exercise1

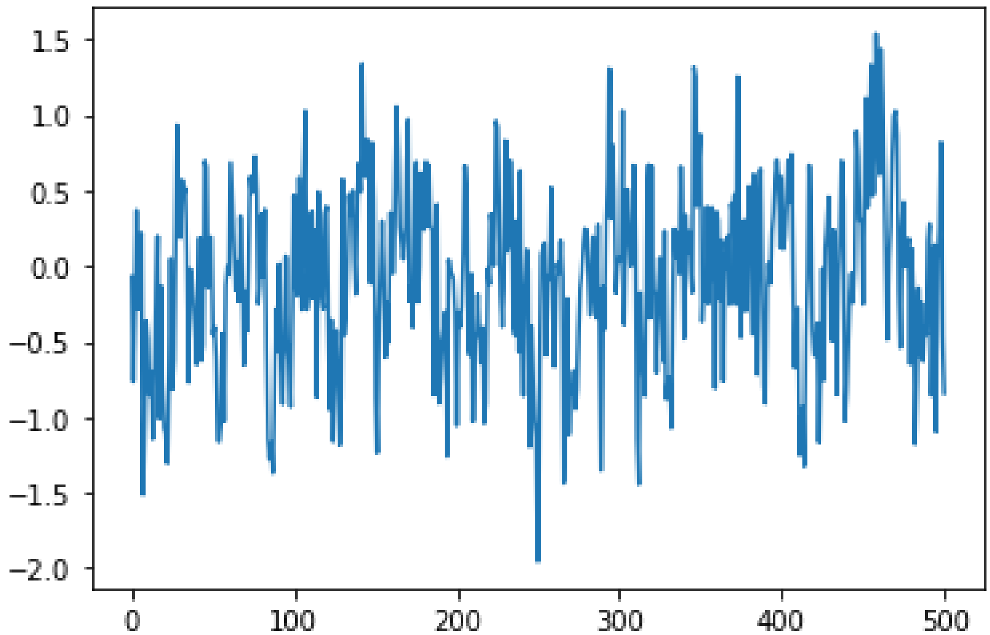
June 10, 2020

[1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

from copy import deepcopy
from sklearn.linear_model import LinearRegression

%matplotlib inline
```

[2]:
```
data = []

with open("ARdata.txt", "r") as f:
    for line in f:
        data.append(float(line))

data = np.array(data[:])
```

[3]:
```
plt.plot(data)
```

[3]:
```
[<matplotlib.lines.Line2D at 0x7fc08528d9d0>]
```



1 a

Find the orderpand fit the parameters: $\beta_i(i = 1, ..., p)$, α_0 using the linear regression method. In order to find p and the best fit use the "AIC" and "BIC" criteria.

[4]:
```
def dividing_data_into_subseries_II(data, k=2):
    X = []
    Y = []
    for i in range(len(data)-k):
        tmp = []
        for j in range(k):
            tmp.append(data[j+i])
        X.append(tmp)
        Y.append(data[k+i] )

    return np.array(X), np.array(Y)
```

1.0.1 Akaike information criterion (AIC)

$$\mathrm{AIC} = N \cdot \ln\left(\frac{\sum_{i=1}^{N} \epsilon^2}{N}\right) + 2k$$

where: * $N = T-k$ - lenght of data series * k - number of parameters / β_i ceofficients * $\epsilon_i = (y_t - \hat{y}_t)^2$

1.0.2 Bayesian information criterion (BIC)

$$\text{BIC} = N \cdot \ln\left(\frac{\sum_{i=1}^{N}\epsilon^2}{N}\right) + \ln(N) \cdot k$$

where: * $N = T - k$ - lenght of data series * k - number of parameters / β_i ceofficients * $\epsilon_i = (y_t - \hat{y}_t)^2$

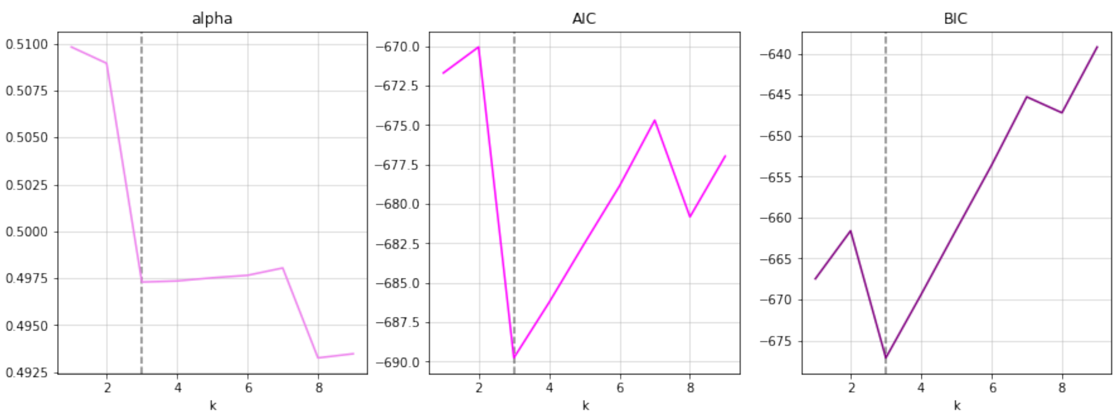
[5]:
```
alpha_tab = []
AIC_tab = []
BIC_tab = []

