott Aaronson. *Quantum computing since Democritus*. Cambridge University Press, 2013.

[AB59] Yakir Aharonov and David Bohm. Significance of electromagnetic potentials in the quantum theory. *Physical review*, 115(3):485, 1959.

[AFM20] David Aasen, Paul Fendley, and Roger SK Mong. Topological defects on the lattice: dualities and degeneracies. *arXiv preprint arXiv:2008.08598*, 2020.

[Ale28] James W Alexander. Topological invariants of knots and links. *Transactions of the American Mathematical Society*, 30(2):275–306, 1928.

[Ali10] Jason Alicea. Majorana fermions in a tunable semiconductor device. *Physical Review B—Condensed Matter and Materials Physics*, 81(12):125318, 2010.

[AMF16] David Aasen, Roger SK Mong, and Paul Fendley. Topological defects on the lattice: I. the ising model. *Journal of Physics A: Mathematical and Theoretical*, 49(35):354001, 2016.

[AMV18] NP Armitage, EJ Mele, and Ashvin Vishwanath. Weyl and dirac semimetals in three-dimensional solids. *Reviews of Modern Physics*, 90(1):015001, 2018.

[ASW84] Daniel Arovas, John R Schrieffer, and Frank Wilczek. Fractional statistics and the quantum hall effect. *Physical review letters*, 53(7):722, 1984.

[ASWZ85] Daniel P Arovas, Robert Schrieffer, Frank Wilczek, and Anthony Zee. Statistical mechanics of anyons. *Nuclear Physics B*, 251:117–126, 1985.

[AT77] PHILIP W Anderson and G Toulouse. Phase slippage without vortex cores: vortex textures in superfluid he 3. *Physical Review Letters*, 38(9):508, 1977.

[AWH22] David Aasen, Zhenghan Wang, and Matthew B Hastings. Adiabatic paths of hamiltonians, symmetries of topological order, and automorphism codes. *Physical Review B*, 106(8):085122, 2022.

[BB23] Arman Babakhani and Parsa Bonderson. G-crossed modularity of symmetry enriched topological phases. *Communications in Mathematical Physics*, 402(3):2979–3019, 2023.

[BBCW19] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. Symmetry fractionalization, defects, and gauging of topological phases. *Physical Review B*, 100(11):115147, 2019.

[BCG+24] Sergey Bravyi, Andrew W Cross, Jay M Gambetta, Dmitri Maslov, Patrick Rall, and Theodore J Yoder. High-threshold and low-overhead fault-tolerant quantum memory. *Nature*, 627(8005):778–782, 2024.

[BDSPV15] Bruce Bartlett, Christopher L Douglas, Christopher J Schommer-Pries, and Jamie Vicary. Modular categories as representations of the 3-dimensional bordism 2-category. *arXiv preprint arXiv:1509.06811*, 2015.

[BGH+17] Paul Bruillard, César Galindo, Tobias Hagge, Siu-Hung Ng, Julia Yael Plavnik, Eric C Rowell, and Zhenghan Wang. Fermionic modular categories and the 16-fold way. *Journal of Mathematical Physics*, 58(4), 2017.

[BGS+09] Kirill I Bolotin, Fereshte Ghahari, Michael D Shulman, Horst L Stormer, and Philip Kim. Observation of the fractional quantum hall effect in graphene. *Nature*, 462(7270):196–199, 2009.

[BH11] Sergey Bravyi and Matthew B Hastings. A short proof of stability of topological order under local perturbations. *Communications in mathematical physics*, 307:609–627, 2011.

[BHM10] Sergey Bravyi, Matthew B Hastings, and Spyridon Michalakis. Topological quantum order: stability under local perturbations. *Journal of mathematical physics*, 51(9), 2010.

[BJQ13] Maissam Barkeshli, Chao-Ming Jian, and Xiao-Liang Qi. Twist defects and projective non-abelian braiding statistics. *Physical Review B—Condensed Matter and Materials Physics*, 87(4):045130, 2013.

[BK98] Sergey B Bravyi and A Yu Kitaev. Quantum codes on a lattice with boundary. *arXiv preprint quant-ph/9811052*, 1998.

[BK+01] Bojko Bakalov, Alexander A Kirillov, et al. *Lectures on tensor categories and modular functors*, volume 21. American Mathematical Society Providence, RI, 2001.

[BK05] Sergey Bravyi and Alexei Kitaev. Universal quantum computation with ideal clifford gates and noisy ancillas. *Physical Review A—Atomic, Molecular, and Optical Physics*, 71(2):022316, 2005.

[BK13] Sergey Bravyi and Robert König. Classification of topologically protected gates for local stabilizer codes. *Physical review letters*, 110(17):170503, 2013.

