# stoch_calc_pset_q2

November 18, 2018

## 1 Stochastic Calculus Problem Set 2 Question 2

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        import seaborn as sns
        sns.set_style('whitegrid')
```

### 1.1 Pre-Question

*How does the SDE tell you that the Hull-White interest rate is not a martingale?*
Because it has a $dt$ term.

### 1.2 Part (a)

#### 1.2.1 Hull-White

Here, we have

$$\beta(u,x) = a_t - b_t R_t^{HW}$$

and

$$\gamma(u,x) = \sigma_t$$

Thus,

$$g_t(t,x) + (a_t - b_t R_t^{HW}) g_x(t,x) + \frac{1}{2}\sigma_t^2 g_{xx}(t,x) = 0$$

#### 1.2.2 Cox-Ingersol-Ross

Here, we have

$$\beta(u,x) = a_t + b_t R_t^{CIR}$$

and

$$\gamma(u,x) = \sigma\sqrt{R_t^{CIR}}$$

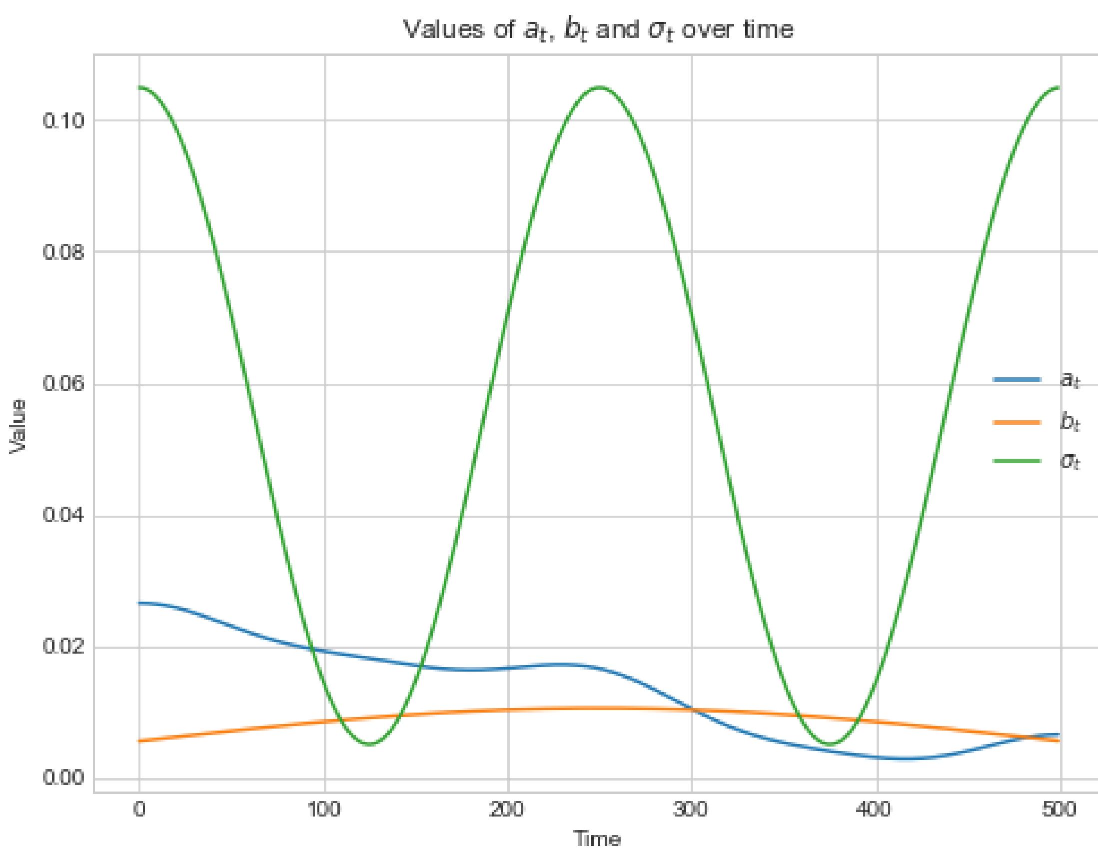
Thus,

$$g_t(t,x) + (a_t - b_t R_t^{CIR}) g_x(t,x) + \frac{1}{2}\sigma^2 R_t^{CIR} g_{xx}(t,x) = 0$$

### 1.3 Part (b)

