stoch_calc_pset_q3+4

November 18, 2018

1 Stochastic Calculus Problem Set 2 Questions 3 and 4

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
```

1.1 Question 3

1.1.1 Part (a)

```
In [2]: delta = 0.01
        N = 250

        m = np.array([0, 0])
        C = np.reshape([1, 0.75, 0.75, 0.9], [2, 2])
```

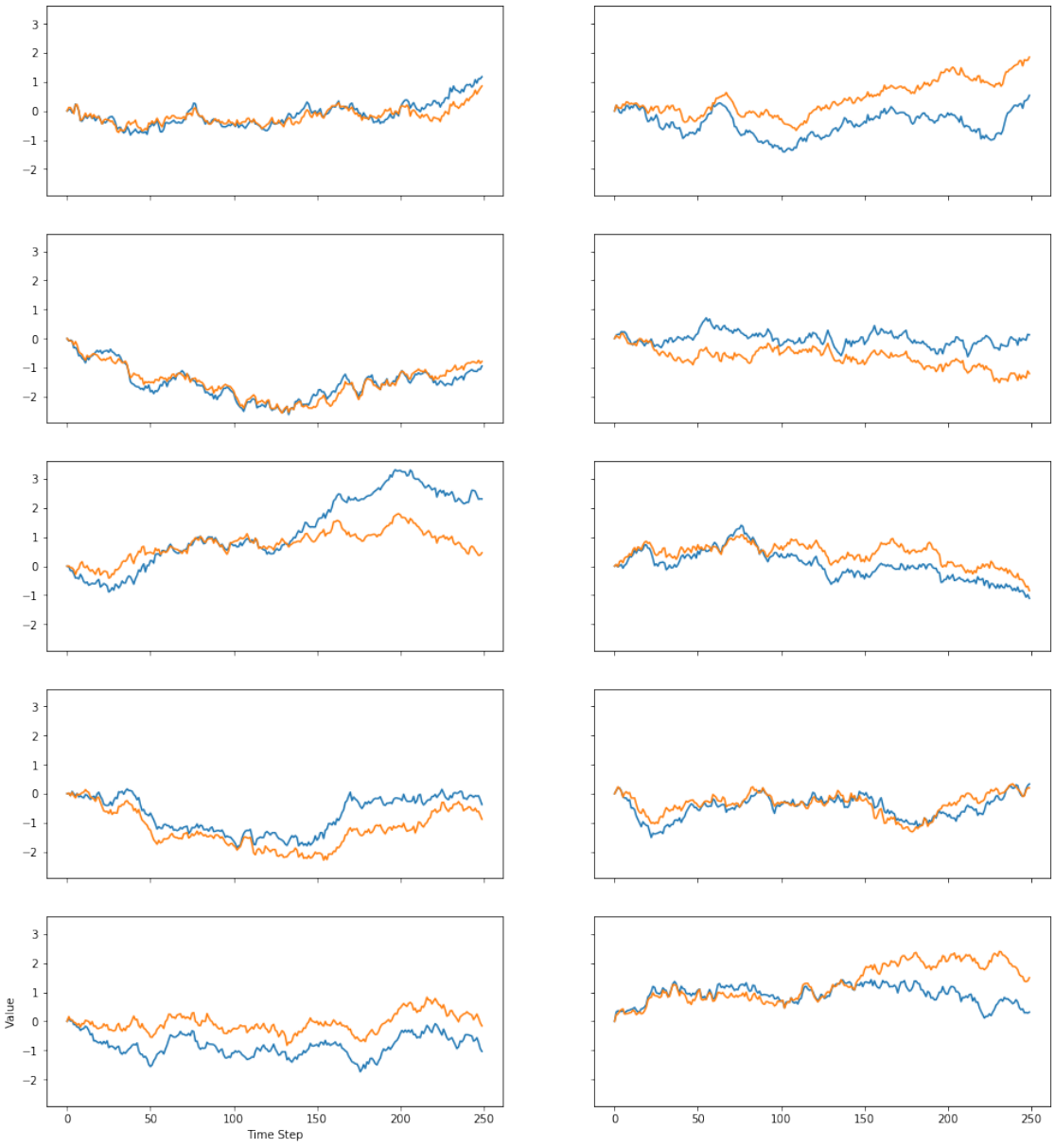
```
In [3]: def sample_gaussian(num_vectors=1):
            return np.random.multivariate_normal(mean=m, cov=delta*C,
                                                 size=num_vectors)
```

1.1.2 Part (b)

