Risk Management - Problems III

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Correlations and Random Matrices

1. Generate several (of order $M = 1000$) large matrices with dimension $N = 8$ and then 16 and 50, populated according to the GOE principle, i.e., first, fill all elements of matrix M from normal distribution $\mathcal{N}(\mu = 0, \sigma^2 = 2)$, add to such matrix its transpose, and divide the sum by the doubled dimension of the matrix, in brief $X = (M + M^\dagger)/(2N)$. Check if the spectrum of X fullfills the Wigner semicircle. What happens if you would neglect the factor N in denominator?

2. Calculate by the brute force the first 10 spectral moments of the above ensemble and compare to the analytic moments obtained from the expression for resolvent $G(z)$, treated as a series expanded around the infinity $z = \infty$. (*) Crosscheck the obtained first 5 non-zero moments with the Sloane encyclopedia of integer sequences, http://oeis.org

3. Check that the replacement of the Gaussian variables by any other iid distribution with finite variance does not change the result for Wigner's semicircle.

4. For sample set (e.g. for $N = 8$ or $N = 16$) of eigenvalues of the GOE ensemble, sort the eigenvalues for each matrix in the increasing order and find the difference between the neightbouring $\lambda_{n+1} - \lambda_n$, for $n \sim N/2$. Plot the histogram of these splittings divided by the mean splitting, with bean size small enough to see the fluctuations. Check the numerics with analytic formula given by the Wigner's surmise . (*) Why not use all the eigenvalues' splittings?

5. Check numerically the spectrum of the Wishart ensemble. Consider several large number of rows N and large number of columns T rectangular random matrices populated from $\mathcal{N}(0, 1)$ and check what happens to the spectrum of matrix $C = \frac{1}{T}XX^\dagger$ where † means transposition. Plot few cases when $r = N/T$ is smaller, equal ot larger than 1. Compare the numerics to Marchenko-Pastur formula.

6. * Fill two large (e.g. $N = 100$) matrices D_1, D_2 in such a way, that on the diagonal the elements are chosen from random binary distribution $\{-1, 1\}$ and all other elements are put to zero. Then rotate one of the matrices by the **random** orthogonal transformation (e.g. $OD_1O^\dagger$) and calculate numerically the spectrum of the ensemble $OD_1O^\dagger + D_2$.

Try to get an analytic result for such operation using the R-transform tricks.

(*) Star means optional exercise (and more difficult...)