stoch_calc_pset_q1

November 18, 2018

1 Stochastic Calculus Problem Set II Question 1

```
In [1]: import numpy as np
        import pandas as pd
        import matplotlib.pyplot as plt
        import seaborn as sns
        sns.set_style('whitegrid')
```

```
In [2]: delta = 0.01
        N = 250

        S_0 = 1
        alpha = 0.1 / N
        sigma = alpha
        r = alpha / 3
```

1.1 Part (a)

```
In [3]: # alpha dt + sigma dW
        S = np.hstack([np.ones(shape=[1000, 1]),
                      alpha*delta + sigma*np.random.normal(loc=0,
                                                            scale=delta,
                                                            size=[1000, N - 1])]).T

        # This accumulates to make the Brownian motion
        for idx, row in enumerate(S[:-1]):
            S[idx + 1] *= row
            S[idx + 1] += row
```

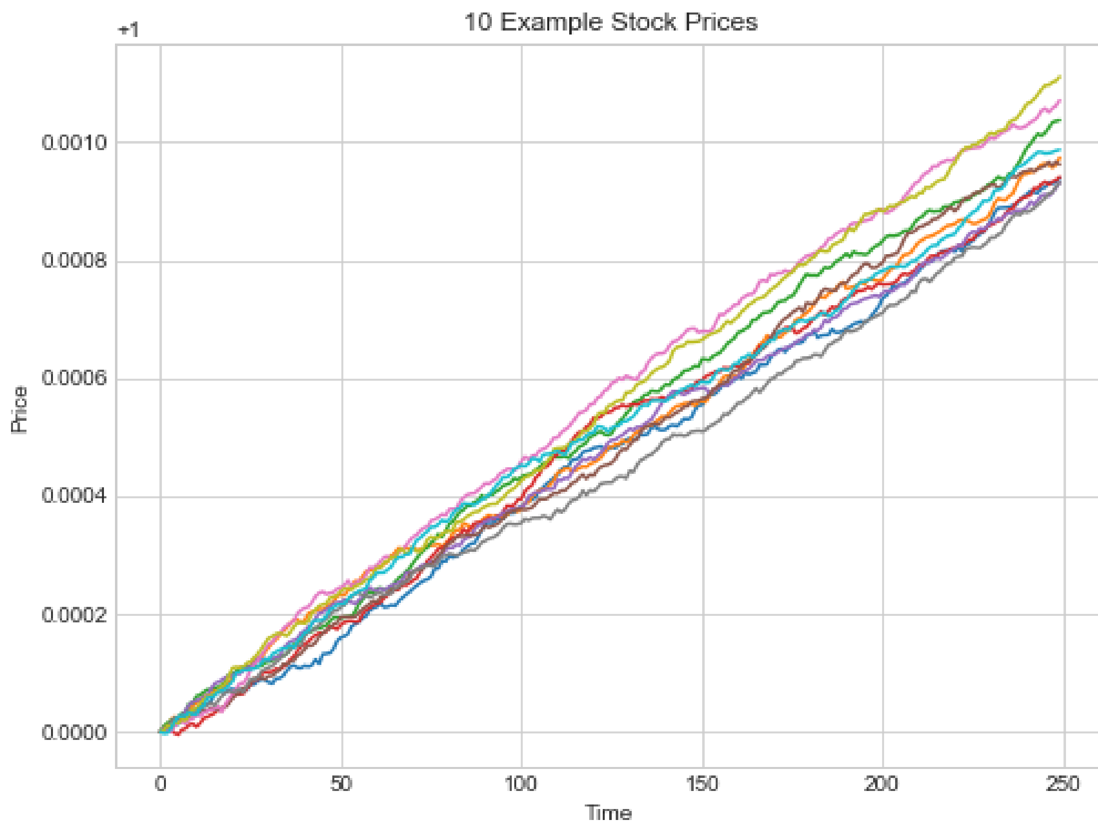
```
In [4]: # Python 0-indexes, so N/2 = 125 = index 124
        # Mean stock price at time t = 125 is ~ 1.0005
        print(np.mean(S[N // 2 - 1]))
```

```
1.0004949287399365
```

1.2 Part (b)

```
In [5]: # 10 paths
        paths = S[:, :10]
```

```
In [6]: fig, ax = plt.subplots(figsize=[8, 6])
        ax.plot(paths)
        ax.set_title('10 Example Stock Prices')
        ax.set_xlabel('Time')
        ax.set_ylabel('Price');
```



1.3 Part (c)

The discounted stock price X_t is a martingale with respect to the risk-neutral measure, and $\tilde{W}_t$ is a standard Wiener process with respect to the risk-neutral measure, so

$$dX_t = \sigma X_t d\tilde{W}_t$$

Furthermore, we can express S_t in terms of X_t:

$$S_t = X_t(1+r)^t$$

In [7]: M = 1000

In [8]:
```
S_N2_hat = np.zeros([10])

for idx, path in enumerate(paths.T):
    S_N2 = path[N // 2]

    # Initial condition
    X_N2 = S_N2 / (1+r)**(N//2)

    # sigma dW_tilde
    X = np.hstack([X_N2 * np.ones(shape=[M, 1]),
                   sigma*np.random.normal(loc=0, scale=delta,
                                          size=[M, N//2 - 1])]).T

    # This accumulates to make the Brownian motion
    for i, row in enumerate(X[:-1]):
        X[i + 1] *= row
        X[i + 1] += row

    S_N2_hat[idx] = np.mean(X[-1] * (1+r)**(N//2))
```

Actually, this becomes a fairly trivial issue, because $dX = 0dt + \sigma X_t d\tilde{W}_t$ and it is clearly obvious, therefore, than $X^{(i)}[N] = X^{(i)}[N/2] + Y$ where Y is a random variable that is independent of $\mathcal{F}_{N/2}$ and (under the risk-neutral measure) has mean value 0, so obviously $\tilde{E}(X_N|\mathcal{F}_{N/2}) = X_{N/2}$.

1.4 Part (d)

This is an estimate of the conditional variance.

In [9]:
```
np.mean((S_N2_hat - S_N2)**2)
```

Out[9]: 2.216343412988674e-09