Lecture 6

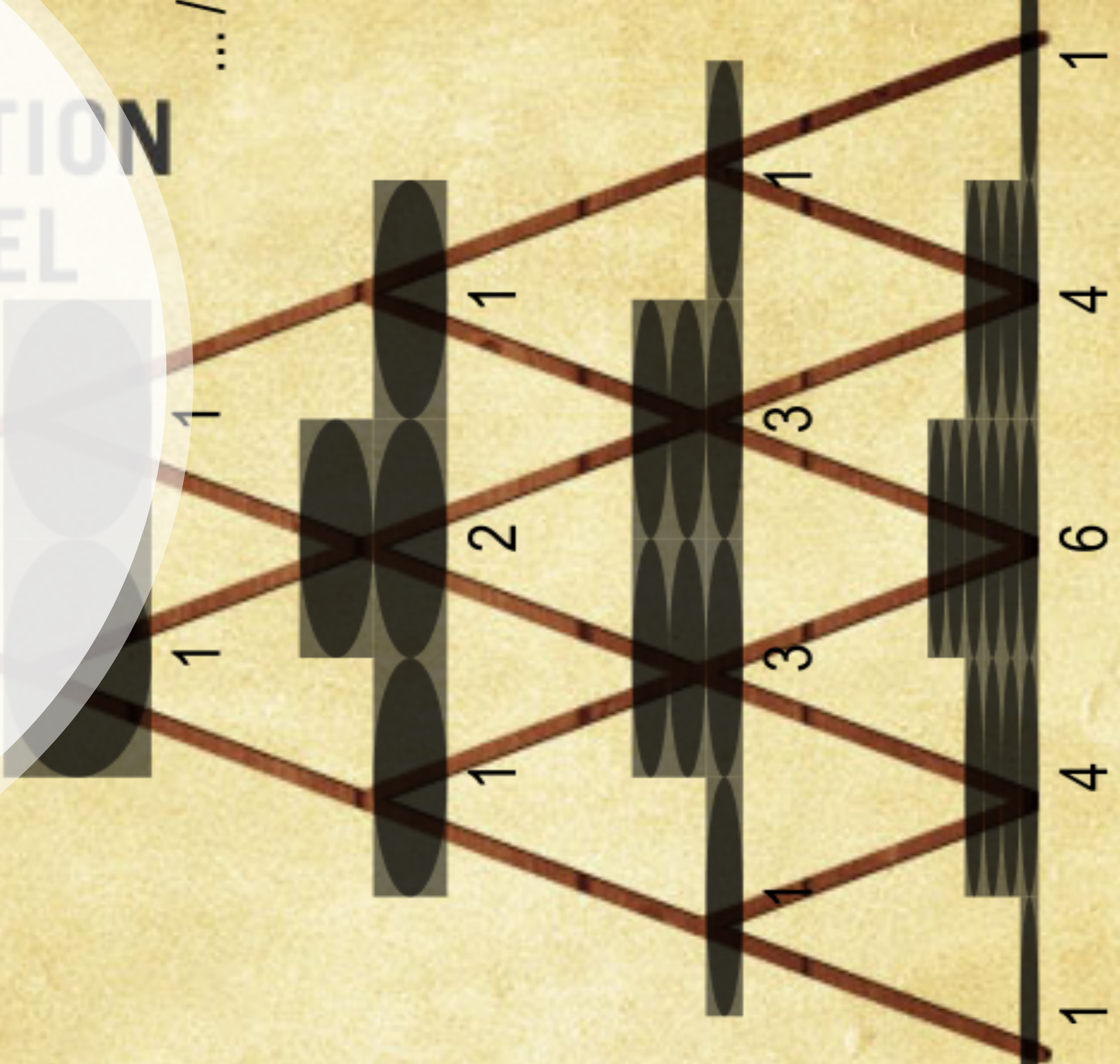
Option Pricing in Binomial Model

Financial instruments and pricing

Fall 2019

Option pricing in Binomial Model

- Stochastic processes
- One-step Binomial Model
- Multi-step Binomial Model
- "Analytic" formula for European options
- Risk-neutral / martingale pricing



Stochastic processes (short introduction)

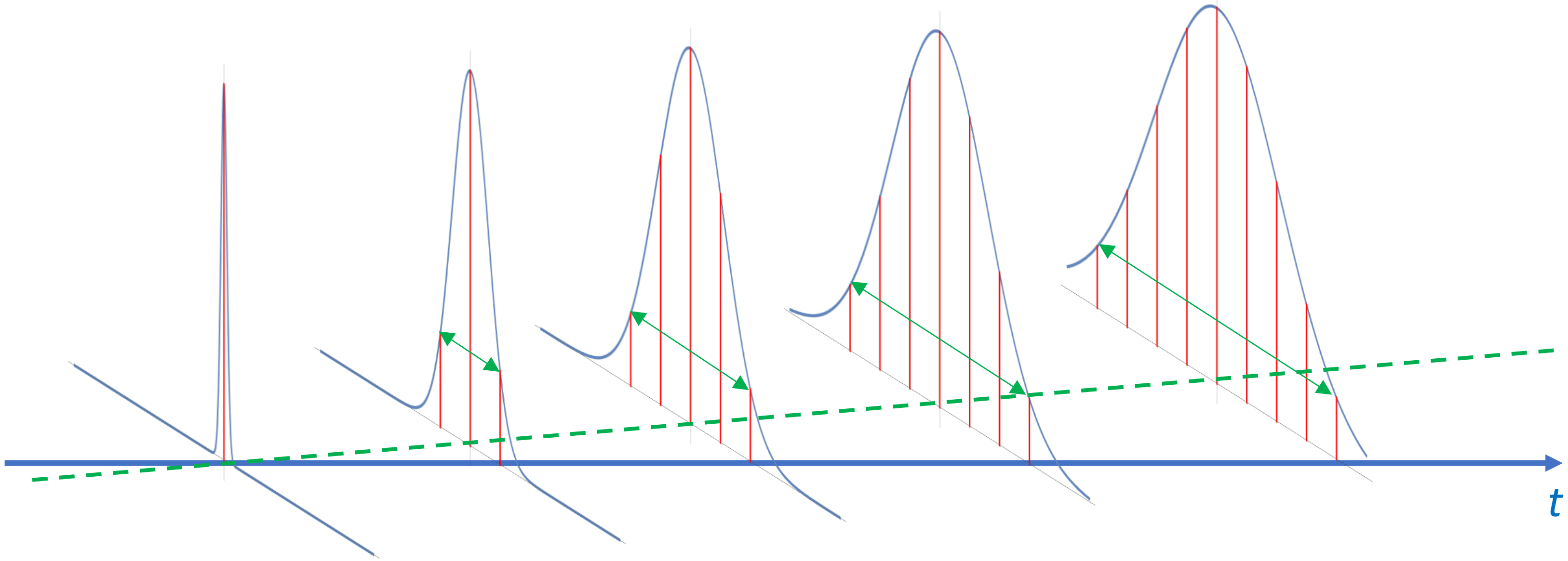
- As already discussed in Lecture 5, in order to price RISK ASSYMETRIC derivatives (OPTIONS) one has to assume some STOCHASTIC PROCESS driving the underlying asset (e.g. share) PRICE EVOLUTION
- If the MARKET IS DYNAMICALLY COMPLETE, i.e. if one can replicate the derivative instrument using some DYNAMICALLY ADJUSTED PORTFOLIO, then (under some additional conditions of risk-neutral / martingale price evolution) one can use ”standard” pricing methods of eqn. (4)

$$\sum_{t} < PV\big(CF(t)\big) > = 0 \qquad (4)$$

- Let’s start by explaining what actually STOCHASTIC PROCESSES are ?

Stochastic processes (short introduction)

❖A stochastic process is a set (family) of random variables indexed by time: $\{S(t)\}_t$

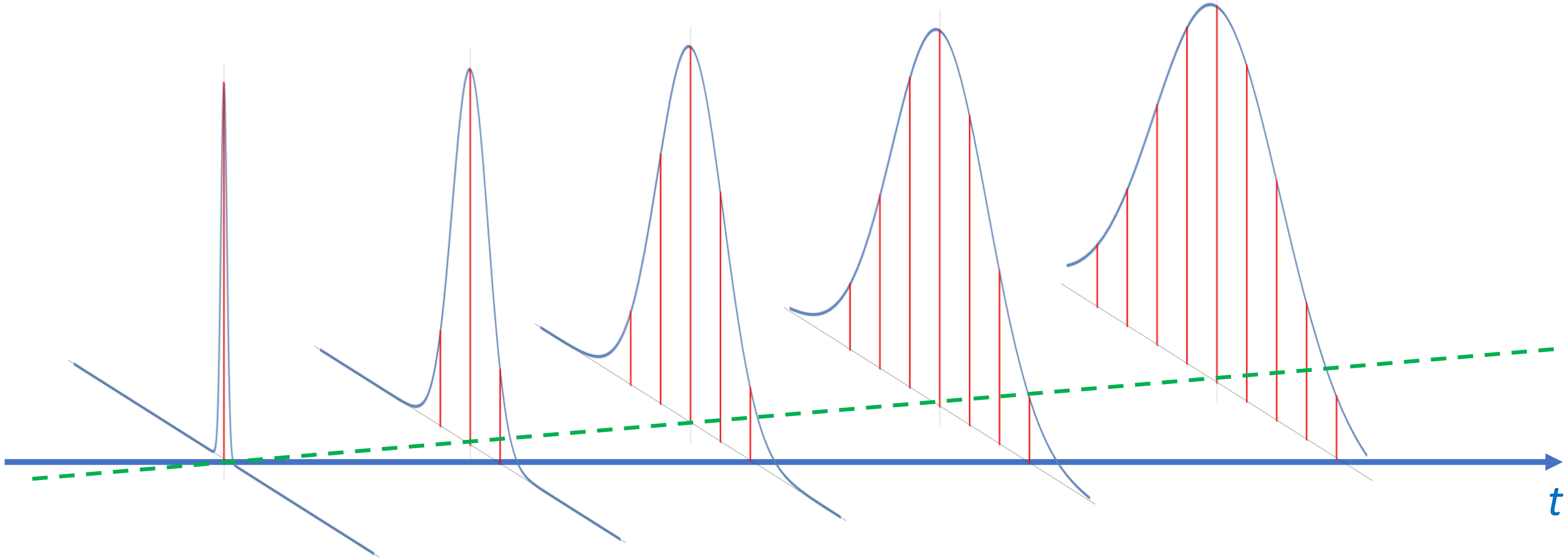


❖Usually the random variables $S(t)$ observed in various moments in time are assumed to have the SAME TYPE of PROBABILITY DISTRIBUTION (FUNCTION), but PARAMETERS (e.g. mean, variance, …) may CHANGE / EVOLVE IN TIME*: $S(t) \sim \mathrm{PDF}(\mu(t), \ldots)$

* If the random variables are identicaly distributed ∀t, then the process is called "stationary". Parameters change for „non-stationary" processes.

Stochastic processes (short introduction)

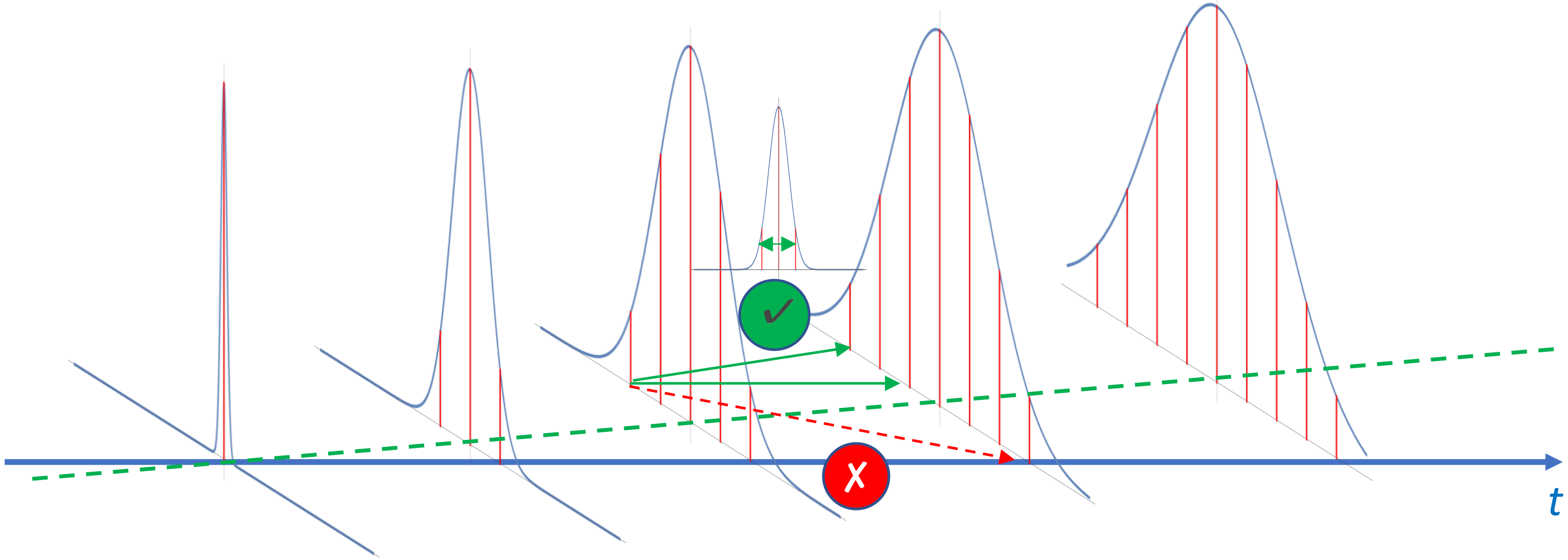
- A stochastic process is a set (family) of random variables indexed by time: $\{S(t)\}_t$



- Both the random variables $S(t)$ and time index t can be either discrete or continuous, so one can have:
  - discrete time - discrete „space” (e.g. Binomial) process
  - continuous time - continuous „space” (e.g. Wiener ⇐ will be discussed in Lecture 7) process,
  - continuous time - discrete „space” (e.g. Poisson) process
  - discrete time - continuous „space” (e.g. ARMA, GARCH) process

Stochastic processes (short introduction)

