exercise2

June 10, 2020

[1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from copy import deepcopy
from sklearn.linear_model import LinearRegression

from sympy.solvers import solve
from sympy import *

%matplotlib inline
```

1 MA(q) processes

$$y_t = \alpha_0\eta_t + \alpha_1\eta_{t-1} + ... + \alpha_q\eta_{t-q}$$

$$y_t = \left(\sum_{i=0}^{g} \alpha_i B^i\right)\eta_t$$

where: * η is a White Gaussian Noise * $< \eta_t >= 0$ * $< \eta^2 >= 1$ * B - backshift operator ($B^0 = 1$)

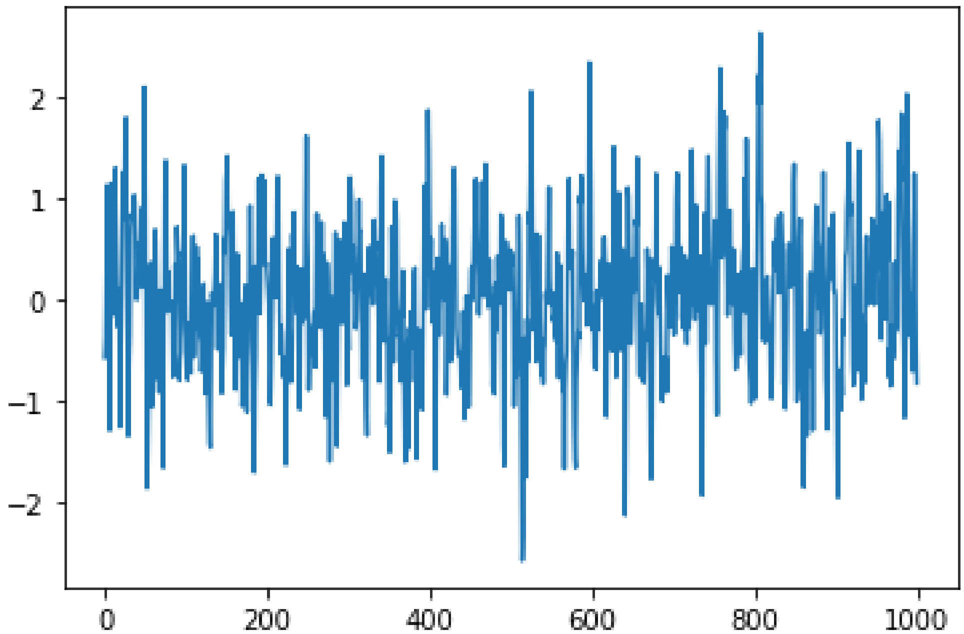
1.1 a)

Find the orderpand fit the parameters: $\beta_i(i = 1, ..., p)$, α_0 using the 2-step linear regression method. In order to find p and the best fit use the "AIC" and "BIC" criteria. Use $m = 5$to estimate the realized noise in 1-st step and remember about normalizing it such that the variance is 1, in 2-nd step use this data to fit regression of $y(t)$ against the (shifted) noise variables.

[2]:
```
data = []

with open("MAdata.txt", "r") as f:
    for line in f:
        data.append(float(line))

data = np.array(data)
```

```
plt.plot(data)
```

[2]: [<matplotlib.lines.Line2D at 0x7fc6e8074790>]



1.1.1 Two step regression: step 1

[3]:
```
def dividing_data_into_subseries_II(data, k=2):
    X = []
    Y = []
    for i in range(len(data)-k):
        tmp = []
        for j in range(k):
            tmp.append(data[j+i])
        X.append(tmp)
        Y.append(data[k+i] )

    return np.array(X), np.array(Y)
```

[4]:
```
X,Y = dividing_data_into_subseries_II(data, k=5)

reg = LinearRegression(fit_intercept = False).fit(X, Y)
prediction = reg.predict(X)

print("fitted coefficients  beta_(t-4)...beta_(t-1): {:}".format(reg.coef_) )
```

