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1 exercise

In the beginning of October the bank account XYZ showed a balance of 100 000 PLN

a) Below is the list of inflows to / outflows from the account in October:

- 5 X: inflow 17 000 PLN
- 12 X: inflow 35 000 PLN
- 21 X: outflow -55 000 PLN

Compute interest added to the account in the end of October, assuming the interest rate is 2% per annum.

In November there were some additional inflows to / outflows from the account:

- 9 XI: outflow -25 000 PLN
- 30 XI: inflow 100 000 PLN

Compute interest added to the account in the end of November.

Solution:

We have the initial balance on day one $P_1 = 100\ 000$ PLN and the interest rate per annum $y = 0.02$. Our goal is to calculate interest added to the account in the end of October and November. We use the simple interest formula

$$P(1 + r \cdot T) = P + I \; where \; I = r \cdot T, \tag{1}$$

and also calculate the interest rate per day as $y = \frac{0.02}{365}$. Then we do the calculation of interest for every cash flow:

$$I_1 = 100000 \cdot \frac{0.02}{365} \cdot 30 = 164.384 \; PLN, \tag{2}$$

$$I_2 = 17000 \cdot \frac{0.02}{365} \cdot 26 = 24.2192 \; PLN, \tag{3}$$

and so on. After all we have

$$I_{oct} = 164.384 + 24.2192 + 36.4384 - 30.137 = 194.904\ PLN, \tag{4}$$

and the balance in the end of October is $P_{oct} + I_{oct} =$ 97 194.904 PLN.

In the November we follow the same rules obtaining:

$$I_{nov} = 154.447 - 28.7671 = 125.68\ PLN, \tag{5}$$

and the balance in the and of November is $P_{oct} + I_{oct} + P_{nov} + I_{nov} =$ 172 321 PLN

2 exercise

What time t (in years) is needed to double the principal sum on a bank deposit with interest r % p.a. assuming:

- a) no interest is added (capitalized) before the end of the deposit (simple interest)
- b) interest is capitalized once a month, quarter, year, and in general n times a year (compound interest)
- c) continuous compounding of interest

Derive general formula. Compute t for r = 6% p.a.

- T - time in years needed to double the principal sum on a bank deposit with interest r
- P - principal sum on a bank deposit
- n - number of capitalizations per year

a) simple interest:

$$\begin{aligned} InitialCF(0) &= -P \\ FinalCF(T) &= P + I = P(1 + r \cdot T) \end{aligned} \tag{6}$$

$$CF(T) = 2 \cdot P$$

$$P(1 + r \cdot T) = 2 \cdot P$$

$$(1 + r \cdot T) = 2$$

$$T = \frac{1}{r}$$

b) compound interest - discrete capitalization of interest:

$$\begin{aligned} InitialCF(0) &= -P \\ FinalCF(T) &= P + I = P\left(1 + \frac{r}{n}\right)^{n \cdot T} \end{aligned} \tag{7}$$

$$CF(T) = 2 \cdot P$$

$$P\left(1+\frac{r}{n}\right)^{n \cdot T} = 2P$$

$$\left(1+\frac{r}{n}\right)^{n \cdot T} = 2$$

$$T = \frac{1}{n}\log_{\left(1+\frac{r}{n}\right)} 2$$

- yearly capitalization of interest (n = 1): $T \approx 11.896$ years = 4344 days
- half-yearly capitalization of interest (n = 2): $T \approx 11.725$ years = 4282 days
- quarterly capitalization of interest (n = 4): $T \approx 11.639$ years = 4251 days
- monthly capitalization of interest (n = 12): $T \approx 11.581$ years = 4230 days

c) compound interest: continuous capitalization of interest:

* initial $CF(0) = -P$
* final $CF(T) = P \cdot e^{r \cdot T}$

$$CF(T) = 2 \cdot P$$

$$P \cdot e^{r \cdot T} = 2P$$

$$T = \frac{1}{r}\ln 2$$

continuous capitalization of interest: $T \approx 11.552$ years = 4219 days

3 exercise

What should be the nominal interest rate r offered by a bank to ensure that savings of its clients rise at least as fast as the inflation rate i = 2.5% p.a. Assume yearly, half-yearly, quarterly, monthly, and continuous capitalization of interest.

