

Axioms for the category of vector spaces

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The plan

- ① Motivation
- ② The axioms
- ③ The main result
- ④ Future work

Motivation

Motivation

It has been proven that if a **dagger** category $(\mathcal{C}, (-)^\dagger)$ satisfies certain axioms, it is equivalent to Hilb_k , with $k \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ [HK22], [LT25], [PV25].

This work is a **non-dagger** analogue of the above.

Question: What axioms must a category \mathcal{C} satisfy to guarantee:

$$\mathcal{C} \simeq \text{Vect}_k ?$$

Compare with [Freyd64] characterizing Mod_R for a ring R .

The axioms

The category Vect_k

Let k be a division ring. The category Vect_k has:

- k -vector spaces as objects,
- a morphism $A \rightarrow B$ is a k -linear function.

Let us study Vect_k .

Observation 1

The 0-dimensional vector spaces $\{0\}$ has the property that for any other vector spaces A, B , we have:

$$A \xrightarrow{\exists!} \{0\} \xrightarrow{\exists!} B$$

The category Vect_k

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Definition

Let \mathcal{C} be a category. A *zero object* is an object $0 \in \mathcal{C}$ with the property that for all A, B :

$$A \xrightarrow{\exists!} 0 \xrightarrow{\exists!} B$$

The composite above is then called the *zero map* $0 : A \rightarrow B$.

Biproducts (1/2)

Observation 2

When A, B are k -vector spaces, we may form their *biproduct* $A \oplus B$. Its elements are ordered pairs (a, b) with $a \in A, b \in B$.

$$\begin{array}{ccccc} & & A \oplus B & & \\ & \xleftarrow{\pi_A} & & \xrightarrow{\pi_B} & \\ A & & & & B \\ & \xrightarrow{\iota_A} & & \xleftarrow{\iota_B} & \end{array}$$

- We have $\pi_A(a, b) = a$, $\iota_A(a) = (a, 0)$,
- Any linear map $\varphi : A \oplus B \rightarrow C$ corresponds to $(\varphi' : A \rightarrow C, \varphi'' : B \rightarrow C)$,
- Any linear map $\psi : C \rightarrow A \oplus B$ corresponds to $(\psi' : C \rightarrow A, \psi'' : C \rightarrow B)$.

Biproducts (1/2)

Definition

Let A, B be objects in a category \mathcal{C} with a zero object 0 . Their *biproduct* is a diagram:

$$\begin{array}{ccccc} & & A \oplus B & & \\ & \xleftarrow{\pi_A} & & \xrightarrow{\pi_B} & \\ A & \xrightarrow{\iota_A} & A \oplus B & \xleftarrow{\iota_B} & B \end{array}$$

Such that:

- $\pi_A \circ \iota_B = 0, \quad \pi_B \circ \iota_A = 0,$
- (ι_A, ι_B) is the coproduct of A, B in \mathcal{C} ,
- (π_A, π_B) is the product of A, B in \mathcal{C} .

Biproducts (2/2)

Observation 3

For every injective linear function $i : A \rightarrow B$ we can find $r : B \rightarrow A$ such that $r \circ i = 1_A$.

We say that i is a *split monomorphism* or a *section*, with r being its *retraction*.

In Vect_k , any such pair can be completed into a biproduct:

$$\begin{array}{ccccc} & & r & & \\ & \swarrow & & \searrow & \\ A & & B & & A' \\ & \searrow & & \swarrow & \\ & i & & & \end{array}$$

Proof: consider $p := i \circ r : B \rightarrow B$, we have $B \cong A \oplus \text{Ker}(p)$.

Simple object

Observation 4

The 1-dimensional k -vector space I has a special property:
An injective linear function $U \rightarrow I$ is either zero, or it is an isomorphism.

Definition

An object $I \in \mathcal{C}$ is called *simple* when any nonzero monomorphism $A \rightarrow I$ is an isomorphism.

In addition, for a nonzero k -vector space A :

- there exists a nonzero linear function $I \rightarrow A$,
- any such function is a split monomorphism.

Directed colimits (1/2)

Denote by $(\text{Vect}_k)_{\text{mono}} \subseteq \text{Vect}_k$ the subcategory of injective linear maps.

Let (P, \leqslant) be a directed poset. Consider a functor $P \rightarrow (\text{Vect}_k)_{\text{mono}}$. It is a directed diagram of k -vector spaces:

$$\begin{array}{ccccccc}
 A_1 & \xrightarrow{\varphi_{12}} & A_2 & \xrightarrow{\varphi_{23}} & A_3 & \longrightarrow & \dots \\
 & \searrow \iota_1 & \searrow \iota_2 & \searrow \iota_3 & \searrow \ddots & & \downarrow \\
 & & & & \ast & & C
 \end{array}$$

In $(\text{Vect}_k)_{\text{mono}}$ this exists. Also $\text{Span}(\bigcup_{i \in P} \text{Im}(\iota_i)) = C$.

