Lecture 5

Arbitrage(-free) pricing
&
Financial engineering

Financial instruments and pricing

Fall 2019

Arbitrage pricing & Financial engineering

- ❖Arbitrage
- ❖Financial engineering
- ❖Arbitrage-free & risk-neutral pricing
- ❖Static vs dynamical arbitrage

Arbitrage

- In Lecture 4 I tried to convince You that one can use a simple universal formula to price (almost) all interest rate instruments with given CFs:

$$\sum_t \underbrace{PV(CF(t))}_{\text{Present Value}} = \sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \quad (2)$$

$$PV = -CF(0) = \sum_{t>0} \frac{CF(t)}{(1+y(t))^t}$$

$$\big(1+\mathrm{f}(t_1,t_2)\big)^{t_2-t_1} = \frac{\big(1+y(t_2)\big)^{t_2}}{\big(1+y(t_1)\big)^{t_1}} \quad (3)$$

- The same formula can be used to "predict" future (unknown) CFs based on floating interest rates, which can be "forecasted" using forward yields (3): $\mathrm{f}(t_1, t_2)$
- The question remains why formula (2) is so robust ???, i.e. why real market prices of various financial instruments (e.g. deposits, bonds, FRAs, SWAPs of the same credit risk class) adapt to it ?

Arbitrage

- The question remains why formula (2) is so robust ???, i.e. why real market prices of various financial instruments adapt to it ?

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- Note that in (2) the yield curve y(t) is itself determined by market prices (one obtains it by using e.g. the bootstrapping procedure), so eqn. (2) tells in fact that market prices of various instruments are very strongly related, i.e. prices of some instruments determine prices of other instruments and vice versa
- This simply results from the "law of one price", stating that prices of identical (or equivalent) goods must be the same
- Thus if one can "create" financial "goods" using other financial "goods" the prices of the former and the later must be related

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

Arbitrage

- "Law of one price": prices of identical (or equivalent) goods must be the same
  - E.g. consider two standardized identical goods trading in two different markets. If current price in one market is higher than the price in the other market, then traders will buy in the cheaper market and sell in the more expensive market
  - As a result of the increased demand (people are buying) the price in the cheap market ↑
  - As a result of the increased supply (people are selling) the price in the expensive market ↓
  - This will stop when both prices are equal
  - In practice: prices can vary by some (small) amount dictated by transaction costs, transport costs (usually not important for financial assets), …

- If this law is not fulfilled one can make profitable, risk-free* transactions (preferably without even investing his own money**), which is called "arbitrage"

* Arbitrage is market (price) risk-free, but always some transaction / settlement / operational risks remain
**One can alternatively invest his own money with return much higher than „standard" return for a given risk class

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Arbitrage

- If this law is not fulfilled one can make profitable, risk-free transactions (preferably without even investing his own money), which is called "arbitrage"
- Arbitrage is (apart from speculation and hedging) one of the most important motivations of trading in financial markets – there are specialized firms using arbitrage opportunities to make profits (they are usually "big" specialized financial institutions, like banks, hedge funds, etc.)
- Arbitrage transactions will drive market prices towards equilibrium where arbitrage is no-longer possible, the "law of one price" differences in equilibrium are usually dictated by investors with the lowest transaction costs (lowest commissions and BID-ASK spreads)
- The non-arbitrage equilibrium, as we will see in a moment, will agree with eqn. (2). Thus it must automatically apply to real financial instruments*

This is an example of "arbitrage(-free) pricing"

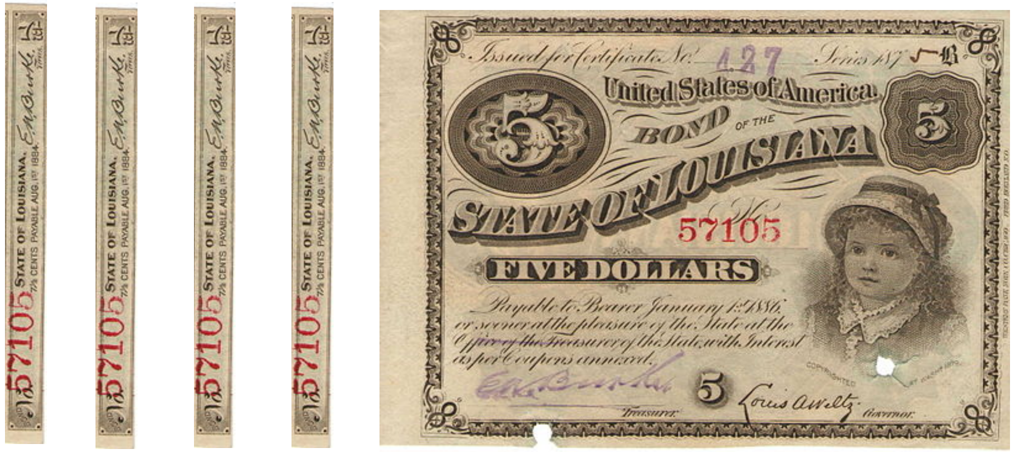
* Here we assume that investors always take advantage of profiatatable arbitrage opportunities if they exist, market limitations (e.g. short-selling not allowed, very low liquidity, etc.) may prevent arbitrageurs from driving prices to the equilibrium.

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ **ARBITRAGE !!!**

Arbitrage example: physical bond

- Consider (a theoretical example) of a physical coupon paying bond where the bond issuer permits to cut-off coupons
- Assume that the present price of the bond PV = \$5 but the total present price of the coupons and principal (all treated as ZERO-COUPON bonds) is \$5.5.
- In this case one can BUY the bond, cut it into pieces and immediately SELL the resulting ZERO-COUPON bonds making a \$0.5 arbitrage profit on each such a trade



both Coupons and Principal can be treated as INDEPENDENT ZERO-COUPON Bonds !

$$\sum_t PV(CF(t)) = 0 \quad (2)$$
IF NOT ⇒ ARBITRAGE !!!

Arbitrage example: physical bond

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- Assume that the present price of the bond PV = $5 but the total present price of the coupons and principal (all treated as ZERO-COUPON bonds) is $5.5.
- In this case one can BUY the bond, cut it into pieces and immediately SELL the resulting ZERO-COUPON bonds making a $0.5 arbitrage profit on each such a trade
  - Arbitrage is possible as in this case eqn. (2) > 0. As a result of arbitrage transactions the bond price will ↑ and the ZERO-COUPON bond prices ↓
  - If instead: PV = $6.0 (i.e. (2) < 0) then one could make the arbitrage profit from BUYING the coupons and principal, gluing them back together (assume it is possible) and SELLING the bond back. As a result the bond price will ↓ and the ZERO-COUPON bond prices ↑
  - Arbitrage transactions are no-longer profitable if PV = $5.5 (i.e. (2) = 0) so real market price of the bond should be very close to the price of the coupons + principal

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Arbitrage example: FX cross-rates

❖ Consider the following FX rates:

	BID(SELL)	ASK(BUY)	
❑ **EUR/USD**:	1.4000	1.4010	How much $ for 1 €
EUR/PLN:	4.3900	4.3950	How much PLN for 1 €

❑ One can easily calculate the **USD/PLN CROSS** rate: **USD/PLN = (EUR/PLN) / (EUR/USD)** *

BID(SELL) ASK(BUY)

$$\frac{4.3900}{1.4010} = 3.1335 - 3.1393 = \frac{4.3950}{1.4000}$$

SPREAD

To BUY $ vs PLN : one BUYS € vs PLN and SELLS € vs $

*** RULE OF THE THUMB: one should use such prices that maximize the BID-ASK spread ! (always: BID < ASK)**
(e.g. for calculating BID: one has BID in nominator and ASK in denominator)

$$\sum_t PV(CF(t)) = 0 \quad (2)$$
IF NOT ⇒ ARBITRAGE !!!