```
In [4]: fig, axarr = plt.subplots(nrows=5, ncols=2, figsize=[16, 18], sharex=True, sharey=True)
        for idx, ax in enumerate(axarr.flatten()):
            # Generate pair of Wiener processes
            W = np.zeros([N, 2])
            for idx, row in enumerate(W[:-1]):
                W[idx + 1] = row + sample_gaussian()

            ax.plot(W);

        axarr[4, 0].set_xlabel('Time Step')
        axarr[4, 0].set_ylabel('Value');
```



1.1.3 Part (c)

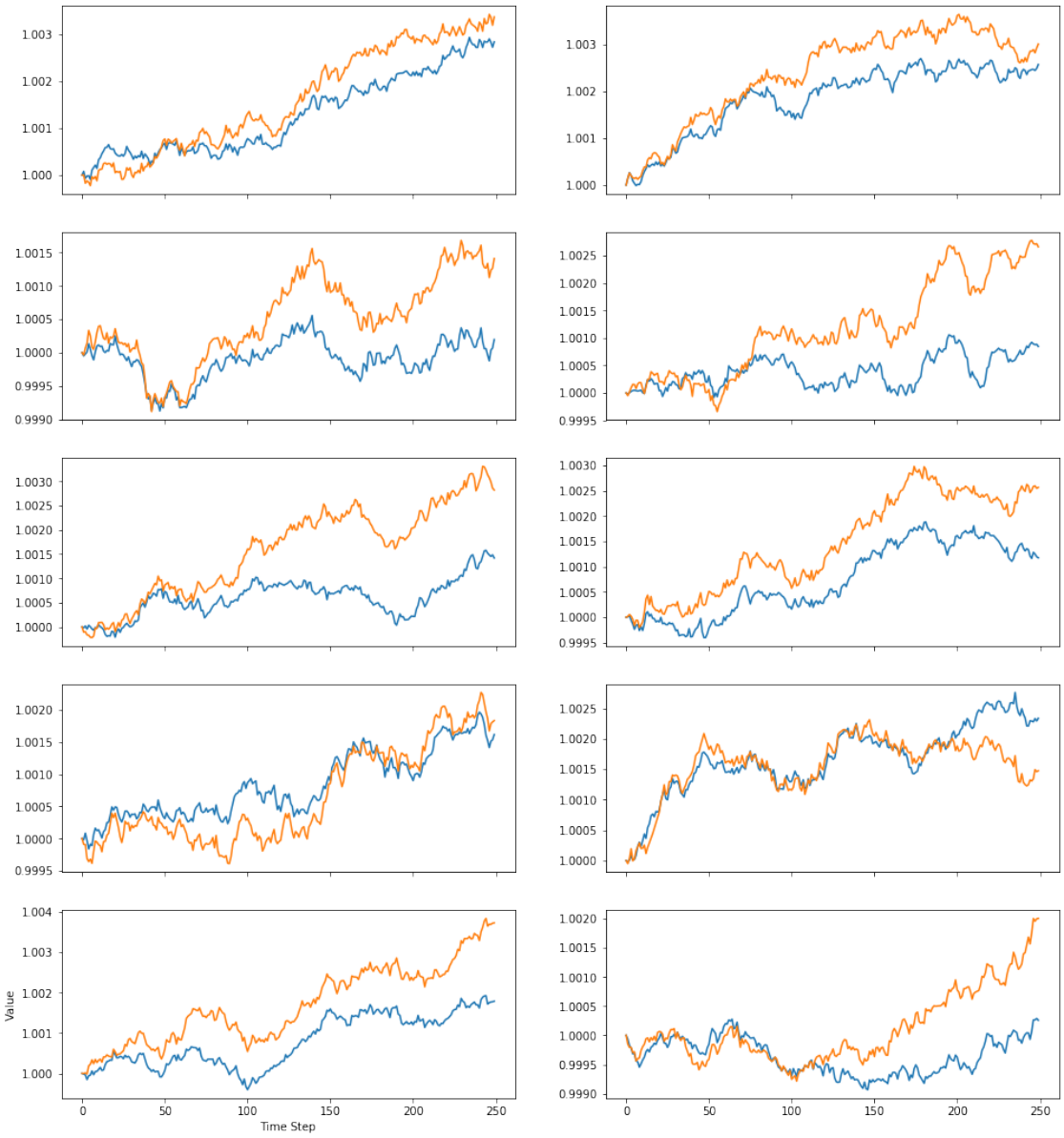
```
In [5]: alpha = np.array([0.1/N, 0.2/N])
        sigma = np.reshape([0.1/N, 0.05/N, 0.1/N, 0.2/N], [2, 2])