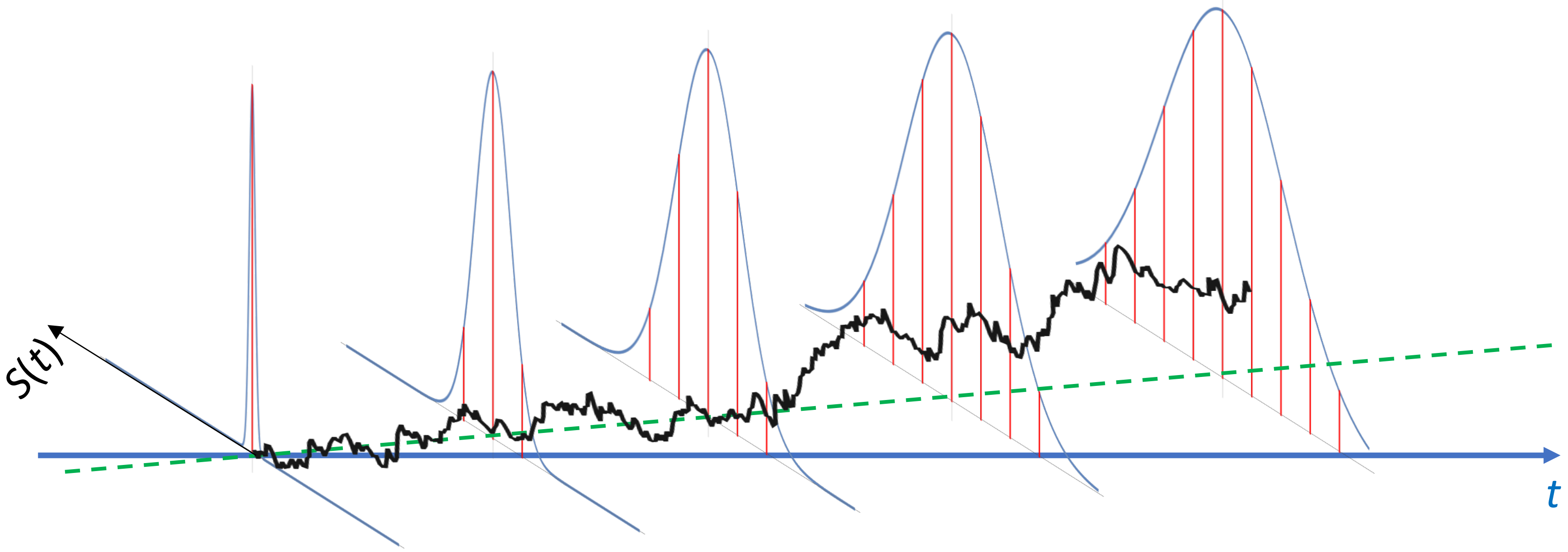
- A stochastic process is a set (family) of random variables indexed by time: $\{S(t)\}_t$



- The price change (e.g. $S(t') - S(t)$ *or* $S(t') / S(t)$) from time t to t' is also a random variable (usually with the same PDF)
  - ☐ E.g. in the Binomial process in each time step the price $S(t)_i$ can change only to $S(t+1)_{i\pm1}$
  - ☐ Therefore: the random variables in times t and $t' > t$ are usually (but not always) correlated, the observed random variable REALIZATION $S(t')_{i'}$ in time t' depends on the realization $S(t)_i$ in time t

Stochastic processes (short introduction)

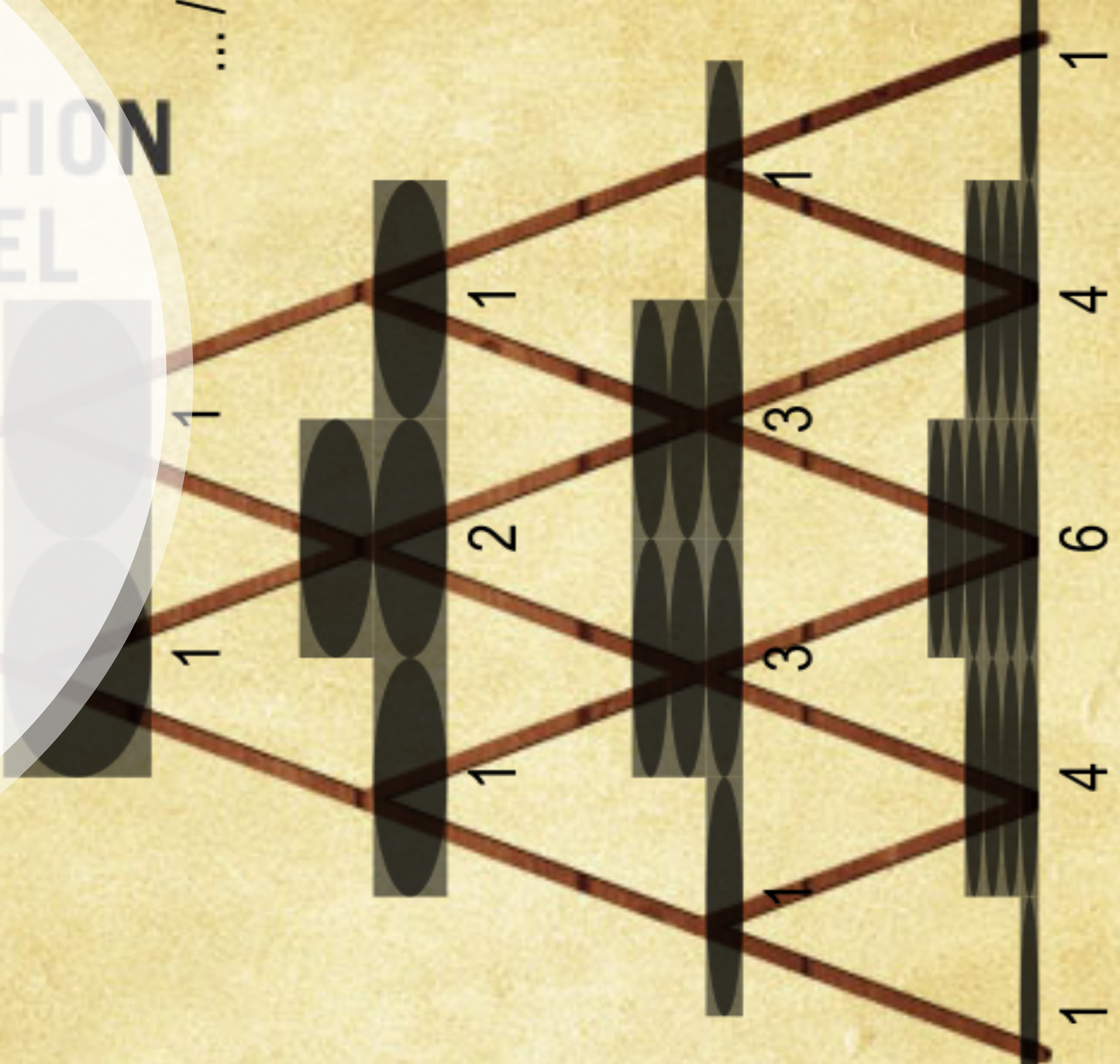
- A stochastic process is a set (family) of random variables indexed by time: $\{S(t)\}_t$



- Sometimes some additional regularity / continuity / smoothness conditions apply
  - E.g. in the Wiener process (Brownian motion) $\{S(t)\}_t$ should have continuous trajectories, i.e. observed realizations $S(t)$ should be continuous functions of time t (However, interestingly: they are nowhere differentiable !)

Option pricing in Binomial Model

- Stochastic processes
- One-step Binomial Model
- Multi-step Binomial Model
- “Analytic” formula for European options
- Risk-neutral / martingale pricing



Binomial model: options notation

❖To remind the options notation:

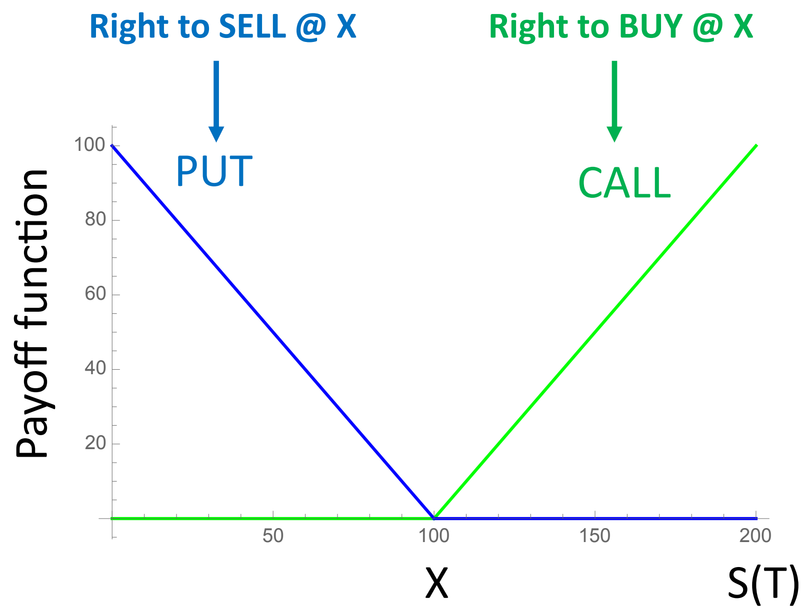
- S(t) – the „underlying asset” (e.g. Share) price in time t , S(0) = S
- X - the eXercise price*
- T - expiration date
- c(t) - European “call” option value/price at time t , c(0) = c is the premium paid in t=0
- p(t) - European “put” option value/price, p(0) = p
- C(t) - American “Call” option value/price, C(0) = C
- P(t) - American “Put” option value/price P(0) = P

Exercised only at EXPIRATION

CAN be EXERCISED EARLY

❖The value of the option at expiration T is:

- $c(T) = C(T) = \max(S(T) – X ; 0)$
- $p(T) = P(T) = \max(X – S(T) ; 0)$



❖These are also “payoff functions” (always paid by option SELLER to option BUYER**) of options settled in cash (i.e. with no physical delivery)

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* In the literature the eXercise price is sometimes denoted: K (striKe price)

**Assymetric risk ! But in exchange of option price (premium) which is always paid by the option BUYER to the option SELLER !

Binomial model (one step)

- Let us start with a simple* Binomial model introduced in late 70's by J. Cox, S. Ross and M. Rubinstein (sometimes called the CRR model)
- Let us assume that the European call option expires in N=1 time steps (time is counted in discrete steps)
- The current share price is $S(0) = S$ with probability one (this is the observed current market price in t=0)
- In next step, i.e. on the option expiry, the share price can either change to $S(1)_u = S \times u$ with probability q or to $S(1)_d = S \times d$ with probability (1-q) (one assumes: u > d)
- By adjusting parameters u, d and q one can manipulate the future (@ option's expiry):
  - Expected share price: $E(S(1)) \equiv \langle S(1) \rangle = q\, S(1)_u + (1-q)\, S(1)_d = S(qu + (1-q)d)$
  - and Standard Deviation: $\sigma(S(1)) \equiv (\langle S(1)^2 \rangle - \langle S(1) \rangle^2)^{1/2} = S\, q^{1/2}\, (1-q)^{1/2}\, (u-d)$

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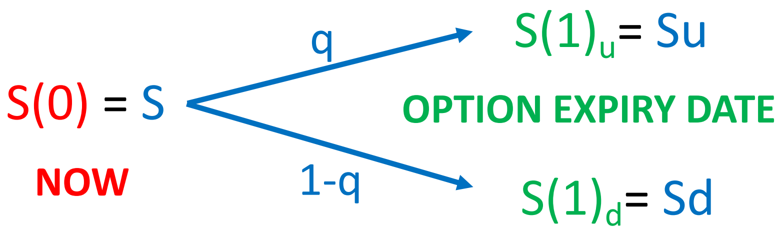
* Although the Binomial model is very simple it has all properties of other (more complicated) pricing models, and, as will be shown in Lecture 7, it tends to the, so called, Black-Scholes model if the number of discrete time steps: N → ∞.

Binomial model (one step)

- Binomial model parameters: S, u, d (and q) are assumed to be known



$S(0) = S$

NOW

q

1-q

$S(1)_u = Su$

OPTION EXPIRY DATE

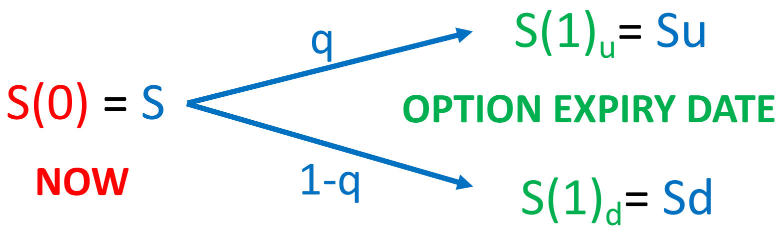
$S(1)_d = Sd$

- One additionally assumes the following:
  - One can buy / sell any non-integer fraction of shares (not a problem for “big” portfolios)
  - Short selling is possible (one can have negative number of shares)
  - One can borrow/deposit any cash amount with (the same) effective “RISK-FREE”* interest rate R%
    NOTE: here interest rate R% is the yield per discrete time period, one does not have to adjust it by any DayCountFactor, i.e. if one borrows/deposits B, the paid/received interest is simply: $INT(1)=B\times R$
  - No transaction costs, margins, taxes, …

*Here „RISK-FREE” refers to market (price) risk. Shares are „risky” as their prices are random variables with a priori unknown (realized) returns. Deposits / loans are „risk-free” as they have a priori fixed/known returns (yields) R%

Binomial model (one step)

- Binomial model parameters: S, u, d (and q) are assumed to be known



$S(0) = S$

NOW

q

1-q

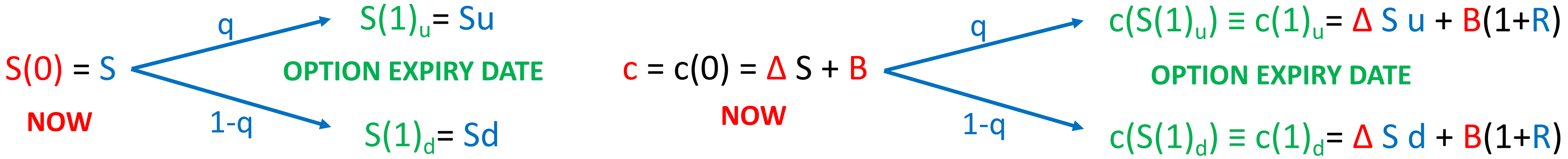
$S(1)_u = Su$

OPTION EXPIRY DATE

$S(1)_d = Sd$

- Assume that the European CALL option can be REPLICATED by:
  - BUYING (in t=0) Δ shares (Δ is a real number from [-1 ; 1], when Δ < 0 one sells shares short)
  - And making a deposit of B (when B < 0 one takes a loan)

Binomial model (one step)



❖Assume that the European CALL option can be REPLICATED by:

- ○ BUYING (in t=0) Δ shares (Δ is a real number from [-1 ; 1], when Δ < 0 one sells shares (short)
- ○ And making a deposit of B (when B < 0 one takes a loan)

❑ Therefore NOW (in t=0) one has:

$$c \equiv c(0) = \Delta \times S + B$$

❑ And on the option EXPIRY (in t=1 time steps):

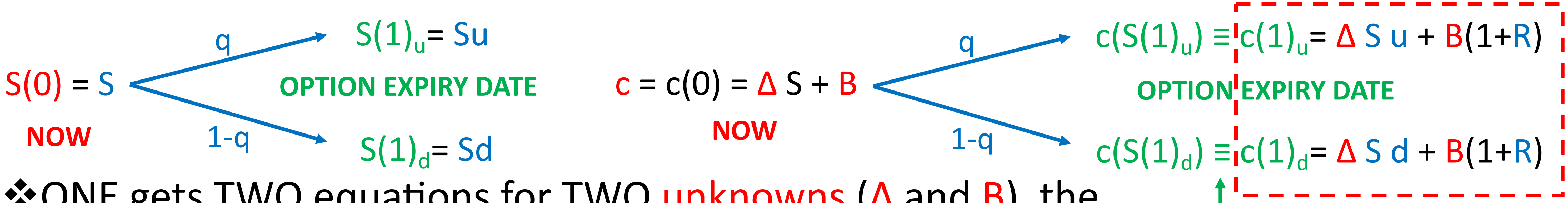
$c(1)_u = \Delta\ S\ u + B(1+R)$ with probability q or $c(1)_d = \Delta\ S\ d + B(1+R)$ with probability (1-q)

❑ But on the EXPIRY date, the (random) option price is simply given by its PAYOFF function:

$$c(1) \equiv c(S(1)) = \max(S(1) - X\ ;\ 0)$$

❑ Thus both: $c(1)_u = c(S(1)_u) = \max(S\ u - X\ ;\ 0)$ and $c(1)_d = c(S(1)_d) = \max(S\ d - X\ ;\ 0)$ are known (can be computed from the model parameters: S, u & d) !!!