```
eps = Y - reg.predict(X)
eps = eps/eps.std()

print("\nmean= {:2f}\t std = {:2f}".format( eps.mean(), eps.std()))
```

```
fitted coefficients  beta_(t-4)…beta_(t-1): [-0.11996622  0.36024784
-0.37833758 -0.08663265  0.80459043]

mean= 0.013197   std = 1.000000
```

[5]:
```
YY = np.vstack((Y,eps)).transpose()
np.hstack((YY, X))
df = pd.DataFrame(np.hstack((X,YY)), columns = ['y(t5)', 'y(t4)', 'y(t3)',␣
↪'y(t2)', 'y(t1)', 'y(t)', 'eps'])

df['eps+1'] = df['eps'].shift(1)
df['eps+2'] = df['eps'].shift(2)
df['eps+3'] = df['eps'].shift(3)
df['eps+4'] = df['eps'].shift(4)
df['eps+5'] = df['eps'].shift(5)

df.dropna(inplace = True)
df
```

[5]:
```
        y(t5)     y(t4)     y(t3)     y(t2)     y(t1)      y(t)       eps  \
5   -0.192600 -1.299081 -0.226935  0.065571  1.144725  0.265750 -0.563038
6   -1.299081 -0.226935  0.065571  1.144725  0.265750 -0.153007 -0.614121
7   -0.226935  0.065571  1.144725  0.265750 -0.153007  0.284510  1.575092
8    0.065571  1.144725  0.265750 -0.153007  0.284510  1.295828  1.452625
9    1.144725  0.265750 -0.153007  0.284510  1.295828  0.755126 -0.540861
..        …         …         …         …         …         …         …
991 -0.363077  0.036522  0.048544 -0.697383 -0.713475  0.160522  1.231984
992  0.036522  0.048544 -0.697383 -0.713475  0.160522  1.239079  1.494245
993  0.048544 -0.697383 -0.713475  0.160522  1.239079  0.618735 -0.730864
994 -0.697383 -0.713475  0.160522  1.239079  0.618735 -0.401085 -1.080186
995 -0.713475  0.160522  1.239079  0.618735 -0.401085 -0.831914 -0.252345

        eps+1     eps+2     eps+3     eps+4     eps+5
5    1.353333  0.103533  1.123592 -1.602866 -0.475611
6   -0.563038  1.353333  0.103533  1.123592 -1.602866
7   -0.614121 -0.563038  1.353333  0.103533  1.123592
8    1.575092 -0.614121 -0.563038  1.353333  0.103533
9    1.452625  1.575092 -0.614121 -0.563038  1.353333
..        …         …         …         …         …
991  0.122370 -1.847775  0.172484  0.156206 -0.317439
992  1.231984  0.122370 -1.847775  0.172484  0.156206
```

```
993  1.494245  1.231984  0.122370 -1.847775  0.172484
994 -0.730864  1.494245  1.231984  0.122370 -1.847775
995 -1.080186 -0.730864  1.494245  1.231984  0.122370

[991 rows x 12 columns]
```

1.1.2 Two step regression: step 2

[6]:
```
def get_AIC_BIC(df, x_cols):

    k=len(x_cols)
    Y = np.array(df['y(t)'])
    X = np.array(df[x_cols])

    reg = LinearRegression(fit_intercept = False).fit(X, Y)
    prediction = reg.predict(X)
    N = len(X)

    alpha = np.sqrt(((Y-prediction)**2).mean())
    AIC = N*np.log(((Y-prediction)**2).mean()) + 2*k
    BIC = N*np.log(((Y-prediction)**2).mean()) + np.log(N)*k

    print("\n\nk = {:}, alpha = {:}\t AIC = {:}\t BIC = {:}\n".format(k,alpha,␣
↪AIC, BIC))
    print("intercept = {:}\t reg.coef_ = {:}".format(reg.intercept_, reg.coef_)␣
↪)

    return [k, alpha, AIC, BIC]
```

[7]:
```
tab = []

tab.append(get_AIC_BIC(df, ['eps+1'] ))
tab.append(get_AIC_BIC(df, ['eps+1', 'eps+2'] ))
tab.append(get_AIC_BIC(df, ['eps+1', 'eps+2', 'eps+3'] ))
tab.append(get_AIC_BIC(df, ['eps+1', 'eps+2', 'eps+3', 'eps+4'] ))
tab.append(get_AIC_BIC(df, ['eps+1', 'eps+2', 'eps+3', 'eps+4', 'eps+5'] ))
```