We say that the family $\{\iota_i, i \in P\}$ is *jointly epic*.

Directed colimits (2/2)

Theorem

The category $(\text{Vect}_k)_{\text{mono}}$ admits directed colimits and the inclusion functor $(\text{Vect}_k)_{\text{mono}} \rightarrow \text{Vect}_k$ preserves jointly epic families of morphisms.

The main result

Summary

Observation

The category Vect_k has the following properties:

- (V1) it has finite biproducts.
- (V2) every pair (s, r) of a section and a retraction can be completed to a biproduct,
- (V3) the 1-dimensional vector space I is simple and for any nonzero vector space A :
 - (V3)(a) there exists a nonzero morphism $I \rightarrow A$,
 - (V3)(b) every nonzero morphism $I \rightarrow A$ is injective.
- (V4) $(\text{Vect}_k)_{\text{mono}}$ admits directed colimits and the inclusion $(\text{Vect}_k)_{\text{mono}} \rightarrow \text{Vect}_k$ preserves jointly epic families of morphisms.

Theorem

Let \mathcal{C} be a category. Assume the following:

- (V1) \mathcal{C} has finite biproducts,
- (V2) every pair (s, r) of a section and a retraction can be completed to a biproduct,
- (V3) there is a simple object I and for any nonzero object A we have:
 - (V3)(a) there exists a nonzero morphism $I \rightarrow A$,
 - (V3)(b) every nonzero morphism $I \rightarrow A$ is a split monomorphism,
- (V4) the subcategory $\mathcal{C}_{\text{split mono}}$ admits directed colimits and the inclusion $\mathcal{C}_{\text{split mono}} \rightarrow \mathcal{C}$ preserves jointly epic families of morphisms.

Then there exists a division ring k and an equivalence:

$$\mathcal{C} \simeq \text{Vect}_k.$$

Future work

The dagger version

A *dagger category* \mathcal{C} is a category equipped with an involutive functor $(-)^{\dagger} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ that is the identity on objects.

Example: a $*$ -monoid M , Rel, $\text{Hilb}_{\mathbb{C}}$.

Given $f : \mathcal{H} \rightarrow \mathcal{K}$, its adjoint $f^{\dagger} : \mathcal{K} \rightarrow \mathcal{H}$ is uniquely determined by the property that for all $u \in \mathcal{H}, v \in \mathcal{K}$:

$$\langle f(u), v \rangle = \langle u, f^{\dagger}(v) \rangle.$$

A biproduct $\mathcal{H} = U \oplus V$ is a *dagger-biproduct* if $U^{\perp} = V$, $V^{\perp} = U$.

A *dagger-monomorphism* f is a morphism satisfying $f^{\dagger} \circ f = 1_{\mathcal{H}}$.
Isometries.

Theorem

Let \mathcal{C} be a category. Assume the following:

- (V1) \mathcal{C} has finite biproducts,
- (V2) every pair (s, r) of a section and a retraction can be completed to a biproduct,
- (V3) there is a simple object I and for any nonzero object A we have:
 - (V3)(a) there exists a nonzero morphism $I \rightarrow A$,
 - (V3)(b) every nonzero morphism $I \rightarrow A$ is a split monomorphism,
- (V4) the subcategory $\mathcal{C}_{\text{split mono}}$ admits directed colimits and the inclusion $\mathcal{C}_{\text{split mono}} \rightarrow \mathcal{C}$ preserves jointly epic families of morphisms.

Then there exists a division ring k and an equivalence:

$$\mathcal{C} \simeq \text{Vect}_k.$$

Theorem [PV25]

Let \mathcal{C} be a **dagger** category. Assume the following:

- (H1) \mathcal{C} has finite **dagger** biproducts,
- (H2) every **dagger monomorphism** can be completed to a **dagger** biproduct,
- (H3) there is a simple object I and for any nonzero object A we have:
 - (H3)(a) there exists a nonzero morphism $I \rightarrow A$,
 - (H3)(b) every nonzero morphism $I \rightarrow A$ is **isomorphic to a dagger monomorphism**,
- (H4) the subcategory $\mathcal{C}_{\text{dagger mono}}$ admits directed colimits and the inclusion $\mathcal{C}_{\text{dagger mono}} \rightarrow \mathcal{C}$ preserves jointly epic families of morphisms.
- (H5) Every **dagger automorphism** has a strict square root.

Then there exists a **dagger** equivalence $\mathcal{C} \simeq \text{Hilb}_{\mathbb{C}}$.

Axioms are similar, but why?

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Thank you.