[BKK+24] Jan Balewski, Milan Kornjaca, Katherine Klymko, Siva Darbha, Mark R Hirsbrunner, Pedro Lopes, Fangli Liu, and Daan Camps. Engineering quantum states with neutral atoms. *arXiv preprint arXiv:2404.04411*, 2024.

[BKKK22] Sergey Bravyi, Isaac Kim, Alexander Kliesch, and Robert Koenig. Adaptive constant-depth circuits for manipulating non-abelian anyons. *arXiv preprint arXiv:2205.01933*, 2022.

[BKP17] Parsa Bonderson, Christina Knapp, and Kaushal Patel. Anyonic entanglement and topological entanglement entropy. *Annals of Physics*, 385:399–468, 2017.

[BMD06] Hector Bombin and Miguel Angel Martin-Delgado. Topological quantum distillation. *Physical review letters*, 97(18):180501, 2006.

[BMD08] Hector Bombin and MA Martin-Delgado. Family of non-abelian kitaev models on a lattice: Topological condensation and confinement. *Physical Review B—Condensed Matter and Materials Physics*, 78(11):115421, 2008.

[BMD11] H Bombin and MA Martin-Delgado. Nested topological order. *New Journal of Physics*, 13(12):125001, 2011.

[BNQ13] Parsa Bonderson, Chetan Nayak, and Xiao-Liang Qi. A time-reversal invariant topological phase at the surface of a 3d topological insulator. *Journal of Statistical Mechanics: Theory and Experiment*, 2013(09):P09016, 2013.

[BNRW16] Paul Bruillard, Siu-Hung Ng, Eric C Rowell, and Zhenghan Wang. On classification of modular categories by rank. *International Mathematics Research Notices*, 2016(24):7546–7588, 2016.

[Bom15] Héctor Bombín. Gauge color codes: optimal transversal gates and gauge fixing in topological stabilizer codes. *New Journal of Physics*, 17(8):083002, 2015.

[Bon21] Parsa Bonderson. Measuring topological order. *Physical Review Research*, 3(3):033110, 2021.

[BPZ84] Alexander A Belavin, Alexander M Polyakov, and Alexander B Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Physics B*, 241(2):333–380, 1984.

[BR22] Matthew Buican and Rajath Radhakrishnan. Galois orbits of tqfts: symmetries and unitarity. *Journal of High Energy Physics*, 2022(1):1–73, 2022.

[Bru00] Alain Bruguieres. Catégories prémodulaires, modularisations et invariants des variétés de dimension 3. *Mathematische Annalen*, 316(2):215–236, 2000.

[BRWZ18] Parsa Bonderson, Eric Rowell, Zhenghan Wang, and Qing Zhang. Congruence subgroups and super-modular categories. *Pacific Journal of Mathematics*, 296(2):257–270, 2018.

[Bur18] Fiona J Burnell. Anyon condensation and its applications. *Annual Review of Condensed Matter Physics*, 9(1):307–327, 2018.

[BW18] Wade Bloomquist and Zhenghan Wang. On topological quantum computing with mapping class group representations. *Journal of Physics A: Mathematical and Theoretical*, 52(1):015301, 2018.

[Car89] John L Cardy. Boundary conditions, fusion rules and the verlinde formula. *Nuclear Physics B*, 324(3):581–596, 1989.

[CCW16] Iris Cong, Meng Cheng, and Zhenghan Wang. Topological quantum computation with gapped boundaries. *arXiv preprint arXiv:1609.02037*, 2016.

[CCW17] Iris Cong, Meng Cheng, and Zhenghan Wang. Universal quantum computation with gapped boundaries. *Physical Review Letters*, 119(17):170504, 2017.

[CDGG21] Thomas Creutzig, Tudor Dimofte, Niklas Garner, and Nathan Geer. A qft for non-semisimple tqft. *arXiv preprint arXiv:2112.01559*, 2021.

[CDH+20] Shawn X Cui, Dawei Ding, Xizhi Han, Geoffrey Penington, Daniel Ranard, Brandon C Rayhaun, and Zhou Shangnan. Kitaev's quantum double model as an error correcting code. *Quantum*, 4:331, 2020.

[CG93] Antoine Coste and Terry Gannon. Remarks on galois symmetry in rational conformal field theories. Technical report, P00019855, 1993.

[CGH23] Sebastiano Carpi, Tiziano Gaudio, and Robin Hillier. From vertex operator superalgebras to graded-local conformal nets and back. *arXiv preprint arXiv:2304.14263*, 2023.

[CGPM14] Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. *Journal of Topology*, 7(4):1005–1053, 2014.