```
k = 1, alpha = 0.5983398034602126        AIC = -1015.948170249943        BIC =
-1011.049455715613

intercept = 0.0  reg.coef_ = [0.40763142]


k = 2, alpha = 0.5112762579620876        AIC = -1325.6152114266736       BIC =
```

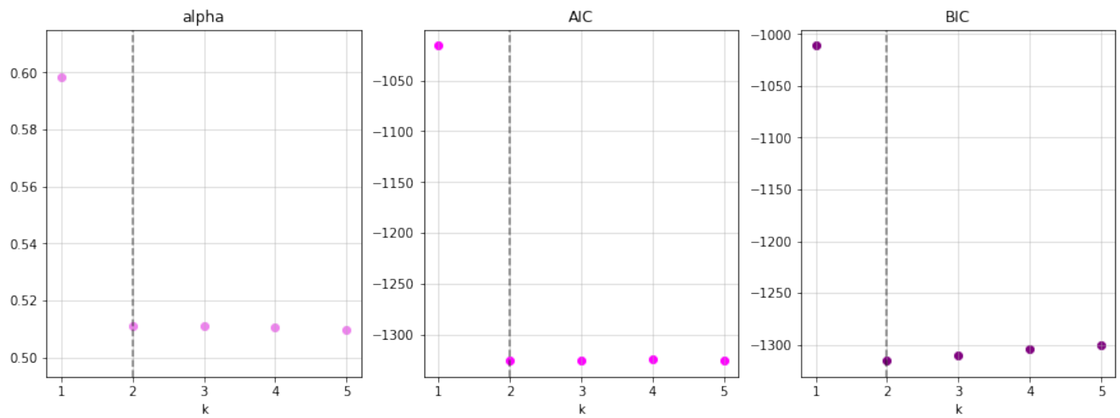
```
-1315.8177823580136

intercept = 0.0  reg.coef_ = [0.41227231 0.31086954]


k = 3, alpha = 0.510873738399572         AIC = -1325.1762226617864       BIC =
-1310.4800790587965

intercept = 0.0  reg.coef_ = [ 0.41314462  0.3105654  -0.02029883]


k = 4, alpha = 0.5108607704161228        AIC = -1323.2265342505132       BIC =
-1303.6316761131932

intercept = 0.0  reg.coef_ = [ 0.41330546  0.31040558 -0.02024874  0.00364592]


k = 5, alpha = 0.5098490840077213        AIC = -1325.1554923366602       BIC =
-1300.6619196650104

intercept = 0.0  reg.coef_ = [ 0.41343387  0.3117848  -0.02167565  0.00412761
0.03220715]
```

[8]:

```python
k_tab, alpha_tab, AIC_tab, BIC_tab = np.array(tab).transpose()

fig, axs = plt.subplots(nrows=1, ncols=3, figsize = (15,5))
for ax in axs:
    ax.set_xlabel('k')
    ax.grid(True, alpha = 0.5)
    ax.axvline(x = 2, linestyle='dashed', color = 'black', alpha = 0.5)

axs[0].set_title('alpha')
axs[1].set_title('AIC')
axs[2].set_title('BIC')
axs[0].scatter(k_tab, alpha_tab, color = 'violet')
axs[1].scatter(k_tab, AIC_tab, color = 'fuchsia')
axs[2].scatter(k_tab, BIC_tab, color = 'purple')
```

[8]:

```
<matplotlib.collections.PathCollection at 0x7fc6e7689490>
```



According to the 2-step regression method, q=2 is the order of MA(q) process with the coefficients: * $\beta_1 = 0.4112644$ (coefficient at eps+1) * $\beta_2 = 0.3131448$ (coefficient at eps+2) * $alpha_0 = 0.5112762$

[9]:
```
q=2
Y = np.array(df['y(t)'])
X = np.array(df[['eps+1','eps+2']])

reg = LinearRegression(fit_intercept = False).fit(X, Y)
prediction = reg.predict(X)
N = len(Y)

alpha_0 = np.sqrt(((Y-prediction )**2).mean())
AIC = N*np.log(((Y-prediction)**2).mean() )+ 2*q
BIC = N*np.log(((Y-prediction)**2).mean()) + np.log(N)*q

print("\n\nq = {:}, alpha = {:}\t AIC = {:}\t BIC = {:}\n".format(q,alpha_0,␣
↪AIC, BIC))
print("reg.coef_ = {:}\tintercept = {:}".format(reg.coef_, reg.intercept_) )
```

