Differential Geometry of Curves and Surfaces.

A solution manual.

M^2

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Contents

Chapter 1

Curves

1.1 PARAMETRIZED CURVES

Problem 1. An interval means a nonempty conected subset of $\mathbb{R}$. Prove that every interval has one of the following forms:

$$(a,b),[a,b],(a,b],[a,b),(-\infty,b),(-\infty,b],(a,\infty),[a,\infty),(-\infty,\infty)$$

Problem 2. If $f:[a,b]\to\mathbb{R}$ is smooth, show that $f^{(k)}:[a,b]\to\mathbb{R}$ can be defined without reference to any extension.

Proof. A simple way to define $f^{(k)}$ is as follows: We say that $f\in C^k([a,b])$ if

$f\in C^k((a,b))$ and

$$f^{(i)}(a)=\lim_{h\to 0+}\frac{f^{(i-1)}(a+h)-f^{(i-1)}}{h},$$

$$f^{(i)}(b)=\lim_{h\to 0-}\frac{f^{(i-1)}(b+h)-f^{(i-1)}(b)}{h}$$

for $i=1,...,k$. □

Problem 3. Show that a logarithmic spiral restricted to the interval $[0,\infty)$ has finite arc length.

Proof. First, we have that

$$\dot{\gamma}(t)=ce^{\lambda t}\left([\cos(t)-\sin(t)],[\sin(t)+\cos(t)]\right)$$

and under the standard euclidean norm

$$||\dot{\gamma(t)}|| = ce^{\lambda t}\sqrt{2},$$

so we must show that

$$\int_0^\infty ce^{\lambda t}\sqrt{2}dt,$$

converges. To do this, first notice that for $x > 0$

$$\int_0^x e^{\lambda t}\sqrt{2}dt = \frac{c\sqrt{2}}{\lambda}\left(e^{\lambda x} - 1\right),$$

and since $\lambda < 0$,

$$\lim_{x\to\infty} e^{\lambda x} = 0,$$

therefore

$$\int_0^\infty ce^{\lambda t}\sqrt{2}dt = \lim_{x\to\infty}\frac{c\sqrt{2}}{\lambda}\left(e^{\lambda x} - 1\right) = -\frac{c\sqrt{2}}{\lambda}.$$

So γ has finite arc length. □

Problem 4. For the curve

$$\gamma(t) = (sin(t), cos(t) + ln(tan(t/2))), t \in (\pi/2, \pi)$$

show that any point p in its image, the segment of the tangent line at p and the y-axis has length 1.

Problem 5. If all three component functions of a space curve γ are quadratic functions, prove that the image γ is contained in a plane.

Problem 6. Prove that the arc length, L, of the graph of the polar coordinate function $r(\theta), \theta \in [a, b]$, is

$$L = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\ \mathrm{d}\theta$$

Proof. If $\gamma : [a, b] \to \mathbb{R}^n$ is a parametrized curve, its arc length is given by

$$\int_a^b \|\gamma'(t)\|\, dt,$$

and if we write γ in component form we have $\|\gamma'(t)\| = \sqrt{\sum_{i=1}^{n}(\gamma_i(t)')^2}$, which means that the arc length can be written as

$$\int_a^b \left[\sum_{i=1}^{n}\left(\frac{d\gamma_i(t)}{dt}\right)^2\right]^{1/2} dt.$$

So given a polar coordinate function $f(\theta)$, we know its graph is given by

$$\gamma(\theta) = (f(\theta)\cos(\theta), f(\theta)\sin(\theta))$$

and in this way:

$$\gamma(\theta)' = (f(\theta)'\cos(\theta) - f(\theta)\sin(\theta), f(\theta)'\sin(\theta) + f(\theta)\cos(\theta)),$$

therefore

$$\|\gamma(\theta)'\| = \sqrt{(f(\theta)')^2 + (f(\theta))^2},$$

as one can easily check. Thus, the arc length of the graph of $f(\theta)$ is given by

$$L = \int_a^b \sqrt{(f(\theta)')^2 + (f(\theta))^2} dt,$$

as expected. □

Problem 7. Let $\gamma : [a, b] \to \mathbb{R}$ be a regular curve. Suppose P is a partition $a = t_0 < t_1 < ... < t_k = b$ of $[a, b]$, and define $\delta = \max\{t_{i+1} - t_i\}$, show that

$$L = \sum_{i=0}^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\|$$

converges to the arc length of γ for every sequence of partitions for which $\delta \to 0$.