Arbitrage example: FX cross-rates

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$$\frac{4.3900}{1.4010} = 3.1335 - 3.1393 = \frac{4.3950}{1.4000}$$

❖If the current (“**direct**”) **USD/PLN** rate is: 3.1310 – 3.1320 (BID(SELL) – ASK(BUY))

❑One can **BUY € @ 3.1320 PLN** and immediatelly **SELL € @ 3.1335 PLN** making an **arbitrage profit**

❑Assume one has 100 PLN** and one can BUY $: 100 PLN / 3.1320 = 31.93$

❑One then can BUY € using these $: 31.93 $ / 1.4010 = 22.79 €

❑One then can SELL € getting back PLN: 22.79 € x 4.3900 = 100**.05 PLN**

Arbitrage profit !!!

** In fact one does not even need PLN in cash as these transactions will simply clear on the SPOT date !

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Arbitrage example: FX cross-rates

❖Consider the following FX rates:

How much $ for 1 € — How much PLN for 1 €

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BID(SELL) ASK(BUY)

To BUY $ vs PLN : one BUYS € vs PLN and SELLS € vs $

$$\downarrow \frac{4.3900}{1.4010} = 3.1335 - 3.1393 = \frac{4.3950}{1.4000}$$

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Arbitrage profit !!!

❖ Due to arbitrage transactions:

❑ “**direct**” **USD/PLN** ↑ **and** EUR/USD ↑ , EUR/PLN ↓ ⇒ **CROSS USD/PLN** ↓ **(arbitrage profits ↓↓)**

❑**Transactions stop when CROSS BID (SELL) < “direct” ASK(BUY) !**

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Arbitrage example: FX cross-rates

❖Consider the following FX rates:

How much $ for 1 € — **How much PLN for 1 €**

BID(SELL) ASK(BUY) — **BID(SELL) ASK(BUY)**

- **EUR/USD:** ↓ 1.4000 – 1.4010 **EUR/PLN:** 4.3900 – 4.3950 ↑
- One can easily calculate the **USD/PLN CROSS** rate: **USD/PLN = (EUR/PLN) / (EUR/USD)** *

BID(SELL) ASK(BUY)

To BUY $ vs PLN : one BUYS € vs PLN and SELLS € vs $

$$\frac{4.3900}{1.4010} = 3.1335 \; - \; 3.1393 = \frac{4.3950}{1.4000} \uparrow$$

BID(SELL) ASK(BUY)

❖If the current (“**direct**”) **USD/PLN** rate is: ↓ 3.1400 – 3.1410

- One can **BUY € @ 3.1393 PLN** and immediatelly **SELL € @ 3.1400 PLN** making an **arbitrage profit**
- Assume one has 100 $** and one can **SELL $**: 100 $ x 3.1400 = 314 PLN
- One then can **BUY €** using these PLN: 314 PLN / 4.3950 = 71.44 €
- One then can **SELL €** getting back $: 71.44 € x 1.4000 = 100.**02 $** **Arbitrage profit !!!**

❖ **Due to arbitrage transactions:**

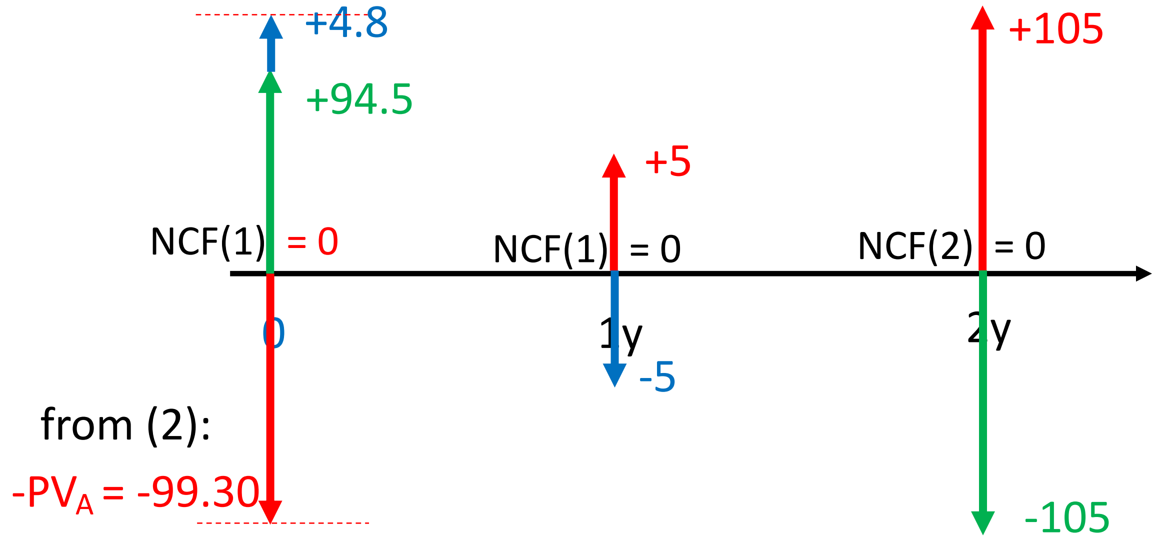
- “**direct**” **USD/PLN** ↓ **and** EUR/USD ↓ , EUR/PLN ↑ ⇒ **CROSS USD/PLN** ↓ (**arbitrage profits** ↓↓)
- **Transactions stop when “direct” BID (SELL) < CROSS ASK(BUY) !**

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Arbitrage example: many bonds

- NOTE that in previous examples all CFs (buy and sell transactions) were done in t=0
- In more complicated cases, including derivative instruments, one has to consider CFs in many points in time t
- For example consider the following bonds of the same issuer (all with PAR val. = 1 $):
  - fixed coupon bond AAA (T=2 yrs, r% = 5%) trading @ PV_A (% of PAR)
  - zero-coupon bond BBB (T=1 yrs) trading @ PV_B = 96 (% of PAR)
  - zero-coupon bond CCC (T=2 yrs) trading @ PV_C = 90 (% of PAR)



- Assume we BUY: 100 bond AAA and SELL (short) N_B bonds BBB & N_C bonds CCC *
- Let's try to match future CFs, such that NCF(t) = 0 for all t ⇒ $CF_B(1) = -100r = -5$, $CF_C(2) = -100(1+r) = -105$
- $N_B = 100r = 5$, $N_C = 100(1+r) = 105$
- $CF_B(0) = N_B \times PV_B = 5 \times .96 = 4.80$
- $CF_C(0) = N_C \times PV_C = 105 \times .90 = 94.50$
- Therefore (eqn. (2)): $PV_A = - CF_A(0) = 4.8 + 94.5 = 99.30$

* We assume the bonds are divisible and one can buy or sell (short) any fraction of a bond.