```
In [2]: epsilon = 0.01  # Keep interest rate bounded away from 0

        delta = 0.01  # Time step
        T = 500  # Roughly 2 years
        K_a = 0.01
        K_b = 0.005
        K_sigma = 0.05

        t = np.arange(0, T)
        b = K_b * (1.1 + np.sin(np.pi*t / T))
        sigma = K_sigma * (1.1 + np.cos(4*np.pi*t / T))
        a = 0.5 * sigma**2 + K_a * (1.1 + np.cos(np.pi*t / T))
```

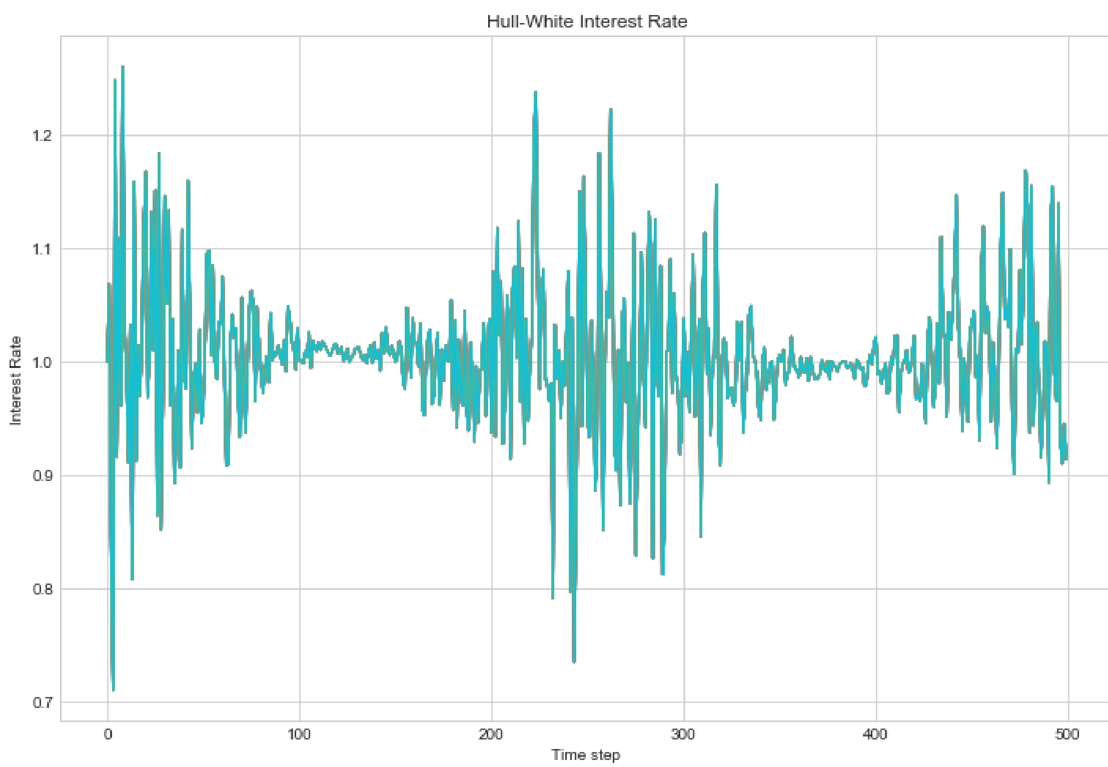
```
In [3]: fig, ax = plt.subplots(figsize=[8, 6])
        ax.plot(a, label='$a_t$')
        ax.plot(b, label='$b_t$')
        ax.plot(sigma, label='$\sigma_t$')
        ax.set_title('Values of $a_t$, $b_t$ and $\sigma_t$ over time')
        ax.set_xlabel('Time')
        ax.set_ylabel('Value')
        ax.legend();
```



### 1.3.1 Sub-Part 1