Binomial model (one step)



- ONE gets TWO equations for TWO unknowns (Δ and B), the solution is given by the model parameters:

$$\begin{cases} c(1)_u = \Delta S u + B(1+R) \\ c(1)_d = \Delta S d + B(1+R) \end{cases} \Rightarrow \begin{cases} \Delta = \dfrac{c(1)_u - c(1)_d}{S(u-d)} \\ B = -\dfrac{d\,c(1)_u - u\,c(1)_d}{(u-d)(1+R)} \end{cases}$$

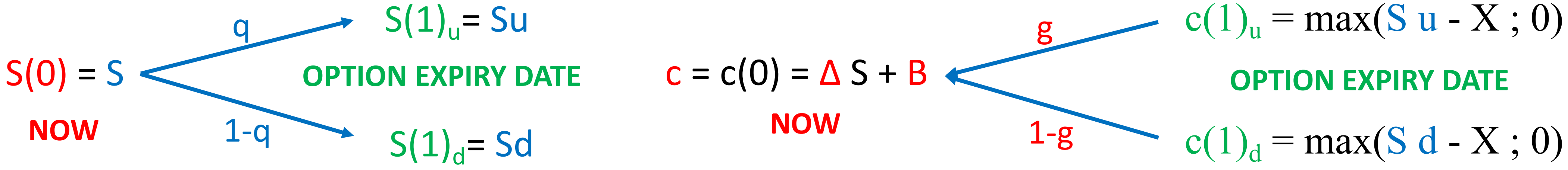
$$\begin{cases} c(1)_u = \max(S\,u - X\,;0) \\ c(1)_d = \max(S\,d - X\,;0) \end{cases}$$

- Knowing Δ and B one computes (current) option "fair" price (based on arbitrage-free arguments: the option can be replicated so its price must equal the portfolio price):

$$c \equiv c(0) = \Delta \times S + B$$

$$c = \frac{g\,c(1)_u + (1-g)c(1)_d}{(1+R)}, \quad \text{where: } g = \frac{(1+R)-d}{u-d}$$

Binomial model (one step)



$S(0) = S$

NOW

q

$S(1)_u = Su$

$1-q$

$S(1)_d = Sd$

OPTION EXPIRY DATE

$c = c(0) = \Delta S + B$

NOW

g

$c(1)_u = \max(S\,u - X\,;\,0)$

$1-g$

$c(1)_d = \max(S\,d - X\,;\,0)$

OPTION EXPIRY DATE

$$c = \frac{g\, c(1)_u + (1-g)c(1)_d}{(1+R)} \quad , \quad \text{where: } g = \frac{(1+R)-d}{u-d}$$

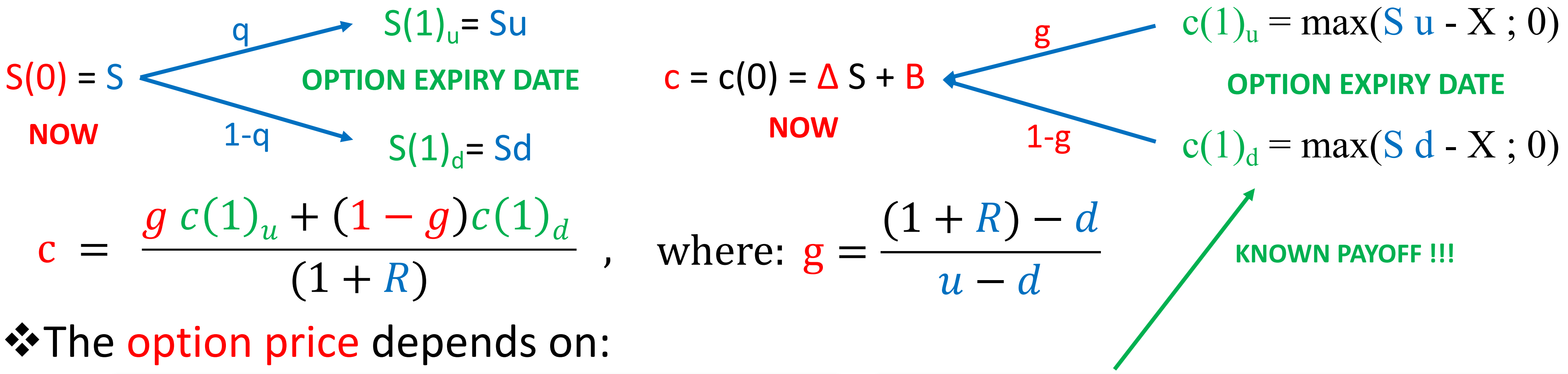
KNOWN PAYOFF !!!

- The option price depends on:
  - the current share price: S (through the PAYOFF FUNCTION)
  - the eXercise price: X (through the PAYOFF FUNCTION)
  - NOTE: using other PAYOFF FUNCTIONS one can also price PUTs and other options
  - the “RISK FREE” interest rate R%
  - Binominal model parameters: u & d (NOTE: $(u\text{-}d) \propto \sigma(S(1)) = S\, q^{1/2}(1\text{-}q)^{1/2}\,(u\text{-}d) \Leftarrow$”Volatility”
  - option type, i.e. for American options one simply sets: $C = \max(S - X\,;\,c\,)$

CF FROM (immediate) EARLY EXERCISE

- BUT it DOES NOT DEPEND ON probability q, i.e. DOES NOT DEPEND ON THE EXPECTED FUTURE SHARE PRICE: $E(S(1)) = S(q\,u + (1\text{-}q)\,d)$ (the expected return on shares) !!!

Binomial model (one step)



$S(0) = S$

NOW

q

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1-q

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OPTION EXPIRY DATE

$c = c(0) = \Delta S + B$

NOW

g

$c(1)_u = \max(S\,u - X\,;\,0)$

1-g

$c(1)_d = \max(S\,d - X\,;\,0)$

OPTION EXPIRY DATE

KNOWN PAYOFF !!!

$$c = \frac{g\,c(1)_u + (1-g)c(1)_d}{(1+R)}\ , \quad \text{where: } g = \frac{(1+R)-d}{u-d}$$

❖The option price depends on:

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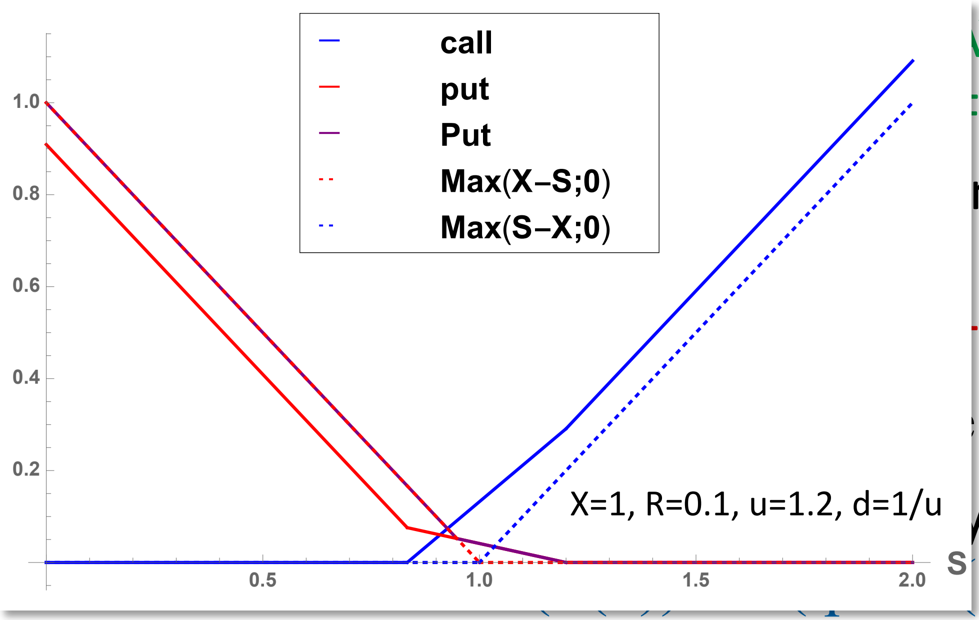

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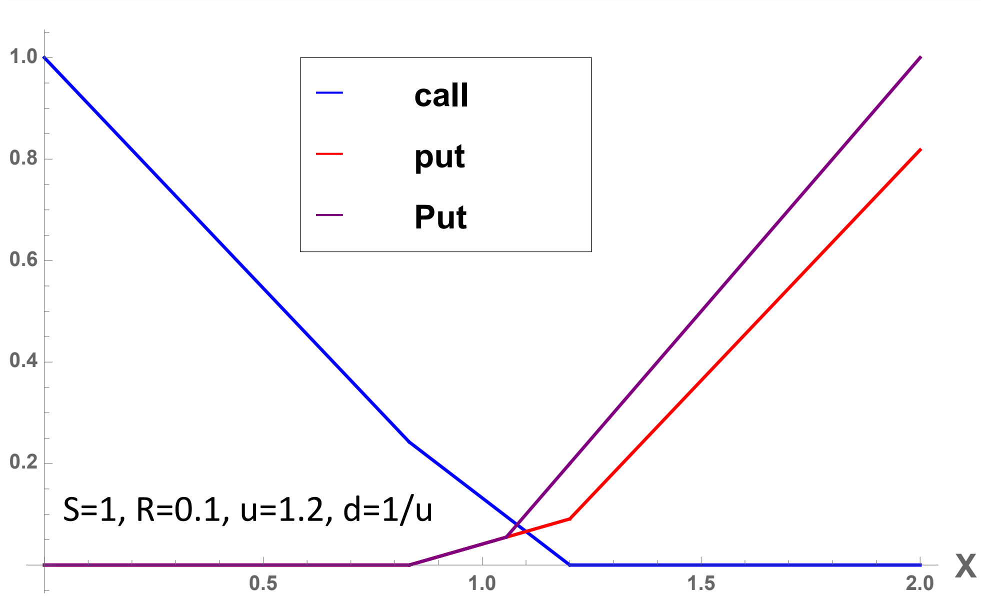

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Binomial model (one step)

$$S(0) = S \xrightarrow{q} S(1)_u = Su$$

$$S(0) = S \xrightarrow{1-q} S(1)_d = Sd$$

NOW **OPTION EXPIRY DATE**

$$c = c(0) = \Delta S + B \xleftarrow{g} c(1)_u = \max(S\,u - X\,;\,0)$$

$$c = c(0) = \Delta S + B \xleftarrow{1-g} c(1)_d = \max(S\,d - X\,;\,0)$$

NOW **OPTION EXPIRY DATE**

KNOWN PAYOFF !!!

$$c = \frac{g\,c(1)_u + (1-g)c(1)_d}{(1+R)}, \quad \text{where: } g = \frac{(1+R)-d}{u-d}$$

❖The option price depends on:

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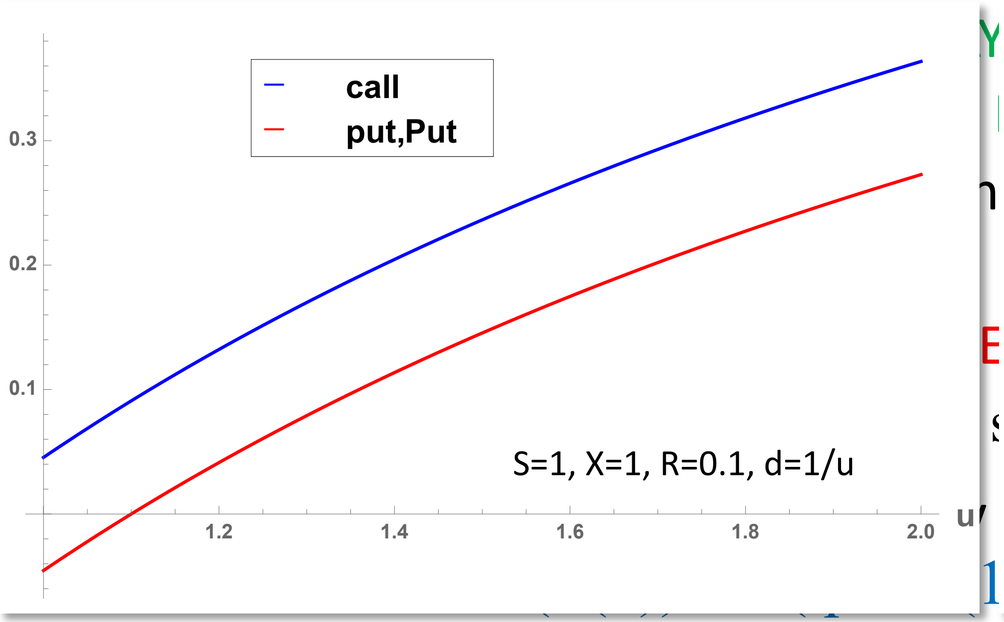

