Problem compilation

M^2

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Chapter 1

Topology

1.1 Metric spaces

Problem 1. *Let* $X = \{f : [a,b] \to \mathbb{R} | f \in C[a,b]\}$*, and define* $d : X \times X \to X$ *by*

$$d(f,g) = \int_a^b |f(t) - g(t)| dt.$$

for $f, g \in X$*. Prove that* (X, d) *is a metric space.*

Proof. It is certainly the case that

$$0 \leq |f(t) - g(t)|,$$

and integrating both sides

$$0 \leq \int_a^b |f(t) - g(t)| dt = d(f,g),$$

so positive definiteness holds.

Moreover, suppose $d(f,g) = 0$, then $\int_a^b |f(t) - g(t)| dt = 0$, so $f(t) = g(t)$ on every subset with non-zero measure of $[a,b]$. However, suppose $\lambda \in [a,b]$ and $f(\lambda) \neq g(\lambda)$, then, there exists an open set $B(f(\lambda))$ such that, for any $x \in [a,b]$, $g(x) \notin B(f(\lambda))$, but this means that $f(t) \neq g(t)$ in some set of non-zero measure, which contradicts our initial assumption, hence $f(t) = g(t)$ everywhere on $[a,b]$. Conversely, if $f(t) = g(t)$, on $[a,b]$, then $|f(t) - g(t)| = 0$ and $d(f,g) = 0$.

Finally, we know that $|a+b| \leq |a| + |b|$, so letting $a = g(t) - f(t)$, $b = f(t) - h(t)$ and integrating, we show that the triangle inequality holds.

Consequently, (X, d) is a metric space. □

Problem 2. *Let* $(X_i, d_i), (Y_i, d_i)$ *and* $f_i : X_i \to Y_i$, $(i = 1, 2, ..., n)$ *be metric spaces and continuous functions.*

Let $X = \prod_{i=1}^n X_i$ *and* $Y = \prod_{i=1}^n Y_i$ *and convert* X *and* Y *into metric spaces in the standard manner. Define the function* $F : X \to Y$ *by*

$$F(x_1, x_2, ..., x_n) = (f_1(x_1), f_2(x_2), ..., f_n(x_n)).$$

Prove that F *is continuous.*

Proof. Let $\varepsilon > 0$ be given. Since each function f_i is continuous, there must exists $\delta_i > 0$, such that, if $a_i, b_i \in X_i$, and

$$d_i(a_i, b_i) < \delta_i,$$

then

$$d_i'(f_i(a_i), f_i(b_i)) < \varepsilon.$$

Let $\delta = \min_{1 \leq i \leq n}\{\delta_1, ..., \delta_n\}$, and suppose $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in X$. If

$$d(a, b) = \max_{1 \leq i \leq n}\{d_i(a_i, b_i)\} < \delta,$$

then for each $i = 1, ..., n$, $d_i(a_i, b_i) < \delta$, which means $d_i'(f_i(a_i), f_i(b_i)) < \varepsilon$, and especially:

$$\max_{1 \leq i \leq n}\{d_i'(f_i(a_i), f_i(b_i))\} < \varepsilon.$$

Therefore:

$$d(F(a), F(b)) < \varepsilon,$$

so F is a continuous function. □

Chapter 2

Algebra

2.1 Linear algebra

Problem 1. *Find $p \in P_5(\mathbb{R})$ that makes:*

$$\int_{-\pi}^{\pi} |sin(x) - p(x)|^2 dx$$

as small as posible.

Proof. Define the inner product $\langle u, v\rangle$ by

$$\langle u, v\rangle = \int_{-\pi}^{\pi} (uv)(x)\, dx$$

Since $\{1, x, x^2, x^3, x^4\}$ is a basis of $P_5(\mathbb{R})$, applying Gram-Schmidt procedure using the inner product we just defined yields the orthonormal basis:

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{\sqrt{\frac{3}{2}}x}{\pi^{3/2}}, \frac{3\sqrt{\frac{5}{2}}\left(x^2 - \frac{\pi^2}{3}\right)}{2\pi^{5/2}}, \frac{5\sqrt{\frac{7}{2}}\left(x^3 - \frac{3\pi^2 x}{5}\right)}{2\pi^{7/2}}, \frac{105\left(x^4 - \frac{6}{7}\pi^2\left(x^2 - \frac{\pi^2}{3}\right) - \frac{\pi^4}{5}\right)}{8\sqrt{2}\pi^{9/2}} \right)$$

which we denote by $(e_1, e_2, e_3, e_4, e_5)$. Let $v = sin(x)$, we want to find $u \in P_5(\mathbb{R})$ such that

$$||v - u||$$

is as small as posible.