```
q = 2, alpha = 0.5112762579620876        AIC = -1325.6152114266736       BIC =
-1315.8177823580136

reg.coef_ = [0.41227231 0.31086954]     intercept = 0.0
```

2 b)

Compute the sample autocorrelation function (SACF), $\rho(t)$ and the sample partial autocorrelation function (SPACF),$\phi(t)$ and plot them for $t = 1, ...10$. Based on the plot of SPACF check the orderp (on the plot include Gaussian N(0,(T−t)−1) bands for 95% confidence level to check when SPACF becomes statistically zero).

[10]:
```python
def dividing_data_into_subseries(data, k=2, k_max=11):
    data_k = []
    for i in range(len(data)-k_max):
        tmp = [ ]
        for j in range(k):
            tmp.append(data[j+i])
        tmp.append(data[k+i])
        data_k.append(tmp)
    return np.array(data_k).T
```

[11]:
```python
def get_autocorrelation_coeff(data, delta):

    data_k = dividing_data_into_subseries(data, k=delta)
    denominator = (data**2).mean()
    nominator = (data_k[0]*data_k[-1]).mean()

    return nominator/denominator
```

[12]:
```python
def get_autocorrelation(p,data):

    autocorrelation_tab = []

    for delta in range(1,p+1):
        autocorrelation = get_autocorrelation_coeff(data, delta)
        autocorrelation_tab.append(autocorrelation)

    return autocorrelation_tab
```

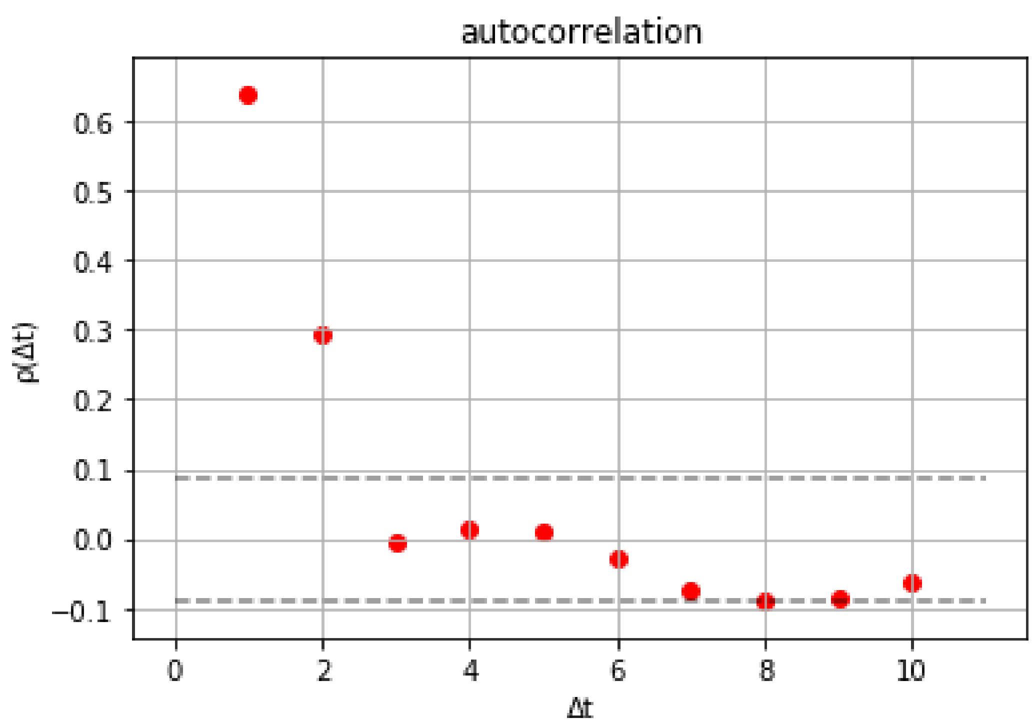
2.0.1 sample autocorrelation function (SACF)

[13]:
```python
p=10
p_tab = list(range(1,p+1))
autocorrelation_tab = get_autocorrelation(p,data)

for delta_t, corr in zip(p_tab, autocorrelation_tab):
    print(u'\u0394t = {:}\t \u03C1(\u0394t)={:2f}'.format(delta_t,corr ))

N = 490
confidence_level = 1.96/np.sqrt(N)


plt.title('autocorrelation')
```

```python
plt.scatter(p_tab, autocorrelation_tab, color ='red')
plt.ylabel(u'\u03C1(\u0394t)')
plt.xlabel(u'\u0394t')
plt.grid()

plt.hlines(y = confidence_level, xmin=0, xmax = p+1, linestyles='dashed', color = 'black', alpha = 0.5)
plt.hlines(y = -confidence_level, xmin=0, xmax = p+1, linestyles='dashed', color = 'black', alpha = 0.5)
```