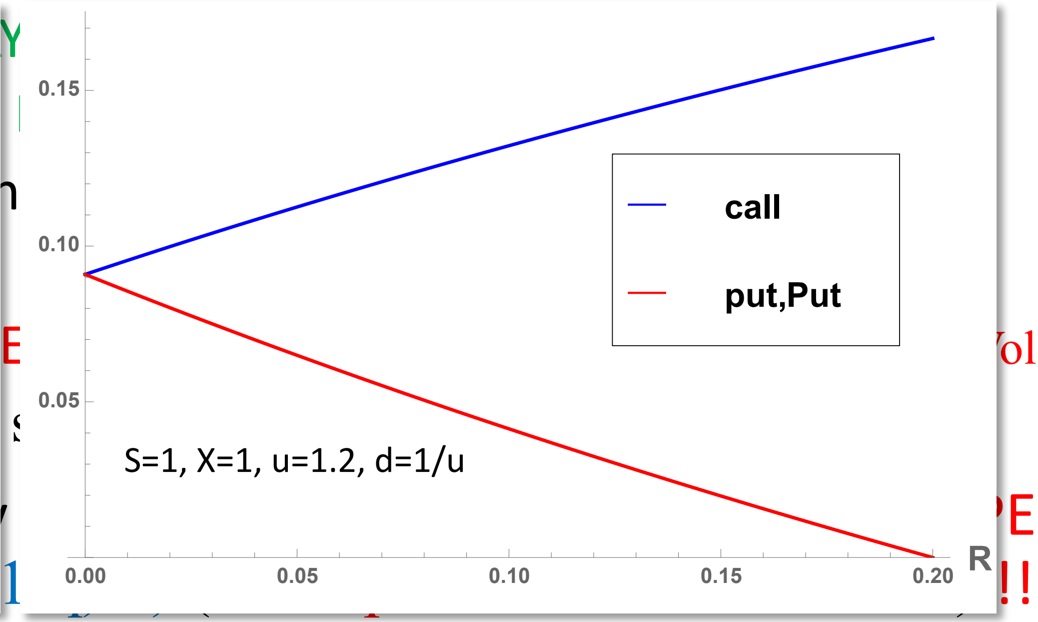

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Option pricing in Binomial Model

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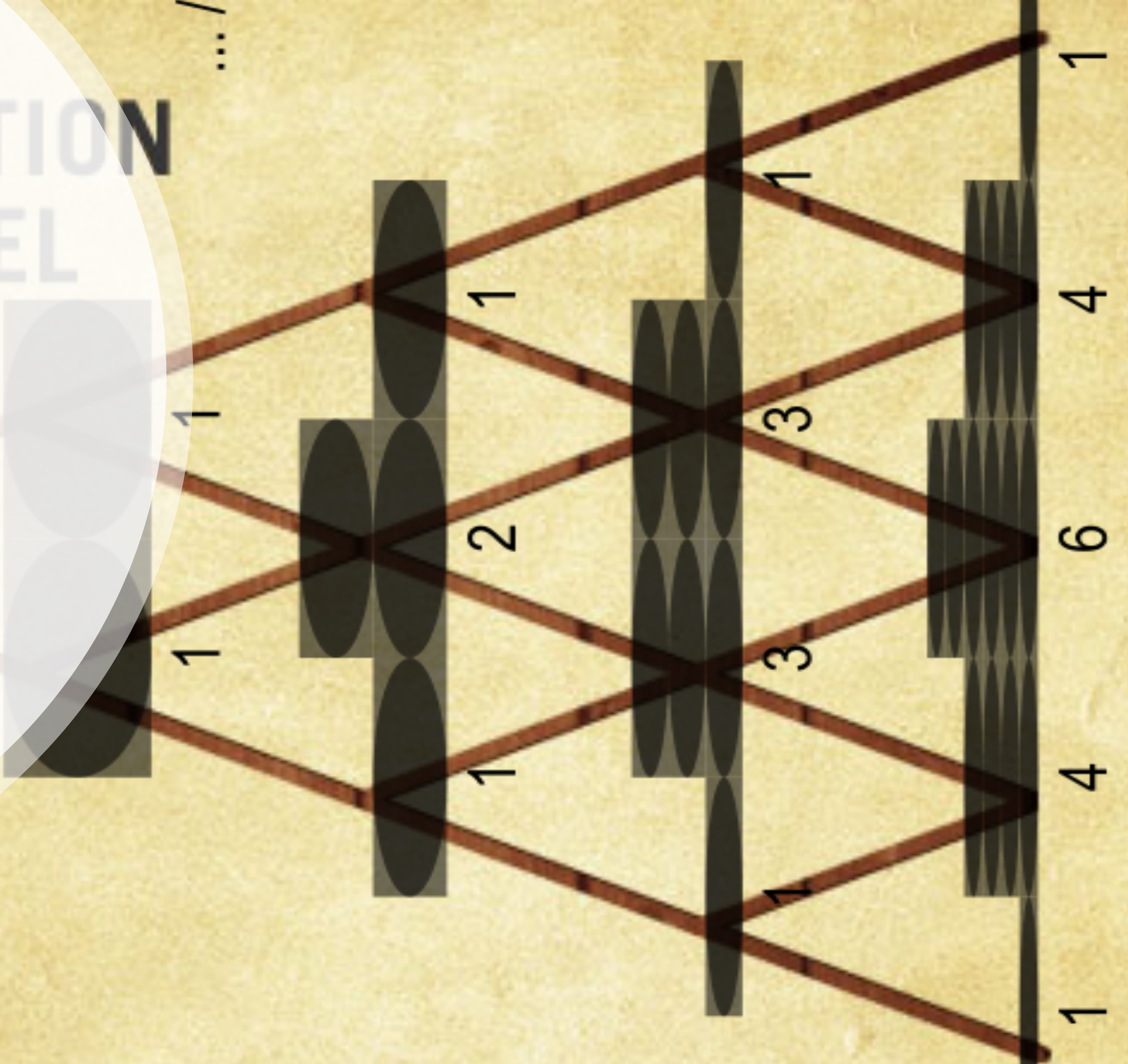
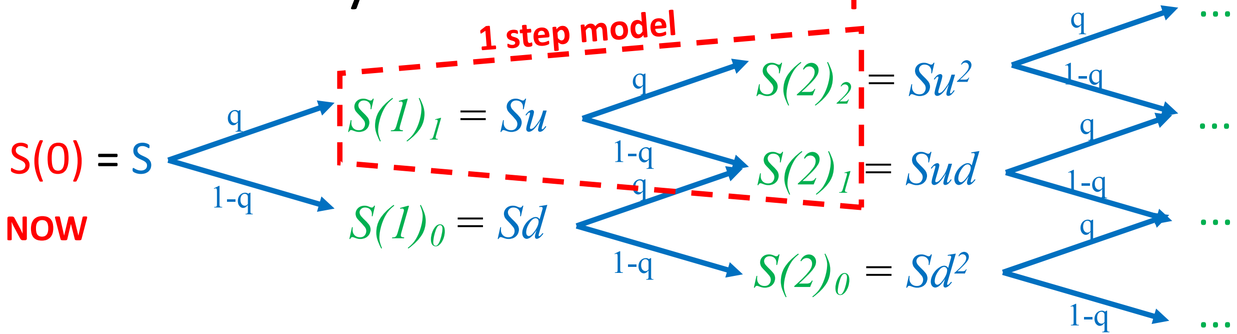
- ❖Risk-neutral / martingale pricing



Binomial model (N steps)

- The model can be easily extended to N steps



1 step model

S(0) = S

NOW

q

1-q

$S(1)_1 = Su$

$S(1)_0 = Sd$

q

1-q

q

1-q

$S(2)_2 = Su^2$

$S(2)_1 = Sud$

$S(2)_0 = Sd^2$

q

1-q

q

1-q

q

1-q

…

…

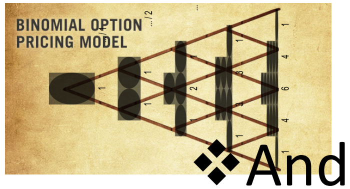
…

…

OPTION EXPIRY DATE (n=N)

$$S(N)_j = S\,u^{j}\,d^{N\text{-}j}$$

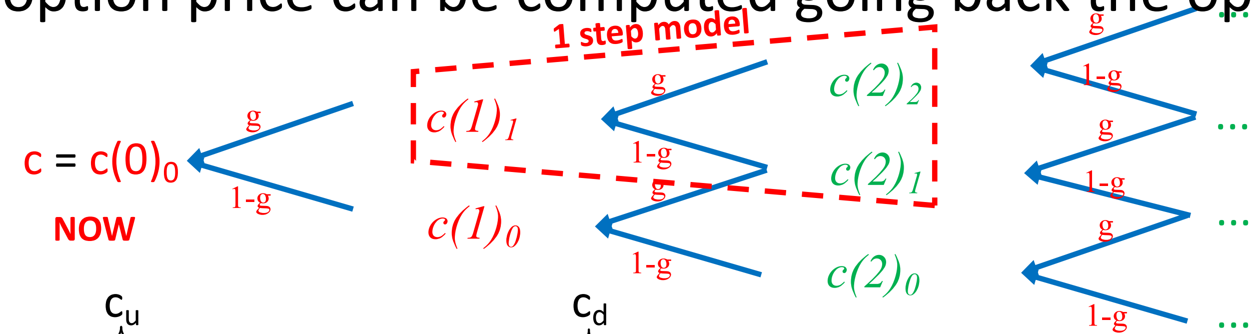
- After n steps the share price: $S(n)_j = S\,u^{j}\,d^{n\text{-}j}$ has Binomial probability distribution



How many paths lead to j-th price

$$\mathcal{P}_q(n)_j = \binom{n}{j} q^j (1-q)^{n-j} \; , \qquad j = 0,1,\dots,n$$

- And the option price can be computed going back the option Binomial tree



1 step model

c = c(0)$_0$

NOW

g

1-g

$c(1)_1$

$c(1)_0$

g

1-g

g

1-g

$c(2)_2$

$c(2)_1$

$c(2)_0$

g

1-g

g

1-g

g

1-g

…

…

…

…

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=N)

$$c(N)_j = \max(S\,u^{j}\,d^{N\text{-}j} - X\,;\,0)$$

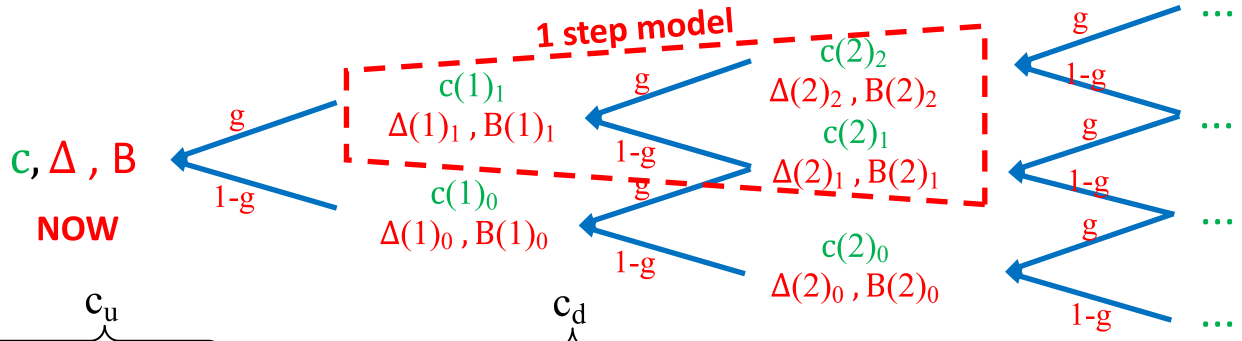
$$c(n)_j = \frac{g\,\overbrace{c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)} \qquad g = \frac{(1+R)-d}{u-d}$$

Binomial model (N steps)

- After n steps the share price: $S(n)_j = S\,u^{j}\,d^{\,n\text{-}j}$ has Binomial probability distribution
- For each step n one can easily compute the number of shares $\Delta\,(n)_j$ and the value of deposit/loan $\mathrm{B}\,(n)_j$ used to create the replicating portfolio: $c(n)_j = \Delta\,(n)_j \times S(n)_j + \mathrm{B}\,(n)_j$

$$\Delta(\mathrm{n})_{\mathrm{j}} = \frac{\overbrace{c(n+1)_{j+1}}^{c_u} - \overbrace{c(n+1)_{j}}^{c_d}}{S(u-d)} \qquad \mathrm{B}(\mathrm{n})_{\mathrm{j}} = -\frac{d\,\overbrace{c(n+1)_{j+1}}^{c_u} - u\,\overbrace{c(n+1)_{j}}^{c_d}}{(u-d)(1+R)}$$

- Note that now these depend on the evolution of the share price $S(n)_j$ and change in each step, thus the replicating portfolio has to be constantly adjusted



KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=N)

$$c(\mathrm{N})_{\mathrm{j}} = \max(S\,u^{j}\,d^{\,N\text{-}j} - \mathrm{X}\,;\,0)$$

$$c(n)_j = \frac{g\,\overbrace{c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_{j}}^{c_d}}{(1+R)} \qquad g = \frac{(1+R)-d}{u-d}$$

Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps

S(0) = 1

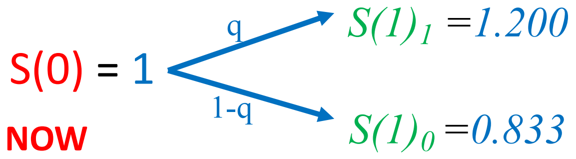
NOW

c = ???

NOW

Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



$S(0) = 1$

NOW

q

$1\text{-}q$

$S(1)_1 = 1.200$

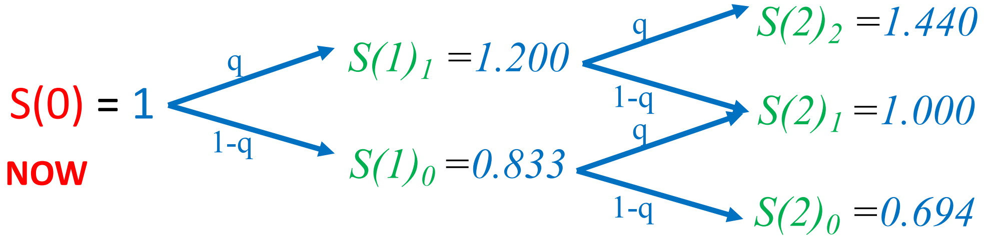
$S(1)_0 = 0.833$

c = ???

