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Real Analysis

Fundamental Facts

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Part I
Sequences and Series

Chapter 1
Sequences

Definition 1.1 Sequence
A sequence on a set X is a map $f : \mathbb{N} \to X$.

For $n \in \mathbb{N}$ we adopt the notation $f(n) = x_n$ [1], and call x_n *the n'th term of the sequence*. The set of values of a sequence is usually denoted by

$$\{x_n : n \in \mathbb{N}\},$$

when the sequence is understood, or only $\{x_n\}$.

1.1 Cauchy criterion of convergence

Theorem 1: Cauchy Criterion

A sequence of real numbers converges if and only if it is a Cauchy sequence.

1.2 Monotonic sequences and Weierstrass theorem

Definition 1.2 A sequence $\{x_n\}$ is increasing if $m < n$ implies $x_m \leq x_n$ and strictly increasing if $m < n$ implies $x_m < x_n$.

Theorem 2: Weierstrass

An increasing sequence of real numbers converges if and only if it is bounded above.

[1] where x is a dummy variable.

1.3 Subsequences

Definition 1.3 Let $\{x_n\}$ be a sequence and suppose $n_1 < n_2 < \ldots < n_k < \ldots$ is an increasing sequence of natural numbers. The sequence $x_{n_1}, x_{n_2}, \ldots$, (indexed by the sequence above), is called a *subsequence* of $\{x_n\}$.

Example 1.1 Consider the sequence $\{\frac{n}{n^2}\}$ whose terms are:

$$1, \frac{2}{4}, \frac{3}{9}, \frac{4}{16}, \ldots$$

if we index only using even integers, we get the subsequence

$$\frac{2}{4}, \frac{4}{16}, \frac{6}{36}, \frac{8}{64}, \ldots$$

Chapter 2
Series

Definition 2.1 A *series* or *infinite sum* is a pair of sequences on $\mathbb{R}$:

$$(\{s_n\}, \{a_n\}),$$

which are connected by the relation

$$\sum_{k=1}^{n} a_k = s_n.$$

Definition 2.2 If the sequence $\{s_n\}$ converges to $\lambda \in \mathbb{R}$, we call λ *the limit of the infinite sum* $\sum_{k=1}^{\infty} a_k$, i.e:

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} s_n = \lambda.$$

2.1 Cauchy convergence criterion

Theorem 3: Cauchy convergence for series

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$ there exists and index $N \in \mathbb{N}$ such that the inequalities $m > n > N$ imply:

$$|a_n + \ldots + a_m| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| < \varepsilon.$$

Proof Apply *cauchy convergence criterion* to s_n.

Corollary 2.1 *Changing a finite amount of terms to the sequence* $\{a_n\}$ *doesnt affect the convergence or divergence of the sequence* $\{s_n\}$.

Corollary 2.2 *If* $\{s_n\}$ *converges, then* $\lim_{k\to\infty} a_k = 0$.

Proof Let $s_n \to \lambda$ as $n \to \infty$. By Cauchy, we know that there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k\right| = |a_{n+1}| = d(a_{n+1}, 0) < \varepsilon,$$

for all $\varepsilon > 0$ and whenever $n > N$, as desired. □

2.2 Absolute convergence and the comparison theorem

Definition 2.3 The series $\sum_{k=1}^{\infty} a_k$ is *absolutely convergent* if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Suppose $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then $\sum_{k=1}^{\infty} |a_k|$ converges, so *Cauchy convergence criterion for series* implies

$$\varepsilon > ||a_n| + \ldots + |a_m|| = |a_n| + \ldots + |a_m| \geq |a_n + \ldots + a_m|,$$

so the series $\sum_{k=1}^{\infty} a_k$ also converges. This means

$$\text{Absolute convergence} \implies \text{Convergence}$$

but the converse is not necesarily true.

The following theorems allow us to determine whether a series converges absolutely or not.

Lemma 2.1 *A series with non-negative terms converges if and only if its sequence of partial sums is bounded above.*

Proof Let $\{s_n\}$ be the sequence of partial sums. Clearly $\{s_n\}$ is increasing, thus, apply Weierestrass to it. □

Theorem 4: The comparison theorem

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series with nonnegative terms. If there exists and index $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n > N$, then the convergence of $\sum_{k=1}^{\infty} b_k$ implies the convergence of $\sum_{k=1}^{\infty} a_k$, and the divergence of $\sum_{k=1}^{\infty} a_k$ implies the divergence of $\sum_{k=1}^{\infty} b_k$

Proof Let $N \in \mathbb{N}$ be the desired index.

The convergence of $\sum_{k=1}^{\infty} b_k$ implies the equation below, which in turn implies the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$ is bounded above. Together with lemma 2.1, this implies $\sum_{k=1}^{\infty} a_k$ converges.

Now, suppose $\sum_{k=1}^{\infty} a_k$ diverges and let $\{A_n\}$ be its sequence of partial sums (respecively let $\{B_n\}$ the sequence of partial sums of $\sum_{k=1}^{\infty} b_k$). If it were the case that $\sum_{k=1}^{\infty} b_k$ converges, then[1] $\{B_n\}$ would be bounded above say, by $\lambda \in \mathbb{R}$.

Then for any $n > N$:

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \lambda.$$

which means $\{A_n\}$ is bounded above and hence, convergent, yielding a contradiction. Therefore $\sum_{k=1}^{\infty} b_k$ diverges. □

This theorem gives us a sufficient condition for the absolute convergence of an arbitrary series.

Theorem 5: The Weierstrass M-Test for absolute convergence

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series. Suppose there exists an index $N \in \mathbb{N}$ such that $|a_n| \leq b_n$ for all $n > N$.
If $\sum_{k=1}^{\infty} b_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Proof After some index $N \in \mathbb{N}$, the series $\sum_{k=1}^{\infty} b_k$ will contain nonnegative terms , and by the comparison theorem its convergence implies the convergence of $\sum_{k=1}^{\infty} |a_k|$, making $\sum_{k=1}^{\infty} a_k$ absolutely convergent. □

Theorem 6: Limit comparison test

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series with $a_k \geq 0, b_k > 0$, for all $k \in \mathbb{N}$. Suppose

$$\lim_{k\to\infty} \frac{a_k}{b_k} = c > 0.$$

$\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

Proof By hypothesis, for all $\varepsilon > 0$, there exists an index $N \in \mathbb{N}$ for which

$$\left|\frac{a_n}{b_n} - c\right| < \varepsilon$$

whenever $n > N$. Clearly this means

$$(c - \varepsilon)b_n < a_n < (c + \varepsilon)b_n,$$

and for small enough ε, such that $(c - \varepsilon) > 0$:

$$b_n < (c - \varepsilon)b_n < a_n < (c + \varepsilon)b_n.$$

[1] by lemma 2.1

If $\sum_{k=1}^{\infty} a_k$ converges, the equation above together with the comparison theorem imply $\sum_{k=1}^{\infty} b_k$ converges, and if it diverges, then by the comparion theorem so does $\sum_{k=1}^{\infty} b_k$. □

Theorem 7: d'Alembert's test (Ratio test)

Suppose that the limit $\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| = \alpha$ exists for the series $\sum_{k=1}^{\infty} a_k$. Then,

1. If $\alpha < 1$, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.
2. If $\alpha > 1$, the series $\sum_{k=1}^{\infty} a_k$ diverges.
3. There exists both absolutely convergent and divergent series for which $\alpha = 1$.