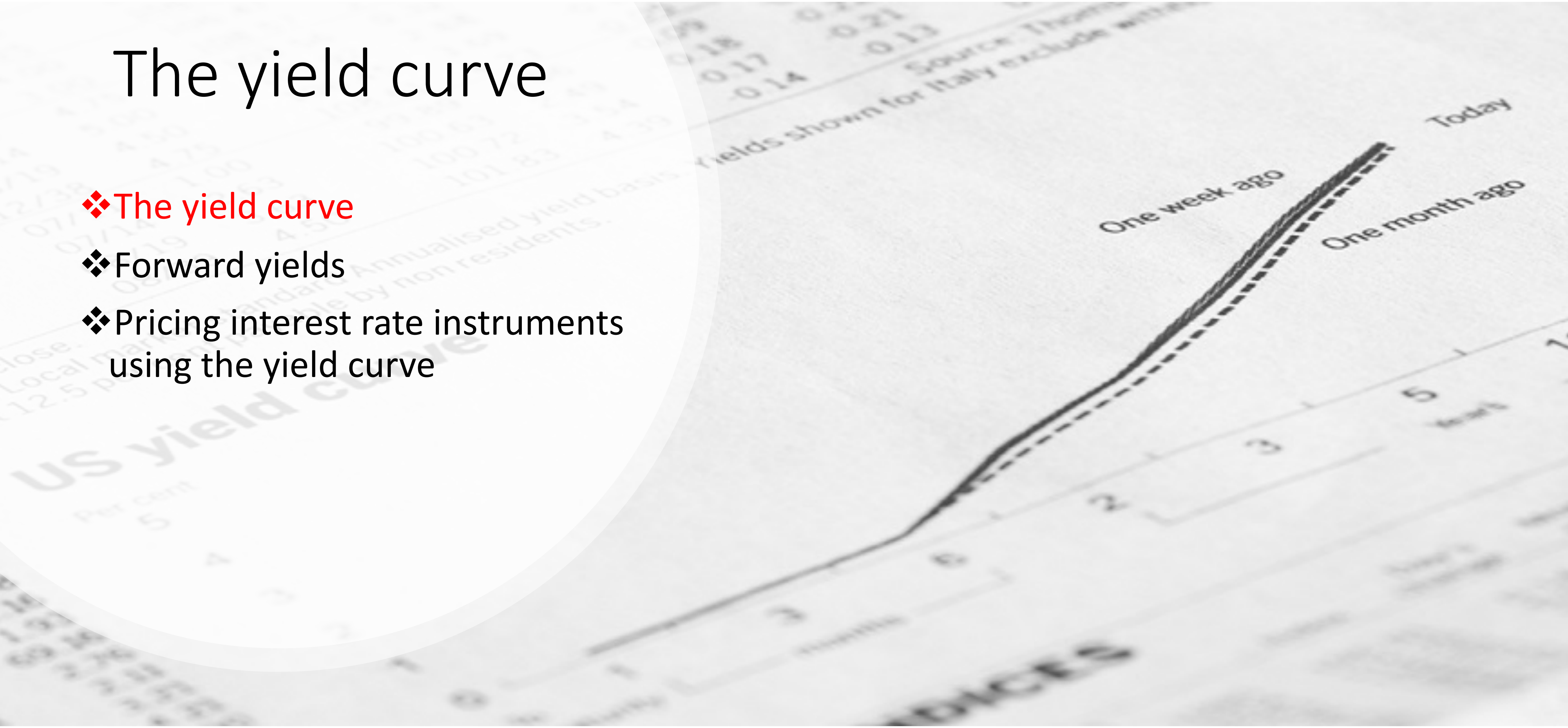
Lecture 4

The yield curve

Financial instruments and pricing

Fall 2019

The yield curve



- The yield curve
- Forward yields
- Pricing interest rate instruments using the yield curve

The yield curve

- The (interest rate) instruments pricing formula used so far assumed a constant effective rate of return (the same yield y was used to discount all future CFs, independent of t)*

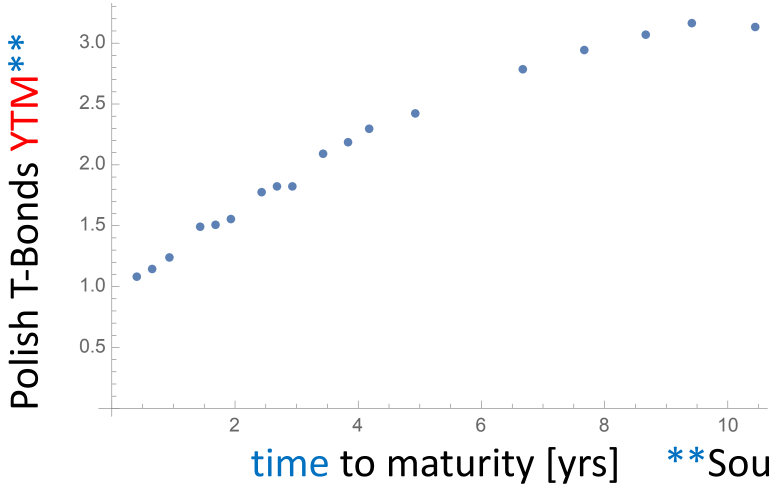
$$\sum_t CF(t)DF(t) = \sum_t \frac{CF(t)}{(1+y)^t} = 0 \qquad (1)$$

$$PV = -CF(0) = \sum_{t>0} \frac{CF(t)}{(1+YTM)^t}$$

- Real market statistics shows that the yield y=YTM is NOT constant over time but it is rather some function of time ⇒ the yield curve: y(t)

*NOTE: instead of the anually compounded yield y, one alternatively can (and sometimes does) use the continuously compounded yield y_c, then:

$DF(t) = \exp(- y_c t)$



**Source: BondSpot fixing on 23rd Nov 2018

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- Real market statistics shows that the yield y=YTM is NOT constant over time but it is rather some function of time ⇒ the yield curve: y(t)
  - From Lecture 2 we know, that for fixed coupon bonds, the effective lifetime of a bond is usually shorter than the time to maturity (e.g. duration D < maturity T) ⇒ problem: one can solve (1) for y=YTM, BUT which time t is appropriate in y(t) ?-maturity, (modified) duration, average lifetime, … ?

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CCC	99,40	4	1,5	5,84	1,46
DDD	97,84	5	2,0	6,18	1,95
EEE	99,18	5	2,5	6,46	2,36
FFF	82,20	-	3,0	6,75	3,00
GGG	100,28	6	3,5	6,89	3,17
HHH	76,00	-	4,0	7,10	4,00
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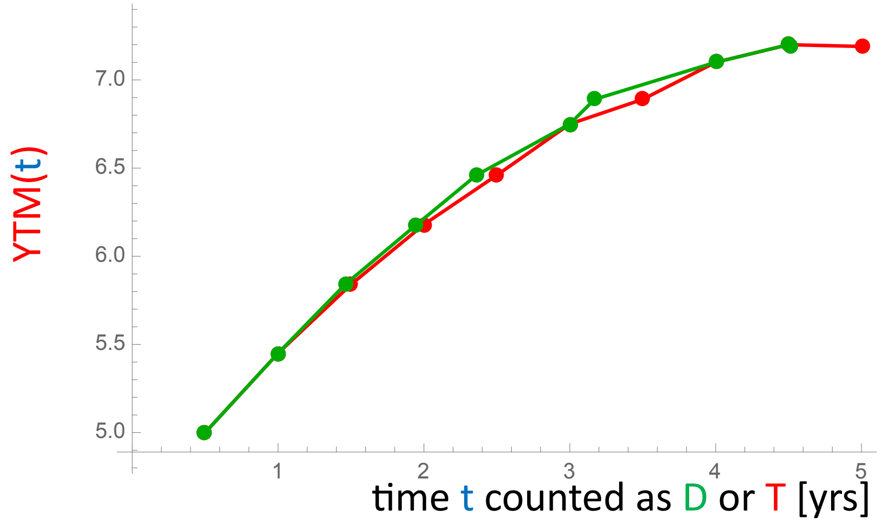

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This can be solved by assuming that EACH CASH FLOW is treated INTEPENDENTLY !
For a given bond, EACH CF(t) is discounted using a different yield: YTM ⟶ y(t) !

The yield curve: the zero-coupon curve

❖This can be solved by assuming that EACH CASH FLOW is treated INTEPENDENTLY ! For a given bond, EACH CF(t) is discounted using a different yield: YTM → y(t), such as a BOND was cut (“stripped”) into a SUM OF ZERO-COUPON bonds !

