Axioms for the category of vector spaces

Miloslav Štěpán
with Jan Paseka, Thomas Vetterlein

Institute for Mathematical Methods in Medicine und Data-Based Modelling
Johannes Kepler University

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JOHANNES KEPLER
UNIVERSITY LINZ

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The plan

1. Motivation
2. The axioms
3. The main result
4. Future work

Motivation

Motivation

It has been proven that if a **dagger** category $(\mathcal{C}, (-)^\dagger)$ satisfies certain axioms, it is equivalent to Hilb_k, with $k \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ [HK22], [LT25], [PV25].

This work is a **non-dagger** analogue of the above.

Question: What axioms must a category $\mathcal{C}$ satisfy to guarantee:

$$\mathcal{C} \simeq \mathrm{Vect}_k \ ?$$

Compare with [Freyd64] characterizing Mod_R for a ring R.

The axioms

The category Vect_k

Let k be a division ring. The category Vect_k has:

- k-vector spaces as objects,
- a morphism $A \to B$ is a k-linear function.

Let us study Vect_k.

Observation 1

The 0-dimensional vector spaces $\{0\}$ has the property that for any other vector spaces A, B, we have:

$$A \xrightarrow[\exists!]{} \{0\} \xrightarrow[\exists!]{} B$$

The category Vect_k

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- k-vector spaces as objects,
- a morphism $A \to B$ is a k-linear function.

Let us study Vect_k.

Definition

Let $\mathcal{C}$ be a category. A *zero object* is an object $0 \in \mathcal{C}$ with the property that for all A, B:

$$A \xrightarrow[\exists!]{} 0 \xrightarrow[\exists!]{} B$$

The composite above is then called the *zero map* $0 : A \to B$.

Biproducts (1/2)

Observation 2

When A, B are k-vector spaces, we may form their *biproduct* $A \oplus B$. Its elements are ordered pairs (a, b) with $a \in A$, $b \in B$.

$$\begin{tikzcd} A \arrow[r, shift right, "\iota_A"'] & A \oplus B \arrow[l, shift right, "\pi_A"'] \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "\iota_B"] \end{tikzcd}$$

- We have $\pi_A(a, b) = a, \qquad \iota_A(a) = (a, 0)$,
- Any linear map $\varphi : A \oplus B \to C$ corresponds to $(\varphi' : A \to C, \varphi'' : B \to C)$,
- Any linear map $\psi : C \to A \oplus B$ corresponds to $(\psi' : C \to A, \psi'' : C \to B)$.

Biproducts (1/2)

Definition

Let A, B be objects in a category $\mathcal{C}$ with a zero object 0. Their *biproduct* is a diagram:

$$\begin{tikzcd} A \arrow[r, shift right, "\iota_A"'] & A \oplus B \arrow[l, shift right, "\pi_A"'] \arrow[r, shift left, "\pi_B"] & B \arrow[l, shift left, "\iota_B"] \end{tikzcd}$$

Such that:

- $\pi_A \circ \iota_B = 0, \qquad \pi_B \circ \iota_A = 0,$
- (ι_A, ι_B) is the coproduct of A, B in $\mathcal{C}$,
- (π_A, π_B) is the product of A, B in $\mathcal{C}$.

Biproducts (2/2)

Observation 3

For every injective linear function $i : A \to B$ we can find $r : B \to A$ such that $r \circ i = 1_A$.

We say that i is a *split monomorphism* or a *section*, with r being its *retraction*.

In Vect_k, any such pair can be completed into a biproduct:

$$\begin{tikzcd} A \arrow[r, shift right, "i"'] & B \arrow[l, shift right, "r"'] \arrow[r, dashed, shift left] & A' \arrow[l, dashed, shift left] \end{tikzcd}$$

Proof: consider $p := i \circ r : B \to B$, we have $B \cong A \oplus \mathrm{Ker}(p)$.

Simple object

Observation 4

The 1-dimensional k-vector space I has a special property:
An injective linear function $U \to I$ is either zero, or it is an isomorphism.

Definition

An object $I \in \mathcal{C}$ is called *simple* when any nonzero monomorphism $A \to I$ is an isomorphism.

In addition, for a nonzero k-vector space A:

- there exists a nonzero linear function $I \to A$,
- any such function is a split monomorphism.

Directed colimits (1/2)

Denote by $(\mathrm{Vect}_k)_{\mathrm{mono}} \subseteq \mathrm{Vect}_k$ the subcategory of injective linear maps.

Let $(P, \leqslant)$ be a directed poset. Consider a functor $P \to (\mathrm{Vect}_k)_{\mathrm{mono}}$. It is a directed diagram of k-vector spaces:

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_{12}"] \arrow[drrr, dashed, "\iota_1"] & A_2 \arrow[r, "\varphi_{23}"] \arrow[drr, dashed, "\iota_2"] & A_3 \arrow[r] \arrow[dr, dashed, "\iota_3"] & \dots \arrow[d, dashed] \\
& & & C
\end{tikzcd}$$

In $(\mathrm{Vect}_k)_{\mathrm{mono}}$ this exists. Also $\mathrm{Span}(\bigcup_{i \in P} \mathrm{Im}(\iota_i)) = C$.