Arbitrage example: many bonds

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

- ❖NOTE that in previous examples all CFs (buy and sell transactions) were done in t=0
- ❖In more complicated cases, including derivative instruments, one has to consider CFs in many points in time t
- ❖For example consider the following bonds of the same issuer (all with PAR val. = 1 $):
  - ❑ fixed coupon bond AAA (T=2 yrs, r% = 5%) trading @ PV_A (% of PAR)
  - ❑ zero-coupon bond BBB (T=1 yrs) trading @ $PV_B = 96$ (% of PAR)
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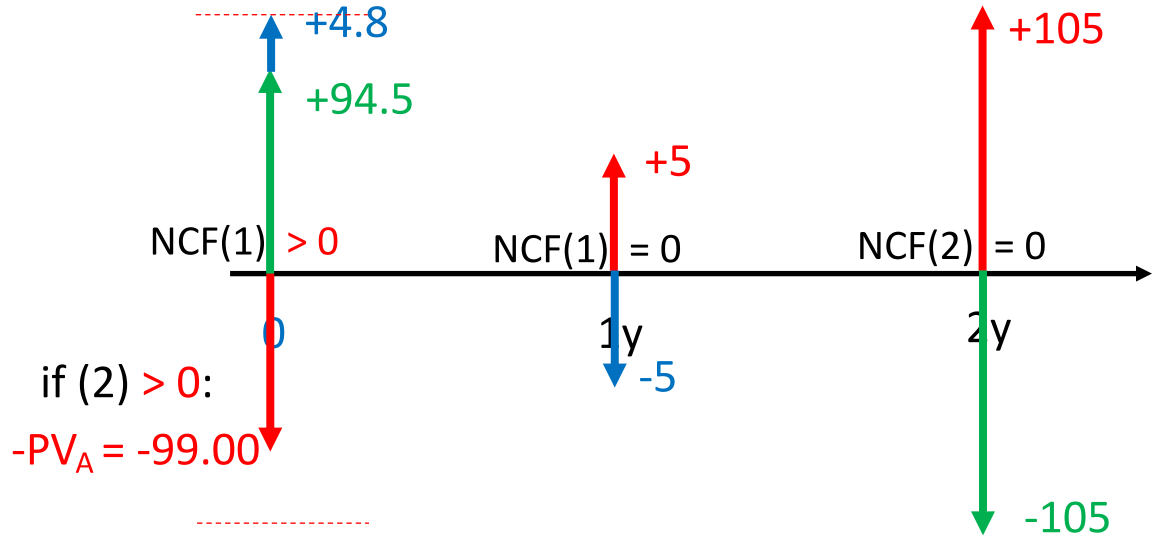

- ❖ From eqn. (2): $PV_A = -CF_A(0) = 4.8 + 94.5 = 99.30$
- ❖ IF real $PV_A < 99.30$, e.g. $PV_A = 99.00$
- ❖ For each trade, where we BUY: 100 bond AAA and SELL (short) N_B bonds BBB & N_C bonds CCC, we make a profit of 0.30 $
- ❖ We don't invest anything and get profits ⇒ ARBITRAGE
- ❖ These trades will drive PV_A↑, PV_B↓,PV_C↓ until (2) = 0

Arbitrage example: many bonds

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

- NOTE that in previous examples all CFs (buy and sell transactions) were done in t=0
- In more complicated cases, including derivative instruments, one has to consider CFs in many points in time t
- For example consider the following bonds of the same issuer (all with PAR val. = 1 $):
  - fixed coupon bond AAA (T=2 yrs, r% = 5%) trading @ PV_A (% of PAR)
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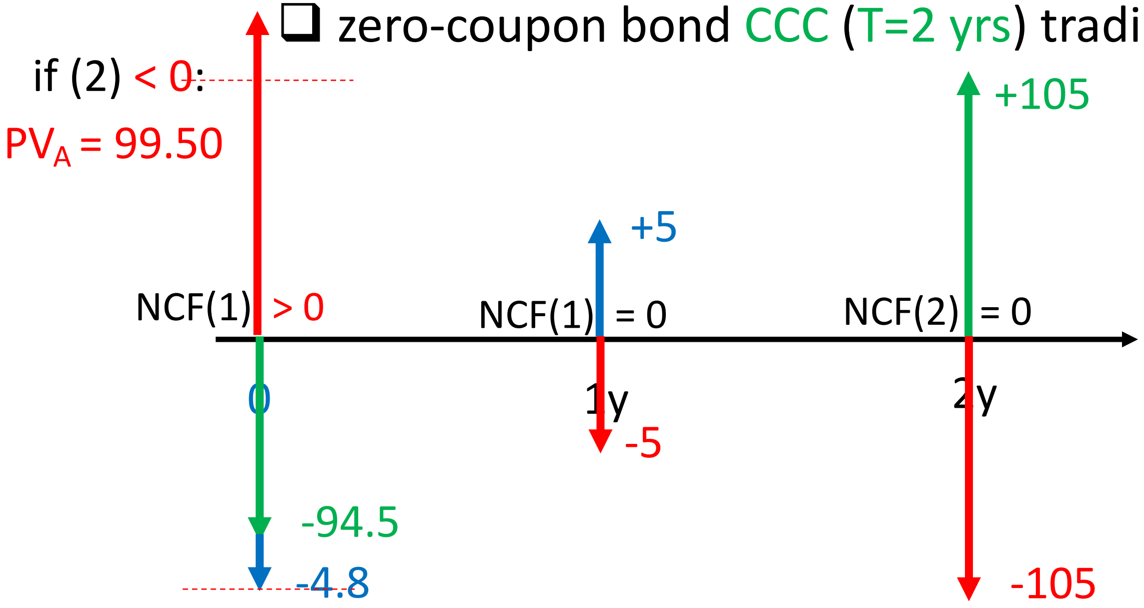


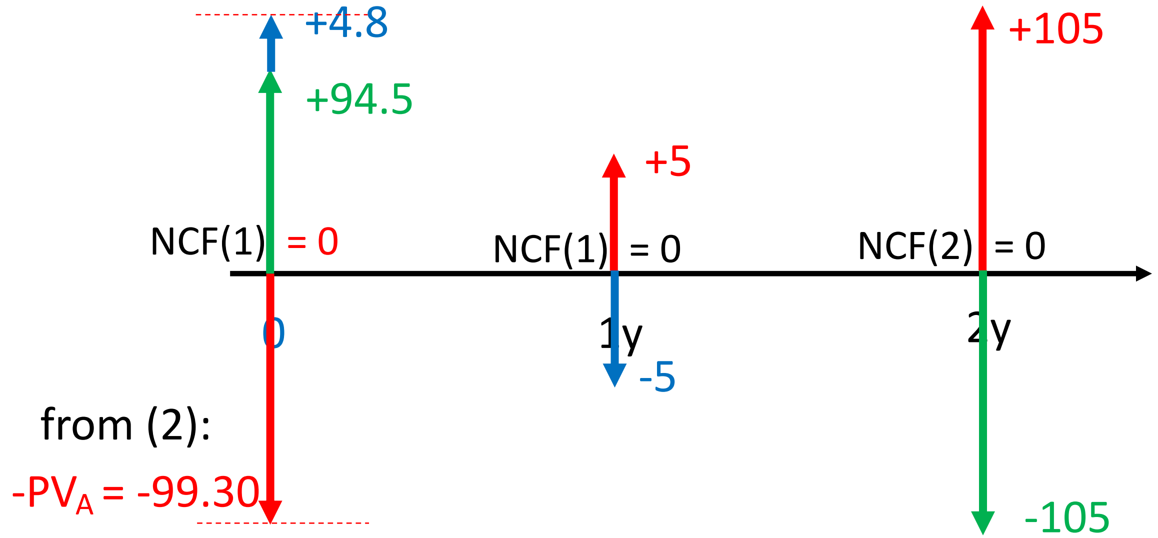
- From eqn. (2): PV_A = - $CF_A(0)$ = 4.8 + 94.5 = 99.30
- IF real PV_A > 99.30 , e.g. PV_A = 99.50
- We inverse BUY ↔ SELL transactions
- For each trade, where we SELL (short) : 100 bond AAA and BUY N_B bonds BBB & N_C bonds CCC, we make a profit of 0.20 $
- We don't invest anything and get profits ⇒ ARBITRAGE
- These trades will drive PV_A↓, PV_B↑, PV_C↑ until (2) = 0

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Arbitrage example: many bonds

- NOTE that in previous examples all CFs (buy and sell transactions) were done in t=0
- In more complicated cases, including derivative instruments, one has to consider CFs in many points in time t
- For example consider the following bonds of the same issuer (all with PAR val. = 1 $):
  - fixed coupon bond AAA (T=2 yrs, r% = 5%) trading @ PV_A (% of PAR)
  - zero-coupon bond BBB (T=1 yrs) trading @ PV_B = 96 (% of PAR) ⇒ $1+y(1) = 100/PV_B$
  - zero-coupon bond CCC (T=2 yrs) trading @ PV_C = 90 (% of PAR) ⇒ $(1+y(2))^2 = 100/PV_C$