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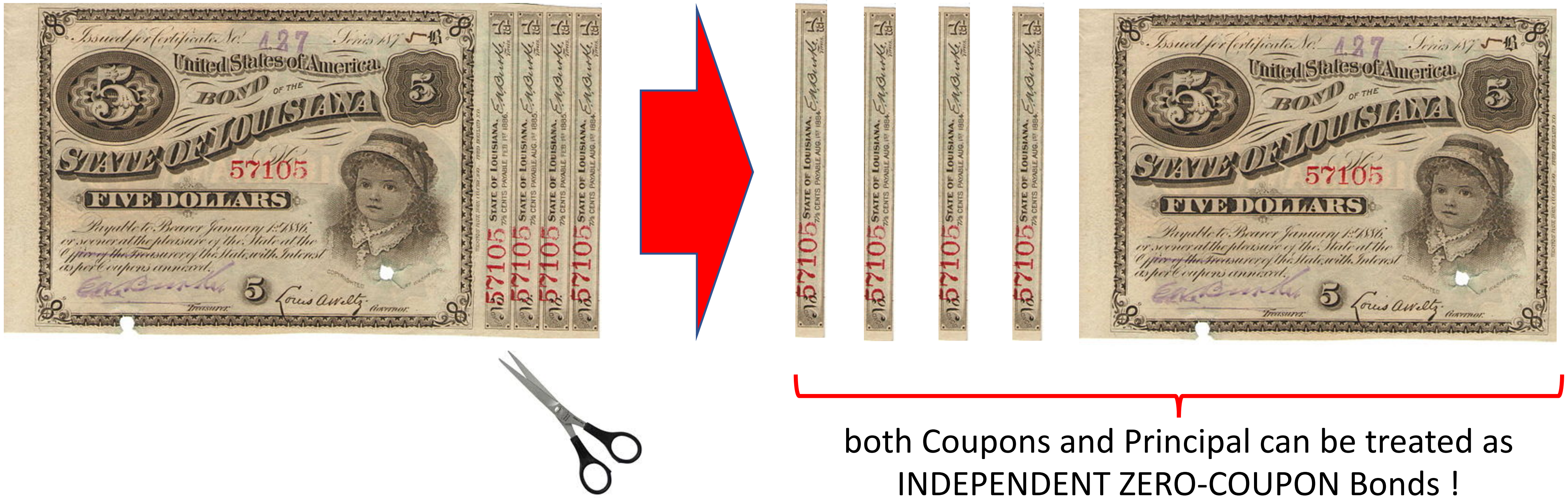

both Coupons and Principal can be treated as INDEPENDENT ZERO-COUPON Bonds !

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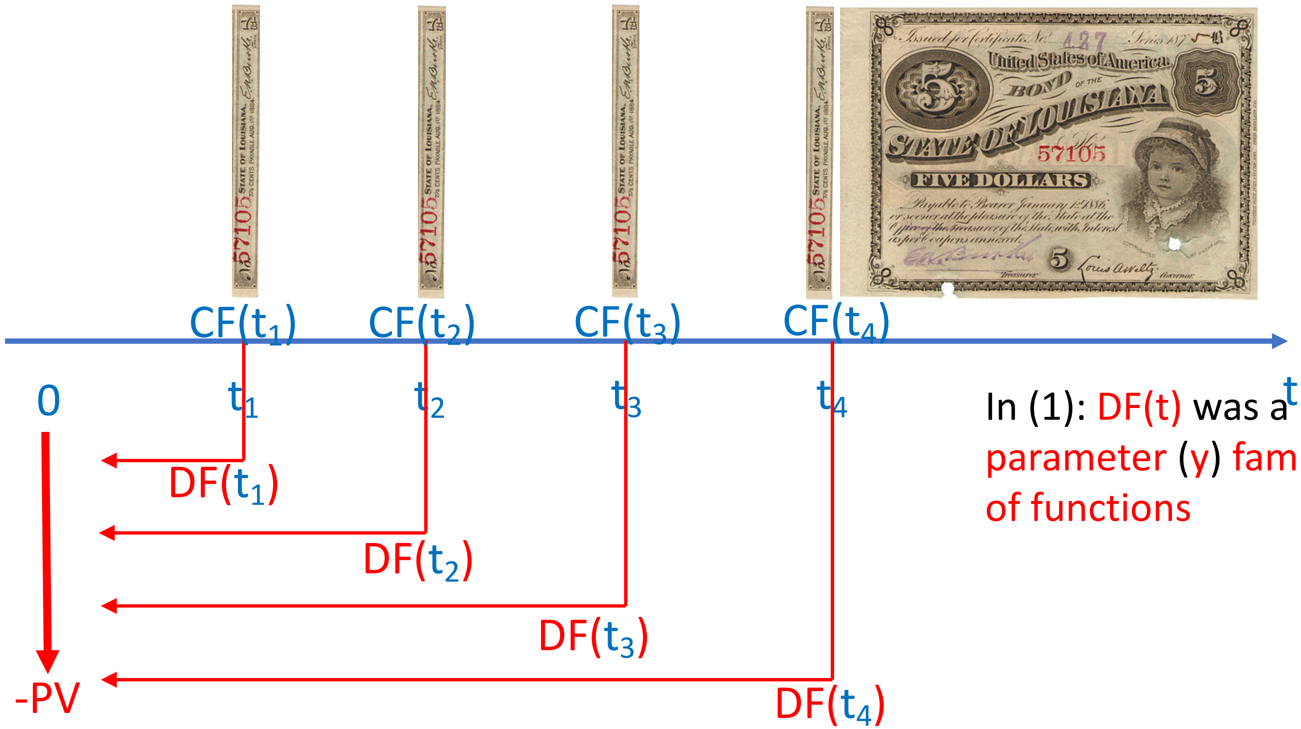

In (1): DF(t) was a one-parameter (y) family of functions

- ☐ The zero-(coupon)-yield is in one-to-one correspondence with the Discount Factor
  y(t) ⟷DF(t)
- ☐ Previously: each DF(t) was globally constrained:
  DF(t) = $(1+y)^{-t}$, y = YTM = const.!
- ☐ NOW: various DF(t) get independent !
  DF(t) = $(1+y(t))^{-t}$

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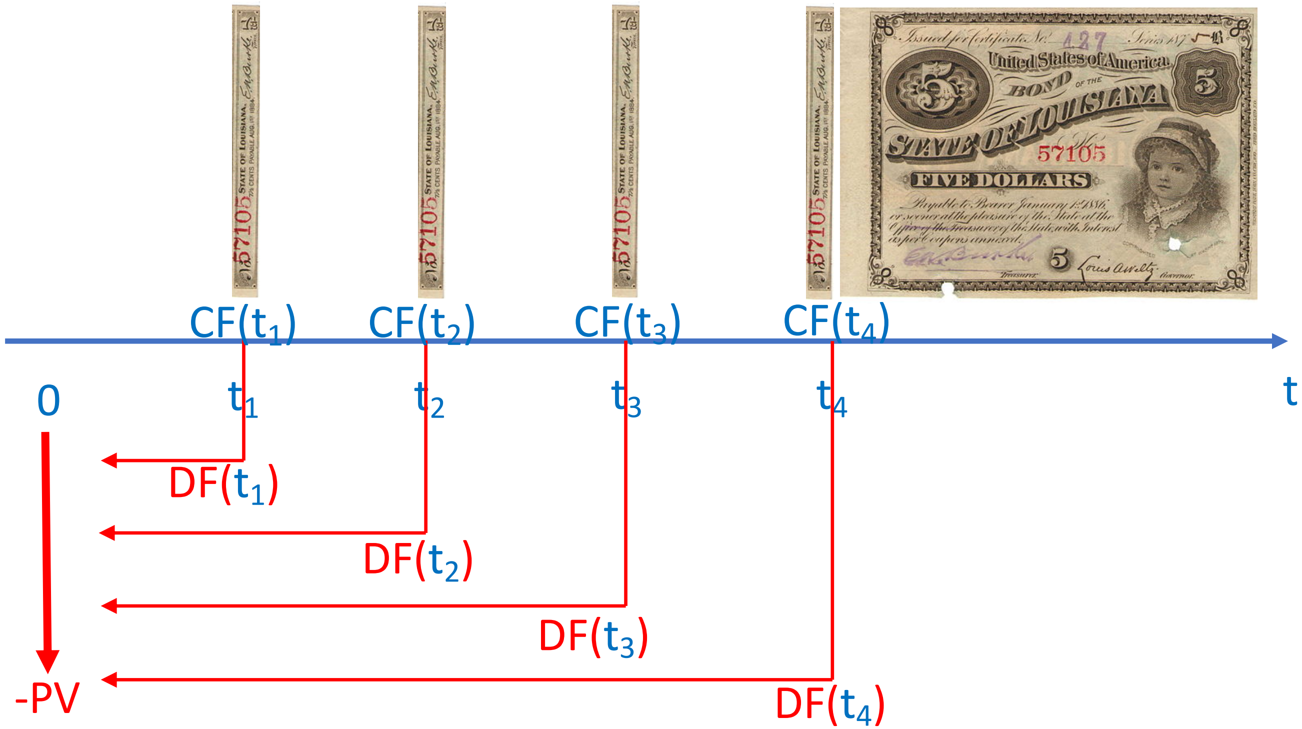

- ☐ For an individual bond with future #CFs > 1, y(t) of course does NOT have a unique solution!
- ☐ It is: ONE eqn. for #CFs unknown variables:DF(t_i)
- ☐ In example (left) one can e.g. fix DF(t_1), …, DF(t_3) at ANY level and DF(t_4) will be simply given by:

$$DF(t_4) = \frac{PV - \sum_{i=1}^{3} CF(t_i)DF(t_i)}{CF(t_4)}$$

Infinitely many solutions !!! (1 eqn. with 4 vars)

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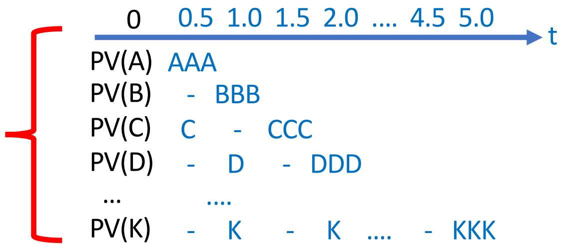
Bonds with matching CFs !