[CK18] Bob Coecke and Aleks Kissinger. Picturing quantum processes: A first course on quantum theory and diagrammatic reasoning. In *Diagrammatic Representation and Inference: 10th International Conference, Diagrams 2018, Edinburgh, UK, June 18-22, 2018, Proceedings 10*, pages 28–31. Springer, 2018.

[CKWZ24] Liang Chang, Quinn T Kolt, Zhenghan Wang, and Qing Zhang. Modular data of non-semisimple modular categories. *arXiv preprint arXiv:2404.09314*, 2024.

[CLRL17] Ching-Kit Chan, Netanel H Lindner, Gil Refael, and Patrick A Lee. Photocurrents in weyl semimetals. *Physical Review B*, 95(4):041104, 2017.

[Cop97] B Jack Copeland. The church-turing thesis. 1997.

[COT24] Kevin Coulembier, Victor Ostrik, and Daniel Tubbenhauer. Growth rates of the number of indecomposable summands in tensor powers. *Algebras and Representation Theory*, 27(2):1033–1062, 2024.

[Cur94] Pierre Curie. Sur la symétrie dans les phénomènes physiques, symétrie d'un champ électrique et d'un champ magnétique. *Journal de physique théorique et appliquée*, 3(1):393–415, 1894.

[CZW19] Shawn Xingshan Cui, Modjtaba Shokrian Zini, and Zhenghan Wang. On generalized symmetries and structure of modular categories. *Science China Mathematics*, 62:417–446, 2019.

[Dav11] Kelly J Davis. Axiomatic tqft, axiomatic dqft, and exotic 4-manifolds. *arXiv preprint arXiv:1106.2358*, 2011.

[Del02] Pierre Deligne. Catégories tensorielles. *Mosc. Math. J*, 2(2):227–248, 2002.

[Del19] Colleen Delaney. *A categorical perspective on symmetry, topological order, and quantum information*. PhD thesis, UC Santa Barbara, 2019.

[DGNO10] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. On braided fusion categories i. *Selecta Mathematica*, 16:1–119, 2010.

[DGP+21] Colleen Delaney, César Galindo, Julia Plavnik, Eric C Rowell, and Qing Zhang. Braided zesting and its applications. *Communications in Mathematical Physics*, 386:1–55, 2021.

[DGP+24] Colleen Delaney, César Galindo, Julia Plavnik, Eric C Rowell, and Qing Zhang. G-crossed braided zesting. *Journal of the London Mathematical Society*, 109(1):e12816, 2024.

[DHW13] Orit Davidovich, Tobias Hagge, and Zhenghan Wang. On arithmetic modular categories. *arXiv preprint arXiv:1305.2229*, 2013.

[Dir31] Paul Adrien Maurice Dirac. Quantised singularities in the electromagnetic field. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 133(821):60–72, 1931.

[DKLP02] Eric Dennis, Alexei Kitaev, Andrew Landahl, and John Preskill. Topological quantum memory. *Journal of Mathematical Physics*, 43(9):4452–4505, 2002.

[DKP21] Colleen Delaney, Sung Kim, and Julia Plavnik. Zesting produces modular isotopes and explains their topological invariants. *arXiv preprint arXiv:2107.11374*, 2021.

[DM19] Tanmay Deshpande and Swarnava Mukhopadhyay. Crossed modular categories and the verlinde formula for twisted conformal blocks. *arXiv preprint arXiv:1909.10799*, 2019.

[DMNO13] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik. The witt group of non-degenerate braided fusion categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013(677):135–177, 2013.

[DR22] Marco De Renzi. *Non-semisimple extended topological quantum field theories*, volume 277. American Mathematical Society, 2022.

[Dri86] Vladimir Gershonovich Drinfeld. Quantum groups. *Zapiski Nauchnykh Seminarov POMI*, 155:18–49, 1986.

[EC24] Tyler Ellison and Meng Cheng. Towards a classification of mixed-state topological orders in two dimensions. *arXiv preprint arXiv:2405.02390*, 2024.

[EGNO16] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205. American Mathematical Soc., 2016.

[EK09] Bryan Eastin and Emanuel Knill. Restrictions on transversal encoded quantum gate sets. *Physical review letters*, 102(11):110502, 2009.

[EM45] Samuel Eilenberg and Saunders MacLane. General theory of natural equivalences. *Transactions of the American Mathematical Society*, 58(2):231–294, 1945.

[ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. *Annals of mathematics*, pages 581–642, 2005.

[ENO10] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Fusion categories and homotopy theory. *Quantum topology*, 1(3):209–273, 2010.

[ENO11] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Weakly group-theoretical and solvable fusion categories. *Advances in Mathematics*, 226(1):176–205, 2011.

[Fal94] Gerd Faltings. A proof for the verlinde formula. *Journal of Algebraic Geometry*, 3(2):347, 1994.