for k in range(1,10):
    X,Y = dividing_data_into_subseries_II(data, k=k)
    reg = LinearRegression(fit_intercept = False).fit(X, Y)
    prediction = reg.predict(X)
    N = len(X)

    alpha = np.sqrt(1/len(X)*sum((prediction - Y)**2))
    alpha_tab.append(alpha)

    AIC = N* np.log( sum( (Y-prediction)**2)/N) + 2*k
    AIC_tab.append(AIC)

    BIC = N*np.log(((Y-prediction)**2).mean()) + np.log(N)*k
    BIC_tab.append(BIC)

    print("k = {:}, alpha = {:}\t AIC = {:}\t BIC = {:}\n".format(k, alpha,␣
↪AIC, BIC))
```

```
k = 1, alpha = 0.5098213793872195        AIC = -671.6948511065393        BIC =
-667.4802430081169

k = 2, alpha = 0.5089534806801068        AIC = -670.0478628437139        BIC =
-661.6226506522112

k = 3, alpha = 0.49730099043593373       AIC = -689.7655823502289        BIC =
-677.133782119155

k = 4, alpha = 0.497358470939916         AIC = -686.2535779182602        BIC =
-669.4192178138736

k = 5, alpha = 0.4975269543361798        AIC = -682.5206999266912        BIC =
-661.4878202930672

k = 6, alpha = 0.4976581485857435        AIC = -678.8634674312794        BIC =
-653.6361208558675

k = 7, alpha = 0.4980495877486443        AIC = -674.6909655247008        BIC =
```

```
-645.273216904385

k = 8, alpha = 0.4932606726330802        AIC = -680.8234519856328        BIC =
-647.2193785932913

k = 9, alpha = 0.49347956141858257       AIC = -676.9734551469284        BIC =
-639.1871466984973
```

[6]:
```python
k_tab = list(range(1,10))

fig, axs = plt.subplots(nrows=1, ncols=3, figsize = (15,5))

for ax in axs:
    ax.set_xlabel('k')
    ax.grid(True, alpha = 0.5)
    ax.axvline(x=3, color = 'black', alpha = 0.5, ls = '--')

axs[0].set_title('alpha')
axs[1].set_title('AIC')
axs[2].set_title('BIC')

axs[0].plot(k_tab, alpha_tab, color = 'violet')
axs[1].plot(k_tab, AIC_tab, color = 'fuchsia')
axs[2].plot(k_tab, BIC_tab, color = 'purple')
```

[6]:
```
[<matplotlib.lines.Line2D at 0x7fc084953250>]
```



1.0.3 Best prediction: k=3

[7]:
```
AIC_index= AIC_tab.index(min(AIC_tab))
BIC_index= BIC_tab.index(min(BIC_tab))

AIC_best = AIC_tab[AIC_index]
BIC_best = BIC_tab[BIC_index]

print("best AIC: k = {:}\t AIC = {:}".format(AIC_index+1, AIC_best))
print("best BIC: k = {:}\t BIC = {:}".format(BIC_index+1, BIC_best))
```

```
best AIC: k = 3  AIC = -689.7655823502289
best BIC: k = 3  BIC = -677.133782119155
```

[8]:
```
k=3

X,Y = dividing_data_into_subseries_II(data, k=k)
reg = LinearRegression(fit_intercept=False).fit(X, Y)
prediction = reg.predict(X)
N = len(X)

alpha = np.sqrt(((prediction - Y)**2).mean())
AIC = N*np.log(((Y-prediction)**2).mean()) + 2*k
BIC = N*np.log(((Y-prediction)**2).mean()) + np.log(N)*k
print("k = {:}, alpha = {:}\t AIC = {:}\t BIC = {:}\n".format(k, alpha, AIC,␣
↪BIC))
print("model parameters: intercept: {:} coefficients: {:}".format(reg.
↪intercept_, reg.coef_))
```