NOW

Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



$S(0) = 1$

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

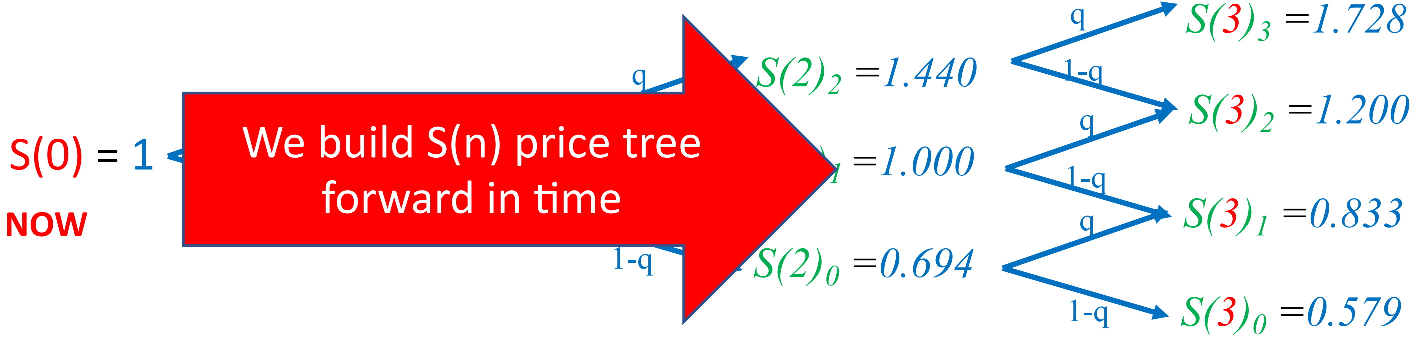
$S(2)_0 = 0.694$

c = ???

NOW

Binomial model (N steps): Example

❖Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



S(0) = 1

NOW

We build S(n) price tree forward in time

$S(2)_2 = 1.440$

$\ldots_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

$S(3)_0 = 0.579$

OPTION EXPIRY DATE (n=3)

$$S(3)_j = 1 \times 1.2^{\,j} \times 1/1.2^{\,3\text{-}j}$$

with probability

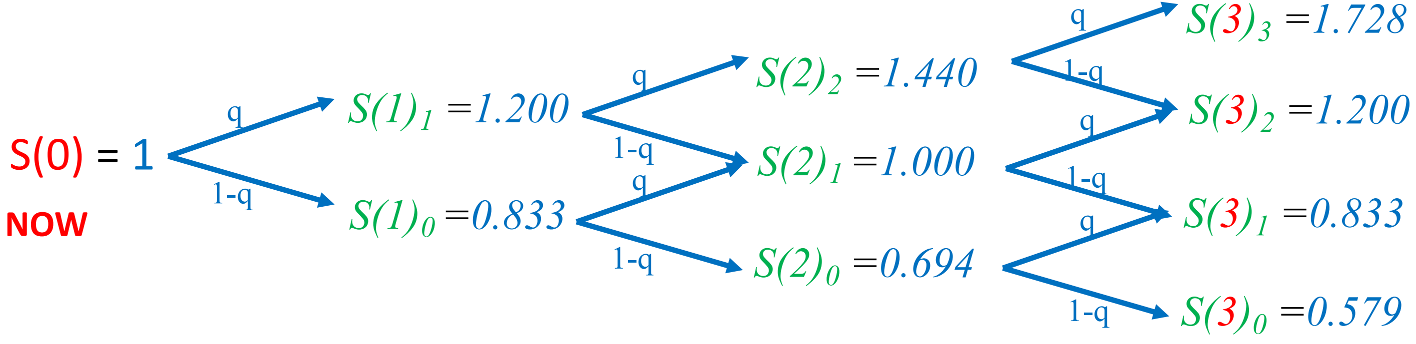
$$\mathcal{P}_q(3)_j = \binom{3}{j} q^j (1-q)^{3-j}$$

c = ???

NOW

Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



S(0) = 1

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

$S(3)_0 = 0.579$

OPTION EXPIRY DATE (n=3)

$$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3-j}$$

Compute payoff

$c(3)_3 = 0.728$

$c(3)_2 = 0.200$

$c(3)_1 = 0$

$c(3)_0 = 0$

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

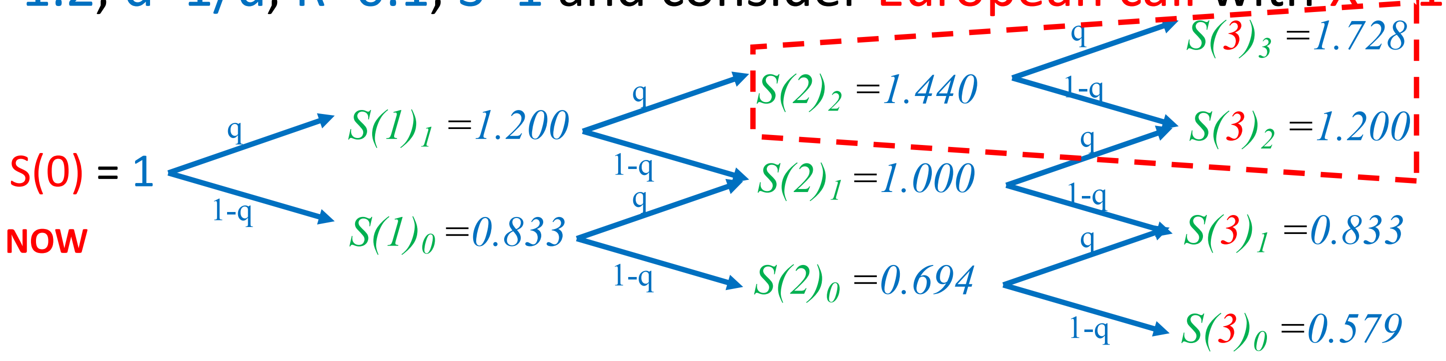
$$c(3)_j = \max(S(3)_j - X\ ;\ 0) = \max(1 \times 1.2^{j} \times 1/1.2^{3-j} - 1\ ;\ 0)$$

c = ???

NOW

Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



S(0) = 1

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

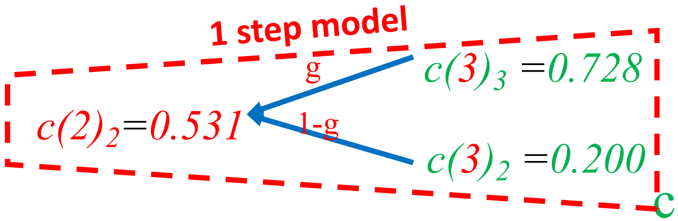
$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

$S(3)_0 = 0.579$

OPTION EXPIRY DATE (n=3)

$$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3-j}$$



$c(2)_2 = 0.531$

$c(3)_3 = 0.728$

$c(3)_2 = 0.200$

$c(3)_1 = 0$

$c(3)_0 = 0$

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

$$c(3)_j = \max(S(3)_j - X\ ;\ 0) = \max(1 \times 1.2^{j} \times 1/1.2^{3-j} - 1\ ;\ 0)$$

c = ???

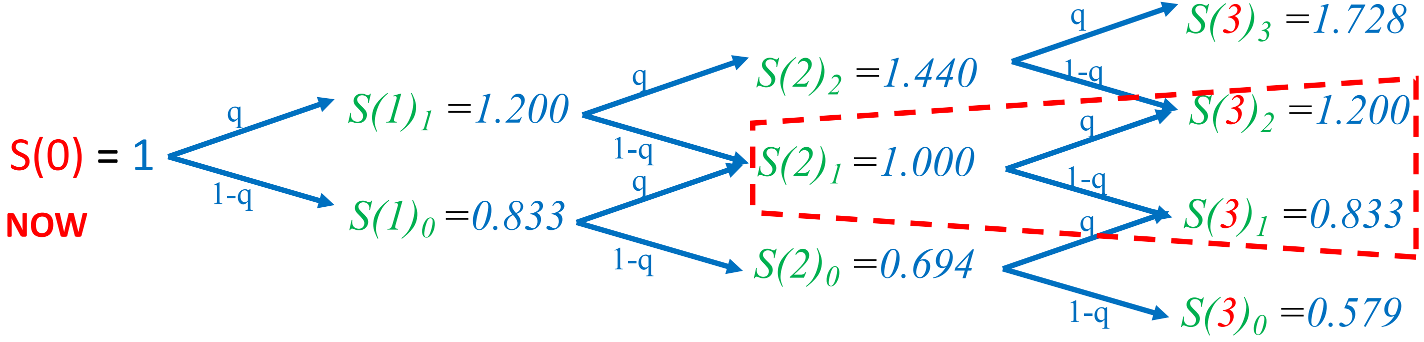
NOW

$$c(n)_j = \frac{g\,\overbrace{c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)}$$

$$g = \frac{(1+R) - d}{u - d} = 0.7273$$

Binomial model (N steps): Example

- ❖Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



$S(0) = 1$

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

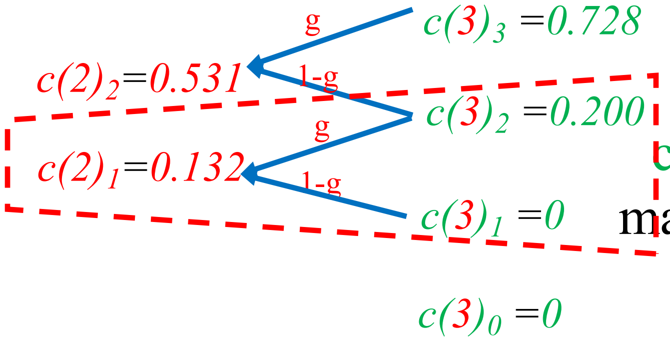
$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

$S(3)_0 = 0.579$

OPTION EXPIRY DATE (n=3)

$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3\text{-}j}$



$c(3)_3 = 0.728$

$c(2)_2 = 0.531$

$c(3)_2 = 0.200$

$c(2)_1 = 0.132$

$c(3)_1 = 0$

$c(3)_0 = 0$

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

$c(3)_j = \max(S(3)_j - X ; 0) = \max(1 \times 1.2^{j} \times 1/1.2^{3\text{-}j} - 1 ; 0)$

c = ???

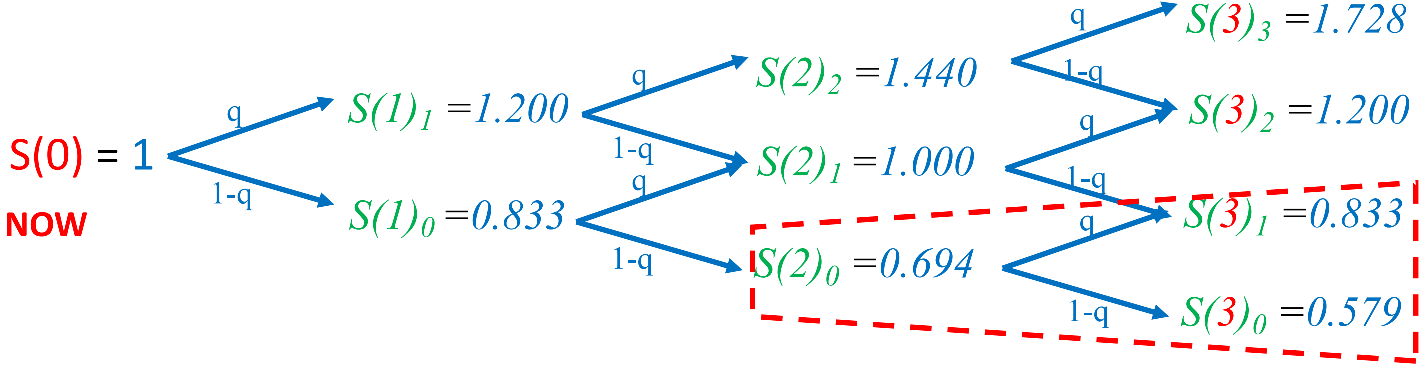
NOW

$$c(n)_j = \frac{g\,\overbrace{c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)}$$

$$g = \frac{(1+R)-d}{u-d} = 0.7273$$

Binomial model (N steps): Example

❖Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



S(0) = 1

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

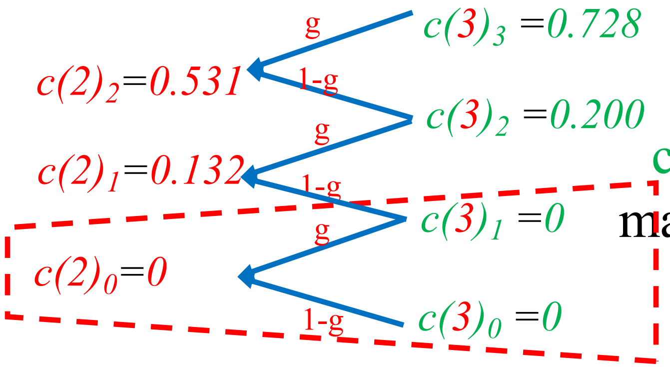
$S(3)_0 = 0.579$

OPTION EXPIRY DATE (n=3)

$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3-j}$

c = ???

NOW



$c(2)_2 = 0.531$

$c(2)_1 = 0.132$

$c(2)_0 = 0$

$c(3)_3 = 0.728$

$c(3)_2 = 0.200$

$c(3)_1 = 0$

$c(3)_0 = 0$

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

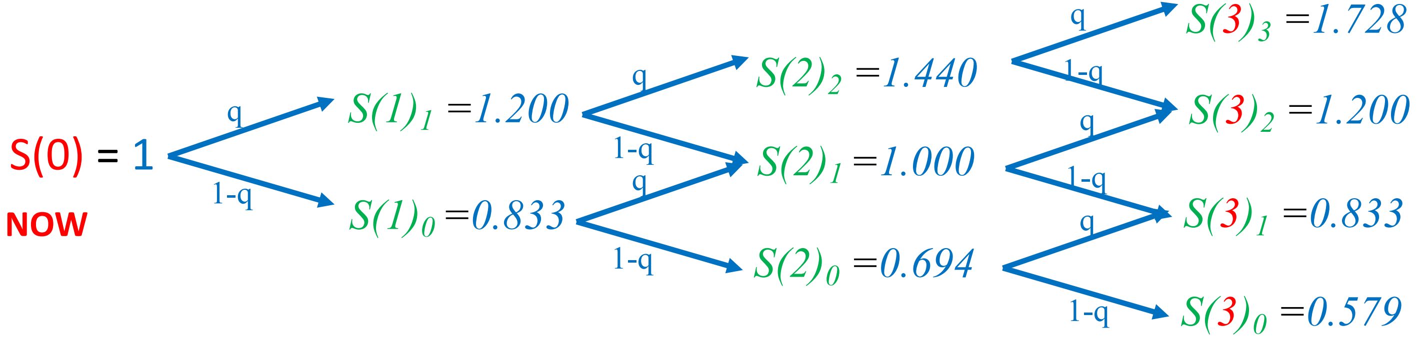
$$c(3)_j = \max(S(3)_j - X ; 0) = \max(1 \times 1.2^{j} \times 1/1.2^{3-j} - 1 ; 0)$$

$$c(n)_j = \frac{\overbrace{g\, c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)}$$

$$g = \frac{(1+R)-d}{u-d} = 0.7273$$

Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



S(0) = 1

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

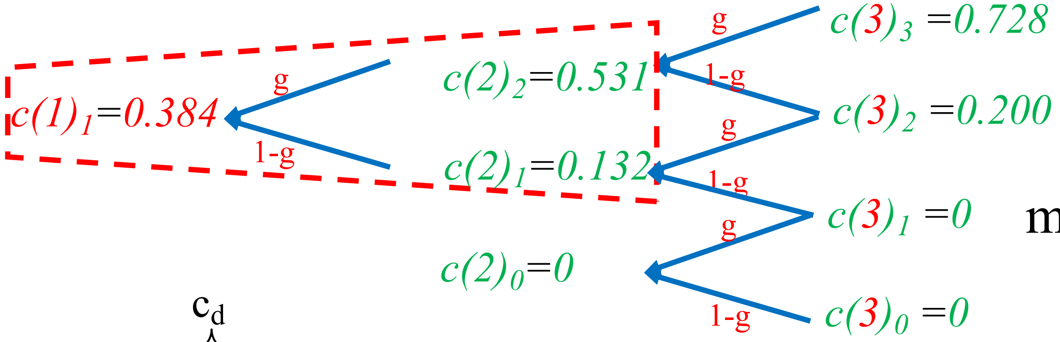
$S(3)_0 = 0.579$

OPTION EXPIRY DATE (n=3)

$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3-j}$

c = ???