[FEC+21] Michael S Fuhrer, Mark T Edmonds, Dimitrie Culcer, Muhammad Nadeem, Xiaolin Wang, Nikhil Medhekar, Yuefeng Yin, and Jared H Cole. Proposal for a negative capacitance topological quantum field-effect transistor. In *2021 IEEE International Electron Devices Meeting (IEDM)*, pages 38–2. IEEE, 2021.

[FFRS05] Jurg Frohlich, Jurgen Fuchs, Ingo Runkel, and Christoph Schweigert. Picard groups in rational conformal field theory. *Contemporary Mathematics*, 391:85, 2005.

[FK08] Liang Fu and Charles L Kane. Superconducting proximity effect and majorana fermions¡? format?¿ at the surface of a topological insulator. *Physical review letters*, 100(9):096407, 2008.

[FK11] Lukasz Fidkowski and Alexei Kitaev. Topological phases of fermions in one dimension. *Physical Review B—Condensed Matter and Materials Physics*, 83(7):075103, 2011.

[FKLW03] Michael Freedman, Alexei Kitaev, Michael Larsen, and Zhenghan Wang. Topological quantum computation. *Bulletin of the American Mathematical Society*, 40(1):31–38, 2003.

[FKW02] Michael H Freedman, Alexei Kitaev, and Zhenghan Wang. Simulation of topological field theories by quantum computers. *Communications in Mathematical Physics*, 227:587–603, 2002.

[FLW02] Michael H Freedman, Michael Larsen, and Zhenghan Wang. A modular functor which is universal for quantum computation. *Communications in Mathematical Physics*, 227:605–622, 2002.

[FM01] Michael H Freedman and David A Meyer. Projective plane and planar quantum codes. *Foundations of Computational Mathematics*, 1:325–332, 2001.

[FM11] Benson Farb and Dan Margalit. *A primer on mapping class groups (pms-49)*, volume 41. Princeton university press, 2011.

[Fre70] Peter Freyd. Homotopy is not concrete. In *The Steenrod Algebra and Its Applications: A Conference to Celebrate NE Steenrod's Sixtieth Birthday: Proceedings of the Conference held at the Battelle Memorial Institute, Columbus, Ohio March 30th–April 4th, 1970*, pages 25–34. Springer, 1970.

[Fre98] Michael H Freedman. P/np, and the quantum field computer. *Proceedings of the National Academy of Sciences*, 95(1):98–101, 1998.

[FS01] Jürgen Fuchs and Christoph Schweigert. Category theory for conformal boundary conditions. *arXiv preprint math/0106050*, 2001.

[FS19] Brendan Fong and David I Spivak. *An invitation to applied category theory: seven sketches in compositionality.* Cambridge University Press, 2019.

[Gal14] César Galindo. On braided and ribbon unitary fusion categories. *Canadian Mathematical Bulletin*, 57(3):506–510, 2014.

[GJ19] Terry Gannon and Corey Jones. Vanishing of categorical obstructions for permutation orbifolds. *Communications in Mathematical Physics*, 369:245–259, 2019.

[GN08] Shlomo Gelaki and Dmitri Nikshych. Nilpotent fusion categories. *Advances in Mathematics*, 217(3):1053–1071, 2008.

[goo23] Suppressing quantum errors by scaling a surface code logical qubit. *Nature*, 614(7949):676–681, 2023.

[Got97] Daniel Gottesman. *Stabilizer codes and quantum error correction.* California Institute of Technology, 1997.

[GPMR21] Nathan Geer, Bertrand Patureau-Mirand, and Matthew Rupert. Some remarks on relative modular categories. *arXiv preprint arXiv:2110.15518*, 2021.

[GS19] Terry Gannon and Andrew Schopieray. Algebraic number fields generated by frobenius-perron dimensions in fusion rings. *arXiv preprint arXiv:1912.12260*, 2019.

[GW14] Zheng-Cheng Gu and Xiao-Gang Wen. Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory. *Physical Review B*, 90(11):115141, 2014.

[GWW14] Zheng-Cheng Gu, Zhenghan Wang, and Xiao-Gang Wen. Lattice model for fermionic toric code. *Physical Review B*, 90(8):085140, 2014.

[GWW15] Zheng-Cheng Gu, Zhenghan Wang, and Xiao-Gang Wen. Classification of two-dimensional fermionic and bosonic topological orders. *Physical Review B*, 91(12):125149, 2015.

[Hal83] F Duncan M Haldane. Nonlinear field theory of large-spin heisenberg antiferromagnets: semiclassically quantized solitons of the one-dimensional easy-axis néel state. *Physical review letters*, 50(15):1153, 1983.

