NE 155 Spring 2016
Homework #2 Due Feb. 12, 2016

SSID:

On homework:

- If you work with anyone else, document what you worked on together.
- Show your work.
- Always clearly label plots (axis labels, a title, and a legend if applicable).
- Homework should be done "by hand" (i.e. not with a numerical program such as MATLAB, Python, or Wolfram Alpha) unless otherwise specified. You may use a numerical program to check your work.
- If you use a numerical program to solve a problem, submit the associated code, input, and output (email submission is fine).
- If using Python, be aware of `copy` vs. `deep copy`: https://docs.python.org/2/library/copy.html

1. (20 points) Using four points on the interval $[x_0, x_3]$, do the following:
   (a) (4 points) Construct all of the Lagrange polynomials $L_j(x)$ that correspond to the points x_0, x_1, x_2, and x_3 by hand.
   (b) (4 points) Use these Lagrange polynomials to construct the interpolating polynomial, $P_3(x)$, that interpolates the function $f(x)$ at the points x_0, x_1, x_2, and x_3 by hand.
   (c) (8 points) Using the $P_3(x)$ you derived, create an interpolant for

   $$f(x) = \cos(\frac{\pi}{2}x) + \frac{x^2}{2}$$

   over $[x_0, x_3]$ with $x_0 = 0, x_1 = 2, x_2 = 3$, and $x_3 = 4$. You may do this using something like Python or MATLAB, but *write your own functions rather than using the built in ones.*

   Plot the actual function and your interpolant using 100 equally spaced points for x between -0.5 and 4.5.

   (d) (4 points) Repeat what you did in part c but instead use $x_0 = 0, x_1 = 1, x_2 = 2.5$, and $x_3 = 4$.

   Discuss the differences in how well the function is interpolated using the different point sets.

2. (15 points) Using the interpolant $P_3(x)$ derived in question 1:
   (a) (3 points) Write the general expression for the error term, $err(x) = |f(x) - P_3(x)|$.
   (b) (4 points) Given
   $$f(x) = \cos(\frac{\pi}{2}x) + \frac{x^2}{2},$$
   use information about the function to bound the error expression.
   (c) (8 points) Use the values $x_0 = 0, x_1 = 2, x_2 = 3$, and $x_3 = 4$ to get the upper bound of $err(x)$ over this interval. That is, insert the points into the expression from part b, find the x that maximizes the expression, and present one final number. You may use a mathematical package to assist you in this calculation.

3. (15 points) We have the following data:
   $$x = [1, 2, 3, 4, 5, 6],$$
   $$f(x) = [1, 3, 15, 12, 7, 3].$$
   (a) (10 points) Using *built in* Python or MATLAB functions, interpolate this data using
   - Piecewise linear interpolation
   - Lagrange polynomial interpolation
   - Spline interpolation

   Create a subplot for each of your interpolants over $[0.75, 6.25]$ using a fine mesh spacing, e.g. 0.05 (note that to use scipy's piecewise linear polynomial interpolation you will need to restrict the range to the exact endpoints, 1.0 and 6.0). Include the data points on the interpolation plots.
   (b) (5 points) Briefly discuss the differences between the resulting interpolations.

4. (20 points) The errors generated by a numerical method on a test problem with various grid resolutions have been recorded in the following table:

Grid Spacing (h)	Error (E)
5.00000e-02	1.036126e-01
2.50000e-02	3.333834e-02
1.25000e-02	1.375409e-02
6.25000e-03	4.177237e-03
3.12500e-03	1.103962e-03
1.56250e-03	2.824698e-04
7.81250e-04	7.185644e-05
3.90625e-04	1.813937e-05

For this numerical method, the error should be of the form

$$E = kh^p$$

(a) (3 points) Write this problem as a linear system $\mathbf{A}\vec{x} = \vec{b}$, where $x = \begin{pmatrix} \log(k) \\ p \end{pmatrix}$ is the vector of unknowns.

(b) (5 points) Derive the normal equations for this over-determined system.

(c) (9 points) Solve the normal equations to obtain a least squares estimate to the parameters k and p.

(d) (3 points) Make a log-log plot that displays both the input data and the function $E = kh^p$. (Checkout Python's `matplotlib.pyplt.loglog` or MATLAB's `loglog` command)