Selected problems in Classical Differential Geometry.

M^2

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Problem 1. The logarithmic spiral

$$\gamma(t) = c(e^{\lambda t}\cos(t), e^{\lambda t}\sin(t)),$$

(where $\lambda, c \in \mathbb{R}$ and $c \neq 0$, $\lambda < 0$), restricted to the interval $[0, \infty)$ has finite arc length.

Proof. We start by noticing that

$$\gamma'(t) = c(e^{\lambda t}[\lambda\cos(t) - \sin(t)], e^{\lambda t}[\lambda\sin(t) + \cos(t)]),$$

which means that under the standard Euclidean norm:

$$||\gamma'(t)|| = ce^{\lambda t}\sqrt{\lambda^2 + 1}.$$

In this way, the arc length of γ is given by

$$\int_0^\infty ||\gamma'(t)||dt = c\sqrt{\lambda^2+1}\int_0^\infty e^{\lambda t}dt,$$

and naturally

$$c\sqrt{\lambda^2+1}\int_0^\infty e^{\lambda t}dt = c\sqrt{\lambda^2+1}\lim_{x\to\infty}\int_0^x e^{\lambda t}dt = c\sqrt{\lambda^2+1}\lim_{x\to\infty}\frac{1}{\lambda}\left(e^{\lambda x} - 1\right),$$

but given that $\lambda < 0$, then it is the case that

$$\lim_{x\to\infty} e^{\lambda x} = 0,$$

therefore:

$$\int_0^\infty ||\gamma'(t)||dt = -\frac{c}{\lambda}\sqrt{\lambda^2+1}.$$

And this completes the demonstration.

$\square$