[Hal88] F Duncan M Haldane. Model for a quantum hall effect without landau levels: Condensed-matter realization of the" parity anomaly". *Physical review letters*, 61(18):2015, 1988.

[Hal13] Brian C Hall. *Quantum theory for mathematicians*, volume 267. Springer Science & Business Media, 2013.

[Has11] Matthew B Hastings. Topological order at nonzero temperature. *Physical review letters*, 107(21):210501, 2011.

[HDSHL24] Yifan Hong, Elijah Durso-Sabina, David Hayes, and Andrew Lucas. Entangling four logical qubits beyond break-even in a nonlocal code. *arXiv preprint arXiv:2406.02666*, 2024.

[HI19] Diego M Hofman and Nabil Iqbal. Goldstone modes and photonization for higher form symmetries. *SciPost Physics*, 6(1):006, 2019.

[HK10] M Zahid Hasan and Charles L Kane. Colloquium: topological insulators. *Reviews of modern physics*, 82(4):3045–3067, 2010.

[HRW08] Seung-Moon Hong, Eric Rowell, and Zhenghan Wang. On exotic modular tensor categories. *Communications in Contemporary Mathematics*, 10(supp01):1049–1074, 2008.

[Hua05] Yi-Zhi Huang. Vertex operator algebras, the verlinde conjecture, and modular tensor categories. *Proceedings of the National Academy of Sciences*, 102(15):5352–5356, 2005.

[Hua08] Yi-Zhi Huang. Vertex operator algebras and the verlinde conjecture. *Communications in Contemporary Mathematics*, 10(01):103–154, 2008.

[Hua21] Yi-Zhi Huang. Representation theory of vertex operator algebras and orbifold conformal field theory. In *Lie Groups, Number Theory, and Vertex Algebras*, volume 768, pages 221–252. Amer. Math. Soc. Providence, RI, 2021.

[ITV$^+$24] Mohsin Iqbal, Nathanan Tantivasadakarn, Ruben Verresen, Sara L Campbell, Joan M Dreiling, Caroline Figgatt, John P Gaebler, Jacob Johansen, Michael Mills, Steven A Moses, et al. Non-abelian topological order and anyons on a trapped-ion processor. *Nature*, 626(7999):505–511, 2024.

[JFR24] Theo Johnson-Freyd and David Reutter. Minimal nondegenerate extensions. *Journal of the American Mathematical Society*, 37(1):81–150, 2024.

[JMNR21] Corey Jones, Scott Morrison, Dmitri Nikshych, and Eric C Rowell. Rank-finiteness for g-crossed braided fusion categories. *Transformation Groups*, 26(3):915–927, 2021.

[Jon97] Vaughan FR Jones. A polynomial invariant for knots via von neumann algebras. In *Fields Medallists' Lectures*, pages 448–458. World Scientific, 1997.

[JVW90] François Jaeger, Dirk L Vertigan, and Dominic JA Welsh. On the computational complexity of the jones and tutte polynomials. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 108, pages 35–53. Cambridge University Press, 1990.

[KBRY15] Vadym Kliuchnikov, Alex Bocharov, Martin Roetteler, and Jon Yard. A framework for approximating qubit unitaries. *arXiv preprint arXiv:1510.03888*, 2015.

[KBS14] Vadym Kliuchnikov, Alex Bocharov, and Krysta M Svore. Asymptotically optimal topological quantum compiling. *Physical review letters*, 112(14):140504, 2014.

[KDP80] K v Klitzing, Gerhard Dorda, and Michael Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance. *Physical review letters*, 45(6):494, 1980.

[Kit97] A Yu Kitaev. Quantum computations: algorithms and error correction. *Russian Mathematical Surveys*, 52(6):1191, 1997.

[Kit01] A Yu Kitaev. Unpaired majorana fermions in quantum wires. *Physics-uspekhi*, 44(10S):131, 2001.

[Kit03] A Yu Kitaev. Fault-tolerant quantum computation by anyons. *Annals of physics*, 303(1):2–30, 2003.

[Kit06] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2–111, 2006.

[KJ96] Alexander Kirillov Jr. On an inner product in modular tensor categories. *Journal of the American Mathematical Society*, 9(4):1135–1169, 1996.

[KJ04] Alexander Kirillov Jr. On g–equivariant modular categories. *arXiv preprint math/0401119*, 2004.

[KK12] Alexei Kitaev and Liang Kong. Models for gapped boundaries and domain walls. *Communications in Mathematical Physics*, 313(2):351–373, 2012.

[KKO+22] Justin Kaidi, Zohar Komargodski, Kantaro Ohmori, Sahand Seifnashri, and Shu-Heng Shao. Higher central charges and topological boundaries in 2+ 1-dimensional tqfts. *SciPost Physics*, 13(3):067, 2022.