```
k = 3, alpha = 0.49730099043593373       AIC = -689.7655823502289        BIC =
-677.133782119155

model parameters: intercept: 0.0 coefficients: [ 0.21790417 -0.0712842
0.48597427]
```

2 b)

Compute the sample autocorrelation function (SACF, $\rho(t)$ and the sample partial autocorrelation function (SPACF,$\phi(t)$) and plot them for $t = 1, ...10$. Based on the plot of SPACF check the orderp (on the plot include Gaussian N(0,(T−t)−1) bands for 95% confidence level to check when SPACF becomes statistically zero).

[9]:
```
def dividing_data_into_subseries(data, k=2):
    data_k = []
    for i in range(len(data)-k):
        tmp = [ ]
```

```
        for j in range(k):
            tmp.append(data[j+i])
        tmp.append(data[k+i])
        data_k.append(tmp)

    return np.array(data_k).T
```

[10]:
```
def get_autocorrelation_coeff(data, delta):

    data_k = dividing_data_into_subseries(data, k=delta)
    denominator = (data**2).mean()
    nominator = (data_k[0]*data_k[-1]).mean()

    return nominator/denominator
```

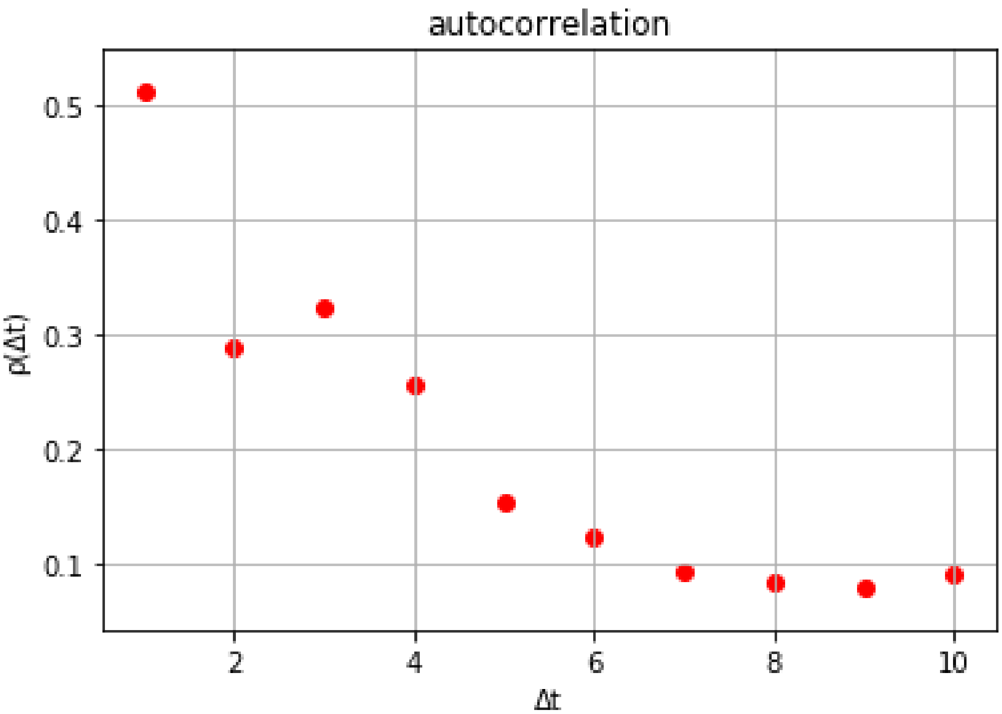
[11]:
```
def get_autocorrelation(p,data):

    autocorrelation_tab = []

    for delta in range(1,p+1):
        autocorrelation = get_autocorrelation_coeff(data, delta)
        autocorrelation_tab.append(autocorrelation)

    return autocorrelation_tab
```

2.0.1 sample autocorrelation function (SACF)

[12]:
```
p=10
p_tab = list(range(1,p+1))
autocorrelation_tab = get_autocorrelation(p,data)

for delta_t, corr in zip(p_tab, autocorrelation_tab):
    print(u'\u0394t = {:}\t \u03C1(\u0394t)={:2f}'.format(delta_t,corr ))

plt.title('autocorrelation')
plt.scatter(p_tab, autocorrelation_tab, color ='red')
plt.ylabel(u'\u03C1(\u0394t)')
plt.xlabel(u'\u0394t')
plt.grid()
```