In [6]: fig, axarr = plt.subplots(nrows=5, ncols=2, figsize=[16, 18], sharex=True)
        for idx, ax in enumerate(axarr.flatten()):
            # Generate pair of Wiener processes
            S = np.ones([N, 2])
            for idx, row in enumerate(S[:-1]):
```

```
            S[idx + 1] = row + row * alpha * delta + row @ sigma * sample_gaussian()

        ax.plot(S)

    axarr[4, 0].set_xlabel('Time Step')
    axarr[4, 0].set_ylabel('Value');
```



1.2 Question 4

1.2.1 Constrained optimization problem

$$\min_{w} w^T \sigma C \sigma^T w \text{ subject to } \alpha^T w = \mu_V,\ \mathbf{1}^T w = 1,\ w \geq 0$$

where μ_V is some target mean return.

1.2.2 Equivalent unconstrained problem

$$\min_{w,\lambda} w^T \sigma C \sigma^T w + \lambda_1(\alpha^T w - \mu_V) + \lambda_2(\mathbf{1}^T w - 1)$$

Setting the gradient of the Lagrangian equal to 0 gives us:

$$2\sigma C \sigma^T w + \lambda_1 \alpha + \lambda_2 \mathbf{1} = 0$$

$$\alpha^T w = \mu_V$$

$$\mathbf{1}^T w = 1$$

```
In [7]: mu_V = 1
```

```
In [8]: mu_tilde = np.array([mu_V, 1])
        m_tilde = np.vstack([alpha, np.ones_like(alpha)]).T
        cov_inv = np.linalg.inv(sigma @ C @ sigma.T)
        B = m_tilde.T @ cov_inv @ m_tilde

        # Formula from portfolio theory
        w = cov_inv @ m_tilde @ np.linalg.inv(B) @ mu_tilde
```

```
In [9]: if (w < 0).any():
            print('w short sells. Reject.')
        else:
            print('w does not short sell. Consider.')
```

```
w short sells. Reject.
```

```
In [10]: one_zero = np.array([1, 0])
         zero_one = np.array([0, 1])

         one_zero_vol = one_zero.T @ sigma @ C @ sigma.T @ one_zero
         zero_one_vol = zero_one.T @ sigma @ C @ sigma.T @ zero_one
```

```
In [11]: w_max = one_zero if one_zero_vol > zero_one_vol else zero_one
         w_min = one_zero if one_zero_vol < zero_one_vol else zero_one
```

```
In [12]: # Since w_max = [0, 1] and w_min = [1, 0], this problem is just a reproduction
         # of question 3c. Indeed, the blue line appears to have lower variance than
         # the orange line.

         fig, axarr = plt.subplots(nrows=5, ncols=2, figsize=[16, 18], sharex=True)
         for idx, ax in enumerate(axarr.flatten()):
             # Generate pair of Wiener processes
```

```
        S = np.ones([N, 2])
        for idx, row in enumerate(S[:-1]):
            S[idx + 1] = row + row * alpha * delta + row @ sigma * sample_gaussian()

        ax.plot(S)

    axarr[4, 0].set_xlabel('Time Step')
    axarr[4, 0].set_ylabel('Value');
```