NOW



$c(1)_1 = 0.384$

$c(2)_2 = 0.531$

$c(2)_1 = 0.132$

$c(2)_0 = 0$

$c(3)_3 = 0.728$

$c(3)_2 = 0.200$

$c(3)_1 = 0$

$c(3)_0 = 0$

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

$$c(3)_j = \max(S(3)_j - X\ ;\ 0) = \max(1 \times 1.2^{j} \times 1/1.2^{3-j} - 1\ ;\ 0)$$

$$c(n)_j = \frac{g\,\overbrace{c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)}$$

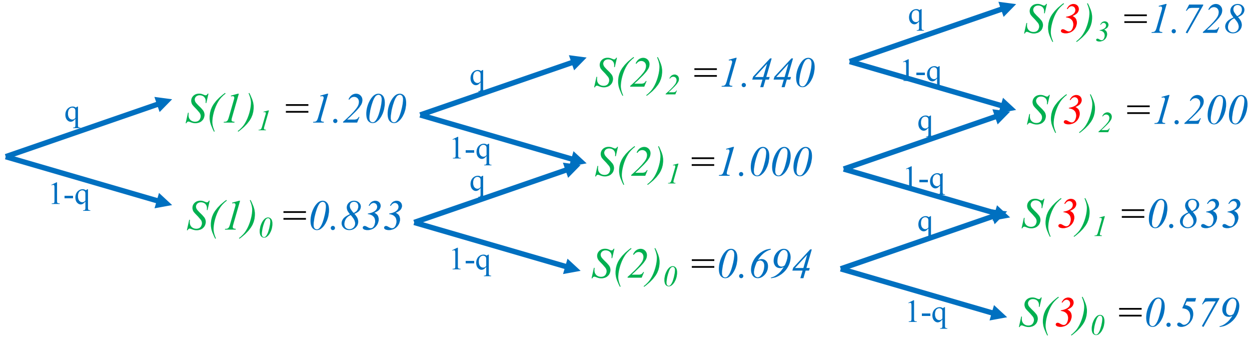
$$g = \frac{(1+R)-d}{u-d} = 0.7273$$

Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps

$S(0) = 1$

NOW



$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

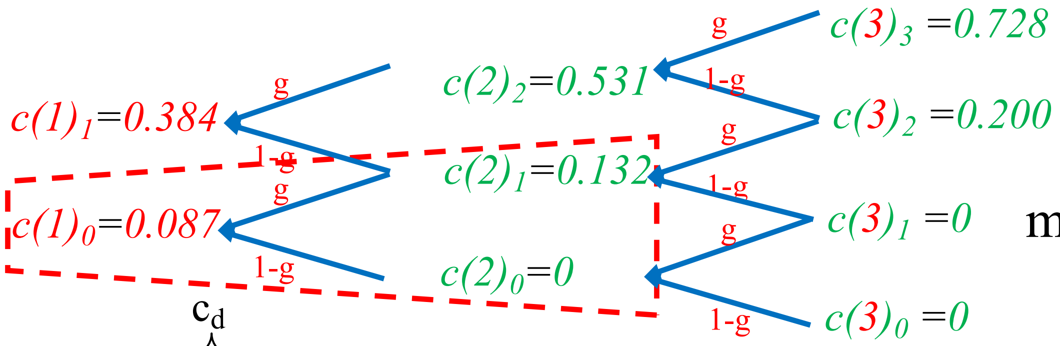
$S(3)_0 = 0.579$

OPTION EXPIRY DATE (n=3)

$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3-j}$

c = ???

NOW



$c(1)_1 = 0.384$

$c(1)_0 = 0.087$

$c(2)_2 = 0.531$

$c(2)_1 = 0.132$

$c(2)_0 = 0$

$c(3)_3 = 0.728$

$c(3)_2 = 0.200$

$c(3)_1 = 0$

$c(3)_0 = 0$

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

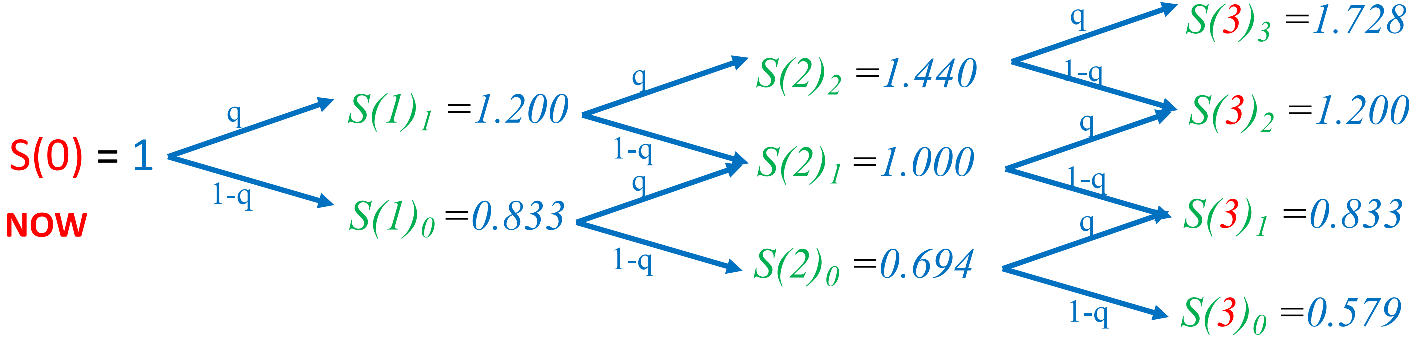
$$c(3)_j = \max(S(3)_j - X\ ;\ 0) = \max(1 \times 1.2^{j} \times 1/1.2^{3-j} - 1\ ;\ 0)$$

$$c(n)_j = \frac{g\,\overbrace{c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)}$$

$$g = \frac{(1+R)-d}{u-d} = 0.7273$$

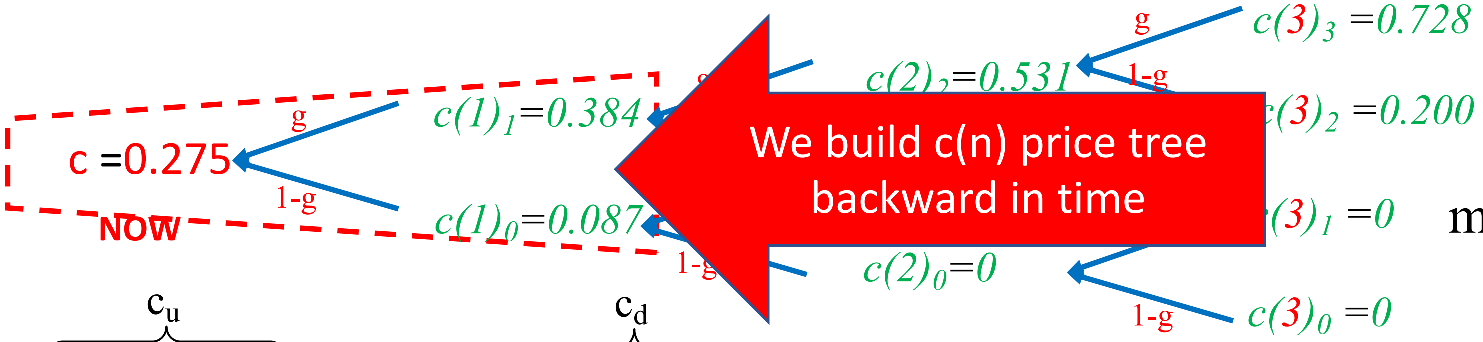
Binomial model (N steps): Example

- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



OPTION EXPIRY DATE (n=3)

$$S(3)_j = 1 \times 1.2^{\,j} \times 1/1.2^{\,3\text{-}j}$$



KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

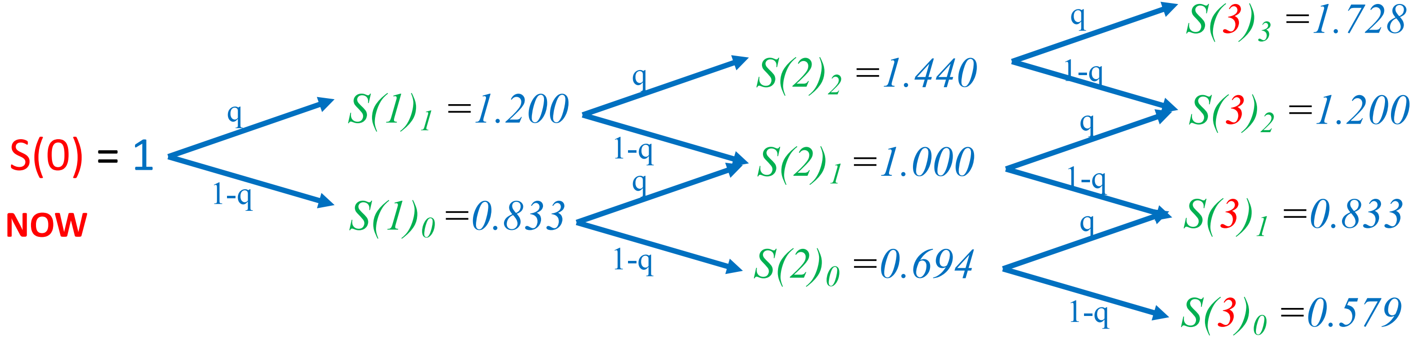
$$c(3)_j = \max(S(3)_j - X\ ;\ 0) = \max(1 \times 1.2^{\,j} \times 1/1.2^{\,3\text{-}j} - 1\ ;\ 0)$$

$$c(n)_j = \frac{\overbrace{g\,c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)}$$

$$g = \frac{(1+R) - d}{u - d} = 0.7273$$

Binomial model (N steps): Example

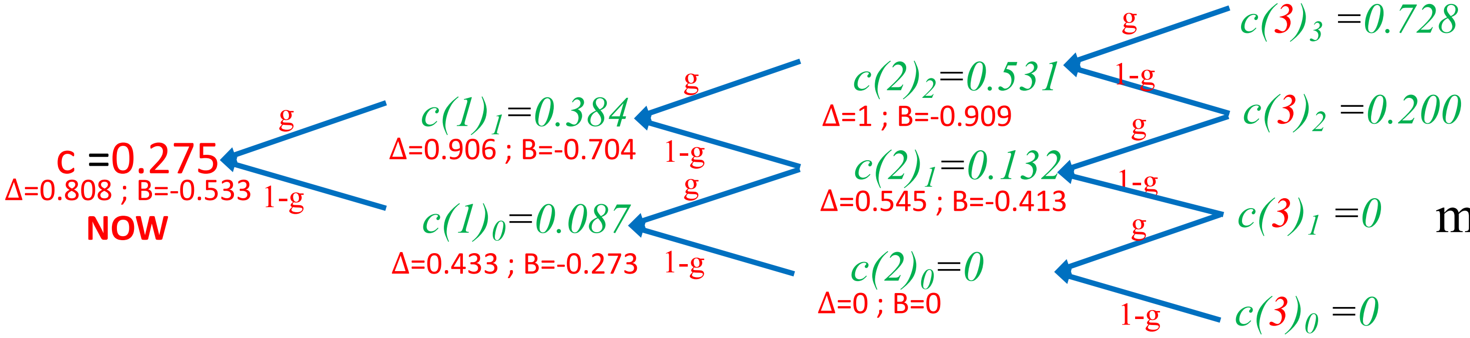
- Assume: u=1.2, d=1/u, R=0.1, S=1 and consider European call with X = 1, expiring in N = 3 steps



OPTION EXPIRY DATE (n=3)

$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3-j}$

- Performing the same way one can also compute $\Delta(n)_j$ and $B(n)_j$
  - ❑ NOTE that for the call option $\Delta(n)_j \geq 0$ and $B(n)_j \leq 0$, so to replicate the option we BUY shares and take a LOAN
  - ❑ AND they vary in each time step so the REPLICATING PORTFOLIO must be DYNAMICALLY ADJUSTED !!!



KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

$c(3)_j = \max(S(3)_j - X\ ; 0) = \max(1 \times 1.2^{j} \times 1/1.2^{3-j} - 1\ ; 0)$

$$\Delta(n)_j = \frac{c(n+1)_{j+1} - c(n+1)_j}{S(n)_j(u-d)} \qquad B(n)_j = -\frac{d\, c(n+1)_{j+1} - u\, c(n+1)_j}{(u-d)(1+R)}$$

Binomial model: Dynamical hedging/arbitrage

- We (NOW: n=0) SELL the call option from the previous Example

S(0) = 1

NOW

- In order to HEDGE the SHORT OPTION position we (NOW) BUY THE REPLICATING PORTFOLIO: $\Delta \times S + B$, i.e. we BUY Δ =0.808 SHARES@S=1 and take a LOAN of B =-0.533@R=10% (per step)

c =0.275

Δ=0.808 ; B=-0.533

NOW

CURRENT POSITIONS (n=0):

SHORT call: -c

LONG shares: +0.808 S

Loan: B=-0.533

CURRENT NET WORTH*= 0 !

