Risk Management - Problems II

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Spring 2020
Jagiellonian University WFAIS.IF-Y491.0

Risk measures

1. Show that the Expected Shortfall of a continuous random variable X

$$ES_\alpha(X) \equiv E(-X|X \leq -VaR_\alpha(X)) = -\frac{1}{\alpha}\int_{-\infty}^{-VaR_\alpha(X)} xp(x)dx,$$

where $VaR_\alpha(X)$ is the Value at Risk defined as:

$$Pr(X \leq -VaR_\alpha(X)) = \int_{-\infty}^{-VaR_\alpha(X)} p(x)dx = \alpha \quad , \quad \alpha \in (0,1),$$

can be alternatively calculated as:

$$ES_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR\gamma(X)d\gamma \tag{1}$$

Note: This formula is useful, e.g. when calculating ES_α from empirical data: one can easily compute VaR_α from the empirical CDF and then use (1) to compute ES_α - see Exercise 4 and 5.

2. Derive analytic formula for $VaR_\alpha(X)$ and $ES_\alpha(X)$ for

   (a) $X \sim$ exponential distribution, i.e. $p(x) = \lambda e^{-\lambda x} \quad , \quad x \geq 0$

   (b) $X \sim$ normal distribution with mean μ and standard deviation σ

   Make a plot of $VaR_\alpha(X)$ and $ES_\alpha(X)$ as a function of $\alpha \in (0,1)$ for a standard exponential distribution ($\lambda = 1$) and a standard normal distribution ($\mu = 0, \sigma = 1$).

   Note 1: For the normal distribution you can use, e.g the (inverse) error function $Erf^{-1}(x)$ or the standard Gaussian quantile $\Phi^{-1}(x)$

   Note 2: You can use some symbolic algebra software (e.g. Wolfram Mathematica) to solve the exercise, you do not have to calculate it "by hand"

3. Using results of Exercise 2 and assuming that the share price in time t is normally distributed, according to the (approximate) formula

$$S(t) = S(0) + S(0)\mu t + S(0)\sigma\sqrt{t}\,\xi\ , \tag{2}$$

where ξ is a standard Gaussian random variable (mean: 0, variance: 1)

(a) Derive a functional relation between volatility σ and VaR_α and ES_α

(b) Compute daily VaR_α and ES_α for Gaussian share prices (2). Current share price is $S(0) = 100$ PLN, $\mu = 10\%/year$ and "annual" volatility $\sigma = 20\%/year$. Assume that a year has 250 business days, and assume a possibility of observing only one loss exceeding VaR_α in a one-year perspective: $\alpha = 1/250$.

(c) Compute daily VaR_α and ES_α if "annual volatility" increases to $30\%/year$

(d) Compute daily VaR_α and ES_α if one assumes observing 2, 3, losses $> VaR_\alpha$ in one year, choose the confidence level α accordingly

(e) Compute daily VaR_α and ES_α if one increases time-length of the investment to 2, 3, years (we consider 1 loss $> VaR_\alpha$ in that time), choose the confidence level α accordingly

(f) Compute weekly VaR_α and ES_α. Choose the confidence level α such that one can (statistically) expect 1 weekly loss exceeding VaR_α during one-year investment scope

4. Data file *dat_St.txt* contains a sample of 1000 daily share prices $S(t)$ generated for some geometric Brownian motion process (log rates of return: $R(t) = \ln(S(t)/S(0))$ are normally distributed).

   (a) Based on this empirical data compute: (annualized) historical Volatility: $\sigma\sqrt{T} = \sqrt{250} \cdot sd$ (sd - standard deviation of daily log rates of return and we assume a year T has 250 business days) and (annualized) mean return $\mu T = 250 \cdot \langle . \rangle$ (where $\langle . \rangle$ is the mean daily log rate of return)

   (b) Based on results of point (a) and Exercise 3 (i.e. using the Gaussian approximation (2)) compute daily $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares.

   (c) Based on results of point (a) and Exercise 3 (i.e. using the Gaussian approximation (2)) compute weekly $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares.