[KL09] Louis H Kauffman and Samuel J Lomonaco. Topological quantum information theory. In *Proceedings of Symposia in Applied Mathematics*, volume 68, 2009.

[KL20] Kyle Kawagoe and Michael Levin. Microscopic definitions of anyon data. *Physical Review B*, 101(11):115113, 2020.

[KM05] Charles L Kane and Eugene J Mele. Quantum spin hall effect in graphene. *Physical review letters*, 95(22):226801, 2005.

[KPEB18] Markus S Kesselring, Fernando Pastawski, Jens Eisert, and Benjamin J Brown. The boundaries and twist defects of the color code and their applications to topological quantum computation. *Quantum*, 2:101, 2018.

[KR08] Liang Kong and Ingo Runkel. Morita classes of algebras in modular tensor categories. *Advances in Mathematics*, 219(5):1548–1576, 2008.

[KT73] JM Kosterlitz and David James Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *J. Phys. C*, 6:1181–1203, 1973.

[KT18] John Michael Kosterlitz and David James Thouless. Ordering, metastability and phase transitions in two-dimensional systems. In *Basic Notions Of Condensed Matter Physics*, pages 493–515. CRC Press, 2018.

[KY15] Vadym Kliuchnikov and Jon Yard. A framework for exact synthesis. *arXiv preprint arXiv:1504.04350*, 2015.

[KZ22] Liang Kong and Zhi-Hao Zhang. An invitation to topological orders and category theory. *arXiv preprint arXiv:2205.05565*, 2022.

[KZL+16] Christina Knapp, Michael Zaletel, Dong E Liu, Meng Cheng, Parsa Bonderson, and Chetan Nayak. How quickly can anyons be braided? or: How i learned to stop worrying about diabatic errors and love the anyon. *arXiv preprint arXiv:1601.05790*, 2016.

[Lan95] Rolf Landauer. Is quantum mechanics useful? *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences*, 353(1703):367–376, 1995.

[Lee97] David M Lee. The extraordinary phases of liquid 3 he. *Reviews of Modern Physics*, 69(3):645, 1997.

[LM77] JM Leinaas and J Myrheim. On the theory of identical particles. *Il nuovo cimento*, 37:132, 1977.

[Lur08] Jacob Lurie. On the classification of topological field theories. *Current developments in mathematics*, 2008(1):129–280, 2008.

[Lyu95] Volodymyr V Lyubashenko. Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity. *Communications in mathematical physics*, 172:467–516, 1995.

[Mer79] N David Mermin. The topological theory of defects in ordered media. *Reviews of Modern Physics*, 51(3):591, 1979.

[ML13] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[Moc03] Carlos Mochon. Anyons from nonsolvable finite groups are sufficient for universal quantum computation. *Physical Review A*, 67(2):022315, 2003.

[Moc04] Carlos Mochon. Anyon computers with smaller groups. *Physical Review A*, 69(3):032306, 2004.

[Mou84] J Moussouris. *Quantum models of space-time based on recoupling theory.* PhD thesis, University of Oxford, 1984.

[MR65] Ward D Maurer and John L Rhodes. A property of finite simple non-abelian groups. *Proceedings of the American Mathematical Society*, 16(3):552–554, 1965.

[MS88] Gregory Moore and Nathan Seiberg. Polynomial equations for rational conformal field theories. *Physics Letters B*, 212(4):451–460, 1988.

[MS89] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Communications in Mathematical Physics*, 123:177–254, 1989.

[MS90] Gregory Moore and Nathan Seiberg. Lectures on rcft. In *Physics, geometry and topology*, pages 263–361. Springer, 1990.

[MS01] Roderich Moessner and Shivaji L Sondhi. Resonating valence bond phase in the triangular lattice quantum dimer model. *Physical Review Letters*, 86(9):1881, 2001.

[MS12] Scott Morrison and Noah Snyder. Non-cyclotomic fusion categories. *Transactions of the American Mathematical Society*, 364(9):4713–4733, 2012.

[MSP02] G Misguich, D Serban, and V Pasquier. Quantum dimer model on the kagome lattice: Solvable dimer-liquid and ising gauge theory. *Physical review letters*, 89(13):137202, 2002.

[Müg00] Michael Müger. Galois theory for braided tensor categories and the modular closure. *Advances in Mathematics*, 150(2):151–201, 2000.

[Müg02] Michael Müger. On the structure of modular categories. *arXiv preprint math/0201017*, 2002.

[MZ21] CH Marrows and K Zeissler. Perspective on skyrmion spintronics. *Applied Physics Letters*, 119(25), 2021.

[NC10] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.

[Nik04] Dmitri Nikshych. Semisimple weak hopf algebras. *Journal of Algebra*, 275(2):639–667, 2004.

