Financial Instruments and Pricing

Fall 2019

Set 6&7
(Option pricing)

1. **<u>N-step Binomial model</u>**
   In the N=3 step Binomial model example discussed in Lecture 6, p. 32, with parameters: S=1, u=1.2, d=1/u, R=0.1:
   a. Derive the Binomial tree with the B, Δ and the option price for the <u>American Call</u> option C with the exercise price X=1
   b. Derive the Binomial tree with the B, Δ and the option price for the <u>European put</u> option p with the exercise price X=1
   c. Derive the Binomial tree with the B, Δ and the option price for the <u>American Put</u> option P with the exercise price X=1
   d. Check (numerically) if the "call-put parity" (see Lecture 5, p. 36):

$$c\text{-}p = S - X\,(1+R)^{-N}$$
$$S - X \;\leq\; C - P \;\leq\; S - X\,(1+R)^{-N}$$

   is satisfied.

2. **<u>Index options (with continuous dividends) / foreign exchange options / options of futures (forwards), "cost of carry"</u>.**
   Let's make the Binominal model slightly more complicated. Assume that after each (discrete) time step the shareowner receives a (small) percentage R* of the share price as a dividend (e.g. if the share price after one time step is S(1) then he will receive S(1)×R* as a dividend for each single share he owns, we assume that that payment does not affect the share price, i.e. it is still S(1) after paying the dividend. Using a notation from Lecture 6 and assuming that the option can be replicated by a position in Δ shares and some deposit/loan of value B compute:
   a. B, Δ and the option price,
   b. the martingale probability g,
   c. what would be the model for 2,3,… n time steps,
   d. Compute (<u>ONLY initial, i.e. in n=0</u>) values of B, Δ and (ONLY initial) option prices for the European call option: c, the American Call option: C, the European put option: p and the American Put option: P for the parameters from Problem 1 and R*=0.05

   <u>NOTE:</u> The continuous dividend model is applicable for the underlying asset being a portfolio/basket of many shares (**<u>e.g. a stock exchange index</u>**) with dividend payments „spread up" approximately uniformly in time.
   It can also be used for **<u>foreign exchange (F/X)</u>** options. Here the „share" is replaced by some foreign currency, and the investor who buys the „share" (currency) will put it into a deposit paying interest R* after each time step (in such a model R is the interest on a „domestic" currency and R* is the interest on a „foreign" currency).
   In general (R - R*) is the net **<u>cost of carry</u>** of the **underlying asset** of the option. Buying the underlying asset is an alternative of making a deposit or it is financed

by a loan (cost R for each time step). The owner of the asset receives profits R* for each time step. For example, one can price **options on futures** **(forwards)**, assuming that there is no margin deposits (or margin deposits bare standard interest) one simply has: R*=R - the cost of carry is then zero.

3. **N-step Binomial model. Analytical pricing of European options.**
   Assume that one wants to price the European option, which **can be exercised only at expiration (i.e. after N time steps)**. Let (in general) the payoff function be: $f(S(N))$ (e.g. for the call option: $f(S(N)) = \max(S(N)–X , 0)$). Using the analytic formula from Lecture 6, p. 44:
   $$f = \frac{1}{(1+R)^N} \sum_{j=0}^{N} \binom{N}{j} g^j (1-g)^{N-j} f\left(Su^j d^{N-j}\right)$$
   a. Price (numerically) **the European binary option: b**. Make calculations for: S = 1, X = 1, u =1.05, d = 1/u , R = 0.025, N = 12.
   b. Price (numerically) **the European call option: c**. Make calculations for: S = 1, X = 1, u =1.05, d = 1/u , R = 0.025, N = 12.
   c. Price (numerically) **the European put option: p**. Make calculations for: S = 1, X = 1, u =1.05, d = 1/u , R = 0.025, N = 12.
   d. * Check (numerically) how the (call and put) option prices depend on: S, X, R, u (d=1/u) – draw plots similar to those in Lecture 6, pp. 16-17 (i.e. assume that one parameter is changed, and the rest is kept constant at values from point a).

4. We have shown (see Lecture 7, p. 61) that the **Binomial model converges to the B-S model** for: $\Delta t \to 0$, $N \to \infty$ ($N \Delta t = T$). For finite N & Δt and for the following Binomial model parameters:
   $$(1+R) = e^{r\,T/N} \qquad u = e^{\sigma\sqrt{\frac{T}{N}}} \qquad d = 1/u$$
   one gets an approximation of the B-S model.
   a. Compare value of a **European call** option from the Binomial model with that from the B-S model. Make comparison in function of a number of the Binomial model steps N = 10, 20, 30,…. , 100 (show the results and also plot them in the function of N). Make calculations for the option with parameters: S=1, X=1, T=1, σ=0.2, r=0.1.
   b. Repeat point (a) for a **European put** option with the same parameters.
   c. * Use the **Binomial model** do compute the value of American Call and American Put options with the same parameters as in point (a).

5. Using (numerical) **Monte-Carlo method**:
   a. Price the **European call option** with parameters S=1, X=1, T=1, σ =0.2, r=0.1 (for 1k, 10k and 100k trajectory realizations). Compute the option price (mean value of the discounted ! payoff) and its statistical error (standard deviation of the mean). Compare the results with the B-S price calculated analytically.
   b. For the same parameters price the **European Binary call** option, whose payoff is $b(T) = \Theta(S(T)-X)$ and compare it with the price computed analytically (see Lecture 7, p. 67)

c. For the same parameters price a **“strange sinusoidal”** **European** **option**, whose payoff function is $V(T) = \sin S(T)$ for $S(T) \leq \pi$ and $V(T) = 0$ for $S(T) > \pi$
d. * Price the **Asian call option** (in payoff function of the call option the share price is replaced by its arithmetic average during life of the option: $V(T) = \max(\langle S(t)\rangle_t - X , 0)$ where $0 \leq t \leq T$)

NOTE: For the European options in points (a) and (b) there is no need to generate the whole trajectory but one can generate just S(T) at the option expiration T (remember about the risk neutral probability measure !).
For point (c) generate trajectories (compute the average price $\langle S(t)\rangle_t$) with time step $\Delta t = 1/250$ (i.e. approximately daily prices)

*NOTE: THESE ARE NOT OBLIGATORY EXERCISES

Jakub Gizbert-Studnicki