```
Δt = 1    (Δt)=0.512124
Δt = 2    (Δt)=0.288868
Δt = 3    (Δt)=0.323168
Δt = 4    (Δt)=0.255728
Δt = 5    (Δt)=0.153287
```

```
Δt = 6    (Δt)=0.124052
Δt = 7    (Δt)=0.092280
Δt = 8    (Δt)=0.083226
Δt = 9    (Δt)=0.079486
Δt = 10   (Δt)=0.091703
```



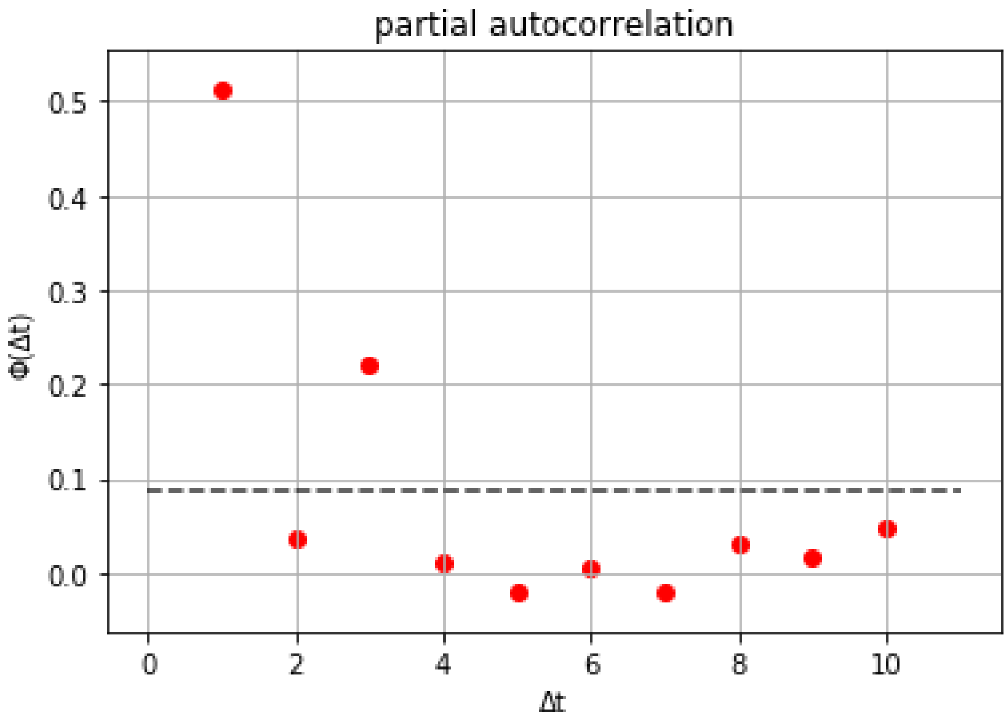
2.0.2 Partial autocorrelation function (PACF)

[13]:
```python
def get_partial_autocorrelation(p, data):

    autocorrelation_tab = get_autocorrelation(p,data)

    M = np.identity(p)*0.5
    for i in range(p):
        for j in range(i+1, p):
            M[i][j] = autocorrelation_tab[j-i-1]

    A = np.matrix(M.T + M)
    A_inv = np.linalg.inv(A)

    psi_vec = np.array(np.dot(A_inv,autocorrelation_tab))[0]
```

```
    return psi_vec[-1]
```

[14]:
```
p=10
p_tab = list(range(1,p+1))
partial_corr_tab_tab = []

for k in range(1,p+1):
    partial_corr_tab_tab.append(get_partial_autocorrelation(k,data))


for delta_t, p_corr in zip(p_tab, partial_corr_tab_tab):
    print(u'\u0394t = {:}\t \u03A6(\u0394t)={:2f}'.format(delta_t, p_corr ))