0 0.5 1.0 1.5 2.0 .... 4.5 5.0 t

PV(A) AAA
PV(B) - BBB
PV(C) C - CCC
PV(D) - D - DDD
... .
PV(K) - K - K .... - KKK

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PV_N vector

CF_{NxT} matrix

DF_T vector

☐ For an individual bond with future #CFs > 1, y(t) of course does NOT have a unique solution!

☐ BUT one can assume that the same yield curve y(t) applies to ALL INSTRUMENTS within the given credit risk class (e.g. all T-Bonds)

☐ If #instruments with matching CFs = #CFs one can get a unique solution !

Single solution if #indep. eqns = #variables

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$$\begin{pmatrix}102{,}47\\99{,}57\\99{,}40\\97{,}84\\99{,}18\\82{,}20\\100{,}28\\76{,}00\\73{,}13\\91{,}08\end{pmatrix} = \begin{array}{c|cccccccccc} & \text{CFs in } t=0.5 & & & & & & & & & \text{CFs in } t=5.0 \\ & \multicolumn{10}{c}{\xrightarrow{\text{time}}} \\ \text{AAA CFs}\rightarrow & 105 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 4 & 0 & 104 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 5 & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 5 & 0 & 5 & 0 & 105 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 & 0 \\ \text{Bonds}\downarrow & 6 & 0 & 6 & 0 & 6 & 0 & 106 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 \\ \text{KKK CFs}\rightarrow & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 105 \end{array} \bullet \begin{pmatrix}DF(0.5)\\DF(1.0)\\DF(1.5)\\DF(2.0)\\DF(2.5)\\DF(3.0)\\DF(3.5)\\DF(4.0)\\DF(4.5)\\DF(5.0)\end{pmatrix}$$

$$\underset{\text{vector}}{\boldsymbol{PV_N}} = \underset{\text{matrix}}{\boldsymbol{CF_{NxT}}} \bullet \underset{\text{vector}}{\boldsymbol{DF_T}}$$

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If N=T and rank(***CF***) = T, then:

$$\begin{bmatrix} DF(0.5) \\ DF(1.0) \\ DF(1.5) \\ DF(2.0) \\ DF(2.5) \\ DF(3.0) \\ DF(3.5) \\ DF(4.0) \\ DF(4.5) \\ DF(5.0) \end{bmatrix} = \begin{bmatrix} 105 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 104 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 5 & 0 & 105 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 & 0 & 0 \\ 6 & 0 & 6 & 0 & 6 & 0 & 106 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 105 \end{bmatrix}^{-1} \bullet \begin{bmatrix} 102{,}47 \\ 99{,}57 \\ 99{,}40 \\ 97{,}84 \\ 99{,}18 \\ 82{,}20 \\ 100{,}28 \\ 76{,}00 \\ 73{,}13 \\ 91{,}08 \end{bmatrix}$$

$$\boldsymbol{DF_T} = (\boldsymbol{CF_{NxT}})^{-1} \bullet \boldsymbol{PV_N}$$

vector = matrix • vector

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⬇

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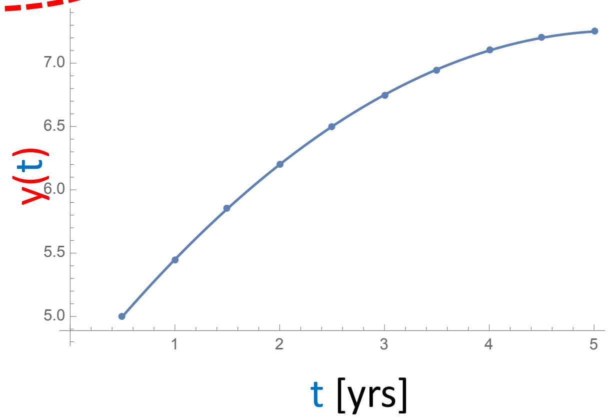
If N=T and rank(***CF***) = T, then:

		DF(t)		$y(t)=DF(t)^{-1/t}-1$
DF(0.5)		0,976		5,00
DF(1.0)		0,948		5,45
DF(1.5)		0,918		5,85
DF(2.0)		0,887		6,20
DF(2.5)	=	0,854	⇔	6,50
DF(3.0)		0,822		6,75
DF(3.5)		0,790		6,95
DF(4.0)		0,760		7,10
DF(4.5)		0,731		7,20
DF(5.0)		0,705		7,25

DF_T
vector



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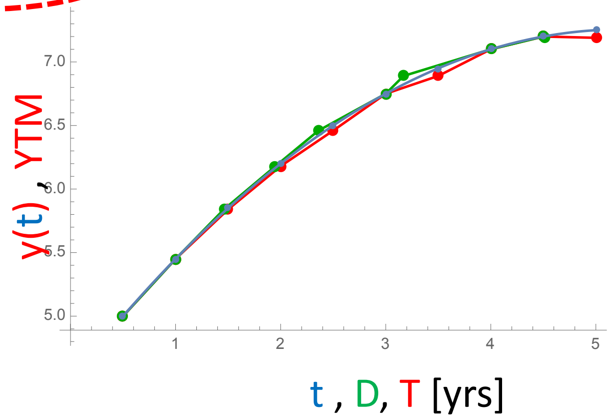
DF_T
vector



Comparison with: YTM(D) and YTM(T)

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The yield curve: bootstrapping

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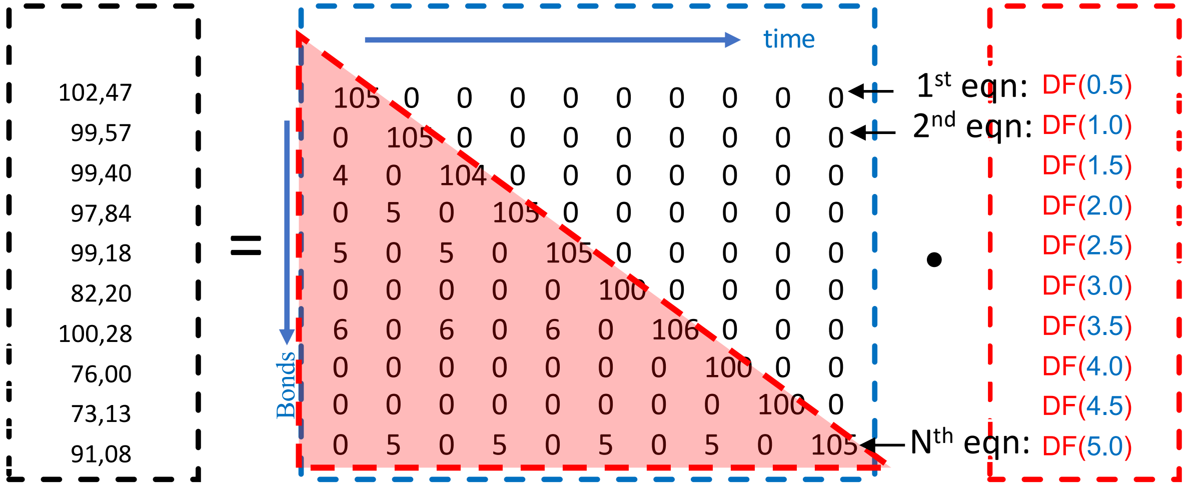
Solving for DF(t) is extremely simple if ***CF*** matrix is **triangular**



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PV_N vector = CF_{NxT} matrix • DF_T vector

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$$\boldsymbol{PV_n = \sum_t CF_{nt} \cdot DF_t}$$

- ○ This is for a special choice of (matching CFs) „benchmark" bonds, with maturities shifted by a constant time step
- ○ In such a case one has an iterative formula:

$$DF(t_n) = \frac{PV_n - \sum_{i=1}^{n-1} CF(t_i)DF(t_i)}{CF(t_n)}$$

- ○ One starts with bonds with 1 CF and solves: $DF(t_1) = \frac{PV_1}{CF(t_1)}$
- ○ Then for bonds with 2 CFs: $DF(t_2) = \frac{PV_2 - CF(t_1)DF(t_1)}{CF(t_2)}$
- ○ …. This is called BOOTSTRAPPING !