[NO09] Zohar Nussinov and Gerardo Ortiz. A symmetry principle for topological quantum order. *Annals of Physics*, 324(5):977–1057, 2009.

[NSS+08] Chetan Nayak, Steven H Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma. Non-abelian anyons and topological quantum computation. *Reviews of Modern Physics*, 80(3):1083–1159, 2008.

[Ost03] Victor Ostrik. Module categories, weak hopf algebras and modular invariants. *Transformation groups*, 8:177–206, 2003.

[PBD17] Benedikt Placke, Stefano Bosco, and David P DiVincenzo. A model study of present-day hall-effect circulators. *EPJ Quantum Technology*, 4:1–14, 2017.

[Pfe09] Hendryk Pfeiffer. Tannaka–kreın reconstruction and a characterization of modular tensor categories. *Journal of Algebra*, 321(12):3714–3763, 2009.

[PSYZ23] Julia Plavnik, Andrew Schopieray, Zhiqiang Yu, and Qing Zhang. Modular tensor categories, subcategories, and galois orbits. *Transformation Groups*, pages 1–26, 2023.

[Ran88] GS Ranganath. On defects in biaxial nematic liquid crystals. *Current Science*, pages 1–6, 1988.

[RB11] Nicolas Regnault and B Andrei Bernevig. Fractional chern insulator. *Physical Review X*, 1(2):021014, 2011.

[Reu23a] David Reutter. Semisimple four-dimensional topological field theories cannot detect exotic smooth structure. *Journal of Topology*, 16(2):542–566, 2023.

[Reu23b] David Reutter. Uniqueness of unitary structure for unitarizable fusion categories. *Communications in Mathematical Physics*, 397(1):37–52, 2023.

[RS20] Arthur P Ramirez and Brian Skinner. Dawn of the topological age? *Physics Today*, 73(9):30–36, 2020.

[RSW09] Eric Rowell, Richard Stong, and Zhenghan Wang. On classification of modular tensor categories. *Communications in Mathematical Physics*, 292(2):343–389, 2009.

[RT91] Nicolai Reshetikhin and Vladimir G Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inventiones mathematicae*, 103(1):547–597, 1991.

[SA17] Masatoshi Sato and Yoichi Ando. Topological superconductors: a review. *Reports on Progress in Physics*, 80(7):076501, 2017.

[SA19] Jun Ho Son and Jason Alicea. Commuting-projector hamiltonians for two-dimensional topological insulators: Edge physics and many-body invariants. *Physical Review B*, 100(15):155107, 2019.

[SBZ22] Alexis Schotte, Lander Burgelman, and Guanyu Zhu. Fault-tolerant error correction for a universal non-abelian topological quantum computer at finite temperature. *arXiv preprint arXiv:2301.00054*, 2022.

[SF18] Brian Skinner and Liang Fu. Large, nonsaturating thermopower in a quantizing magnetic field. *Science advances*, 4(5):eaat2621, 2018.

[SFN15] Sankar Das Sarma, Michael Freedman, and Chetan Nayak. Majorana zero modes and topological quantum computation. *npj Quantum Information*, 1(1):1–13, 2015.

[Sha12] Ramamurti Shankar. *Principles of quantum mechanics*. Springer Science & Business Media, 2012.

[Shi19] Kenichi Shimizu. Non-degeneracy conditions for braided finite tensor categories. *Advances in Mathematics*, 355:106778, 2019.

[Sho94] Peter W Shor. Algorithms for quantum computation: discrete logarithms and factoring. In *Proceedings 35th annual symposium on foundations of computer science*, pages 124–134. Ieee, 1994.

[Sim23] Steven H Simon. *Topological quantum*. Oxford University Press, 2023.

[Sky62] Tony Hilton Royle Skyrme. A unified field theory of mesons and baryons. *Nuclear Physics*, 31:556–569, 1962.

[ŠMYM18] Libor Šmejkal, Yuriy Mokrousov, Binghai Yan, and Allan H MacDonald. Topological antiferromagnetic spintronics. *Nature physics*, 14(3):242–251, 2018.

[Sop23] Nikita A Sopenko. *Topological invariants of gapped quantum lattice systems*. California Institute of Technology, 2023.

[STL+10] Jay D Sau, Sumanta Tewari, Roman M Lutchyn, Tudor D Stanescu, and S Das Sarma. Non-abelian quantum order in spin-orbit-coupled semiconductors: Search for topological majorana particles in solid-state systems. *Physical Review B—Condensed Matter and Materials Physics*, 82(21):214509, 2010.

[SZBV22] Alexis Schotte, Guanyu Zhu, Lander Burgelman, and Frank Verstraete. Quantum error correction thresholds for the universal fibonacci turaev-viro code. *Physical Review X*, 12(2):021012, 2022.