N = 490
confidence_level = 1.96/np.sqrt(N)
N= 500
confidence_level_II = 1.96/np.sqrt(N)


plt.title('partial autocorrelation')
plt.scatter(p_tab, partial_corr_tab_tab, color ='red')
plt.ylabel(u'\u03A6(\u0394t)')
plt.xlabel(u'\u0394t')
plt.hlines(y = confidence_level, xmin=0, xmax = p+1, linestyles='dashed', color␣
↪= 'black', alpha = 0.5)
plt.hlines(y = confidence_level_II, xmin=0, xmax = p+1, linestyles='dashed',␣
↪color = 'black', alpha = 0.5)

plt.grid()
```

```
Δt = 1   Φ(Δt)=0.512124
Δt = 2   Φ(Δt)=0.036053
Δt = 3   Φ(Δt)=0.220016
Δt = 4   Φ(Δt)=0.010044
Δt = 5   Φ(Δt)=-0.020596
Δt = 6   Φ(Δt)=0.004085
Δt = 7   Φ(Δt)=-0.019884
Δt = 8   Φ(Δt)=0.031417
Δt = 9   Φ(Δt)=0.017215
Δt = 10  Φ(Δt)=0.047557
```



Based on the above plot, with confidence level 95%, we may determinate k = 3. The corellation at Δt = 2 is quite small, but the result agrees with the results obtained in a).

3 c)

3.0.1 Autocorrelation matrix and Yule Walker Equations

For the order p established in (b) fit the parameters: $\beta_i(i = 1, ..., p)$ and α_0 using the Yule-Walkermethod (in the matrix form).

[15]:
```
k=3
autocorrelation_tab = get_autocorrelation(3,data)

M = np.identity(k)*0.5
for i in range(k):
    for j in range(i+1, k):
        M[i][j] = autocorrelation_tab[j-i-1]

A = np.matrix(M.T + M)
A_inv = np.linalg.inv(A)
beta_vec = np.array(np.dot(A_inv, autocorrelation_tab))[0]
beta_vec ## beta1, beta2, beta3
```

[15]: array([0.48572791, -0.07255982, 0.22001576])

$$\hat{\alpha_0}^2 = (1 - \rho^T \hat{A}^{-1}\rho) \cdot \frac{1}{T}\sum_{t=1}^{T} y_t^2$$

[16]:
```
alpha_0 = np.sqrt(float((1-np.dot( np.dot(A_inv, autocorrelation_tab),␣
↪autocorrelation_tab))*(data**2).mean()))
alpha_0
```

[16]: 0.4965234301264099

4 d)

Using the data and the fit of point (a) and/or (c) compute the empirical noise

$$\hat{\eta}(t) = \frac{y_t - \hat{y}_t}{\alpha_0}$$

and check if it has standard Gaussian N(0,1) distribution (use e.g.Kolmogorov-Smirnov test).

[17]:
```
## Based on results in a:
coeff_a = np.array([0.48597427, -0.0712842,  0.21790417])   ### beta1, beta2,␣
↪beta3  (yt = b1y_{t-1}+b2 y_{t-2})
alpha_0a = 0.4973009904359337

## Based on results in c:
coeff_c = np.array([ 0.48572791, -0.07255982,  0.22001576])
alpha_0c = 0.4965234301264099
```

[19]:
```
df= pd.DataFrame(dividing_data_into_subseries(data, k=3).transpose(),␣
↪columns=['t3','t2','t1','t0'])
df
```

[19]:

	t3	t2	t1	t0
0	-0.071753	-0.764857	-0.255960	0.369323
1	-0.764857	-0.255960	0.369323	-0.282561
2	-0.255960	0.369323	-0.282561	0.191984
3	0.369323	-0.282561	0.191984	0.218465
4	-0.282561	0.191984	0.218465	-1.514103
..	…	…	…	…
493	-0.403822	0.138601	-1.097626	0.037099
494	0.138601	-1.097626	0.037099	0.251938
495	-1.097626	0.037099	0.251938	0.819219
496	0.037099	0.251938	0.819219	-0.459295
497	0.251938	0.819219	-0.459295	-0.843288