```
Δt = 1    (Δt)=0.639600
Δt = 2    (Δt)=0.294154
Δt = 3    (Δt)=-0.003510
Δt = 4    (Δt)=0.014596
Δt = 5    (Δt)=0.010283
Δt = 6    (Δt)=-0.027150
Δt = 7    (Δt)=-0.075388
Δt = 8    (Δt)=-0.090840
Δt = 9    (Δt)=-0.086528
Δt = 10   (Δt)=-0.062394
```

[13]: <matplotlib.collections.LineCollection at 0x7fc6e7654e90>



2.0.2 Partial autocorrelation function (PACF)

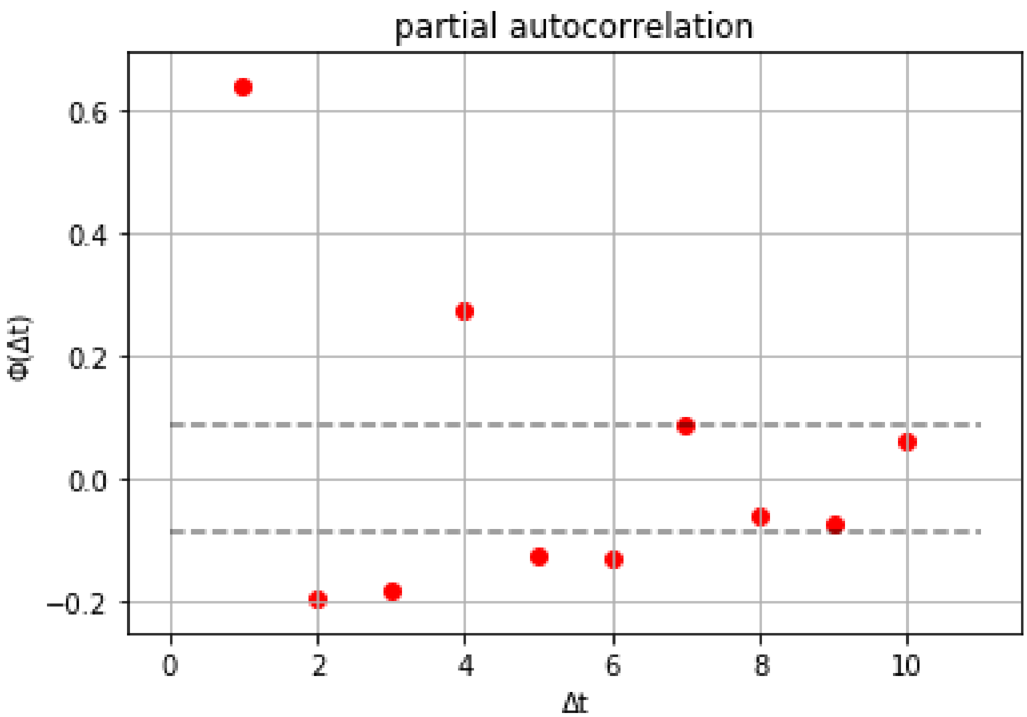
[14]:
```
def get_partial_autocorrelation(p, data):

    autocorrelation_tab = get_autocorrelation(p,data)

    M = np.identity(p)*0.5
    for i in range(p):
        for j in range(i+1, p):
            M[i][j] = autocorrelation_tab[j-i-1]

    A = np.matrix(M.T + M)
    A_inv = np.linalg.inv(A)

    psi_vec = np.array(np.dot(A_inv,autocorrelation_tab))[0]

    return psi_vec[-1]
```

[15]:
```
p=10
p_tab = list(range(1,p+1))
partial_corr_tab_tab = []

for k in range(1,p+1):
    partial_corr_tab_tab.append(get_partial_autocorrelation(k,data))


for delta_t, p_corr in zip(p_tab, partial_corr_tab_tab):
    print(u'\u0394t = {:}\t \u03A6(\u0394t)={:2f}'.format(delta_t, p_corr ))


plt.title('partial autocorrelation')
plt.scatter(p_tab, partial_corr_tab_tab, color ='red')
plt.ylabel(u'\u03A6(\u0394t)')
plt.xlabel(u'\u0394t')
plt.grid()
plt.hlines(y = confidence_level, xmin=0, xmax = p+1, linestyles='dashed', color␣
 ↪= 'black', alpha = 0.5)
plt.hlines(y = -confidence_level, xmin=0, xmax = p+1, linestyles='dashed',␣
 ↪color = 'black', alpha = 0.5)
```