- NOTE that if (2) = 0 is fulfilled (NO ARBITRAGE !) then, in order to match CFs, one simply has for 1 AAA bond :

$$\frac{PV_A}{100} = r\frac{PV_B}{100} + (1+r)\frac{PV_C}{100} = \frac{r}{(1+y(1))^1} + \frac{1+r}{(1+y(2))^2}$$

So one recovers formula from Lecture 4, where y(t) is the zero-coupon yield curve (computed using zero-cpn bonds)

$$PV = -CF(0) = \sum_{t>0} \frac{CF(t)}{(1+y(t))^t}$$

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Arbitrage summary

- Arbitrage opportunities occur when prices of the same or ”equivalent” (portfolio of) instruments differ (the “Law of one price” is broken)
- General non-arbitrage rule: if one can match all future CFs of many instruments, such that NCF(t) = 0 for all t > 0, then also NCF(0) = 0 (this is the sense of eqn. (2))
- If this condition is not fulfilled then one can make profitable arbitrage: by trading these instruments one can make extraordinary profits without incurring (market price) risk and without using his own funds
- The arbitrage transactions will move the prices such that arbitrage becomes less and less profitable and finally arbitrage opportunities will vanish when markets come back to the the equilibrium (2)
- This is the mechanism causing that formula (2) is (closely) fulfilled in real markets

Arbitrage pricing & Financial engineering

- ❖Arbitrage
- ❖Financial engineering
- ❖Arbitrage-free & risk-neutral pricing
- ❖Static vs dynamical arbitrage

Financial engineering

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

- The arbitrage examples discussed so far could be understood as follows:
  - using (a PORTFOLIO of) financial instruments one can REPLICATE (artificially create) another financial instrument(s)
  - the price of any financial instrument that can be "REPLICATED" (i.e. artificially created) must equal the price of the (portfolio of) replicating instruments ("Law of one price")
  - IF NOT ⇒ ARBITRAGE opportunities occur and drive the prices to be equal
  - E.g. in the FX example one has artificially created the "CROSS" rate USD/PLN from a PORTFOLIO of EUR/PLN and EUR/USD
    If the price of the "direct" USD/PLN does not match the price of the PORTFOLIO ⇒ ARBITRAGE
  - E.g. in the "many bonds" example one has artificially created 100 coupon bonds AAA from a PORTFOLIO of N_B zero-coupon bonds BBB and N_C zero-coupon bonds CCC
    If the price of the real bond does not match the price of the PORTFOLIO ⇒ ARBITRAGE
- The markets on which instruments can be replicated by portfolios of other instruments are called "COMPLETE MARKETS"
- From our discussion the prices in such markets must agree with eqn. (2) (in practice one must remember about BID-ASK spreads, transaction costs, etc.)

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Financial engineering

- The term "financial engineering" was coined to describe:
  - In a "broad sense": the use of mathematical techniques to solve financial problems. Financial engineering uses tools and knowledge from the fields of computer science, statistics, economics, and applied mathematics to address current financial issues as well as to devise new and innovative financial products. from Investopedia
  - Here I will use a "narrow sense", i.e. the ART of constructing financial products based on other, already existing financial products
- If the market is COMPLETE then one can artificially create ("REPLICATE") any financial instrument, so eqn. (2) MUST BE FULFILLED ⇒ ARBITRAGE-FREE PRICING
- For derivative products this is a "paradigm change":
  - "Originally" derivatives could be understood as a kind of BET, where the parties "guess" future behaviour of the underlying asset price, and thus derivative's price depends on these BETs
  - Here: derivative products can be artificially CREATED by "US" from cash instruments and "SOLD" to the client (counterparty) ⇒ we sell the (financial) product and HEDGE it using other instruments !
  - Such derivatives should be priced using eqn. (2), if MISSPRICED then the client will make profitable arbitrage and "WE" will loose money ⇒ DERIVATIVE's PRICE "=" COST OF HEDGING !

Financial engineering: FRA

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

- The problem remains to show, that derivative instruments can be REPLICATED !

$$\left(1+f(t_1,t_2)\right)^{t_2-t_1} = \frac{\left(1+y(t_2)\right)^{t_2}}{\left(1+y(t_1)\right)^{t_1}} \quad (3)$$

- Here I will show it for the FRA transaction*
- Recall that in Lecture 3 we discussed that FRA is financially equivalent to a forward deposit (FRA BID) / loan (FRA ASK), so it remains to show, that a FORWARD DEPOSIT / LOAN can be replicated by a PORTFOLIO OF (current) DEPOSITS / LOANS

* If time permits I will discuss the IRS swap example on the blackboard: IRS is equivalent to the portfolio of deposits / loans, or the portfolio of FRAs

Financial engineering: FRA

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

❖ The problem remains to show, that derivative instruments can be REPLICATED !

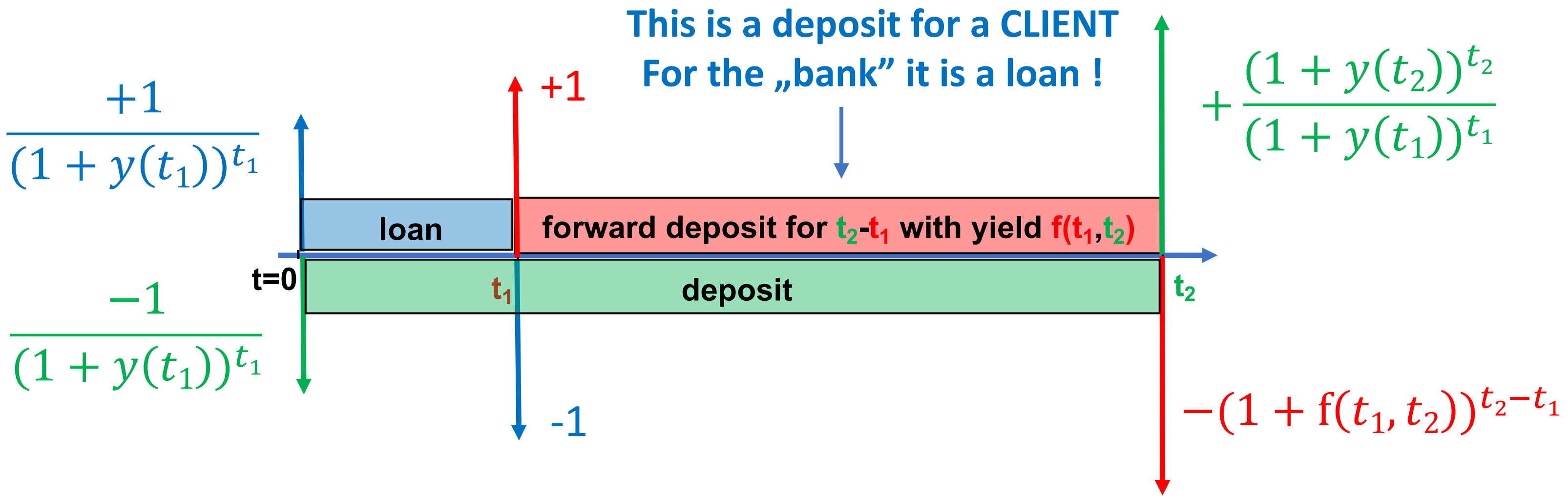
BID(SELL)

$$\underbrace{(1+f(t_1,t_2))^{t_2-t_1}}_{\text{BID(SELL)*}} = \frac{(1+y(t_2))^{t_2}}{(1+y(t_1))^{t_1}} \quad (3)$$

ASK(BUY)

❖ Here I will show it for the FRA transaction*

❖ Recall that in Lecture 3 we discussed that FRA is financially equivalent to a forward deposit (FRA BID) / loan (FRA ASK), so it remains to show, that a FORWARD DEPOSIT / LOAN can be replicated by a PORTFOLIO OF (current) DEPOSITS / LOANS



FRA BID (FWD DEPOSIT) $1@f(t_1,t_2)$

To create F. DEPOSIT for CLIENTS a „bank"

t=0
- ✓ takes a loan for t_1: $\frac{1}{(1+y(t_1))^{t_1}}$ @ $y(t_1)$
- ✓ deposits it for t_2: $\frac{-1}{(1+y(t_1))^{t_1}}$ @ $y(t_2)$

t_1
- ✓ repays the loan for t_1: -1
- ✓ from the forward deposits: $+1@f(t_1,t_2)$

t_2
- ✓ repays the forward deposit
- ✓ from the deposit for t_2

* RULE OF THE THUMB "MAX BID-ASK SPREAD" WORKS !