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- ☐ BUT one can assume that the same yield curve y(t) applies to ALL INSTRUMENTS within the given credit risk class (e.g. all T-Bonds)
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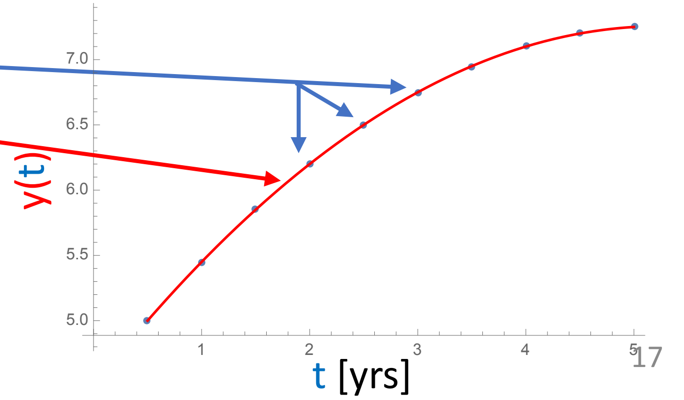
- By applying the same universal yield curve y(t) to a series of „benchmark” bonds / other interest rate instruments and using the bootstrapping method (or, in general, solving: $\boldsymbol{PV_n = \sum_t CF_{nt} \cdot DF_t}$) one gets a NONPARAMETRIC (exact) estimate of $y(t_n)$ at the data („collocation”) points: t_n
- For intermadiate t (i.e. $t_n < t < t_{n+1}$) one can interpolate



The yield curve: parametric fitting

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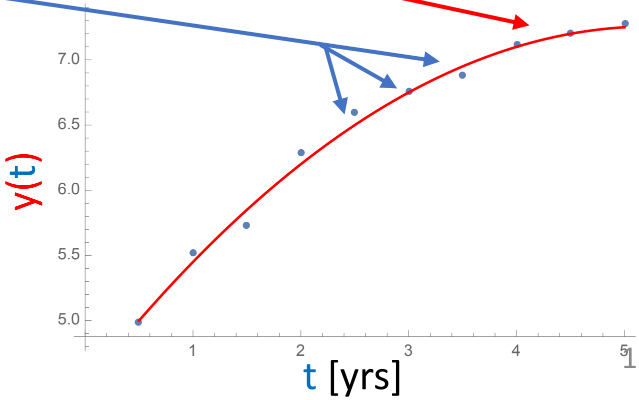
$$\boxed{y(t)=DF(t)^{-1/t}-1}$$

$$\sum_t CF(t)\boxed{DF(t)} = \sum_t \frac{CF(t)}{(1+\boxed{y(t)})^t} = 0 \qquad (2)$$

$$\longrightarrow PV = -CF(0) = \sum_{t>0} \frac{CF(t)}{(1+y(t))^t}$$

- Alternatively one can choose to FIT some function y(t) (PARAMETRIC estimate) to the bond / other instruments data
  - One assumes: $PV_n = \sum_t CF_{nt} \cdot DF_t + \varepsilon_n$, where ε_n is some stochastic noise, and thus $y(t_n)$ is not exact at data points
  - One solves for the MIN. SS = $\sum_n \varepsilon_n^2$ or MAX. likelihood parameters*
  - This is especially usefull when one wants to use more / less data** than „benchmark" bonds or one cannot find bonds with matching CFs



*More about statistical methods in data fitting will be tought in „Risk Management" Lectures.

**One usually weights the bond data, depending on the total nominal issued, liquidity, ...

The yield curve: parametric fitting

- This can be solved by assuming that EACH CASH FLOW is treated INTEPENDENTLY ! For a given bond, EACH CF(t) is discounted using a different yield: YTM → y(t), such as a BOND was cut (“stripped”) into a SUM OF ZERO-COUPON bonds !

$y(t)=DF(t)^{-1/t} -1$

$$\sum_t CF(t)DF(t) = \sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \qquad (2)$$

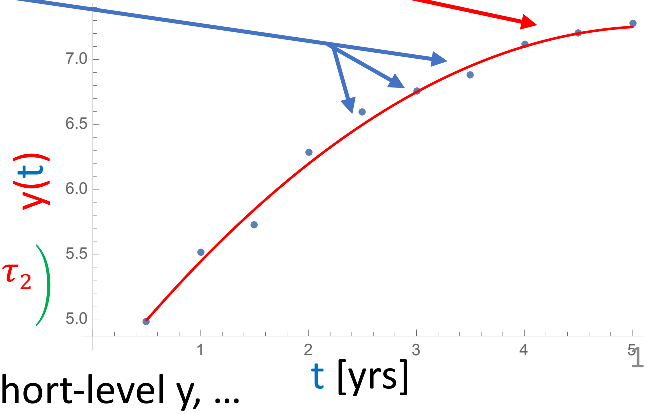
$$PV = -CF(0) = \sum_{t>0} \frac{CF(t)}{(1+y(t))^t}$$

- Alternatively one can choose to FIT some function y(t) (PARAMETRIC estimate) to the bond / other instruments data

- One must of course choose some yield curve model, e.g.:
  - ☐ Polynomial function
  - ☐ Splines (piecewise polynomials)
  - ☐ Nelson-Siegel

$$y(t) = \beta_0 + \beta_1 \frac{1-e^{-t/\tau}}{t/\tau} + \beta_2\left(\frac{1-e^{-t/\tau}}{t/\tau} - e^{-t/\tau}\right) + \beta_3\left(\frac{1-e^{-t/\tau_2}}{t/\tau_2} - e^{-t/\tau_2}\right)$$

  - ☐ Svensson
  - ☐...

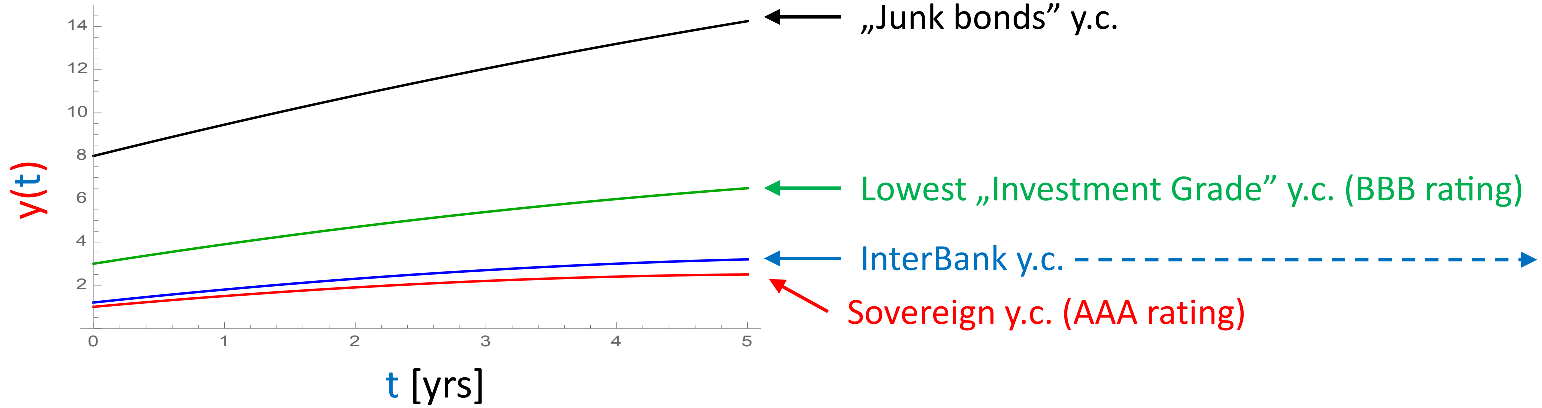


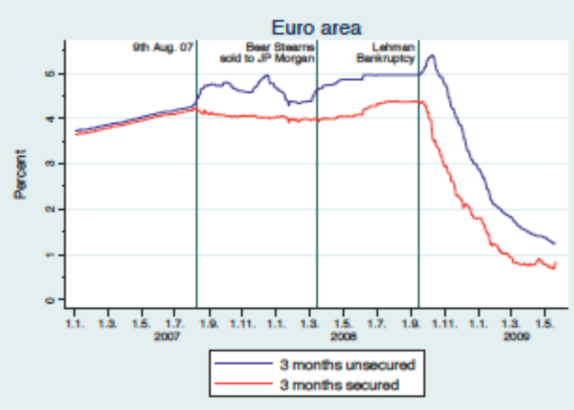
The parameters („factors”) have interpretation, e.g. β_0: long-level y, β_1: short-level y, ...

The yield curve: final notes

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \qquad (2)$$

- ❖Remember that the (zero coupon) yield curve y(t) is universal for a given credit risk ⇒ in fact one has many different yield curves y(t), each one for a different risk class !
  - ❑ e.g. State Treasury, interbank market, …
  - ❑ the credit risk is related to the yield difference („spread”) over the safest (AAA rating) class: usually domestic currency denominated State Treasury („sovereign”) Bills & Bonds
  - ❑ the credit risk spread usually rises with t (as default probability increases with time)