```
Δt = 1   Φ(Δt)=0.639600
Δt = 2   Φ(Δt)=-0.194504
Δt = 3   Φ(Δt)=-0.182638
Δt = 4   Φ(Δt)=0.273179
Δt = 5   Φ(Δt)=-0.125215
Δt = 6   Φ(Δt)=-0.128887
Δt = 7   Φ(Δt)=0.086820
```

```
Δt = 8   Φ(Δt)=-0.060005
Δt = 9   Φ(Δt)=-0.071853
Δt = 10  Φ(Δt)=0.060464
```

[15]: <matplotlib.collections.LineCollection at 0x7fc6e75f7f50>



The autocorrelation drops abruptly to 0 (with confidence level 95%) after $q = 2$, therefore we may assume that the order of MA(q) process is 2.

3 c

For the order q established in (b) fit the parameters: $\beta_i (i = 1, ..., q)$ and α_0 using the Yule-Walkermethod (in the matrix form).

[16]:
```
def dividing_data_into_subseries(data, k=2):
    data_k = []
    for i in range(len(data)-k):
        tmp = [ ]
        for j in range(k):
            tmp.append(data[j+i])
        tmp.append(data[k+i])
        data_k.append(tmp)
    return np.array(data_k).T
```

[17]:
```
data_1 = dividing_data_into_subseries(data, k=1)
data_2 = dividing_data_into_subseries(data, k=2)
data_3 = dividing_data_into_subseries(data, k=3)
```

We are looking for the autocorrelation function $ACF \equiv \rho\Delta t$

$$< y_t y_{\Delta t} >=< y_t^2 > \rho_{\Delta t}$$

$$y_t = \sum_{i=0}^{g} \alpha_i \eta_{t-i}$$

$$y_{t-\Delta t} = \sum_{i=0}^{g} \alpha_j \eta_{t-\Delta t-j}$$

$$< y_t y_{t-\Delta t} >= \sum_{i=0}^{g}\sum_{j=0}^{g} < \eta_{t-i}\eta_{t-\Delta t-h} >= \sum_{i=\Delta t}^{g} \alpha_i \alpha_{i-\Delta t}$$

[18]:
```
cov0 = (data**2).mean()
cov1 = (data_1[0]*data_1[-1]).mean()
cov2 = (data_2[0]*data_2[-1]).mean()
cov3 = (data_3[0]*data_3[-1]).mean()

print("cov0 = {:2f} \t cov1 = {:2f}\t cov2 = {:2f} \tcov3 = {:2f} ".
 ↪format(cov0, cov1,cov2,cov3))
```

```
cov0 = 0.524192          cov1 = 0.333107          cov2 = 0.150836          cov3 =
-0.003968
```

3.0.1 Yule Walker Equations:

- MA(1):

$$< y_t y_t >= \alpha_0^2 + \alpha_1^2$$

$$< y_t y_{t-1} >= \alpha_0 \cdot \alpha_1$$

- MA(2):

$$< y_t y_t >= \alpha_0^2 + \alpha_1^2 + \alpha_2^2$$

$$< y_t y_{t-1} >= \alpha_0\alpha_1 + \alpha_1\alpha_2$$

$$< y_t y_{t-2} >= \alpha_0\alpha_2$$

[19]:
```
alpha0, alpha1, alpha2 = symbols('alpha0 alpha1 alpha2', real=True)
x = symbols('x')
```

[20]:
```
## q=1

f1 = cov0-alpha0**2-alpha1**2
f2 = cov1-alpha0*alpha1

sol_q1 = solve( [f1,f2], set = True)
sol_q1 = sol_q1[1]
sol_q1
```

[20]:
```
set()
```

[21]:
```
## q=2

f1=cov0-alpha0**2-alpha1**2-alpha2**2
f2=cov1-alpha0*alpha1-alpha1*alpha2
f3=cov2-alpha0*alpha2

sol_q2 = solve( [f1,f2,f3], set = True)
sol_q2 = sol_q2[1]
sol_q2
```

[21]:
```
{(-0.521048424658716, -0.410972423453576, -0.289485433038988),
 (-0.289485433038988, -0.410972423453576, -0.521048424658716),
 (0.289485433038988, 0.410972423453576, 0.521048424658716),
 (0.521048424658716, 0.410972423453576, 0.289485433038988)}
```

3.0.2 Invertibility condition: checking the roots of the polynomial:

$$\sum a_i z^i = 0$$

If we take only a_i's such that the roots $|z| < 1$, we are left with:

[22]:
```
X = [1,x,x**2]

for coeffs in list(sol_q2):
    roots = solve(sum(list(map(lambda x,y: x*y, X, coeffs))))
    print("\ncoeffs:\t", coeffs)
    print("roots:\t", roots)
```