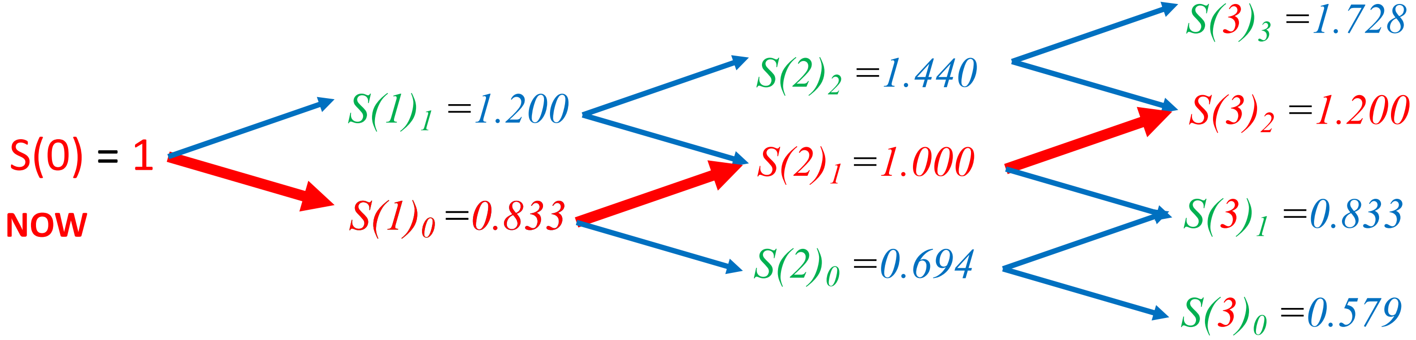
***IF OPTION PRICES ARE FAIR:** $c = \Delta \times S + B$

****If one closes all positions one gets CF=0**

- NOTE that the current CF(0) = 0, as $c = \Delta \times S + B$, so we BUY the shares (CF = -0.808×1) from the INFLOWS OF SELLING OPTION (CF = +0.275) and from the LOAN (CF=+0.533)

Binomial model: Dynamical hedging/arbitrage

- We will follow a (random) share price S trajectory



$S(0) = 1$

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

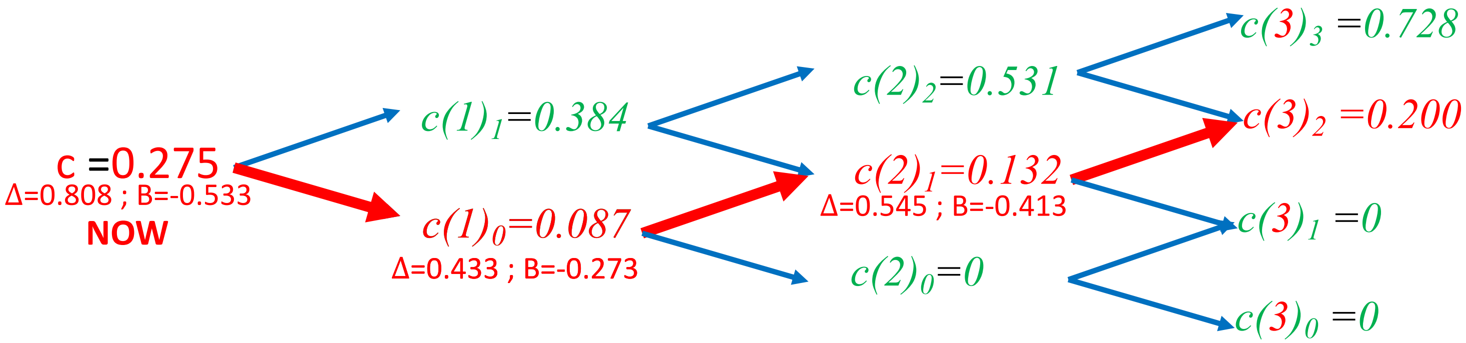
$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

$S(3)_0 = 0.579$



$c = 0.275$

$\Delta=0.808$; $B=-0.533$

NOW

$c(1)_1=0.384$

$c(1)_0=0.087$

$\Delta=0.433$; $B=-0.273$

$c(2)_2=0.531$

$c(2)_1=0.132$

$\Delta=0.545$; $B=-0.413$

$c(2)_0=0$

$c(3)_3 = 0.728$

$c(3)_2 = 0.200$

$c(3)_1 = 0$

$c(3)_0 = 0$

CURRENT POSITIONS (n=0):

SHORT call: -c

LONG shares: +0.808 S

Loan: B=-0.533

CURRENT NET WORTH*= 0 !

***IF OPTION PRICES ARE FAIR:** $c = \Delta \times S + B$

****If one closes all positions one gets CF=0**

Binomial model: Dynamical hedging/arbitrage

❖We follow a (random) share price S trajectory: in NEXT STEP (n=1) the share price ↓

$S(0) = 1 \rightarrow S(1)_0 = 0.833$

NOW

❖We adjust THE REPLICATING PORTFOLIO, as S↓and thus Δ↓, we SELL: 0.433-0.808= -0.376 SHARES@S=0.833 and, as B↑, we REPAY OLD LOAN: -0.533×1.1=-0.586 and we take a NEW (smaller) LOAN: -0.273 @ R=10%

$c = 0.275 \rightarrow c(1)_0 = 0.087$

Δ=0.808 ; B=-0.533

NOW

Δ=0.433 ; B=-0.273

CURRENT POSITIONS (n=1):

SHORT call: -c

LONG shares: +0.433 S

Loan: B=-0.273

CURRENT NET WORTH*= 0 !

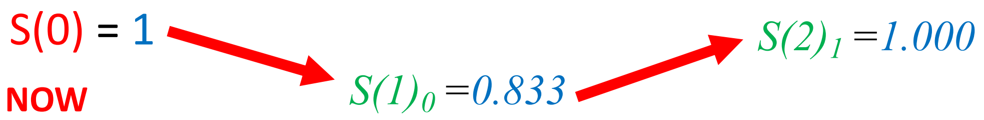
***IF OPTION PRICES ARE FAIR:** $c = \Delta \times S + B$

****If one closes all positions one gets CF=0**

❖NOTE that the current CF(1) = 0, as the INFLOWS from SELLING THE SHARES (CF = +0.376×0.833=+0.313) and NEW LOAN (CF = +0.273) exactly EQAUAL OUTFLOWS FOR REPAYING THE OLD LOAN (CF = -0.586)

Binomial model: Dynamical hedging/arbitrage

❖We follow a (random) share price S trajectory: in NEXT STEP (n=2) the share price ↑



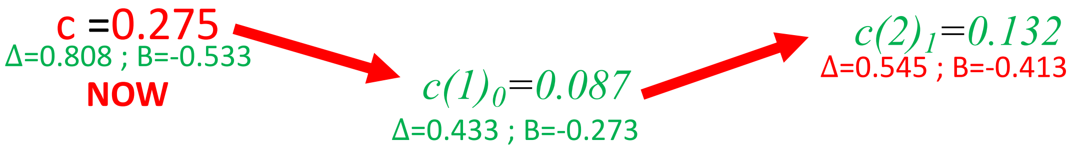
$S(0) = 1$

NOW

$S(1)_0 = 0.833$

$S(2)_1 = 1.000$

❖We adjust THE REPLICATING PORTFOLIO, as S↑ and thus Δ ↑, we BUY: 0.545-0.433= 0.113 SHARES@S=1.000 and, as B↓, we REPAY OLD LOAN: -0.273×1.1=-0.300 and we take a NEW (bigger) LOAN: -0.413 @ R=10%



c =0.275

Δ=0.808 ; B=-0.533

NOW

$c(1)_0 = 0.087$

Δ=0.433 ; B=-0.273

$c(2)_1 = 0.132$

Δ=0.545 ; B=-0.413

CURRENT POSITIONS (n=2):

SHORT call: -c

LONG shares: +0.545 S

Loan: B=-0.413

CURRENT NET WORTH*= 0 !

***IF OPTION PRICES ARE FAIR:** c =Δ×S+B

****If one closes all positions one gets CF=0**

❖NOTE that the current CF(2) = 0, as the INFLOWS from NEW LOAN (CF = +0.413) exactly EQUAL OUTFLOWS FOR BUYING THE SHARES (CF = -0.113×1.000=-0.113) and FOR REPAYING THE OLD LOAN (CF = -0.300)

Binomial model: Dynamical hedging/arbitrage

- We follow a (random) share price S trajectory: in NEXT STEP (n=3) the share price ↑

S(0) = 1

NOW

$S(1)_0 = 0.833$

$S(2)_1 = 1.000$

$S(3)_2 = 1.200$

THIS IS THE OPTION
EXPIRY DATE !!!
(N=3)

- We CLOSE ALL OPEN POSITIONS: we SELL +0.545 SHARES @S=1.200 and we REPAY THE LOAN: -0.413×1.1=-0.454
- As the option SELLER (SHORT) we MUST PAY the PAYOFF: $c(3)_2 = 0.200$

c =0.275
Δ=0.808 ; B=-0.533
NOW

$c(1)_0 = 0.087$
Δ=0.433 ; B=-0.273

$c(2)_1 = 0.132$
Δ=0.545 ; B=-0.413

$c(3)_2 = 0.200$

PREVIOUS POSITIONS (n=2):
SHORT call: -c
LONG shares: +0.545 S
Loan: B=-0.413
CURRENT NET WORTH*= 0 !
***IF OPTION PRICES ARE FAIR:** $c = \Delta \times S + B$
****If one closes all positions one gets CF=0**

- NOTE that the current CF(3) = 0 again !, as the INFLOWS from SELLING THE SHARES (CF = +0.545×1.200=+0.655) exactly EQUAL OUTFLOWS FOR REPAYING THE LOAN (CF = -0.455) AND THE OPTION’S PAYOFF (CF = -0.200)

Binomial model: Dynamical hedging/arbitrage

- We have followed a (random) share price S trajectory

$S(0) = 1$ → $S(1)_0 = 0.833$ → $S(2)_1 = 1.000$ → $S(3)_2 = 1.200$

NOW

THIS IS THE OPTION EXPIRY DATE !!!
(N=3)

- In each time STEP the NET CF from trading our TOTAL POSITION (SHORT OPTION + LONG REPLICATING PORTFOLIO) WAS ZERO ! The result is the same for ANY possible trajectory !
- As the market is DYNAMICALLY COMPLETE, we have successfully managed to construct a DYNAMICALLY ADJUSTED PORTFOLIO of shares and loan which replicated the option perfectly.

$c = 0.275$ → $c(1)_0 = 0.087$ → $c(2)_1 = 0.132$ → $c(3)_2 = 0.200$

$\Delta=0.808$; $B=-0.533$
NOW

$\Delta=0.433$; $B=-0.273$

$\Delta=0.545$; $B=-0.413$

TOTAL RESULT = 0 !!!!
(NET CF = 0, P/L = 0)

- If the option was (NOW) mispriced, one could make ARBITRAGE PROFITS ! ⇐ **BUT: ONE HAS TO KNOW THE BINOMIAL PROCESS PARAMETERS u & d !!!**
  - if c > 0.275 one sells (NOW) the option at a higher price making DYNAMICAL arbitrage profit
  - if c < 0.275 one inverses all transactions (positions) and also makes DYNAMICAL arbitrage profit

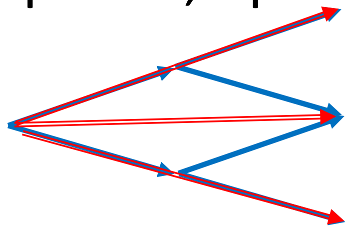
i.e. KNOW FUTURE SHARE VOLATILITY !!!

Binomial model: American options & flexibility

The Binomial* model can be easily adjusted for:

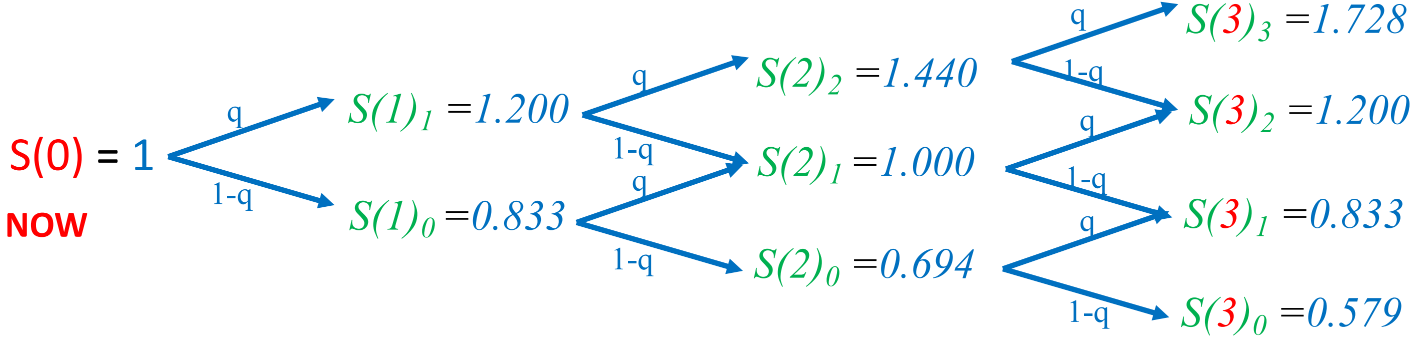
- Various payoff functions – if payoff depends only on the final share price S(N) one simply has to adjust option prices at maturity (before iterative backward procedure)
    - E.g. for put options with the eXercise price X one has: $p(N)_j = \max(X - S\, u^{j}\, d^{N-j}\ ;\ 0)$
- American options and early exercise – in each backward Binomial step one has to check the current payoff from the option (i.e. from early exercise)
    - E.g. for American Put options: $P(n)_j = \max(X - S u^{j} d^{\,n-j}\ ;\ p(n)_j)$, where: $p(n)_j = \dfrac{g\, P(n+1)_{j+1} + (1-g)P(n+1)_j}{(1+R)}$ „standard" pricing formula
- Non-constant risk-free rate R% and volatility (u & d) – parameters can change in each step of the price evolution
- (Continuous) dividend/other payments (this works well e.g. for F/X options, options on futures, …) ⇒ see Problems: Set 6

*One can also construct a (more general) Trinomial model (with three-valence graph)
This way in a single step one can get two Binomial steps ⇒ better convergence.