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ **ARBITRAGE !!!**

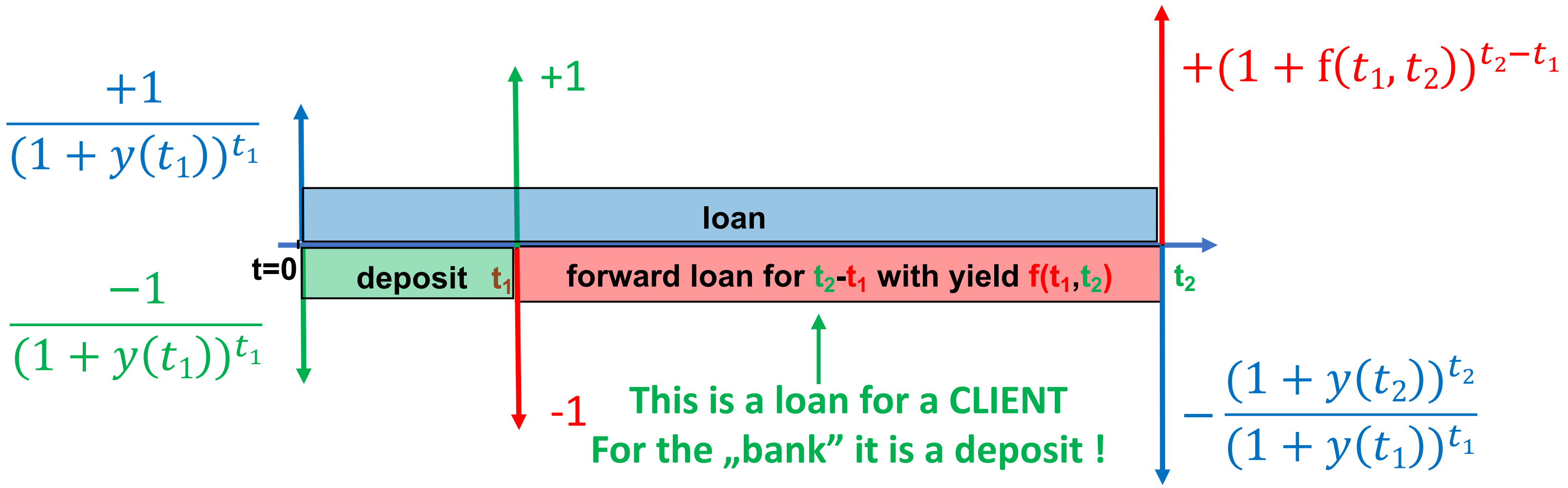
Financial engineering: FRA

❖ The problem remains to show, that derivative instruments can be REPLICATED !

❖ Here I will show it for the FRA transaction*

$$\underset{\text{ASK(BUY)*}}{\left(1+\mathrm{f}(t_1,t_2)\right)^{t_2-t_1}} = \frac{\overset{\text{ASK(BUY)}}{\left(1+y(t_2)\right)^{t_2}}}{\underset{\text{BID(SELL)}}{\left(1+y(t_1)\right)^{t_1}}} \quad (3)$$

❖ Recall that in Lecture 3 we discussed that FRA is financially equivalent to a forward deposit (FRA BID) / loan (FRA ASK), so it remains to show, that a FORWARD DEPOSIT / LOAN can be replicated by a PORTFOLIO OF (current) DEPOSITS / LOANS



FRA ASK (FWD LOAN) 1@$\mathrm{f}(t_1,t_2)$

To create F. LOAN for CLIENTS the „bank"

t=0
- ✓ takes a loan for t_2: $\frac{1}{(1+y(t_1))^{t_1}}$ @ $y(t_2)$
- ✓ deposits it for t_1: $\frac{-1}{(1+y(t_1))^{t_1}}$ @ $y(t_1)$

t_1
- ✓ uses money from the deposit for t_1: +1
- ✓ to the forward loan: -1@$\mathrm{f}(t_1,t_2)$

t_2
- ✓ repays the loan for t_2
- ✓ from the forward loan

*** RULE OF THE THUMB "MAX BID-ASK SPREAD" WORKS !**

Financial engineering: Equity Forward

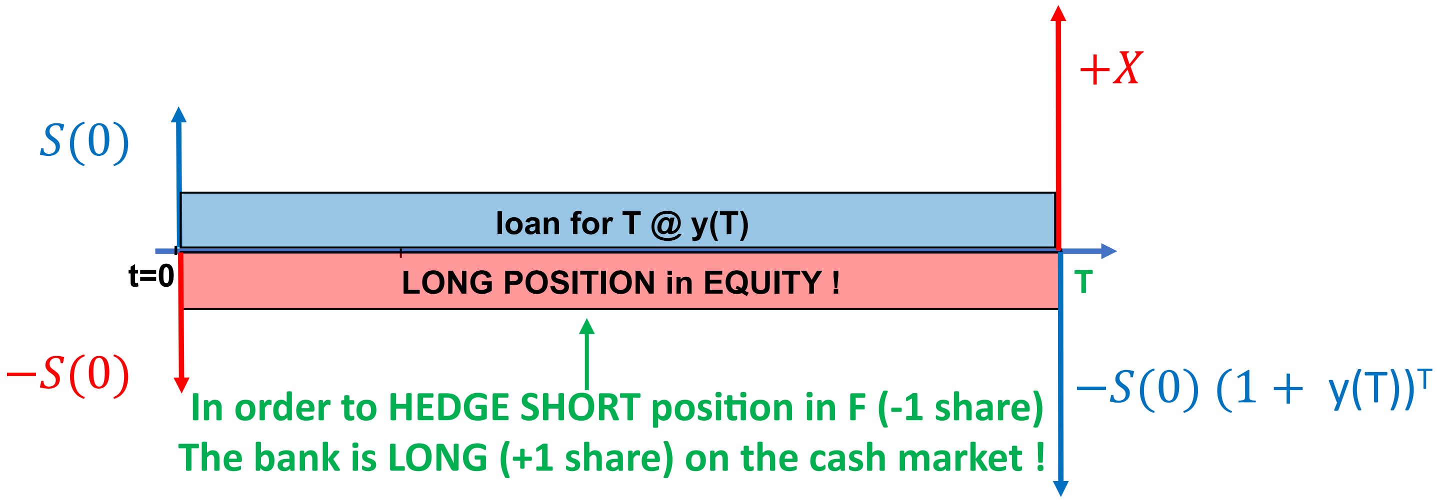
$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

❖ One can use the same techniques to "create" other derivative instruments, also the ones not related to interest rates

❖ Let's "create" and price an Equity Forward, i.e. a forward transaction to buy / sell 1 XYZ share in time T, for the agreed "eXercise" / "forward" price X

❖ Assume we know the CURRENT (zero coupon) yield curve y(t), and the CURRENT share price S(0)



E. FORWARD ASK (client buys @ X)

To SELL 1 share to the CLIENT, the „bank"

t=0
- ✓ takes a LOAN for T: $S(0)$ @ y(T)
- ✓ BUYS +1 share (LONG) : $-S(0)$

T
- ✓ delivers (sells) the share in EF: +X
- ✓ repays the LOAN: $-S(0)\ (1+y(T))^T$

$$X = S(0)\left(1+y(T)\right)^T \quad (4)$$

ASK(BUY)* ASK(BUY) ASK(BUY)

* RULE OF THE THUMB "MAX BID-ASK SPREAD" WORKS !