```
coeffs:  (0.521048424658716, 0.410972423453576, 0.289485433038988)
roots:   [-0.709832648812809 - 1.13844192863999*I, -0.709832648812809 +
1.13844192863999*I]

coeffs:  (-0.521048424658716, -0.410972423453576, -0.289485433038988)
roots:   [-0.709832648812809 - 1.13844192863999*I, -0.709832648812809 +
1.13844192863999*I]
```

```
coeffs:  (0.289485433038988, 0.410972423453576, 0.521048424658716)
roots:   [-0.394370661155689 - 0.632498514735841*I, -0.394370661155689 +
0.632498514735841*I]

coeffs:  (-0.289485433038988, -0.410972423453576, -0.521048424658716)
roots:   [-0.394370661155689 - 0.632498514735841*I, -0.394370661155689 +
0.632498514735841*I]
```

for $q = 2$ we have two possible sets of coefficients $alpha_i$:

- $\alpha_0 = 0.52105$, $\alpha_1 = 0.41097$, $\alpha_2 = 0.28949$
- $\alpha_0 = -0.52105$ $\alpha_1 = -0.41097$, $\alpha_2 = -0.28949$

4 d)

Using the data and the fit of point (a) and/or (c) compute the empirical noise

$$\hat{\eta}(t) = \frac{y_t - \hat{y}_t}{\alpha_0}$$

and check if it has standard Gaussian N(0,1) distribution (use e.g.Kolmogorov-Smirnov test).

[23]:
```
## Based on results in a:

coeff_a = np.array([ 0.41126442, 0.31314482])  ##  a1, a2
alpha_0a = 0.5113506601881491

## Based on results in c:
coeff_c = np.array([0.410972423453576, 0.289485433038988]) ## a1, a2
alpha_0c = 0.521048424658716
```

[24]:
```
df.drop(['y(t5)', 'y(t4)','y(t3)', 'eps+3', 'eps+4', 'eps+5'], inplace = True,␣
 ↪axis = 1)
df.head()
```

[24]:

	y(t2)	y(t1)	y(t)	eps	eps+1	eps+2
5	0.065571	1.144725	0.265750	-0.563038	1.353333	0.103533
6	1.144725	0.265750	-0.153007	-0.614121	-0.563038	1.353333
7	0.265750	-0.153007	0.284510	1.575092	-0.614121	-0.563038
8	-0.153007	0.284510	1.295828	1.452625	1.575092	-0.614121
9	0.284510	1.295828	0.755126	-0.540861	1.452625	1.575092

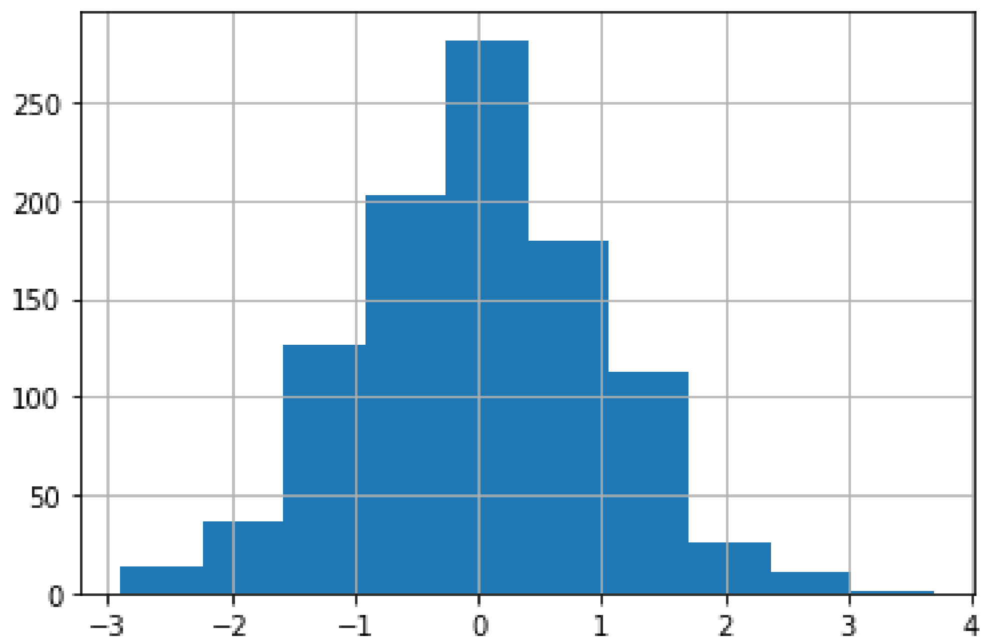
[25]:
```
q=2
Y = np.array(df['y(t)'])
X = np.array(df[['eps+1','eps+2']])
N = len(Y)
```