To do this, we compute:

$$P_{P_5(\mathbb{R})}(v) = \sum_{i=1}^{5} \langle v, e_i \rangle e_i = \left(\int_{-\pi}^{\pi} (sin(x) e_i) dx \right) e_i.$$

Which equals

$$\frac{35\left(\pi^2 - 15\right)\left(x^3 - \frac{3\pi^2 x}{5}\right)}{2\pi^6} + \frac{3x}{\pi^2},$$

thus, for any $u \in P_5(\mathbb{R})$:

$$||v - P_{P_5(\mathbb{R})}(v)|| = \int_{-\pi}^{\pi} \left| sin(x) - \frac{35\left(\pi^2 - 15\right)\left(x^3 - \frac{3\pi^2 x}{5}\right)}{2\pi^6} + \frac{3x}{\pi^2} \right|^2 dx \leq ||v - u||.$$

□

2.2 Group theory

Problem 1. *If K is a field, denote the columns of the $n \times n$ identity matrix E by $\varepsilon_1, ..., \varepsilon_n$.*

A permutation matrix P over K is a matrix obtained from E by permuting its columns; that is, the columns of P are $\varepsilon_{\alpha(1)}, ..., \varepsilon_{\alpha(n)}$ form some $\alpha \in S_n$.

Prove that the set of all permutation matrices over K is a group isomorphic to S_n. (The inverse of P under matrix multiplication is P^t).

Proof. Let $P_M(K)$ the set of all permutation matrices over K. It is clearly the case that $P_M(K) \subset GL_n(K)$, and $E \in P_M(K)$, since it can be obtained from itself permuting with the identity permutation $i \in S_n$.

Now suppose $A, B \in P_M(K)$, then AB is certainly $n \times n$ and

$$\begin{aligned}
(AB)_{\bullet,k} &= A\,(B)_{\bullet,k} \\
&= A\varepsilon_{\alpha(s)} \\
&= \sum_{i=1}^{n} (\varepsilon_{\alpha(s)})_{i,\bullet} \times A_{\bullet,i} \\
&= (\varepsilon_{\alpha(s)})_{\alpha^{-1}(s),\bullet} \times A_{\bullet,\alpha^{-1}(s)} \\
&= \varepsilon_x,
\end{aligned}$$

for some $s, x \in \{1, 2, ...n\}$.

Then AB is indeed a permutation matrix and $AB \in P_M(K)$. Moreover, suppose $A \in P_M(K)$, then $A^t \in P_M(K)$ and $A \times A^t = E$, so $P_M(K)$ has the desired closure and we conclude

$$(P_M(K), \times)$$

is a group.

Moreover, define $\phi : S_n \to P_M(K)$ by

$$\phi(\alpha) = P_\alpha,$$

where P_α is the permutation matrix with columns $\varepsilon_\alpha(1), ..., \varepsilon_\alpha(n)$. Clearly ϕ is injective and suppose $X \in P_M(K)$, then $X_{\bullet,\lambda} = \varepsilon_k$ for some $k \in \{1, ..., n\}$, but then $X = P_\lambda$, for $\lambda \in S_n$, thus, ϕ is surjective.

Finally:

$$\phi(\alpha \circ \beta) = P_{\alpha\circ\beta},$$

but

$$\left(P_{\alpha\circ\beta}\right)_{\bullet,i} = \varepsilon_{\alpha\circ\beta(i)},$$

and

$$\begin{aligned}
\left(P_\alpha \circ P_\beta\right)_{\bullet,i} =& P_\alpha \circ (P_\beta)_{\bullet,i} \\
=& P_\alpha \circ \varepsilon_{\beta(i)} \\
=& \sum_{j=1}^{n} (P_\alpha)_{\bullet,j} \left(\varepsilon_{\beta(i)}\right)_{j,\bullet} \\
=& \sum_{j=1}^{n} \varepsilon_{\alpha(j)} \left(\varepsilon_{\beta(i)}\right)_{j,\bullet},
\end{aligned}$$

and $\left(\varepsilon_{\beta(i)}\right)_{j,\bullet}$ equals 1 when $j = \beta(i)$, so this equals

$$\varepsilon_\alpha(\beta(i)) = \varepsilon_{\alpha\circ\beta(i)} = \left(P_{\alpha\circ\beta}\right)_{\bullet,i}$$

therefore

$$\phi(\alpha \circ \beta) = P_{\alpha\circ\beta} = P_\alpha \circ P_\beta = \phi(\alpha) \circ \phi(\beta).$$