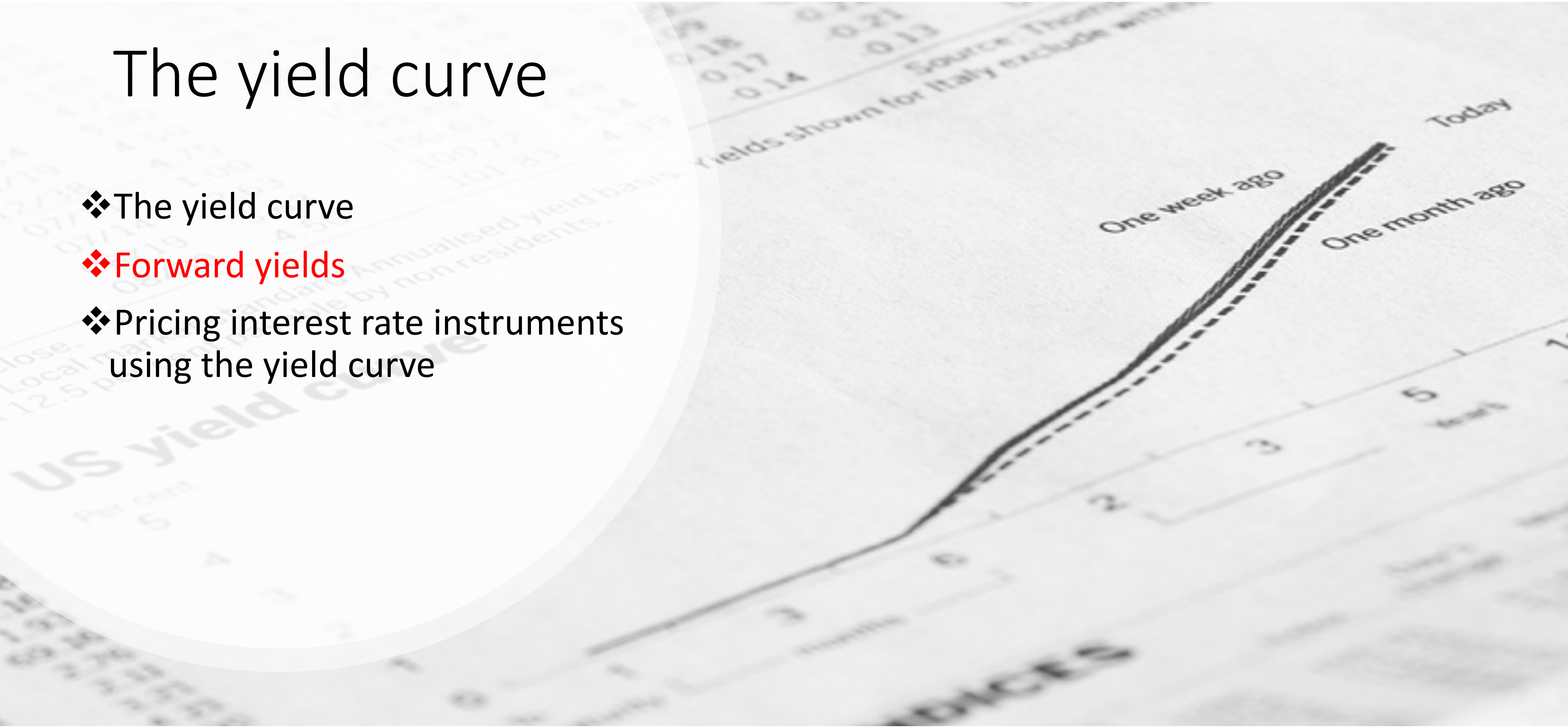




- ❖The yield curves discussed so far were purely deterministic. One can as well assume some STOCHASTIC short rate model which will translate into (longer) Bond prices and thus the yields will become a stochastic proces*

*More about this in the end of this series of Lectures (if time permits).

The yield curve

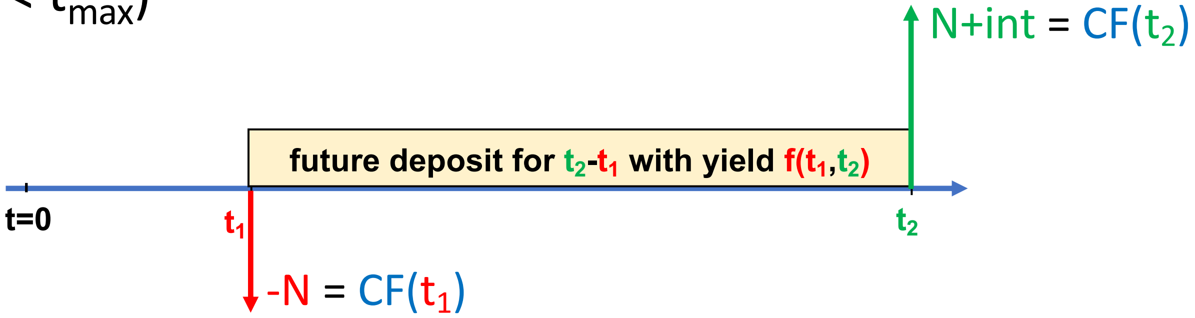


- ❖The yield curve
- ❖Forward yields
- ❖Pricing interest rate instruments using the yield curve

Forward yields

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \quad (2)$$

- Assume that one knows the (zero-coupon) yield curve: $y(t)$ for any $t > 0$ (at least up to some t_{max}), e.g. one has used the bootstrap method and interpolated intermediate points, or one has fitted some function $y(t)$
- Using eqn. (2) the yields $y(t)$ can be used to price any interest rate instrument (of a given credit risk class), e.g. using (2) for any bond one can solve for: $PV = -CF(0) = \sum_{t>0} \frac{CF(t)}{(1+y(t))^t}$
- Let's use the $y(t)$ to compute ("forward") yield: $f(t_1,t_2)$ of a future deposit* starting in t_1 ($t_1 > 0$) and ending in t_2 ($t_1 < t_2 < t_{max}$)**



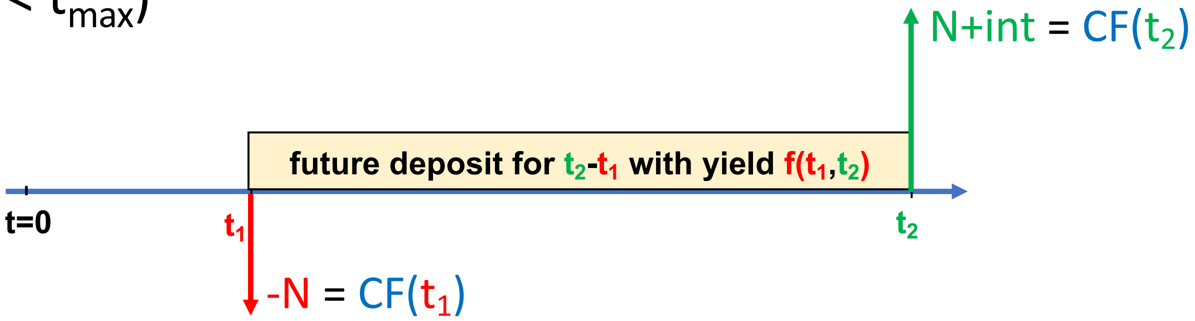
* This can be as well a loan (if we reverse the signs of CFs)

**Recall from Lecture 3 that the interest rate of such a deposit is equivalent to the „t_1 x t_2" FRA rate.

Forward yields

$$\boxed{\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \qquad (2)}$$

- Assume that one knows the (zero-coupon) yield curve: y(t) for any t > 0 (at least up to some t_{max}), e.g. one has used the bootstrap method and interpolated intermediate points, or one has fitted some function y(t)
- Using eqn. (2) the yields y(t) can be used to price any interest rate instrument (of a given credit risk class), e.g. using (2) for any bond one can solve for: $PV = -CF(0) = \sum_{t>0} \frac{CF(t)}{(1+y(t))^t}$
- Let's use the y(t) to compute ("forward") yield: $f(t_1,t_2)$ of a future deposit* starting in t_1 ($t_1 > 0$) and ending in t_2 ($t_1 < t_2 < t_{max}$)**



N+int = $CF(t_2)$

future deposit for t_2-t_1 with yield $f(t_1,t_2)$

t=0

t_1

t_2

-N = $CF(t_1)$

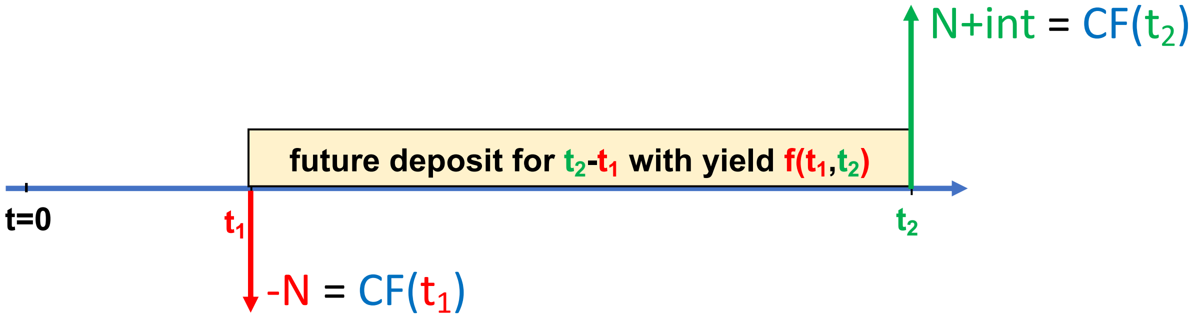
✓From the deposit one has: $N(1+f(t_1,t_2))^{t_2-t_1} = N + \text{int} \;\Rightarrow\; -CF(t_1)\big(1+f(t_1,t_2)\big)^{t_2-t_1} = CF(t_2)$

✓From eqn. (2): $-\frac{CF(t_1)}{(1+y(t_1))^{t_1}} = \frac{CF(t_2)}{(1+y(t_2))^{t_2}} \longrightarrow \boxed{(1+f(t_1,t_2))^{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}}}$