In each case the effective interest rate should be equal to $y = 2.5$ %

* discrete capitalization of interest:
* initial CF(0) = $-P$
* final CF(T) = $P + I$ = $P\left(1+\frac{r}{n}\right)^{n \cdot T}$

$$\sum_t \frac{CF(t)}{(1+y)^t}) = 0$$

$$CF(0) + \frac{CF(T)}{(1+y)^T} = 0$$

$$-P + \frac{P\left(1+\frac{r}{n}\right)^{n\cdot T}}{(1+y)^T} = 0$$

$$\frac{P\left(1+\frac{r}{n}\right)^{n\cdot T}}{(1+y)^T} = P$$

$$P\left(1+\frac{r}{n}\right)^{n\cdot T} = P\cdot(1+y)^T$$

$$\left(1+\frac{r}{n}\right)^{n} = 1+y$$

explicite equation for effective interest rate:

$$y = \left(1+\frac{r}{n}\right)^{n} - 1$$

* yearly capitalization of interest (n = 1): $r = y = 2.5\%$

$$y = (1+r) - 1$$

* half-yearly capitalization of interest: $r = 2.48457\%$

$$y = \left(1+\frac{r}{2}\right)^{2} - 1$$

* quarterly capitalization of interest: $r = 2.4769\%$

$$y = \left(1+\frac{r}{4}\right)^{4} - 1$$

* monthly capitalization of interest: $r = 2.4718\%$

$$y = \left(1+\frac{r}{12}\right)^{12} - 1$$

* continuous capitalization of interest:

* initial $CF(0) = -P$
* final: $CF(T) = P\cdot e^{r\cdot T}$

$$\sum_t \frac{CF(t)}{(1+y)^t}) = 0$$

$$CF(0) + \frac{CF(T)}{(1+y)^T} = 0$$

$$-P + \frac{P\cdot e^{r\cdot T}}{(1+y)^T} == 0$$

$$e^{r\cdot T} = (1+y)^T$$

$$e^{r} = 1+y$$

* continuous capitalization of interest: $r = 2.46926\%$

4 exercise

Assume the annual effective interest rate (yield) $y_0 =$ [% / year]. What is the equivalent half-annual, monthly, weekly, daily, continuous effective interest rate y_n (yield). Derive general formula and make computations for $y_0 = 5\%$ p.a.

* discrete capitalization of interest:
* initial CF(0) $= -P$
* final CF(T) $= P + I = P\left(1+y\right)^T$
* definition of the equivalent half-annual (y_2), monthly (y_{12}), weekly (y_{52}), daily (y_{365}) effective interest rate

$$\sum_t \frac{CF(t)}{\left(1+y_n\right)^{nt}}) = 0$$

$$CF(0) + \frac{CF(T)}{\left(1+y_n\right)^{nT}} = 0$$

$$-P + P \cdot \frac{\left(1+y\right)^T}{\left(1+y_n\right)^{nT}} = 0$$

$$(1+y) = \left(1+y_n\right)^n$$

$$\left(1+y\right)^{\frac{1}{n}} - 1 = y_n$$

* the equivalent half-annual effective interest rate: $y_2 = 2.46951\%$
* the equivalent monthly effective interest rate: $y_{12} = 0.407412\%$
* the equivalent weekly effective interest rate: $y_{52} = 0.0938713\%$
* the equivalent daily effective interest rate: $y_{365} = 0.0133681\%$
* continuous capitalization of interest:
* initial $CF(0) = -P$
* final: $P + I = P\left(1+y\right)^T$
* definition of the equivalent half-annual continuous effective interest rate y_0:

$$\sum_t \frac{CF(t)}{e^{y_0 t}}) = 0$$

$$CF(0) + \frac{CF(T)}{e^{y_0 T}} = 0$$

$$-P + P \cdot \frac{\left(1+y\right)^T}{e^{y_0 T}} = 0$$

$$1 + y = e^{y_0}$$

$$\ln\left(1+y\right) = y_0$$

* the equivalent half-annual continuous effective interest rate $y_0 = 4.87902\%$:

5 exercise

Show that if interest on a deposit is capitalized more often than once a year ($n > 1$) then the effective interest rate $y >$ the nominal rate r (one must assume $r > 0$).