Then ϕ is an isomorphism and

$$S_n \cong P_M(K)$$

□

Chapter 3

Analysis

3.1 Real analysis

Problem 1. *Show that if* $f \in R[a,b]$*, then* $|f|^p \in R[a,b]$*, for* $p \geq 0$*.*

Problem 2. *Starting from Holder's inequality for sums, obtain Holder's inequality for integrals:*

$$\left| \int_a^b (f * g)(x)dx \right| \leq \left(\int_a^b |f|^p(x) \right)^{\frac{1}{p}} * \left(\int_a^b |g|^q(x)dx \right)^{\frac{1}{q}},$$

if $f, g \in R[a,b], p > 1, q > 1,$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*.*

Problem 3. *Starting from Minkwoski's inequality for sums,*

obtain Minkowski's inequailty for integrals:

$$\left(\int_a^b |f+g|^p(x)dx \right)^{\frac{1}{p}} \leq \left(\int_a^b |f|^p(x)dx \right)^{\frac{1}{p}} + \left(\int_a^b |g|^p(x)dx \right)^{\frac{1}{p}},$$

if $f, g \in R[a,b]$ *and* $p \geq 1$*.*

Proof. Let us begin with the inequality:

$$\left(\sum_{k=1}^n |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p \right)^{1/p}. \tag{3.1}$$

Let P be a partition of $[a,b]$, with distinguished points λ_k, and intervals Δ_k, with lenght $\Delta X_k = x_k - x_{k-1}$. Let $a_k = f(\lambda_k)$, and $b_k = g(\lambda_k)$. We now have:

$$\left(\sum_{k=1}^{n} |f+g|^{p}(\lambda_k)\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |f|^{p}(\lambda_k)\right)^{1/p} + \left(\sum_{k=1}^{n} |g|^{p}(\lambda_k)\right)^{1/p}$$

and multiplying both sides by $(\Delta X_k)^{\frac{1}{p}}$:

$$\left(\sum_{k=1}^{n} |f+g|^{p}(\lambda_k)\Delta X_k\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |f|^{p}(\lambda_k)\Delta X_k\right)^{1/p} + \left(\sum_{k=1}^{n} |g|^{p}(\lambda_k)\Delta X_k\right)^{1/p}$$

Finally, since $f, g \in R[a,b]$, then $f+g \in R[a,b]$, moreover, by problem a) it follows that $|f|^p, |g|^p, |f+g|^p \in R[a,b]$, so taking the limit as $\lambda(P) \to 0$ yields:

$$\left(\int_a^b |f+g|^p(x)dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f|^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b |g|^p(x)dx\right)^{\frac{1}{p}}$$

$\square$

Chapter 4

Geometry

4.1 Differential geometry of curves and surfaces

Problem 1. *A logarithmic spiral is a plane curve of the form*

$$\gamma(t) = c\left(e^{\lambda t}\cos(t), e^{\lambda t}\sin(t)\right),$$

where $c, \lambda \in \mathbb{R}$ and $c \neq 0$. Suppose $\lambda < 0$, and γ is restricted to $[0, \infty)$. Show that under this conditions, λ has finite arc length.

Proof. First, we have that

$$\dot{\gamma}(t) = ce^{\lambda t}\left([\cos(t) - \sin(t)], [\sin(t) + \cos(t)]\right)$$

and under the starndard euclidean norm

$$||\gamma\dot{(t)}|| = ce^{\lambda t}\sqrt{2},$$

so we must show that

$$\int_0^\infty ce^{\lambda t}\sqrt{2}dt,$$

converges. To do this, first notice that for $x > 0$

$$\int_0^x e^{\lambda t}\sqrt{2}dt = \frac{c\sqrt{2}}{\lambda}\left(e^{\lambda x} - 1\right),$$

and since $\lambda < 0$,

$$\lim_{x\to\infty} e^{\lambda x} = 0,$$

therefore

$$\int_0^\infty ce^{\lambda t}\sqrt{2}dt = \lim_{x\to\infty} \frac{c\sqrt{2}}{\lambda}\left(e^{\lambda x} - 1\right) = -\frac{c\sqrt{2}}{\lambda}.$$

So γ has finite arc length. $\square$