Forward yields

$$\sum_{t} \frac{CF(t)}{(1+y(t))^{t}} = 0 \quad (2)$$

- Assume that one knows the (zero-coupon) yield curve: y(t) for any t > 0 (at least up to some t_{max}), e.g. one has used the bootstrap method and interpolated intermediate points, or one has fitted some function y(t)
- Forward yields $f(t_1,t_2)$ are completely determined by current yields y(t) (for any $0<t_1<t_2< t_{max}$)

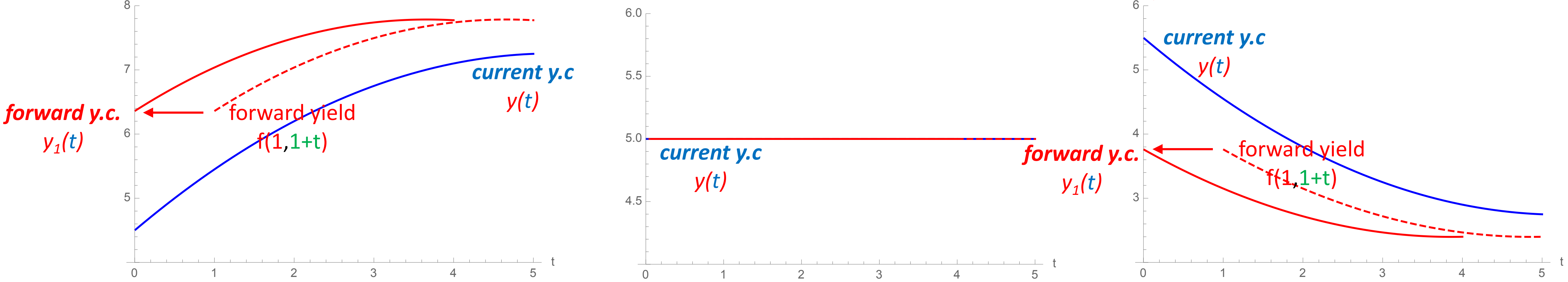


$$(1+\mathrm{f}(t_1,t_2))^{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}}$$

Forward yields: forward yield curve

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \qquad (2)$$

- ❖ Assume that one knows the (zero-coupon) yield curve: $y(t)$ for any $t > 0$ (at least up to some t_{max}), e.g. one has used the bootstrap method and interpolated intermediate points, or one has fitted some function $y(t)$
- ❖ Forward yields $f(t_1,t_2)$ are completely determined by current yields $y(t)$ (for any $0<t_1<t_2< t_{max}$)
  - ❑ Current (i.e. measured in t=0) yield curve $y(t)$ determines the forward yield curve $y_\tau(t) = f(\tau, \tau+t)$
  - ❑ Alternatively: current (investors') EXPECTATIONS about future yields $y_\tau(t)$ determine current y.c. $y(t)$

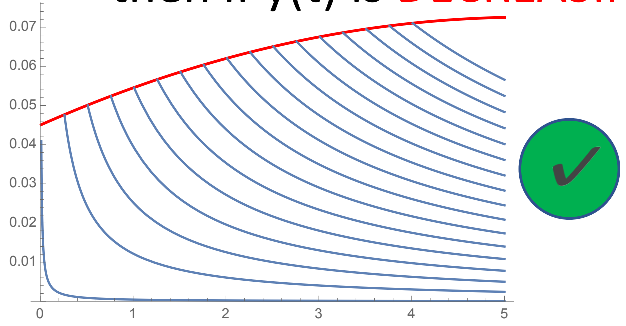


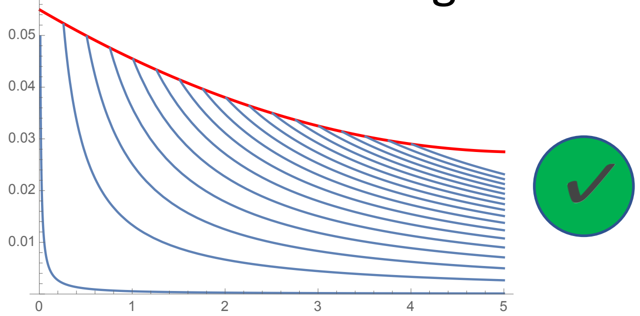
$$(1+f(t_1,t_2))^{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} \Rightarrow y_{t_1}(t_2-t_1) \equiv f(t_1,t_2)$$

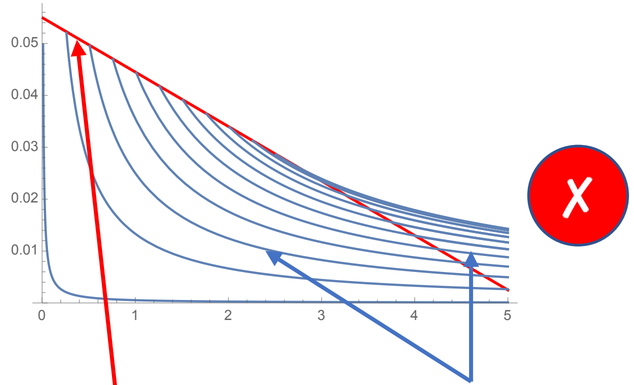
Forward yields: constraints on y(t)

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \qquad (2)$$

- ❖Assume that one knows the (zero-coupon) yield curve: y(t) for any t > 0 (at least up to some t_{max}), e.g. one has used the bootstrap method and interpolated intermediate points, or one has fitted some function y(t)
- ❖Forward yields $f(t_1,t_2)$ are completely determined by current yields y(t) (for any $0<t_1<t_2< t_{max}$)
  - ❑Current (i.e. measured in t=0) yield curve y(t) determines the forward yield curve $y_\tau(t) = f(\tau, \tau+t)$
  - ❑Alternatively: current (investors') EXPECTATIONS about future yields $y_\tau(t)$ determine current y.c. y(t)
  - ❑Relation puts some constraints on current y(t) if one assumes non-negative forward yields $y_\tau(t)>0$* then if y(t) is DECREASING it should be CONVEX enough !



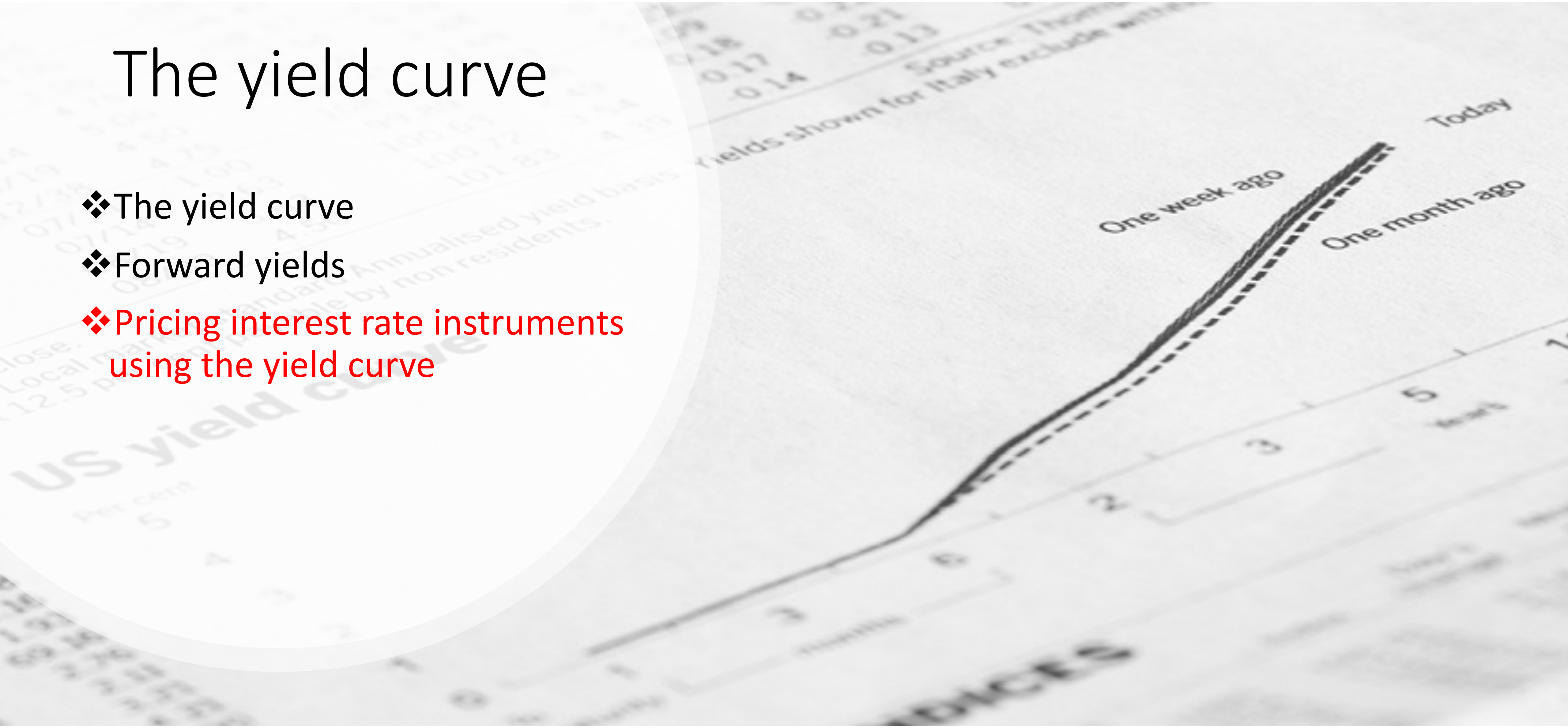




$$\Big(1+\overset{\overset{0}{\wedge}}{f(t_1,t_2)}\Big)^{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} > 1 \Rightarrow y(t_2) > (1+y(t_1))^{t_1/t_2} - 1$$