```
In [4]: # Hull-White
        R_HW = np.ones(shape=[T, 10])

        for idx, row in enumerate(R_HW[1:]):
            if (row < epsilon).any():
                print('Exception occured at iteration {}'.format(idx))
                row[row < epsilon] = epsilon

            dR = (a[idx] - b[idx] * R_HW[idx]) + sigma[idx] * np.random.randn()
            R_HW[idx + 1] = row + dR
```

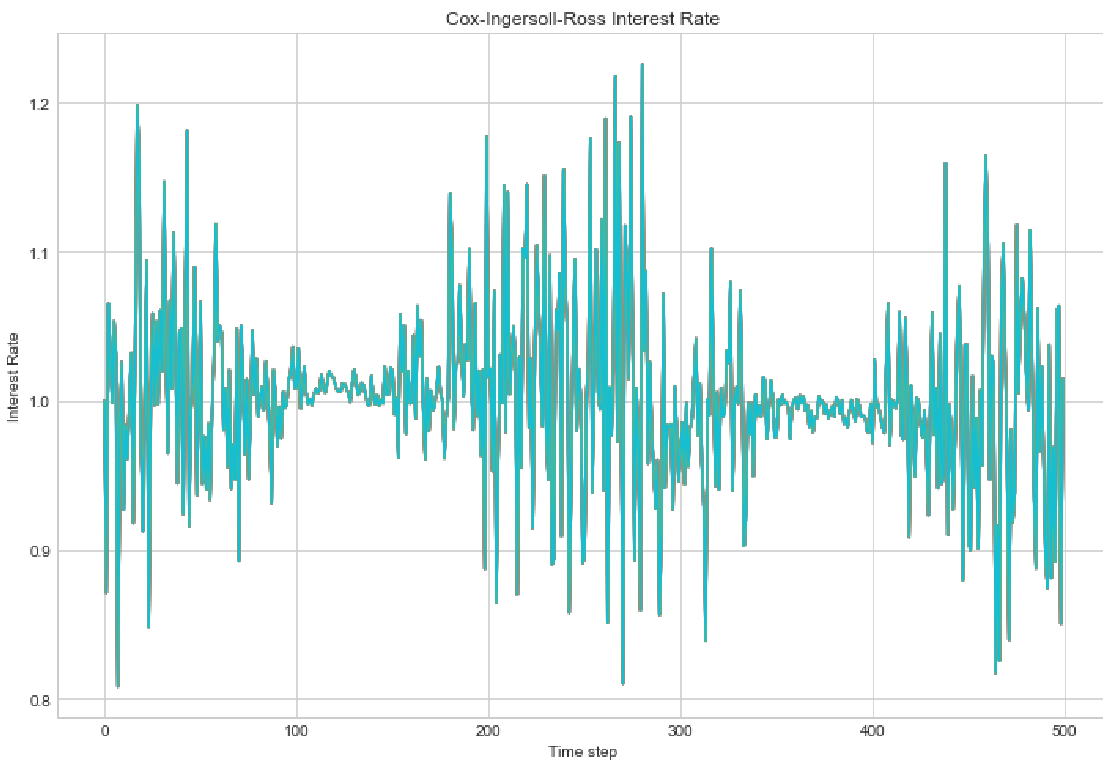
```
In [5]: fig, ax = plt.subplots(figsize=[12, 8])
        ax.plot(R_HW)
        ax.set_title('Hull-White Interest Rate')
        ax.set_xlabel('Time step')
        ax.set_ylabel('Interest Rate');
```




```
In [6]: # Cox-Ingersoll-Ross
        R_CIR = np.ones(shape=[T, 10])
```

```
for idx, row in enumerate(R_CIR[1:]):
    if (row < epsilon).any():
        print('Exception occured at iteration {}'.format(idx))
        row[row < epsilon] = epsilon

    dR = (a[idx] - b[idx] * R_CIR[idx]) + sigma[idx] * np.sqrt(R_CIR[idx]) * np.random
    R_CIR[idx + 1] = row + dR
```