[Tan20] Daniel Tan. *Vertex Operator Algebras, Modular Tensor Categories and a Kazhdan-Lusztig Correspondence at a Non-negative Integral Level.* PhD thesis, UNIVERSITY OF MELBOURNE, 2020.

[Ter15] Barbara M Terhal. Quantum error correction for quantum memories. *Reviews of Modern Physics*, 87(2):307–346, 2015.

[TSG82] Daniel C Tsui, Horst L Stormer, and Arthur C Gossard. Two-dimensional magnetotransport in the extreme quantum limit. *Physical Review Letters*, 48(22):1559, 1982.

[Tur39] Alan Mathison Turing. Systems of logic based on ordinals. *Proceedings of the London Mathematical Society, Series 2*, 45:161–228, 1939.

[Tur92] Vladimir G Turaev. Modular categories and 3-manifold invariants. *International Journal of Modern Physics B*, 6(11n12):1807–1824, 1992.

[Tur10a] Vladimir G Turaev. *Homotopy quantum field theory*, volume 10. European Mathematical Society, 2010.

[Tur10b] Vladimir G Turaev. *Quantum invariants of knots and 3-manifolds.* de Gruyter, 2010.

[Tve80] Helge Tverberg. A proof of the jordan curve theorem. *Bulletin of the London Mathematical Society*, 12(1):34–38, 1980.

[TVV23] Nathanan Tantivasadakarn, Ashvin Vishwanath, and Ruben Verresen. Hierarchy of topological order from finite-depth unitaries, measurement, and feedforward. *PRX Quantum*, 4(2):020339, 2023.

[TY98] Daisuke Tambara and Shigeru Yamagami. Tensor categories with fusion rules of self-duality for finite abelian groups. *Journal of Algebra*, 209(2):692–707, 1998.

[Var04] Veeravalli S Varadarajan. *Supersymmetry for mathematicians: an introduction: an introduction*, volume 11. American Mathematical Soc., 2004.

[VD98] Alfons Van Daele. An algebraic framework for group duality. *Advances in Mathematics*, 140(2):323–366, 1998.

[VD14] Giovanni Viola and David P DiVincenzo. Hall effect gyrators and circulators. *Physical Review X*, 4(2):021019, 2014.

[Ver88] Erik Verlinde. Fusion rules and modular transformations in 2d conformal field theory. *Nuclear Physics B*, 300:360–376, 1988.

[VHF15] Sagar Vijay, Jeongwan Haah, and Liang Fu. A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations. *Physical Review B*, 92(23):235136, 2015.

[vKCK+20] Klaus von Klitzing, Tapash Chakraborty, Philip Kim, Vidya Madhavan, Xi Dai, James McIver, Yoshinori Tokura, Lucile Savary, Daria Smirnova, Ana Maria Rey, et al. 40 years of the quantum hall effect. *Nature Reviews Physics*, 2(8):397–401, 2020.

[Wan10] Zhenghan Wang. *Topological quantum computation*. Number 112. American Mathematical Soc., 2010.

[Wen89] Xiao-Gang Wen. Vacuum degeneracy of chiral spin states in compactified space. *Physical Review B*, 40(10):7387, 1989.

[Wil82a] Frank Wilczek. Magnetic flux, angular momentum, and statistics. *Physical Review Letters*, 48(17):1144, 1982.

[Wil82b] Frank Wilczek. Quantum mechanics of fractional-spin particles. *Physical review letters*, 49(14):957, 1982.

[Wit88] Edward Witten. Topological quantum field theory. *Communications in Mathematical Physics*, 117(3):353–386, 1988.

[Wit89] Edward Witten. Quantum field theory and the jones polynomial. *Communications in Mathematical Physics*, 121(3):351–399, 1989.

[WWW] Z Wang, Z Wu, and Z Wang. Intrinsic mixed-state quantum topological order. *arXiv preprint arXiv:2307.13758*.

[XSW+24] Shibo Xu, Zheng-Zhi Sun, Ke Wang, Hekang Li, Zitian Zhu, Hang Dong, Jinfeng Deng, Xu Zhang, Jiachen Chen, Yaozu Wu, et al. Non-abelian braiding of fibonacci anyons with a superconducting processor. *Nature Physics*, pages 1–7, 2024.

[Yet92] David N Yetter. Framed tangles and a theorem of deligne on braided deformations of tannakian categories. *Contemp. Math*, 134:325–350, 1992.

[ZGHS15] W Zhu, SS Gong, FDM Haldane, and DN Sheng. Fractional quantum hall states at ν= 13/5 and 12/5 and their non-abelian nature. *Physical review letters*, 115(12):126805, 2015.