Financial engineering: Equity Forward

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

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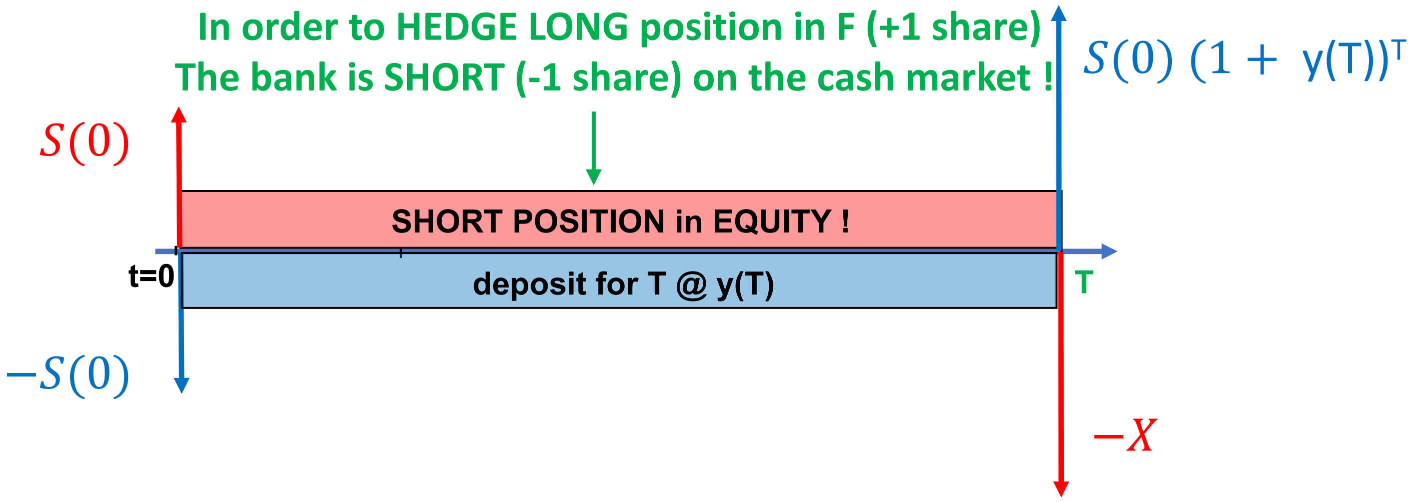


E. FORWARD BID (client sells @ X)

To BUY 1 share from the CLIENT, the „bank"

t=0:
- ✓ SELLS -1 share (SHORT) : $+S(0)$
- ✓ makes deposit for T: $-S(0)$ @ y(T)

T:
- ✓ GETS the share in EF (closes SHORT): $-X$
- ✓ using cash in DEPOSIT: $+S(0)(1+y(T))^T$

$$X = S(0)\big(1 + y(T)\big)^T$$

BID(SELL)* BID(SELL) BID(SELL)

*** RULE OF THE THUMB "MAX BID-ASK SPREAD" WORKS !**

Financial engineering: Equity Forward

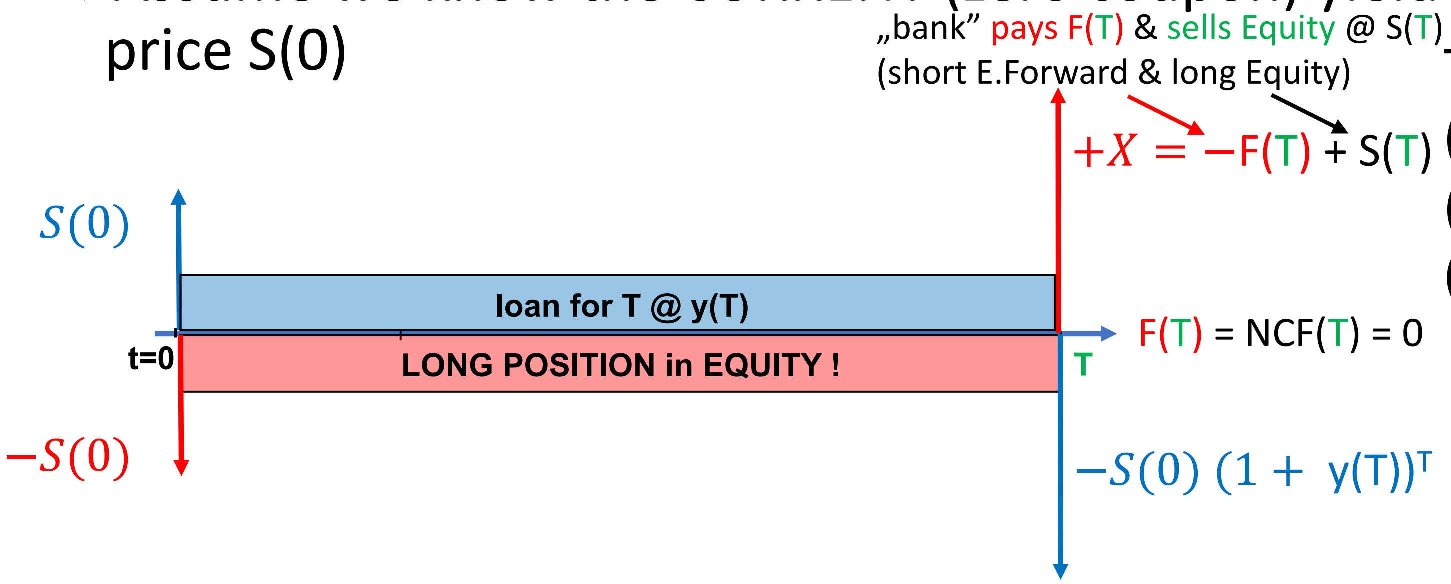
$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

- One can use the same techniques to "create" other derivative instruments, also the ones not related to interest rates

- Let's "create" and price an Equity Forward, i.e. a forward transaction to buy / sell 1 XYZ share in time T, for the agreed "eXercise" / "forward" price X

- Assume we know the CURRENT (zero coupon) yield curve y(t), and the CURRENT share price S(0)



This is of course also valid for the (cash settled) Non-Deliverable Forward (NDF) with the PAYOFF function (i.e. the buyer RECEIVES and the seller PAYS):

F(T) = S(T) - X

$$F(T) = S(T) - S(0)\left(1 + y(T)\right)^T$$

$$X = S(0)\left(1 + y(T)\right)^T$$

Financial engineering: Summary

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

- On the COMPLETE MARKET one can use arbitrage(-free) arguments to compute the (fair) price of derivative instruments, e.g. Forwards and SWAPS
- In general, in order to CREATE ("REPLICATE") a derivative instrument one uses a portfolio of cash instruments to HEDGE the derivative position
  - If "a bank" has LONG derivative position (i.e. its CLIENT has SHORT), the "bank" creates SHORT replicating position in order to HEDGE (compensate) THE PRICE CHANGE RISK
  - If "a bank" has SHORT derivative position (i.e. its CLIENT has LONG), the "bank" creates LONG replicating position in order to HEDGE (compensate) THE PRICE CHANGE RISK
- The derivative PRICE can be computed from the COST OF HEDGING, e.g.:

FORWARD PRICE (X) = SPOT PRICE (S(0)) + "COST OF CARRY"

Interest on „CARRYING" the SPOT LOAN

Interest on „CARRYING" the SPOT position in SHARES

- COST OF CARRY
  - POSITIVE (+): interest paid, storage costs (commodity),…
  - NEGATIVE (-): dividends, interest received,…

Interest from „CARRYING" the SPOT DEPOSIT

$$\underbrace{(1+f(t_1,t_2))^{t_2-t_1}}_{\text{ASK(BUY)*}} = \frac{\overbrace{(1+y(t_2))^{t_2}}^{\text{ASK(BUY)}}}{\underbrace{(1+y(t_1))^{t_1}}_{\text{BID(SELL)}}} \quad (3)$$

$$\underbrace{X}_{\text{ASK(BUY)*}} = \underbrace{S(0)}_{\text{ASK(BUY)}}\underbrace{(1+y(T))^{T}}_{\text{ASK(BUY)}}$$

Arbitrage pricing & Financial engineering

- ❖Arbitrage
- ❖Financial engineering
- ❖Arbitrage-free & risk-neutral pricing
- ❖Static vs dynamical arbitrage

$$\sum_t PV(CF(t)) = 0 \quad (2)$$

IF NOT ⇒ ARBITRAGE !!!