```
[498 rows x 4 columns]
```

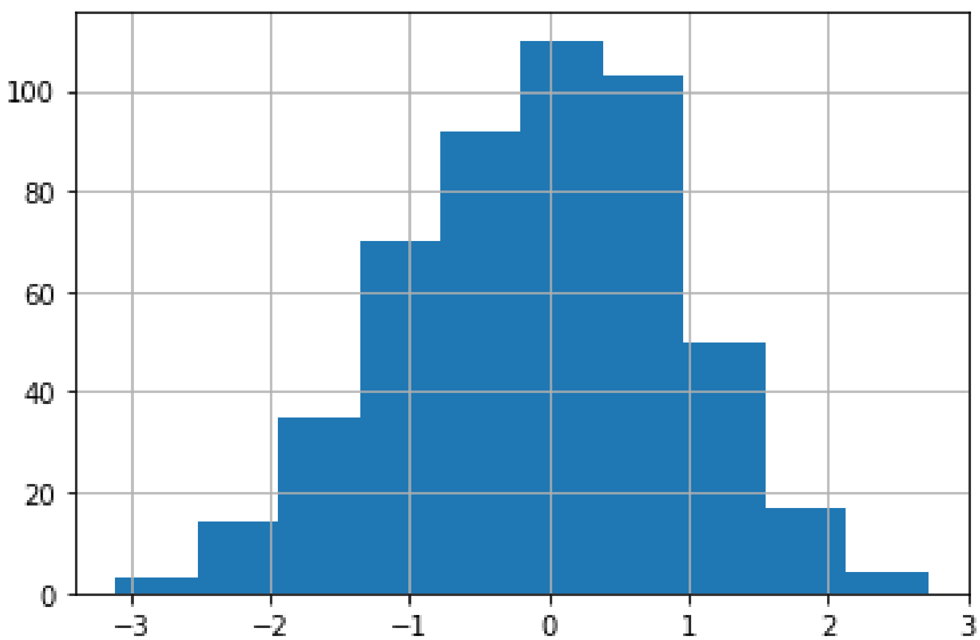
4.0.1 empirical noise

[20]:
```python
df['prediction_a'] = coeff_a[0]*df['t1'] + coeff_a[1]*df['t2'] +␣
↪coeff_a[2]*df['t3']
df['empirical_noise_a'] = (df['t0'] - df['prediction_a'])/alpha_0a
df['empirical_noise_a'].std()
```

[20]:
```
0.9991480116101611
```

[21]:
```python
df['empirical_noise_a'].hist()
```

[21]:
```
<matplotlib.axes._subplots.AxesSubplot at 0x7fc084841310>
```

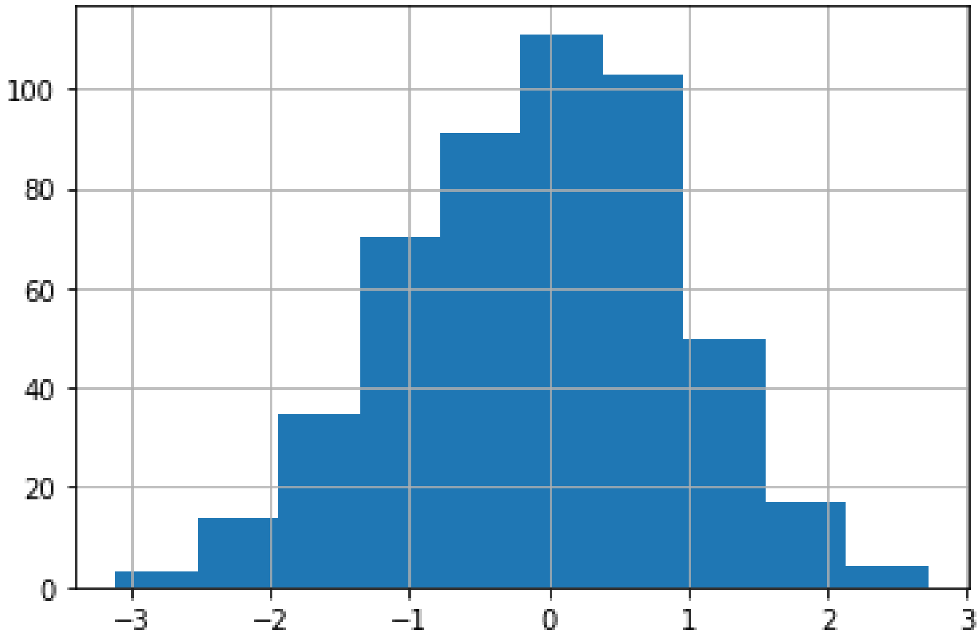


[22]:
```python
df['prediction_c'] = coeff_c[0]*df['t1'] + coeff_c[1]*df['t2'] +␣
↪coeff_c[2]*df['t3']
df['empirical_noise_c'] = (df['t0'] - df['prediction_c'])/alpha_0c
df['empirical_noise_c'].std()
```

[22]:
```
1.0007214617860256
```

[23]:
```python
df['empirical_noise_c'].hist()
```

[23]: <matplotlib.axes._subplots.AxesSubplot at 0x7fc0846f9050>



[24]:
```
from scipy.stats import norm, kstest