*As market data show this does not have to be true (e.g. negative interest rates in Switzerland)

The yield curve



- The yield curve
- Forward yields
- Pricing interest rate instruments using the yield curve

Pricing interest rate instruments

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \quad (2)$$

- ❖Assume that one knows the current (zero-coupon) yield curve: y(t) for any t > 0 (at least up to some t_{max}), e.g. one has used the bootstrap method and interpolated intermediate points, or one has fitted some function y(t)
  - ❑The yields y(t) together with eqn. (2) can be used to price any interest rate instrument, provided future CFs are known, e.g. for a fixed interest rates
- ❖Knowing current yields one can also compute forward yields $f(t_1,t_2)$ (for any $0<t_1<t_2< t_{max}$)

$$(1+\mathrm{f}(t_1,t_2))^{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}}$$

  - ❑The forward yield $f(t_1,t_2)$ is the current expectation (forecast) of the future (unknown) yield given by the current y.c.
  - ❑This can be used to forecast* (unknown) future CFs, e.g. based on floating interest rates like "-BOR"
  - ❑If necessary, one should of course remember to convert the (forward) yield into the nominal rate r_{ZM} using some day count convention:

$$(1+\mathrm{f}(t_1,t_2))^{t_2-t_1} = 1 + r_{ZM}\, DCF_{t_2-t_1}$$

⟵ the Day Count Factor used for r_{ZM}, e.g. $DCF_{t_2-t_1} = (t_2-t_1)/365$

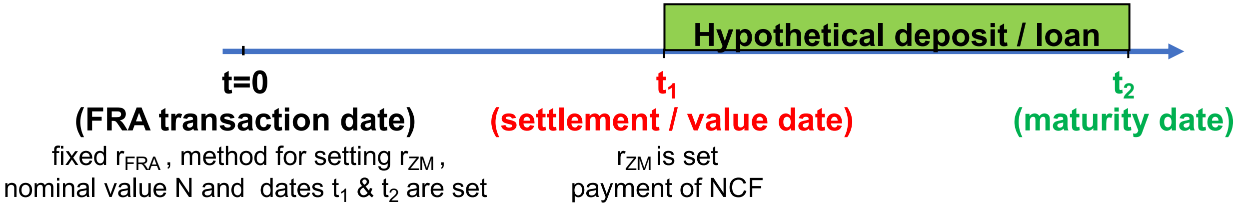
*The exact mechanism why this forecast works (at least for pricing) will be explained in Lecture5

Pricing FRA

$$\left(1+\mathrm{f}(t_1,t_2)\right)^{t_2-t_1} = \frac{\left(1+y(t_2)\right)^{t_2}}{\left(1+y(t_1)\right)^{t_1}} \quad (3)$$

$$\sum_t \frac{CF(t)}{\left(1+y(t)\right)^t} = 0 \quad (2)$$

- ❖Let's price the Forward Rate Agreement, i.e. let's compute r_{FRA}. Recall from Lecture 3 that the counterparties fix the future (FRA) rate r_{FRA} of a (hypothetical) deposit/loan of nominal value N.



Hypothetical deposit / loan

t=0
(FRA transaction date)
fixed r_{FRA}, method for setting r_{ZM},
nominal value N and dates t_1 & t_2 are set

t_1
(settlement / value date)
r_{ZM} is set
payment of NCF

t_2
(maturity date)

- ❖The (only one) CF from "t_1 x t_2" FRA contract is made in the settlement date t_1 and is equal to:

$$CF(t_1) = NCF = \frac{N\,(r_{ZM}-r_{FRA})DCF_{t_2-t_1}}{1+r_{ZM}DCF_{t_2-t_1}}$$

the Day Count Factor used for r_{ZM},
e.g. $DCF_{t_2-t_1} = (t_2-t_1)/365$

- ❖From (2) it is straightforward that r_{FRA} should be set such that $CF(t_1)$=NCF= 0 ⇒ $r_{FRA} = r_{ZM}$
- ❖The future (unknown) floating nominal rate r_{ZM} can be "forecasted" form the forward yield:

$$\left(1+\mathrm{f}(t_1,t_2)\right)^{t_2-t_1} = 1 + r_{ZM}\,DCF_{t_2-t_1}$$

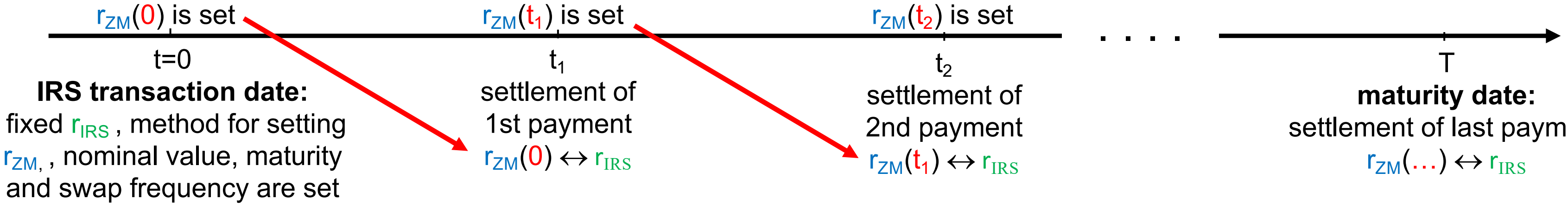
- ❖Which can be computed from (3), thus (as $r_{FRA} = r_{ZM}$): $r_{FRA} = \left(\frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} - 1\right) / \, DCF_{t_2-t_1}$

Pricing IRS

$$(1+\mathrm{f}(t_1,t_2))^{t_2-t_1}=\frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}}\quad(3)$$

$$\sum_t \frac{CF(t)}{(1+y(t))^t}=0\quad(2)$$

- Let's price the Interest Rate Swap, i.e. let's compute r_{IRS}. Recall from Lecture 3 that the counterparties exchange (swap) the future stream of payments (called the "legs" of IRS) computed on the nominal value N of the swap. One leg is calculated using fixed interest rate r_{IRS} while the other leg is based on floating interest rate r_{ZM}.



$r_{ZM}(0)$ is set

$r_{ZM}(t_1)$ is set

$r_{ZM}(t_2)$ is set

t=0

IRS transaction date:
fixed r_{IRS} , method for setting $r_{ZM,}$, nominal value, maturity and swap frequency are set

t_1
settlement of 1st payment
$r_{ZM}(0) \leftrightarrow r_{IRS}$

t_2
settlement of 2nd payment
$r_{ZM}(t_1) \leftrightarrow r_{IRS}$

. . . .

T
maturity date:
settlement of last payment
$r_{ZM}(\ldots) \leftrightarrow r_{IRS}$

- The CFs (made in t_1, t_2, …, T) are equal to:

$$CF(t_i)=NCF(t_i)=N\,(r_{ZM}(t_{i-1})-r_{IRS})DCF_{t_i-t_{i-1}}$$

the Day Count Factor,
e.g. $DCF_{t_2-t_1}=(t_2-t_1)/365$

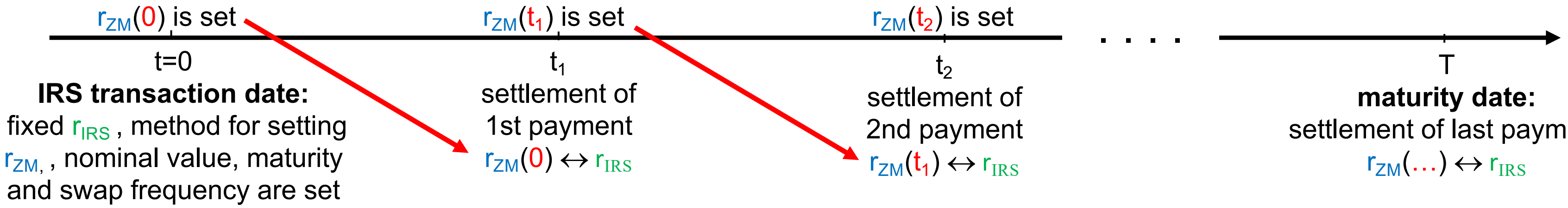
- For simplicity let's assume: YEARLY SWAP payments (e.g. 1Y WIBOR ⟷r_{IRS}) with ACT/ACT ICMA convention (i.e. $t_2-t_1=1$, $DCF_{t_2-t_1}=1$) $\Rightarrow$ $r_{ZM}(t_{i-1})=\mathrm{f}(t_{i-1},t_i)$
- Let's split the net payments into the fixed and the floating leg (and set N = 1):
  - ☐ fixed CFs: $CF_{FIX}(t_i)=\ r_{IRS}$
  - ☐ floating CFs: $CF_{ZM}(t_i)=\ r_{ZM}(t_{i-1})=\mathrm{f}(t_{i-1},t_i)=\text{from }(3)=\frac{(1+y(t_i))^{t_i}}{(1+y(t_{i-1}))^{t_{i-1}}}-1$

Pricing IRS

$$(1+f(t_1,t_2))^{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} \quad (3)$$

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \quad (2)$$

❖Let's price the Interest Rate Swap, i.e. let's compute r_{IRS}. Recall from Lecture 3 that the counterparties exchange (swap) the future stream of payments (called the "legs" of IRS) computed on the nominal value N of the swap. One leg is calculated using fixed interest rate r_{IRS} while the other leg is based on floating interest rate r_{ZM}.