```
In [7]: fig, ax = plt.subplots(figsize=[12, 8])
        ax.plot(R_CIR)
        ax.set_title('Cox-Ingersoll-Ross Interest Rate')
        ax.set_xlabel('Time step')
        ax.set_ylabel('Interest Rate');
```




### 1.3.2 Sub-Part 2

Using WolframAlpha, we have that

$$c(t) = \int_0^t K_b(1.1 + \sin(\frac{\pi t}{T}))du = K_b(1.1t - \frac{T}{\pi}cos(\frac{\pi t}{T}))$$

Thus,

$$\int_0^T \exp[-a(u)(c(T)-c(0))]du = \int_0^T \exp[-1.1K_b T(\frac{1}{2}(K_\sigma(1.1+\cos(4\pi t/T)))^2 + K_a(1.1+\cos(\frac{\pi t}{T})))]du$$

This integral does not appear to have a closed-form solution. Let us call it $I(T)$.
Finally, the integrand of the last integral is

$$\exp[-2(c(T)-c(u))]\sigma(u) = \exp[-2(K_b(1.1T + \frac{T}{\pi} - 1.1u + \frac{T}{\pi}cos(\frac{\pi u}{T})))](K_\sigma(1.1+\cos(4\pi t/T)))$$

$$= \exp[-2(1.41831T - 1.1u + (Tcos((\pi u)/T))/\pi)K_b](1.1 + cos((4\pi t)/T))K_\sigma$$

So, all told, the Hull-White interest rate is

$$R(T) = r\exp[-1.1K_b T] + I(T) + \int_0^T \exp[-2(1.41831T - 1.1u + (Tcos((\pi u)/T))/\pi)K_b](1.1 + cos((4\pi t)/T))K_\sigma du$$

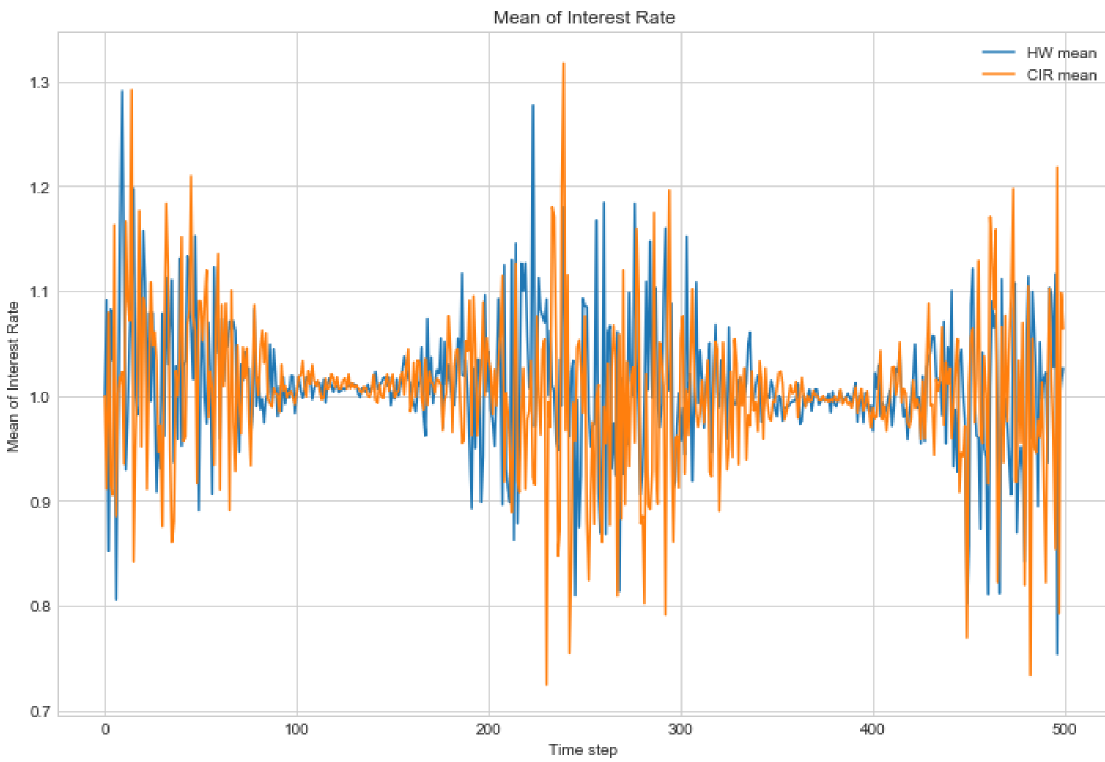
### 1.4 Part (c)