D_N, p_value = kstest(df['empirical_noise_a'], 'norm', args=(0,1))

print("data a: test statistic: D_N = {:},  p-value = {:}".format(D_N, p_value) )
```

```
data a: test statistic: D_N = 0.04233859571377366,  p-value =
0.32615307818129174
```

[25]:
```
from scipy.stats import norm, kstest

D_N, p_value = kstest(df['empirical_noise_c'], 'norm', args=(0,1))

print("data a: test statistic: D_N = {:},  p-value = {:}".format(D_N, p_value) )
```

```
data a: test statistic: D_N = 0.042597049599568976,  p-value =
0.3190418745285941
```

5 e)

Using the fit of point (a) and/or (c) simulate $N = 100$ future(forecast) paths for $t = T+1, T+2, ..., T+10$. Using the generated forecast paths estimate the mean value $< y(t) >$ and the standard deviation $\sigma(y(t))$ of $y(t)$ for each future $t = T+1, T+2, T+10$ and plot them as the continuation of the sample data series (plot of mean with error bars of standard deviation).

[26]:
```
def data_generator(beta1, beta2,beta3, alpha_0, past_data, T=10):

    for i in range(len(past_data), T + len(past_data)):
        noise = np.random.normal(0,1)
        new = alpha_0*noise + beta1*past_data[i-1] + beta2*past_data[i-2] +␣
 ↪beta3*past_data[i-3]
        past_data= np.append(past_data,new)

    return past_data
```

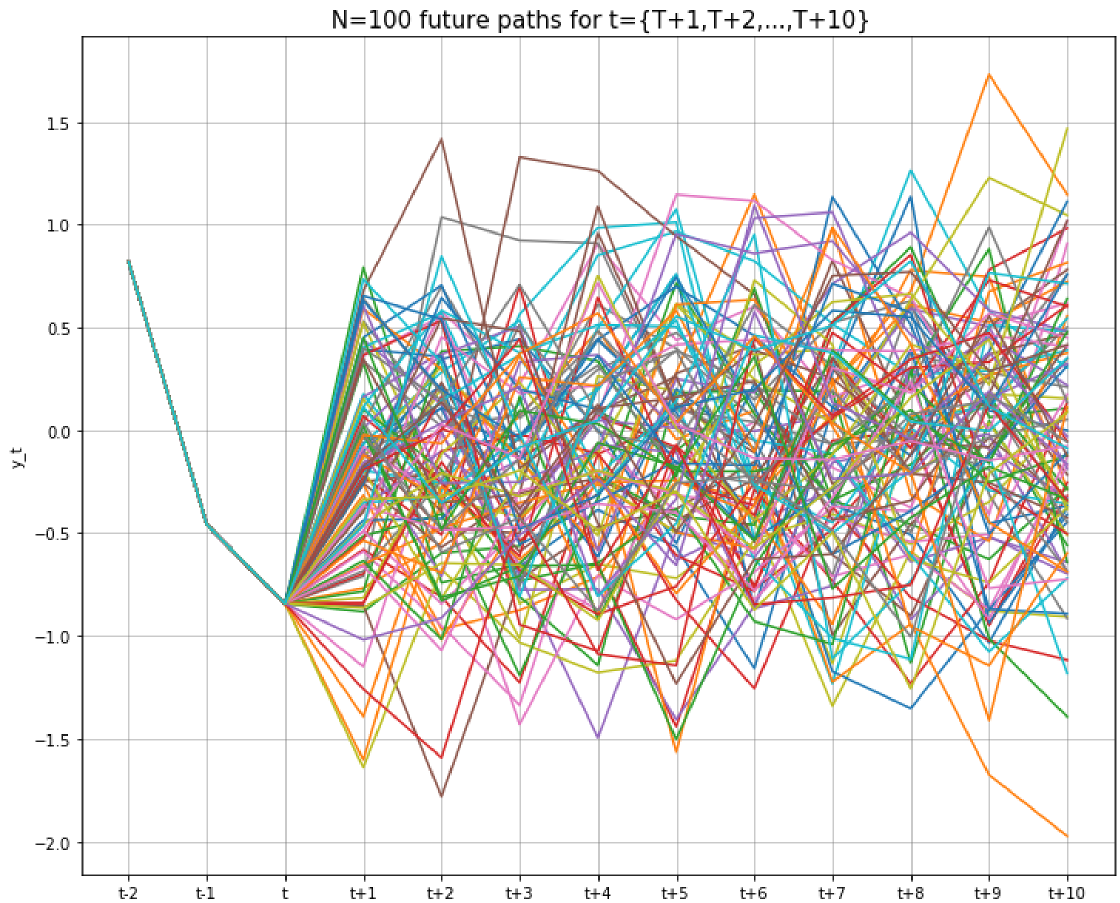
[27]:
```
beta1, beta2, beta3 = [0.48597427, -0.0712842,  0.21790417]
alpha_0 = 0.4973009904359337