We say that the family $\{\iota_i, i \in P\}$ is *jointly epic*.

Directed colimits (2/2)

Theorem

The category $(\mathrm{Vect}_k)_{\mathrm{mono}}$ admits directed colimits and the inclusion functor $(\mathrm{Vect}_k)_{\mathrm{mono}} \to \mathrm{Vect}_k$ preserves jointly epic families of morphisms.

The main result

Summary

Observation

The category Vect_k has the following properties:

(V1) it has finite biproducts.

(V2) every pair (s, r) of a section and a retraction can be completed to a biproduct,

(V3) the 1-dimensional vector space I is simple and for any nonzero vector space A:

- (V3)(a) there exists a nonzero morphism $I \to A$,
- (V3)(b) every nonzero morphism $I \to A$ is injective.

(V4) $(\mathrm{Vect}_k)_{\mathrm{mono}}$ admits directed colimits and the inclusion $(\mathrm{Vect}_k)_{\mathrm{mono}} \to \mathrm{Vect}_k$ preserves jointly epic families of morphisms.

Theorem

Let $\mathcal{C}$ be a category. Assume the following:

(V1) $\mathcal{C}$ has finite biproducts,

(V2) every pair (s, r) of a section and a retraction can be completed to a biproduct,

(V3) there is a simple object I and for any nonzero object A we have:

- (V3)(a) there exists a nonzero morphism $I \to A$,
- (V3)(b) every nonzero morphism $I \to A$ is a split monomorphism,

(V4) the subcategory $\mathcal{C}_{\text{split mono}}$ admits directed colimits and the inclusion $\mathcal{C}_{\text{split mono}} \to \mathcal{C}$ preserves jointly epic families of morphisms.

Then there exists a division ring k and an equivalence:

$$\mathcal{C} \simeq \mathrm{Vect}_k.$$

Future work

The dagger version

A *dagger category* $\mathcal{C}$ is a category equipped with an involutive functor $(-)^\dagger : \mathcal{C}^{op} \to \mathcal{C}$ that is the identity on objects.

Example: a $*$-monoid M, Rel, $\mathrm{Hilb}_{\mathbb{C}}$.

Given $f : \mathcal{H} \to \mathcal{K}$, its adjoint $f^\dagger : \mathcal{K} \to \mathcal{H}$ is uniquely determined by the property that for all $u \in \mathcal{H}$, $v \in \mathcal{K}$:

$$\langle f(u), v \rangle = \langle u, f^\dagger(v) \rangle.$$

A biproduct $\mathcal{H} = U \oplus V$ is a *dagger-biproduct* if $U^\perp = V$, $V^\perp = U$.

A *dagger-monomorphism* f is a morphism satisfying $f^\dagger \circ f = 1_{\mathcal{H}}$. Isometries.

Theorem

Let $\mathcal{C}$ be a category. Assume the following:

(V1) $\mathcal{C}$ has finite biproducts,

(V2) every pair (s, r) of a section and a retraction can be completed to a biproduct,

(V3) there is a simple object I and for any nonzero object A we have:

- (V3)(a) there exists a nonzero morphism $I \to A$,
- (V3)(b) every nonzero morphism $I \to A$ is a split monomorphism,

(V4) the subcategory $\mathcal{C}_{\text{split mono}}$ admits directed colimits and the inclusion $\mathcal{C}_{\text{split mono}} \to \mathcal{C}$ preserves jointly epic families of morphisms.

Then there exists a division ring k and an equivalence:

$$\mathcal{C} \simeq \mathrm{Vect}_k.$$

Theorem [PV25]

Let $\mathcal{C}$ be a **dagger** category. Assume the following:

(H1) $\mathcal{C}$ has finite **dagger** biproducts,

(H2) every dagger monomorphism can be completed to a **dagger** biproduct,

(H3) there is a simple object I and for any nonzero object A we have:

- (H3)(a) there exists a nonzero morphism $I \to A$,
- (H3)(b) every nonzero morphism $I \to A$ is isomorphic to a dagger monomorphism,

(H4) the subcategory $\mathcal{C}_{\textbf{dagger}\ \mathrm{mono}}$ admits directed colimits and the inclusion $\mathcal{C}_{\textbf{dagger}\ \mathrm{mono}} \to \mathcal{C}$ preserves jointly epic families of morphisms.

(H5) Every dagger automorphism has a strict square root.

Then there exists a **dagger** equivalence $\mathcal{C} \simeq \mathrm{Hilb}_{\mathbb{C}}$.

Axioms are similar, but why?

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Thank you.