Risk-neutral pricing

❖Using arbitrage(-free) arguments we have shown that the eXercise price of an Equity Forward must be set at:

$$X = S(0)\left(1 + y(T)\right)^T$$

❖The EXPECTED CF from the Forward at maturity T is simply equal to the EXPECTED payoff function <F(T)>, dependent on the EXPECTED future share price <S(T)> NOTE that shares are risky (the price can either drop or rise) so we can model it by random variables, thus in (2) we replace the fixed CF(T) by the EXPECTATION value !

$$\langle \mathrm{F(T)} \rangle = < S(T) > -X$$

❖As the only CF (the payoff) is at time T one gets:

$$\mathrm{PV}(\langle \mathrm{F(T)} \rangle) = \frac{\langle S(T) \rangle - S(0)\left(1 + y(T)\right)^T}{\left(1 + y(T)\right)^T} = \left\langle \frac{S(T)}{\left(1 + y(T)\right)^T} \right\rangle - S(0) = \langle \mathrm{PV}(S(T)) \rangle - S(0) = 0$$

$$\langle \mathrm{PV}(S(T)) \rangle = S(0)$$

Present Value

$$\sum_t < PV(CF(t)) > = 0 \quad (4)$$

IF NOT ⇒ ARBITRAGE !!!

Risk-neutral pricing

❖The future share price (the random variable S(T)) has some expected value <S(T)> which in terms of the EXPECTED RATE OF RETURN μ(T) can be written as:

$$\langle S(T)\rangle = S(0)\left(1+\mu(T)\right)^T$$

❖BUT in order to get a CORRECT VALUATION:

$$X = S(0)\left(1+y(T)\right)^T$$

❖using eqn. (2) (or its modified version (4) which takes into account EXPECTED futre CFs):

$$\langle \mathrm{F(T)}\rangle = < S(T) > -X$$

❖one has to make sure that the modelled DISCOUNTED SHARE PRICE probability distributions (i.e. the random variable: PV(S(T))) has the, so called, MARTINGALE property:

$$\langle \mathrm{PV}(S(T))\rangle = S(0)$$

EXPECTATION VALUE = CURRENT VALUE („MARTINGALE") !!!

❖In other words: one HAS TO ASSUME:

$$\langle S(T)\rangle = S(0)\left(1+y(T)\right)^T$$

❖ Implying that:

$$\mu(T) = y(T)$$

$$\sum_t < PV(CF(t)) > = 0 \quad (4)$$

IF NOT ⇒ ARBITRAGE !!!

Risk-neutral pricing

- The future share price (the random variable S(T)) has some expected value <S(T)> which in terms of the EXPECTED RATE OF RETURN μ(T) can be written as:

$$\langle S(T)\rangle = S(0)\left(1+\mu(T)\right)^T$$

- BUT in order to get a CORRECT VALUATION from eqn. (4) ONE ASSUMES:

$$\boxed{\mu(T) = y(T)}$$

EXPECTED RATE OF RETURN FROM (risky) SHARES μ(T) = ”RISK-FREE” interest RATE y(T) !!!

- Of course REAL (risky) SHARES should have a higher EXPECTED RETURN μ(T) than that of the (risk-free) deposits / loans y(T) (investors usually require risk premium, otherwise it wouldn't make any sense to invest in shares !)
- BUT JUST FOR THE CORRECT VALUATION one HAS TO ASSUME a FICTITIOUS RISK-FREE RETURN μ(T) = y(T), as if investors didn't care about the risk !

„RISK NEUTRAL” / ”RISK-FREE” / „MARTINGALE” PRICING = „ARBITRAGE(-FREE) PRICING”

- In fact, as already shown, derivative instruments can be "created" and hedged against market risk so investors don't demand extraordinary risk premium !

Arbitrage pricing & Financial engineering

- ❖Arbitrage
- ❖Financial engineering
- ❖Arbitrage-free & risk-neutral pricing
- ❖Static vs dynamical arbitrage

Static arbitrage / hedging

$$\sum_t < PV\left(CF(t)\right) > = 0 \quad (4)$$

IF NOT ⇒ ARBITRAGE !!!

- Using arbitrage-free arguments we have shown that prices on the COMPLETE MARKET must agree with eqn. (4)
  - This is the case for "RISK SYMMETRIC" derivatives, e.g. Forward* and SWAP contracts, which can be STATICALLY REPLICATED using a portfolio of spot instruments (underlying assets & deposits / loans)
  - The "bank" can create such derivatives by buying/selling a replicating portfolio NOW (in t=0) and his net position (i.e. derivative + oposite replicating portfolio) is perfectly HEDGED against market risk: the bank trades only NOW and does not care what happens in the future
  - If the instrument is mispriced one can make a profitable arbitrage by trading the mispriced instrument and (opposite position in) the replicating portfolio. All these transactions are done NOW (in t=0) and ARBITRAGE profits are fixed: one trades only NOW and does not care what happens in the future

⇓

STATIC ARBITRAGE / HEDGING

*For the exchange traded Futures the situation is slightly more complicated due to margin deposits and marking-to-market mechanism (one has to finance margins and daily loses / one can take advantage of daily profits). So financial Futures are NOT exactly equivalent to Forwards, but their "fair" value stays very close (one can e.g. prove that if one knew future interest rates then the price of Futures should exactly equal the price of Forwards with the same parameters).

Dynamical arbitrage / hedging

$$\sum_t < PV(CF(t)) > = 0 \quad (4)$$

IF NOT ⇒ ARBITRAGE !!!

- ❖This is NOT the case for "RISK ASSYMETRIC" instruments, e.g. for OPTIONS
  - ❑The market of those instruments is NOT COMPLETE (in the above sense) and one simply CANNOT create a STATIC REPLICATING portfolio of spot instruments in order to HEDGE / ARBITRAGE
  - ❑However one can still use the STATIC ARBITRAGE arguments to obtain some (wide) bands and general relations for option prices – see next slides
- ❖As we will discuss in Lecture 6, one can however (under some conditions) create a DYNAMICAL REPLICATING PORTFOLIO, i.e. the one that needs to be constantly adjusted depending on the future evolution of spot prices. Such markets are called "DYNAMICALLY COMPLETE", and one is again allowed to use eqn. (4)
  - ❑Such a portfolio can be used in order to make HEDGE / ARBITRAGE transactions, but the situation is NOT fixed a priori. One has to trade NOT ONLY NOW but CONSTANTLY, up to maturity !
  - ❑One also has to ASSUME some properties of the STOCHASTIC PROCESS driving the underlying asset (e.g. the share) price. If these properties are not satisfied or one cannot adjust the replicating portfolio accordingly, then the HEDGE / ARBITRAGE is not perfect and some level of market risk remains !