past_data = data[-3:]
n = 100
data_gen = []

for i in range(n):
    data_gen.append(data_generator(beta1, beta2,beta3, alpha_0, past_data))
```

[28]:
```
cols=['t-2', 't-1', 't', 't+1', 't+2', 't+3', 't+4', 't+5', 't+6', 't+7',␣
 ↪'t+8', 't+9', 't+10']

df_gen = pd.DataFrame(data_gen, columns = cols)
df_gen
```

[28]:
```
         t-2       t-1         t       t+1       t+2       t+3       t+4  \
0   0.819219 -0.459295 -0.843288 -0.060625  0.105148 -0.662794 -0.387318
1   0.819219 -0.459295 -0.843288 -0.262801 -0.967477 -0.874832 -0.712690
2   0.819219 -0.459295 -0.843288 -0.882172 -0.596419 -0.563360  0.091957
3   0.819219 -0.459295 -0.843288 -0.579876 -0.810633 -1.226854 -0.115794
4   0.819219 -0.459295 -0.843288 -0.016041 -0.352885 -0.140407  0.100392
..       …         …         …         …         …         …         …
95  0.819219 -0.459295 -0.843288 -0.173847  0.138743 -0.385575  0.125745
96  0.819219 -0.459295 -0.843288 -0.489809  0.029353 -0.136810  0.715988
97  0.819219 -0.459295 -0.843288  0.041769 -0.593263 -0.424910 -0.878341
98  0.819219 -0.459295 -0.843288 -0.334659 -0.322477 -0.195775 -0.488055
99  0.819219 -0.459295 -0.843288  0.113722 -0.354304 -0.083739  0.048980

         t+5       t+6       t+7       t+8       t+9      t+10
0  -0.568420 -1.157877  0.109984  0.600537  0.095665 -0.001014
1  -0.249705 -0.337580  0.081310  0.779183  0.747044 -0.288359
2  -0.099533  0.193557  0.190802 -1.130053  0.047709 -0.560807
3   0.047137 -0.774542 -0.480248 -0.018579 -0.947523 -0.317453
```

```
4  -0.632398 -0.443456 -0.293422 -0.920294 -0.515898 -0.697755
..       …        …        …        …        …        …
95 -0.496466  0.022482  0.751976  0.771136  0.286693  0.416364
96  0.031524 -0.136141 -0.142362 -0.051534 -0.146534 -0.076450
97 -0.256225 -0.220284 -0.488863 -1.004768  0.174981 -0.055846
98 -0.296405 -0.873059 -0.496270 -1.256553 -0.211743  0.087114
99  0.554645  0.389061  0.510489  0.822222  0.115431 -0.088185

[100 rows x 13 columns]
```

[29]:
```python
fig, ax = plt.subplots(figsize = (12,10))

ax.set_ylabel('y_t')
ax.set_title(' N=100 future paths for t={T+1,T+2,...,T+10}', fontsize = 15)
ax.grid(color = 'gray', alpha = 0.5)

for i in range(n):
    ax.plot(df_gen.loc[i])
```



[30]:
```python
df_gen.mean()
```

[30]:
```
t-2      0.819219
t-1     -0.459295
t       -0.843288
t+1     -0.195539
t+2     -0.148382
t+3     -0.233988
t+4     -0.102112
t+5     -0.069278
t+6     -0.094131
t+7     -0.072410
t+8     -0.031420
t+9     -0.048089
t+10     0.001615
dtype: float64
```

[31]:
```python
df_gen.std()
```

[31]:
```
t-2      3.347448e-16
t-1      5.579081e-16
t        1.115816e-16
t+1      5.134343e-01
t+2      5.445324e-01
t+3      5.126143e-01
t+4      5.630168e-01
t+5      5.824539e-01
t+6      5.441672e-01
t+7      5.712315e-01
t+8      5.707339e-01
t+9      5.977442e-01
t+10     6.103980e-01
dtype: float64
```

[ ]:
```python

```