

# The Verlinde Formula for MTCs

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One of the most powerful equations in the theory of modular tensor categories (MTCs) is the so-called *Verlinde formula*. This formula was first conjectured by Verlinde [Ver88], and proven the following year by Moore-Seiberg [MS89]. There are now many Verlinde-type formulas. Most importantly, there is one for vertex operator algebras [Hua08] and one in algebraic geometry [Fal94].

To begin our exposition, we set notation. Fix a modular tensor category  $\mathcal{C}$ , with set  $\mathcal{L}$  of isomorphism classes of simple objects. The  $S$ -matrix of  $\mathcal{C}$  is the matrix whose rows and columns are labeled by  $\mathcal{L}$ , and whose  $(a, b)$  entry is

$$S_{a,b} = \text{tr}(\beta_{B,A} \circ \beta_{A,B}) = A^* \left( B \left( \right) A \right) B^*$$

where  $\beta$  is the braiding on  $\mathcal{C}$ ,  $\text{tr}$  is the categorical trace on  $\mathcal{C}$ , and  $A, B$  are representatives of  $a, b$ . The fusion coefficients of  $\mathcal{C}$  are the unique non-negative integers  $N_{-}^{a,b}$  such that for all  $a, b \in \mathcal{L}$

$$A \otimes B \cong \bigoplus_{c \in \mathcal{L}} N_c^{a,b} C$$

where  $A, B, C$  are representatives of  $a, b, c$ . The quantum dimension of an object  $a \in \mathcal{L}$  is  $d_a = \text{tr}(\text{id}_A)$ . The Verlinde formula allows one to express the fusion coefficients in terms of the  $S$ -matrix entries:

$$N_c^{a,b} = \sum_{e \in \mathcal{L}} \frac{S_{a,e} S_{b,e} (S^{-1})_{c,e}}{d_e}. \quad (*)$$

This can be restated as saying that the  $S$ -matrix diagonalizes the fusion coefficients in the sense that if we let

$$N^a = (N_c^{a,b})_{(b,c) \in \mathcal{L}^2}$$

be the “fusion matrix” corresponding to  $a \in \mathcal{L}$ , then  $SN^a S^{-1}$  is diagonal and its entries are normalized  $S$ -matrix values. That is,

$$D^a = SN^a S^{-1}$$

is diagonal and its  $(b, b)$  entry is  $s_{a,b}/d_b$ . From this,  $(*)$  follows by expanding the equality  $N^a = S^{-1}D^a S$ .

We now give a proof. Let  $\mathbb{C}[\mathcal{L}]$  be the vector space freely generated by  $\mathcal{L}$ . Let  $K_{\mathbb{C}}(\mathcal{C})$  be algebra structure on  $\mathbb{C}[\mathcal{L}]$  given by

$$|a\rangle \cdot |b\rangle = |a \otimes b\rangle = \sum_{c \in \mathcal{L}} N_c^{a,b} |c\rangle.$$

This is the (complexified) Grothendieck ring of  $\mathcal{C}$ . Let  $\mathbb{C}[\mathcal{L}]^{\text{func}}$  be the  $\mathbb{C}$ -algebra obtained by endowing  $\mathbb{C}[\mathcal{L}]$  with pointwise multiplication. We claim that the map  $\mu : K_{\mathbb{C}}(\mathcal{C}) \rightarrow \mathbb{C}[\mathcal{L}]^{\text{func}}$  sending  $|a\rangle \in K_{\mathbb{C}}(\mathcal{C})$  to

$$\sum_{b \in \mathcal{L}} \left( A^* \left( \bigcap_{B}^B A \right) |b\rangle \right) \in \mathbb{C}[\mathcal{L}]^{\text{func}}$$

is an isomorphism of  $\mathbb{C}$ -algebras. Here, we use Schur's lemma to identify morphisms  $f : B \rightarrow B$  with the unique number  $\lambda \in \mathbb{C}$  such that  $f = \lambda \cdot \text{id}_B$ . Taking trace, we have

$$A^* \left( \bigcap_{B}^B A \right) = \frac{1}{d_b} \text{tr}(\beta_{B,A} \circ \beta_{A,B}).$$

It is a general category theoretic fact that natural transformations commute with direct sums, so it is clear from expanding that  $\mu$  adds over direct sums, making it a group homomorphism. To verify that  $\mu$  is a morphism of algebras, we compute as follows:

$$C^* \otimes A^* \left( \bigcap_{B}^B A \right) A \otimes C = C^* \left( A^* \left( \bigcap_{B}^B A \right) A \right) C = C^* \left( \bigcap_{B}^B C \right) C = \left( A^* \left( \bigcap_{B}^B A \right) A \right) \cdot \left( C^* \left( \bigcap_{B}^B C \right) C \right).$$

We now verify  $\mu$  is an isomorphism. Choosing a simple object  $|a\rangle \in K_{\mathbb{C}}(\mathcal{C})$  it is clear that

$$\mu(|a\rangle) = \sum_{b \in \mathcal{L}} \frac{s_{a,b}}{d_b} |b\rangle.$$

Thus,  $\mu$  is given on the level of vector spaces by a scaled  $S$ -matrix. Since  $S$  is invertible we get that  $\mu$  is bijective as desired. Note the key use of the

fact that quantum dimensions are non-zero in MTCs. The definition of  $K_{\mathbb{C}}(\mathcal{C})$  exactly says that left multiplication is represented by the fusion matrix  $N^a$ . The computation

$$\begin{aligned} [a] \cdot \mu^{-1}(|b\rangle) &= \mu^{-1} \left( \left( \sum_{c \in \mathcal{L}} \frac{s_{a,c}}{d_c} |c\rangle \right) \cdot |b\rangle \right) \\ &= \mu^{-1} \left( \frac{s_{a,b}}{d_b} |b\rangle \right) = \frac{s_{a,b}}{d_b} \cdot \mu^{-1}(|b\rangle). \end{aligned}$$

shows  $\mu^{-1}(|b\rangle)$  is an eigenvector for  $N^a$  with eigenvalue  $s_{a,b}/d_b$ . Since  $\{\mu^{-1}(|b\rangle)\}_{b \in \mathcal{L}}$  is a basis for  $K_{\mathbb{C}}(\mathcal{C})$  this gives a diagonalization of  $N^a$ . Moreover, the formula for  $\mu$  on simple objects tells us that after re-scaling columns the change of basis matrix is exactly  $S$ . Hence,

$$SN^a S^{-1} = D^a$$

with  $D^a$  the diagonal matrix whose  $(b, b)$  entry is  $s_{a,b}/d_b$ , as desired.

## References

- [Fal94] Gerd Faltings. A proof for the verlinde formula. *Journal of Algebraic Geometry*, 3(2):347, 1994.
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