(d) Based on the empirical data compute daily $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares

Note: compute empirical CDF: $\hat{P}_{\leq}(x) = \frac{\#\ sample\ elements\ \leq x}{n}$ (n - sample size) and then empirical $VaR_\alpha(X) = -\inf\{x : P_{\leq}(x) > \alpha\}$ and empirical $ES_\alpha(X)$ (use formula (1))

(e) Based on the empirical data compute weekly $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ for $\alpha = 0.01, 0.05, 0.10, 0.20$. Assume you invest PLN 10 mln in the shares. In order to have weekly $S(t)$ data "decimate" the sample by taking every 5-th element (assume a week has 5 working days)

(f) On the same chart plot approximate and empirical daily $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ (computed as in points (b) and (d), respectively) for $\alpha \in (0, 1)$. On another chart plot weekly $VaR_\alpha(\Delta S)$ and $ES_\alpha(\Delta S)$ (computed as in points (c) and (e), respectively)

5. Data file *dat_DJIA.txt* contains real Dow Jones Industrial Average (the famous NYSE index) daily close prices from a 10-year period. Repeat all calculations of Exercise 4 using this data.

   Note: You can download historical data series of many financial instruments by using *FinancialData[]* function in Wolfram Mathematica.

Markovitz & CAPM Models

6. Consider a portfolio of $M = 3$ risky asserts (e.g. shares) with the following expected rates of return vector and covariance matrix

$$\boldsymbol{\mu} = \begin{bmatrix} 0.05 \\ 0.07 \\ 0.08 \end{bmatrix} \quad , \quad \boldsymbol{C} = \begin{bmatrix} 0.04 & 0.02 & 0.01 \\ 0.02 & 0.05 & 0.03 \\ 0.01 & 0.03 & 0.06 \end{bmatrix}$$

(a) Plot the "bullet like" relation between the variance and the expected rate of return of the portfolio $V(\mu)$ for various combinations of the assets' weights $x_i \in [0, 1]$ (remember that $\sum_i x_i = 1$). Use the "rotated" (V, μ) coordinates

(b) On the previous chart plot the $V(\mu)$ quadratic (parabola) relations for all 3 combinations of the two assets (1-2, 1-3 and 2-3)

(c) Find the optimal portfolios with the expected returns $\mu = 0.065$, $\mu = 0.07$ and $\mu = 0.075$, respectively. For each μ solve the simpified Markovitz problem (i.e. assume $x_i \in \mathcal{R}$), find the optimal portfolio composition (weights x_i, $i = 1, 2, 3$) and calculate its variance $V^{(m)}$

(d) Find the equation for the optimal portfolio barrier (half parabola) $V^{(m)}(\mu)$ and plot it on the previous chart

(e) Find the portfolio with the minimal possible variance $V^* \equiv \min V^{(m)}(\mu)$. Compute the expected rate of return μ^* of such a portfolio and its composition x_i^*

(f) Assume a risk-free asset with the rate of return $r_0 = 0.03$. Find the Capital Market Line (CML):

$$\mu(\sigma) = r_0 + \frac{\mu_M - r_0}{\sigma_M}\sigma \ ,$$

i.e. compute the expected rate of return μ_M and standard deviation σ_M of the "Market" portfolio, as well as its composition x_i^M.

(g) On a new chart plot the (simplified Markovitz) barrier for the optimal portfolios of risky assets, now in (σ, μ) coordinates. On the same chart plot the CML line and mark the "Market" portfolio (σ_M, μ_M)

(h) For each of the risky assets $i = 1, 2, 3$ compute its Beta $\beta_i \equiv C_{iM} \,/\, \sigma_M^2$. Find and plot the Security Market Line(SML):

$$\mu_i(\beta_i) = r_0 + (\mu_M - r_0)\beta_i \ .$$

On the same chart point the positions (β_i, μ_i) for the risky assets $(i = 1, 2, 3)$.

NOTE: You don't have to derive any analytic formulae for the above points (all formulae are given in the Lecture Notes), simply use the numerical data and the given formulae to solve the problems.