```
reg = LinearRegression(fit_intercept = False).fit(X, Y)
df['prediction'] = reg.predict(X)

alpha_0 = np.sqrt(((Y-df['prediction'] )**2).mean())
df['empirical_noise'] = (Y - prediction)/alpha_0

print("reg.coef_ = {:}\t alpha_0 = {:}".format(reg.coef_, alpha_0) )
```

```
reg.coef_ = [0.41227231 0.31086954]      alpha_0 = 0.5112762579620874
```

[26]:
```
df['empirical_noise'].hist()
```

[26]: `<matplotlib.axes._subplots.AxesSubplot at 0x7fc6e74af890>`



[27]:
```
df['empirical_noise'].std()
```

[27]: 1.0004284202360652

[28]:
```
from scipy.stats import norm, kstest

D_N, p_value = kstest(df['empirical_noise'], 'norm', args=(0,1))

print("data: test statistic: D_N = {:},  p-value = {:}".format(D_N, p_value) )
```

```
data: test statistic: D_N = 0.025872930770524993,  p-value = 0.520767196660961
```

5 e)

Using the fit of point (a) and/or (c) simulate $N = 100$ future(forecast) paths for $t = T+1, T+2, ..., T+10$. Using the generated forecast paths estimate the mean value $< y(t) >$ and the standard deviation $\sigma(y(t))$ of $y(t)$ for each future $t = T+1, T+2, T+10$ and plot them as the continuation of the sample data series (plot of mean with error bars of standard deviation).

We may predict only 2 next steps for MA(2) process

[31]:
```
def data_generator(alpha1, alpha2, alpha_0, past_data, T=2):

    for i in range(len(past_data), T + len(past_data)):
        noise = np.random.normal(0,1)
        new = alpha_0*noise + alpha1*past_data[i-1] + alpha2*past_data[i-2]
        past_data= np.append(past_data,new)

    return past_data
```

[32]:
```
alpha1, alpha2 = [ 0.41126442, 0.31314482]
alpha_0 = 0.5113506601881491

past_data = data[-2:]
n = 100
data_gen = []

for i in range(n):
    data_gen.append(data_generator(alpha1, alpha2, alpha_0, past_data))
```

[34]:
```
cols=[ 't-1', 't', 't+1', 't+2']

df_gen = pd.DataFrame(data_gen, columns = cols)
df_gen
```

[34]:
```
         t-1         t       t+1       t+2
0  -0.401085 -0.831914  0.043877 -0.374000
1  -0.401085 -0.831914 -1.344821 -1.238094
2  -0.401085 -0.831914 -0.257514 -0.943468
3  -0.401085 -0.831914 -0.456999 -1.223775
4  -0.401085 -0.831914 -0.704910 -0.630838
..       …         …         …         …
95 -0.401085 -0.831914  0.018417  0.314068
96 -0.401085 -0.831914  0.190134  0.092740
97 -0.401085 -0.831914 -0.861085 -0.987868
98 -0.401085 -0.831914 -0.380093 -0.194630
99 -0.401085 -0.831914  0.676589  0.068266

[100 rows x 4 columns]
```

[36]:
```
fig, ax = plt.subplots(figsize = (12,10))

ax.set_ylabel('y_t')
ax.set_title(' N=100 future paths for t={T+1,T+2}', fontsize = 15)
ax.grid(color = 'gray', alpha = 0.5)

for i in range(n):
    ax.plot(df_gen.loc[i])
```