* initial CF(0) $= -P$ * final CF(T) $= P + I = P\left(1+\frac{r}{n}\right)^{n \cdot T}$

$$\sum_t \frac{CF(t)}{(1+y)^t}) = 0$$

$$CF(0) + \frac{CF(T)}{(1+y)^T} = 0$$

$$-P + P \cdot \frac{\left(1+\frac{r}{n}\right)^{nT}}{(1+y)^T} = 0$$

* effective interest rate:

$$y = \left(1+\frac{r}{n}\right)^n - 1$$

→ Proof:

$a^n - b^n = (a-b) \cdot \left(\sum_i^{n-1} a^{n-1-i} b^i\right)$

We want to proof that for $n > 1$

$$y = \left(1+\frac{r}{n}\right)^n - 1 > r$$

$$\left(1+\frac{r}{n}\right)^n - 1^n > r$$

$$\left(\left(1+\frac{r}{n}\right) - 1\right) \cdot \left(\sum_{i=0}^{n-1} \left(1+\frac{r}{n}\right)^i\right) > r$$

$$\left(\frac{r}{n}\right) \cdot \left(\sum_{i=0}^{n-1} \left(1+\frac{r}{n}\right)^i\right) > r$$

$$\left(\frac{1}{n}\right) \cdot \left(\sum_{i=0}^{n-1} \left(1+\frac{r}{n}\right)^i\right) > 1$$

$$\sum_{i=0}^{n-1} \left(1+\frac{r}{n}\right)^i > n$$

LHS correspond to a sum of n elements, each element $\left(1+\frac{r}{n}\right)^i \geq 1$. Sum over i is greater than n.

6 exercise

91-day ($T = 91$) 'tex-free' deposit with interest rate $r = 0.06$ p.a.
Daily compounding of interest:

$$0.81 \cdot \left(\frac{x}{365} \cdot 91\right) = \left(1 + \frac{r}{365}\right)^T - 1, \tag{8}$$

then $x = 0.0746247$.
Continuous compounding of interest:

$$0.81 \cdot \left(\frac{x}{365} \cdot 91\right) = e^{r \cdot \frac{T}{365}} - 1, \tag{9}$$

then $x = 0.0746309$.

7 exercise

What amount should be put aside on the bank account each quarter in order to accumulate 10 000 PLN after 2 years of saving. Assume interest rate of r = 4% per annum and:

- no capitalization of interest (simple interest, annuity)
- quaterly capitalization of interest (compound interest, annuity due).

Solution:

In the first scenario we use the formula for simple interest (1) and calculate:

$$\begin{aligned} P_1 &= x(1 + r \cdot 2.00), \\ P_2 &= x(1 + r \cdot 1.75), \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ P_8 &= x(1 + r \cdot 0.25), \end{aligned} \tag{10}$$

and from $P_1 + ... + P_8 = 10000$ we have x = 1196.17 PLN.

In the second scenario we have x PLN in the beginning of each quarter and in the end of each quarter we have x + I PLN. Assuming quaterly capitalization of interest we divide our interest rate r by four, getting r = 1% per quarter. We will describe the money in the end of each quarter by y_i, where i numerates quarters from 1 to 8. Calculations are intuitive

$$
\begin{aligned}
y_1 = x(1+r),\ \textit{let us put}\ (1+r) = a,\ \textit{then}\ y_1 = x \cdot a,\\
y_2 = (y_1 + x)a = x \cdot a + x \cdot a^2,\\
y_3 = (y_2 + x)a = x \cdot a + x \cdot a^2 + x \cdot a^3,\\
.\\
.\\
.\\
y_8 = x(a + ... + a^8),
\end{aligned}
\tag{11}
$$

and knowing that y_8 = 10 000 PLN we get x = 1194.95.

Using the formula for the sum of the geometric series we can obtain the simple annuity due formula (compounding periods and the investment periods match):

$$
FV = x\left(\frac{(1+r)^n - 1}{r}\right)(1+r)\,, \tag{12}
$$

where r has to be rescaled for a given number of compoundings (r = $\frac{\textit{interest p. a.}}{\textit{compoundings p. a.}}$) and n is the number of years multiplied by the number of compoundings per time interval (so per year).

8 exercise

We have given three banks: A, B and C. Bank A offers a one-year deposit with monthly capitalization of interest and the interest rate is growing by 1% each month: from 1% per annum in the first month to 12% per annum in the twelfth month. Bank C offers the same, but the interest rate is falling each month (from 12% to 1%). Bank B offers a deposit with constant interest rate 6.5%.

- Which bank should we choose? (compute effective interest rate for each bank)
- Would the above change if the banks were paying out interest on a monthly basis instead of adding it to the principal amount of the deposit?

