The Verlinde Formula for MTCs

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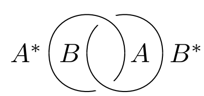
One of the most powerful equations in the theory of modular tensor categories (MTCs) is the so-called *Verlinde formula*. This formula was first conjectured by Verlinde [Ver88], and proven the following year by Moore-Seiberg [MS89]. There are now many Verlinde-type formulas. Most imporantly, there is one for vertex operator algebras and one in algebraic geometry.

To begin our exposition, we set notation. Fix a modular tensor category $\mathscr{C}$, with set $\mathscr{L}$ of isomorphism classes of simple objects. The S-matrix of $\mathscr{C}$ is the matrix whose rows and collumns are labeled by $\mathscr{L}$, and whose (a, b) entry is

$$S_{a,b} = \mathrm{tr}\left(\beta_{B,A} \circ \beta_{A,B}\right) =$$



where β is the braiding on $\mathscr{C}$, tr is the categorical trace on $\mathscr{C}$, and A, B are representatives of a, b. The fusion coefficients of $\mathscr{C}$ are the unique non-negative integers $N_{-}^{-,-}$ such that for all $a, b \in \mathscr{L}$

$$A \otimes B \cong \bigoplus_{c \in \mathscr{L}} N_c^{a,b} C$$

where A, B, C are representatives of a, b, c. The quantum dimension of an object $a \in \mathscr{L}$ is $d_a = \mathrm{tr}(\mathrm{id}_A)$. The Verlinde formula allows one to express the fusion coefficients in terms of the S-matrix entries:

$$N_c^{a,b} = \sum_{e \in \mathscr{L}} \frac{S_{a,e} S_{b,e} (S^{-1})_{c,e}}{d_e}. \tag{$*$}$$

This can be restated as saying that the S-matrix diagonalizes the fusion coefficents in the sense that if we let

$$N^a = (N_c^{a,b})_{(b,c) \in \mathscr{L}^2}$$

be the "fusion matrix" corresponding to $a \in \mathscr{L}$, then SN^aS^{-1} is diagonal and its entries are normalized S-matrix values. That is,

$$D^a = SN^aS^{-1}$$

is diagonal and its (b, b) entry is $s_{a,b}/d_b$. From this, $(*)$ follows by expanding the equality $N^a = S^{-1} D^a S$.

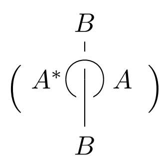
We now give a proof. Let $K(\mathscr{C})$ be the ring whose underlying group is generated by symbols $[a]$ where a is an isomorphism class of simple objects, and whose ring operations are given by

$$\begin{aligned}[]
[a] + [b] &= [a \oplus b], \\
[a] \cdot [b] &= [a \otimes b],
\end{aligned}$$

where $\oplus$ and $\otimes$ are taken over representatives. This is the *Grothendieck ring* of $\mathscr{C}$. Let $K_{\mathbb{C}}(\mathscr{C}) = K(\mathscr{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ be the complexification. Let $\mathbb{C}[\mathscr{L}]$ be the vector space freely generated by $\mathscr{L}$ and let $\mathbb{C}[\mathscr{L}]^{\text{func}}$ be the $\mathbb{C}$-algebra obtained by endowing $\mathbb{C}[\mathscr{L}]$ with pointwise multiplication,

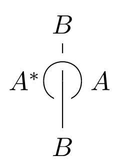
$$\left(\sum_{a\in\mathscr{L}} f_a \left|a\right\rangle\right) \cdot \left(\sum_{a\in\mathscr{L}} g_a \left|a\right\rangle\right) = \sum_{a\in\mathscr{L}} f_a g_a \left|a\right\rangle .$$

We claim that the map $\mu : K_{\mathbb{C}}(\mathscr{C}) \to \mathbb{C}[\mathscr{L}]^{\text{func}}$ sending $[a] \in K_{\mathbb{C}}(\mathscr{C})$ to

$$\sum_{b\in\mathscr{L}}$$


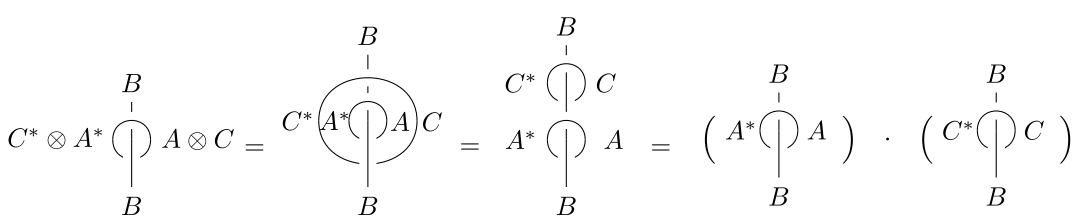
$$\left|b\right\rangle \in \mathbb{C}[\mathscr{L}]^{\text{func}}$$

is an isomorphism of $\mathbb{C}$-algebras. Here, we use Schur's lemma to identify morphisms $f : B \to B$ with the unique number $\lambda \in \mathbb{C}$ such that $f = \lambda \cdot \mathrm{id}_B$. Taking trace, we have



$$= \frac{1}{d_b} \operatorname{tr}(\beta_{B,A} \circ \beta_{A,B}) .$$

It is a general category theoretic fact that natural transformations commute with direct sums, so it is clear from expanding that μ adds over directs sums, making it a group homomorphism. To verify that μ is a morphism of algebras, we compute as follows:



We now verify μ is an isomorphism. Choosing a simple object $[a] \in K_{\mathbb{C}}(\mathscr{C})$ it is clear that

$$\mu([a]) = \sum_{b \in \mathscr{L}} \frac{s_{a,b}}{d_b} |b\rangle .$$

After identifying $K_{\mathbb{C}}(\mathscr{C}) \cong \mathbb{C}[\mathscr{L}]$ by sending $[a]$ to $|a\rangle$, we find that μ is given on the level of vector spaces by a scaled S-matrix. Since S is invertible we get that μ is bijective as desired. Note the key use of the fact that quantum dimensions are non-zero in MTCs. We now observe that

$$[a] \cdot [b] = [a \otimes b] = \sum_{c \in \mathscr{L}} N_c^{a,b}[c],$$

so left multiplication by $[a]$ is represented by the fusion matrix N^a. The computation

$$\begin{aligned} [a] \cdot \mu^{-1}(|b\rangle) &= \mu^{-1}\left(\left(\sum_{c \in \mathscr{L}} \frac{s_{a,c}}{d_c} |c\rangle\right) \cdot |b\rangle\right) \\ &= \mu^{-1}\left(\frac{s_{a,b}}{d_b} |b\rangle\right) = \frac{s_{a,b}}{d_b} \cdot \mu^{-1}(|b\rangle) . \end{aligned}$$

shows $\mu^{-1}(|b\rangle)$ is an eigenvector for N^a with eigenvalue $s_{a,b}/d_b$. Since $\{\mu^{-1}(|b\rangle)\}_{b \in \mathscr{L}}$ is a basis for $K_{\mathbb{C}}(\mathscr{C})$ this gives a diagonalization of N^a. Moreover, the formula for μ on simple objects tells us that after re-scaling collumns the change of basis matrix is exactly S. Hence,

$$SN^aS^{-1} = D^a$$

with D^a the diagonal matrix whose (b, b) entry is $s_{a,b}/d_b$, as desired.

References

[MS89] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Communications in Mathematical Physics*, 123:177–254, 1989.

[Ver88] Erik Verlinde. Fusion rules and modular transformations in 2d conformal field theory. *Nuclear Physics B*, 300:360–376, 1988.