$r_{ZM}(0)$ is set

t=0

IRS transaction date:
fixed r_{IRS} , method for setting r_{ZM}, , nominal value, maturity and swap frequency are set

$r_{ZM}(t_1)$ is set

t_1
settlement of 1st payment
$r_{ZM}(0) \leftrightarrow r_{IRS}$

$r_{ZM}(t_2)$ is set

t_2
settlement of 2nd payment
$r_{ZM}(t_1) \leftrightarrow r_{IRS}$

. . . .

T
maturity date:
settlement of last payment
$r_{ZM}(\ldots) \leftrightarrow r_{IRS}$

❖From (2):

$$\sum_{i=1}^{T} \frac{CF_{ZM}(t_i)}{(1+y(t_i))^{t_i}} = \sum_{i=1}^{T} \frac{CF_{FIX}(t_i)}{(1+y(t_i))^{t_i}}$$

$$\sum_{i=1}^{T} \frac{\frac{(1+y(t_i))^{t_i}}{(1+y(t_{i-1}))^{t_{i-1}}} - 1}{(1+y(t_i))^{t_i}} = \sum_{i=1}^{T} \frac{r_{IRS}}{(1+y(t_i))^{t_i}}$$

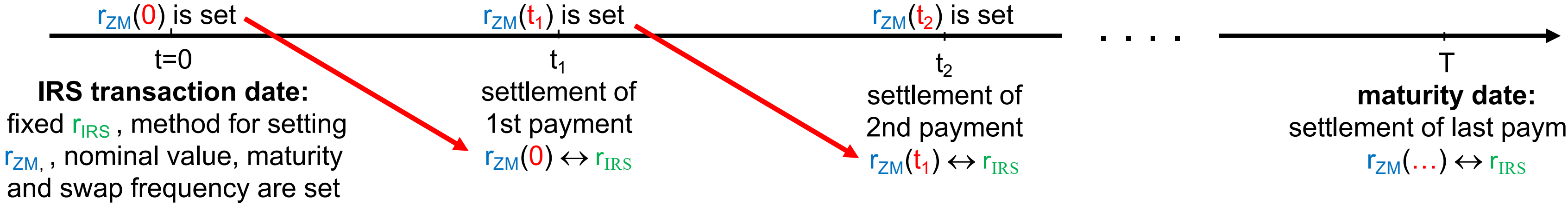
$$\sum_{i=1}^{T} \left(\frac{1}{(1+y(t_{i-1}))^{t_{i-1}}} - \frac{1}{(1+y(t_i))^{t_i}} \right) = \sum_{i=1}^{T} \frac{r_{IRS}}{(1+y(t_i))^{t_i}}$$

Pricing IRS

$$(1+f(t_1,t_2))^{t_2-t_1}=\frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} \quad (3)$$

$$\sum_{t}\frac{CF(t)}{(1+y(t))^{t}}=0 \quad (2)$$

- Let’s price the Interest Rate Swap, i.e. let’s compute r_{IRS}. Recall from Lecture 3 that the counterparties exchange (swap) the future stream of payments (called the “legs“ of IRS) computed on the nominal value N of the swap. One leg is calculated using fixed interest rate r_{IRS} while the other leg is based on floating interest rate r_{ZM}.



$r_{ZM}(0)$ is set

$r_{ZM}(t_1)$ is set

$r_{ZM}(t_2)$ is set

. . . .

t=0

IRS transaction date:
fixed r_{IRS} , method for setting $r_{ZM,}$, nominal value, maturity and swap frequency are set

t_1
settlement of 1st payment
$r_{ZM}(0) \leftrightarrow r_{IRS}$

t_2
settlement of 2nd payment
$r_{ZM}(t_1) \leftrightarrow r_{IRS}$

T
maturity date:
settlement of last payment
$r_{ZM}(\ldots) \leftrightarrow r_{IRS}$

Most terms cancel !

$$\sum_{i=1}^{T}\left(\frac{1}{(1+y(t_{i-1}))^{t_{i-1}}}-\frac{1}{(1+y(t_i))^{t_i}}\right)=\sum_{i=1}^{T}\frac{r_{IRS}}{(1+y(t_i))^{t_i}}$$

$$\frac{1}{(1+y(0))^{0}}-\frac{1}{(1+y(T))^{T}}=\sum_{i=1}^{T}\frac{r_{IRS}}{(1+y(t_i))^{t_i}}$$

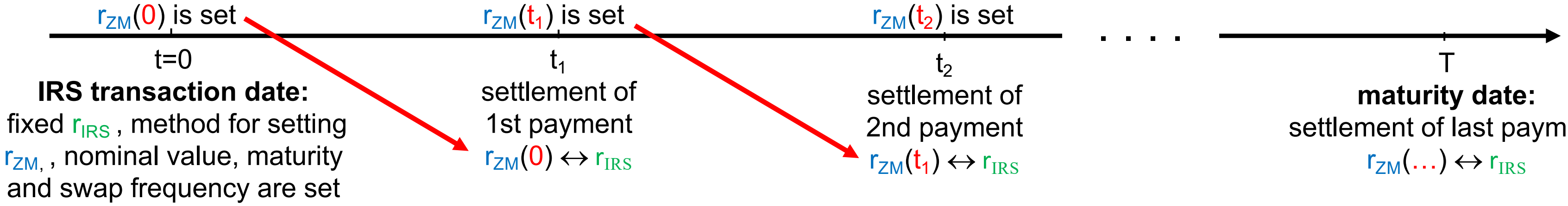
$$1=\sum_{i=1}^{T}\frac{r_{IRS}}{(1+y(t_i))^{t_i}}+\frac{1}{(1+y(T))^{T}}$$

Pricing IRS

$$(1+f(t_1,t_2))^{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} \quad (3)$$

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \quad (2)$$

- Let's price the Interest Rate Swap, i.e. let's compute r_{IRS}. Recall from Lecture 3 that the counterparties exchange (swap) the future stream of payments (called the "legs" of IRS) computed on the nominal value N of the swap. One leg is calculated using fixed interest rate r_{IRS} while the other leg is based on floating interest rate r_{ZM}.



$r_{ZM}(0)$ is set

$r_{ZM}(t_1)$ is set

$r_{ZM}(t_2)$ is set

. . . .

t=0

IRS transaction date:

fixed r_{IRS} , method for setting r_{ZM}, , nominal value, maturity and swap frequency are set

t_1

settlement of 1st payment

$r_{ZM}(0) \leftrightarrow r_{IRS}$

t_2

settlement of 2nd payment

$r_{ZM}(t_1) \leftrightarrow r_{IRS}$

T

maturity date:

settlement of last payment

$r_{ZM}(\ldots) \leftrightarrow r_{IRS}$

Let's put the Nominal val. and the $DCF_{t_2-t_1}$ back

$$N = \sum_{i=1}^{T} \frac{N\, r_{IRS}\, DCF_{t_i-t_{i-1}}}{(1+y(t_i))^{t_i}} + \frac{N}{(1+y(T))^{T}}$$

- The Interest Rate Swap is (in terms of pricing) equivalent to a fixed coupon bond with interest rate r_{IRS} sold at nominal value ("at par") !
- Knowing current yield curve $y(t)$ one can easily solve ⬚ for r_{IRS}

Using FRA and IRS for bootstrapping

$$\sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \quad (2)$$

$$1 + \mathrm{r}_{\mathrm{FRA}}\, DCF_{t_2-t_1} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} \quad (*)$$

$$1 = \sum_{i=1}^{T} \frac{r_{IRS}\, DCF_{t_i-t_{i-1}}}{(1+y(t_i))^{t_i}} + \frac{1}{(1+y(T))^T} \quad (**)$$

- ❖As already discussed the current yield curve y(t) determines forward yields or alternatively the forward yields (i.e. future expectations) determine the current yield curve
- ❖Instead of just using cash / spot instruments data one can also use FRA and IRS market prices, i.e. r_{FRA} and r_{iRS} rates to determine the current yield curve
- ❖This is especially useful for the InterBank yield curve* where standardized OTC interest rate derivatives are mostly traded among market participants. They include FRA with standard t_1xt_2 dates (usually up to 12M, e.g. 1x4, 2x5, 3x6, 4x7, … FRA vs –BOR rate) and IRS with standard maturities T (usually from 1 to 10 yrs vs –BOR rate)
  - ○ First one can look at the standard "–BOR" rates ⇒ y(t) for e.g. t = 1W, 1M, 3M, 6M, 9M, 12M
  - ○ Then one looks at FRA with matching missing maturities, e.g. y(1M) & FRA 1x4 using (*) ⇒ y(5M)
  - ○ Then one looks at matching IRS, e.g. y(1Y) & IRS2Y using (**) ⇒ y(2Y) (like for standard bonds used in bootstrapping)

*For the sovereign (i.e. treasury) y.c. one will rather use „benchmark" T-Bills and T-bonds data

Summary