Solution:

Our way of thinking is similar as in the previous problem - in the beginning of each month we have x PLN, and in the end of each month we have y = x + I PLN, where I means the interest from that month. Capitalization is monthly, so x + I becomes a new x for the following month. The money in the end of $i-th$ month will be our y_i and then, for the banks

A and C we have:

$$
\begin{aligned}
y_1 = x(1+r_1) = x + I, \\
y_2 = y1(1+r_2) = x(1+r_1)(1+r_2), \\
. \\
. \\
. \\
y_{12} = y_{11}(1+r_{12}) = x(1+r_1)...(1+r_{12}) = 1.06692 \; of\ the\ principal\ value.
\end{aligned}
\tag{13}
$$

For the bank B we obtain our result straightforward from the compound interest formula, which is equal to 1.06697 of the principal value.
One can calculate the effective interest rate for each bank from the formula

$$
y = \left(1 + \frac{r}{n}\right)^n - 1. \tag{14}
$$

We get $y_A = y_C = 0.0669719$, which turns out to be equal to y_B. So the effective interest rate for each bank is the same.

For the second case we give our money to the bank for a one-year deposit and every month some interest is paid to us, after one year we get our money back.

$$
\begin{aligned}
y_1 = x(1+r_1) = x + I, \\
y_2 = x(1+r_2) = x(1+r_2), \\
. \\
. \\
. \\
y_{12} = x(1+r_{12}),
\end{aligned}
\tag{15}
$$

so in the end we obtain the same for all three banks:

$$
x(1 + r_1 + r_2 + ... + r_{12}) = 1.065 \; of\ the\ principal\ value. \tag{16}
$$

9 exercise

one-year loan of 10 000 PLN with interest rate 10% p.a. and quaterly amortization. A client can choose between the system of equal principal payments (each quarter 1/4 of the initial value of the loan + accrued interest) or equal total payments (each quarter client pays the same constant amount).

- Compute payments done by the client (CF) in both systems of amortization

- Prepare Amortization Schedule (a table showing current principal value of the loan in the beginning/end of each quarter and a split of the payment (CF) into Principal payment and Interest payment)
- Compute total amount of interest paid to the Bank in both systems
- Which system is more favourable fo the Bank/Client (check what is the effective interest rate in each case)?

In our computations we use the annuity formula.

Figure 1: Equal principal payments

Period	Initial Principal P(t-1)	Interest CFI(t)	Principal CFΔP(t)	Payment CFPMT(t)	Final Principal P(t)
1	10000	250	2500	2750	7500
2	7500	187.5	2500	2687.5	5000
3	5000	125	2500	2625	2500
4	2500	62.5	2500	2562.5	0

Total amount of interest payed to the bank in equal principal payments system: 625 PLN.

Figure 2: Equal total payments

Period	Initial Principal P(t-1)	Interest CFI(t)	Principal CFΔP(t)	Payment CFPMT(t)	Final Principal P(t)
1	10000	250	2408.179	2658.179	7591.821
2	7591.821	189.796	2468.383	2658.179	5123.438
3	5123.438	128.086	2530.093	2658.179	2593.345
4	2593.345	64.834	2593.345	2658.179	0

Total amount of interest payed to the bank in equal total payments system: 632.715 PLN.

In both cases the effective quaterly interest rate is equal to y = 0.025, so both systems are equally favourable for the Bank/Client.

10 exercise

20-year mortgage (t = 20) of 200 000 PLN, which is amortized monthly as an annuity. The interest rate y is changing every 3 months - it is set as the 2% p.a. + WIBOR3M rate. WIBOR3M rate is 4% p.a. at the initial moment:

- compute equal monthly payments for the first three months
- compute the remaining balance of the principal amount after 3 months

- after 3 months the WIBOR3M rate increased to 5% p.a., compute new monthly payments for the next three months.

(a) Using annuity formula:

$$200000\ PLN = \frac{R}{r}\left(1 - \frac{1}{(1+r)^T}\right), \tag{17}$$

where $r = \frac{y}{12}$ per month and $T = t \cdot 12 = 240$ months, which gives the result of R $=$ 1432.86 PLN as the equal monthly payment for the first three months.

(b) We can just substract 3R from 200 000 PLN and get 195 701 PLN

(c) Again from the annuity formula:

$$195701\ PLN = \frac{F}{r_1}\left(1 - \frac{1}{(1+r_1)^{T_1}}\right), \tag{18}$$

where $r_1 = \frac{y_1}{12}$ per month and y is now equal to 0.07 p.a. $= y_1$, $T_1 = t \cdot 12 - 3 = 237$ months, which gives the result F $=$ 1526.11 PLN as the equal monthly payment for the following three months.