```
In [8]: # Hull-White
        R_HW = np.ones(shape=[T, 1000])

        for idx, row in enumerate(R_HW[1:]):
            if (row < epsilon).any():
                print('Exception occured at iteration {}'.format(idx))
                row[row < epsilon] = epsilon

            dR = (a[idx] - b[idx] * R_HW[idx]) + sigma[idx] * np.random.randn()
            R_HW[idx + 1] = row + dR
```

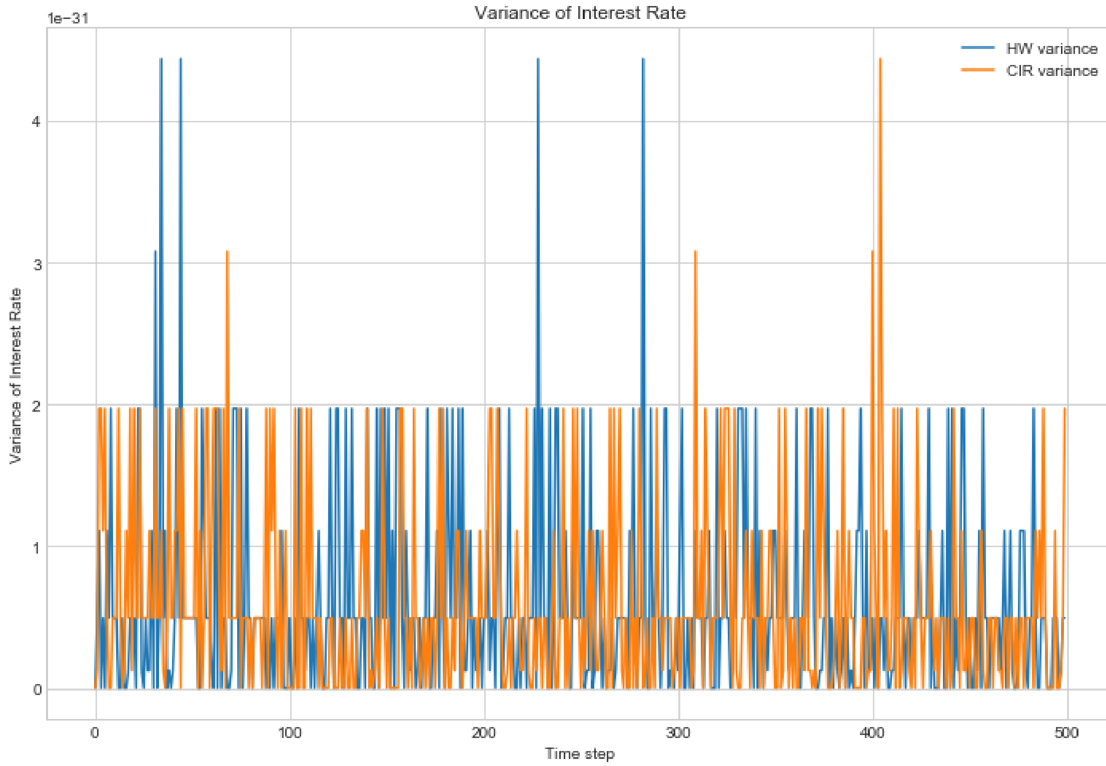
```
In [9]: # Cox-Ingersoll-Ross
        R_CIR = np.ones(shape=[T, 1000])

        for idx, row in enumerate(R_CIR[1:]):
            if (row < epsilon).any():
                print('Exception occured at iteration {}'.format(idx))
                row[row < epsilon] = epsilon

            dR = (a[idx] - b[idx] * R_CIR[idx]) + sigma[idx] * np.sqrt(R_CIR[idx]) * np.random
            R_CIR[idx + 1] = row + dR
```

```
In [10]: R_HW_mean = np.mean(R_HW, axis=1)
         R_CIR_mean = np.mean(R_CIR, axis=1)
         R_HW_var = np.var(R_HW, axis=1)
         R_CIR_var = np.var(R_CIR, axis=1)
```

```
In [11]: fig, ax = plt.subplots(figsize=[12, 8])
         ax.plot(R_HW_mean, label='HW mean')
         ax.plot(R_CIR_mean, label='CIR mean')
         ax.set_title('Mean of Interest Rate')
         ax.set_xlabel('Time step')
         ax.set_ylabel('Mean of Interest Rate')
         ax.legend();
```




```
In [12]: fig, ax = plt.subplots(figsize=[12, 8])
         ax.plot(R_HW_var, label='HW variance')
         ax.plot(R_CIR_var, label='CIR variance')
         ax.set_title('Variance of Interest Rate')
         ax.set_xlabel('Time step')
         ax.set_ylabel('Variance of Interest Rate')
         ax.legend();
```




## 1.5 Part (d)

The exception does not occur at all.