DYNAMICAL ARBITRAGE / HEDGING

Static arbitrage: bands for options

$$\sum_{t} < PV(CF(t)) > = 0 \quad (4)$$

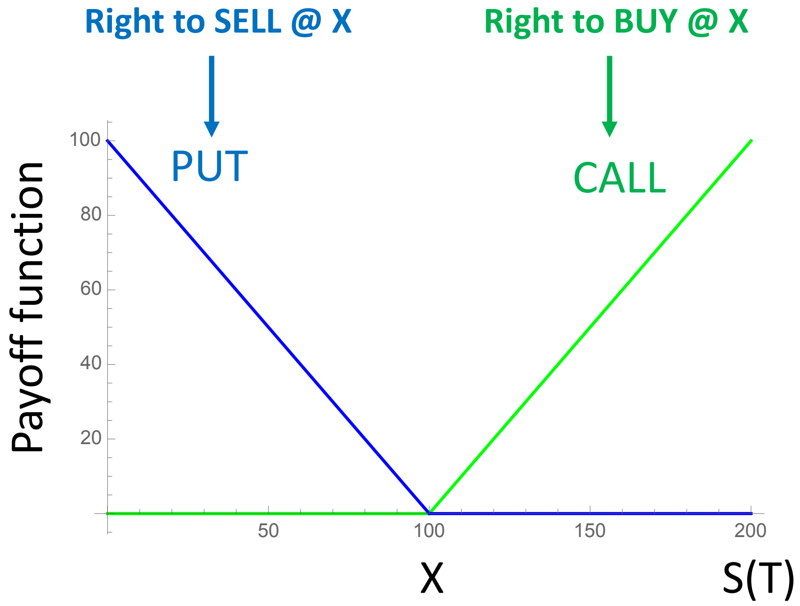
IF NOT ⇒ ARBITRAGE !!!

❖ Let's introduce the following notation:

- ❑ S(t) – the „underlying asset" (e.g. Share) price in time t , S(0) = S
- ❑ X - the eXercise price*
- ❑ T - expiration date
- ❑ c(t) - European "call" option value/price at time t , c(0) = c is the premium paid in t=0
- ❑ p(t) - European "put" option value/price, p(0) = p
- ❑ C(t) - American "Call" option value/price, C(0) = C
- ❑ P(t) - American "Put" option value/price P(0) = P

Exercised only at EXPIRATION

CAN be EXERCISED EARLY



❖ The value of the option at expiration T is:

- ❑ c(T) = C(T) = max(S(T) – X ; 0)
- ❑ p(T) = P(T) = max(X – S(T) ; 0)

❖ These are also "payoff functions" (always paid by option SELLER to option BUYER**) of options settled in cash (i.e. with no physical delivery)

* In the literature the eXercise price is sometimes denoted: K (striKe price)

**Assymetric risk ! But in exchange of option price (premium) which is always paid by the option BUYER to the option SELLER !

Static arbitrage: bands for options

$$\sum_t < PV\big(CF(t)\big) > = 0 \quad (4)$$

IF NOT ⇒ ARBITRAGE !!!

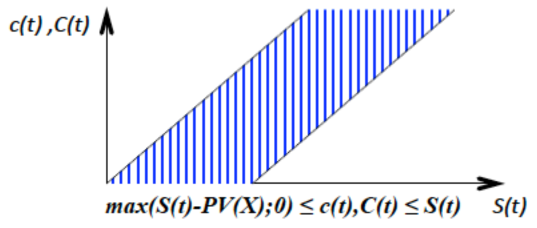
❖ By using the (static) arbitrage-free arguments one can prove e.g the following relations*:

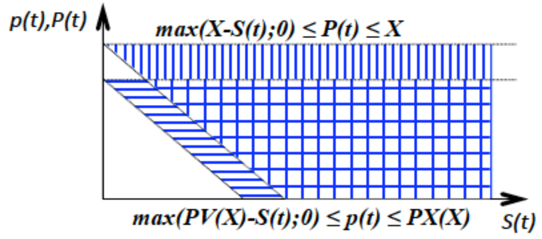
- ○ max (S – PV(X) ; 0) ≤ c ≤ S
  Present Value
- ○ max (S – PV(X) ; 0) ≤ C ≤ S
- ○ max (PV(X) - S ; 0) ≤ p ≤ PV(X)
- ○ max (X - S ; 0) ≤ P ≤ X
- ○ c – p = S – PV(X) (European "call-put parity")
- ○ S - X ≤ C - P ≤ S – PV(X) (American "call-put parity")
- ○ $c_2 \le c_1$ if $X_2 > X_1$
- ○ $p_1 \le p_2$ if $X_2 > X_1$
- ○ $c_2 \le \alpha\, c_1 + (1-\alpha)\, c_3$ if $X_2 = \alpha\, X_1 + (1-\alpha)\, X_3$
- ○ $p_2 \le \alpha\, p_1 + (1-\alpha)\, p_3$ if $X_2 = \alpha\, X_1 + (1-\alpha)\, X_3$
- ○ …

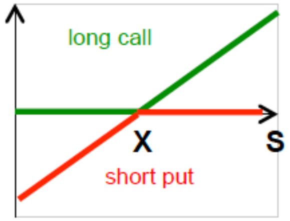
American Call should NOT be exercised early C = c *

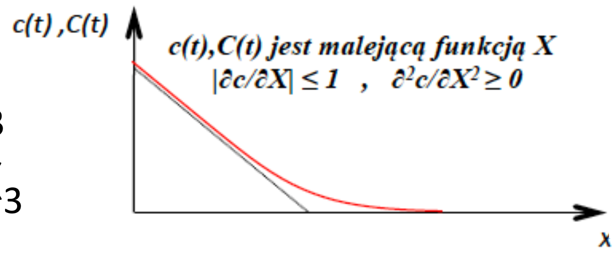
Monotonicity** (as f-ction of X)

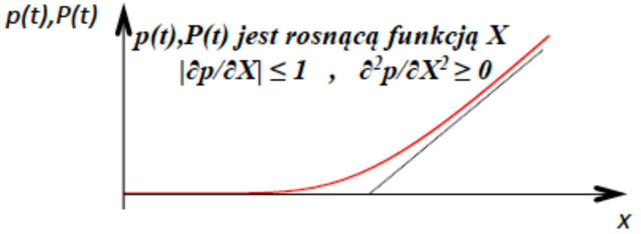
Convexity** (as f-ction of X)











* Dividend payments can change that !

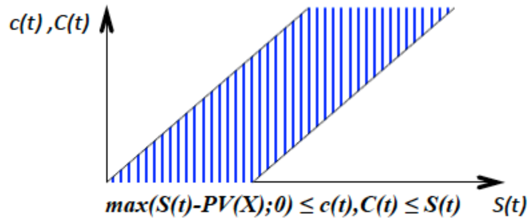
** Some books proove monotonicity & convexity as a f-ction of S (but that is wrong as it would require different S prices ⇒ arbitrage!)

Static arbitrage: bands for options

$$\sum_t < PV\big(CF(t)\big) > \; = 0 \quad (4)$$

IF NOT ⇒ ARBITRAGE !!!

- By using the (static) arbitrage-free arguments one can prove e.g the following relations*:
  - ...
  - max (S – PV(X) ; 0) ≤ C ≤ S
  - ...



- Sketch of a proof:
  - One rewrites it as separate inequalities:
    - + S - C ≥ 0
    - + C ≥ 0
    - + C + PV(X) - S ≥ 0 (*)
  - For each inequality one constructs (NOW) a portfolio with „+": LONG & „-": SHORT pos.
    - E.g. in (*) one BUYS: 1 Call option & deposits PV(X) and one SELLS (SHORT): 1 Share
  - One checks the FUTURE CF from CLOSING the portfolio at options's EXPIRATION
    - E.g. in (*) one has: CF(T) = max(S(T) – X ; 0) + X – S(T) ≥ 0
  - As FUTURE CF ≥ 0 ⇒ CURRENT CF (for the portfolio) ≤ 0 ! (otherwise: STATIC ARBITRAGE !)
    - E.g. in (*) one has: -C - PV(X) + S ≤ 0 (END OF PROOF) ⇐ NOTE: current CF has opposite sign than position !
  - For American options one additionally CHECKS EARLY EXERCISE ! (e.g. C ≥ max(S-X ; 0))

Summary

THE END ;)