Binomial model: Example (European binary)

- Assume: u=1.2, d=1/u, R=0.1, S=1, European BINARY option with X = 1, expiring in N = 3 steps



$S(0) = 1$

NOW

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

$S(3)_0 = 0.579$

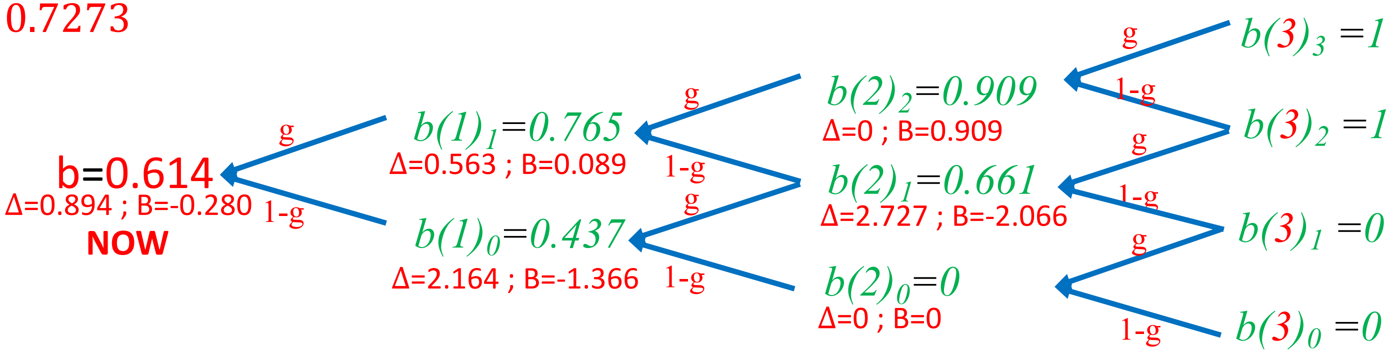
OPTION EXPIRY DATE (n=3)

$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{\,3-j}$

- The BINARY option pays 1 if S(N) > X and 0 otherwise: $b(N)_j = \Theta(S(N)_j - X)$

Heaviside Theta

$$g = \frac{(1+R)-d}{u-d} = 0.7273$$



b=0.614
Δ=0.894 ; B=-0.280
NOW

$b(1)_1 = 0.765$
Δ=0.563 ; B=0.089

$b(1)_0 = 0.437$
Δ=2.164 ; B=-1.366

$b(2)_2 = 0.909$
Δ=0 ; B=0.909

$b(2)_1 = 0.661$
Δ=2.727 ; B=-2.066

$b(2)_0 = 0$
Δ=0 ; B=0

$b(3)_3 = 1$

$b(3)_2 = 1$

$b(3)_1 = 0$

$b(3)_0 = 0$

KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

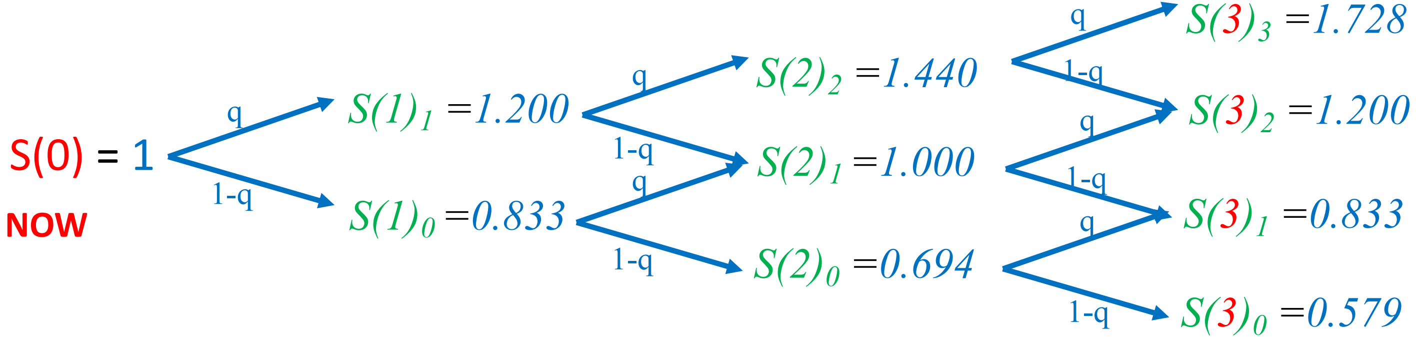
$b(3)_j = \Theta(S(3)_j - X) =$

$\Theta(1 \times 1.2^{j} \times 1/1.2^{\,3-j} - 1)$

$$b(n)_j = \frac{g\, b(n+1)_{j+1} + (1-g)\, b(n+1)_j}{(1+R)} \qquad \Delta(n)_j = \frac{b(n+1)_{j+1} - b(n+1)_j}{S(n)_j (u-d)} \qquad B(n)_j = -\frac{d\, b(n+1)_{j+1} - u\, b(n+1)_j}{(u-d)(1+R)}$$

Binomial model: Example (American binary)

- Assume: u=1.2, d=1/u, R=0.1, S=1, American BINARY option with X = 1, expiring in N = 3 steps



S(0) = 1

NOW

q

1-q

$S(1)_1 = 1.200$

$S(1)_0 = 0.833$

$S(2)_2 = 1.440$

$S(2)_1 = 1.000$

$S(2)_0 = 0.694$

$S(3)_3 = 1.728$

$S(3)_2 = 1.200$

$S(3)_1 = 0.833$

$S(3)_0 = 0.579$

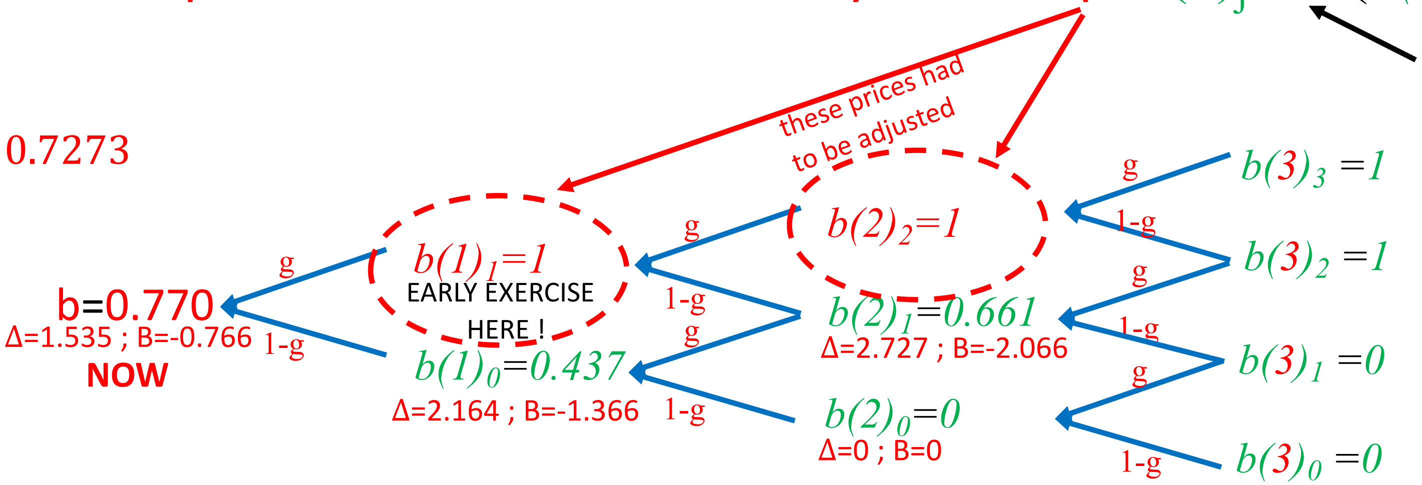
OPTION EXPIRY DATE (n=3)

$S(3)_j = 1 \times 1.2^{j} \times 1/1.2^{3-j}$

- The American option can be exercised at any time step: $b(n)_j \geq \Theta(S(n)_j - X)$ and it is worth more !

Heaviside Theta

$$g = \frac{(1+R)-d}{u-d} = 0.7273$$



these prices had to be adjusted

b=0.770

Δ=1.535 ; B=-0.766

NOW

g

1-g

$b(1)_1 = 1$

EARLY EXERCISE HERE !

$b(1)_0 = 0.437$

Δ=2.164 ; B=-1.366

$b(2)_2 = 1$

$b(2)_1 = 0.661$

Δ=2.727 ; B=-2.066

$b(2)_0 = 0$

Δ=0 ; B=0

$b(3)_3 = 1$

$b(3)_2 = 1$

$b(3)_1 = 0$

$b(3)_0 = 0$

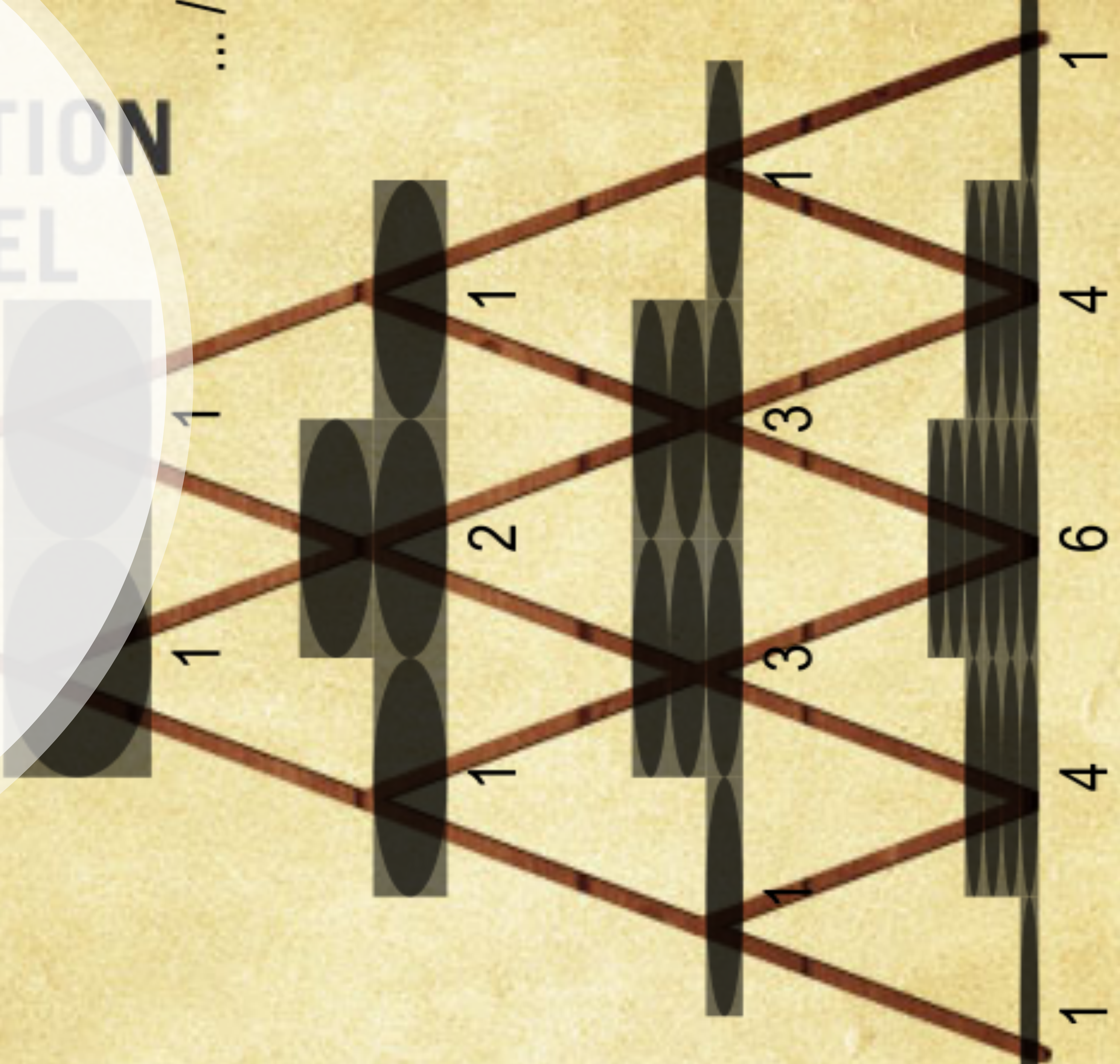
KNOWN PAYOFF !!! on the OPTION EXPIRY DATE (n=3)

$$b(3)_j = \Theta(S(3)_j - X) = \Theta(1 \times 1.2^{j} \times 1/1.2^{3-j} - 1)$$

$$b(n)_j = \frac{g\, b(n+1)_{j+1} + (1-g) b(n+1)_j}{(1+R)} \qquad \Delta(n)_j = \frac{b(n+1)_{j+1} - b(n+1)_j}{S(n)_j (u-d)} \qquad B(n)_j = -\frac{d\, b(n+1)_{j+1} - u\, b(n+1)_j}{(u-d)(1+R)}$$

Option pricing in Binomial Model

- ❖Stochastic processes
- ❖One-step Binomial Model
- ❖Multi-step Binomial Model
- ❖“Analytic” formula for European options
- ❖Risk-neutral / martingale pricing



“Analytic” formula for European options

❖We now want to price a (general) European option: V, with the payoff function depending only on the final share price $S(N)$ at expiration (i.e. after N Binomial time-steps): $V(N) \equiv V(S(N))$

KNOWN PAYOFF FUNCTION !!!

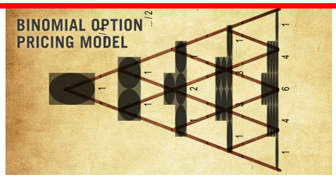
❖We have shown that, in the N-step Binomial model, for such options one has:

- At expiration (n=N): $S(N)_j = S\,u^j d^{N-j} \Rightarrow V(N)_j = V(S\,u^j d^{N-j})$, $j = 0, \dots, N$
- For any 0 ≤ n < N: $V(n)_j = \dfrac{g\,V(n+1)_{j+1} + (1-g)V(n+1)_j}{(1+R)}$, where: $g = \dfrac{(1+R)-d}{u-d}$

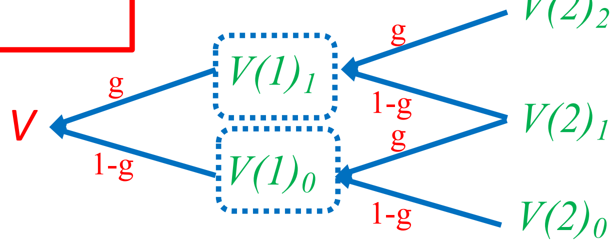
❖It is easy to show* that the current option price is given by:

$$\boxed{V = (1+R)^{-N}\sum_{j=0}^{N}\binom{N}{j} g^j (1-g)^{N-j}\, V(S\,u^j d^{N-j})}$$

How many paths lead to j-th price



*EXAMPLE: for N=2 one has:



$$V = \frac{g\boxed{\dfrac{g\,V(2)_2 + (1-g)V(2)_1}{(1+R)}} + (1-g)\boxed{\dfrac{g\,V(2)_1 + (1-g)V(2)_0}{(1+R)}}}{(1+R)} = \frac{g^2\,V(2)_2 + 2g(1-g)V(2)_1 + (1-g)^2\,V(2)_0}{(1+R)^2}$$

“Analytic” formula for European options

- ❖We now want to price a (general) European option: V, with the payoff function depending only on the final share price $S(N)$ at expiration (i.e. after N Binomial time-steps): $V(N) \equiv V(S(N))$
- ❖We have shown that, in the N-step Binomial model, for such options one has:
  - ○ At expiration (n=N): $S(N)_j = S\, u^j d^{N\text{-}j} \Rightarrow V(N)_j = V(S\, u^j d^{N\text{-}j})$, $j = 0, \ldots, \mathrm{N}$
  - ○ For any 0 ≤ n < N: $V(n)_j = \dfrac{g\, V(n+1)_{j+1} + (1-g) V(n+1)_j}{(1+R)}$, where: $g = \dfrac{(1+R)-d}{u-d}$

KNOWN PAYOFF FUNCTION !!!

- ❖It is easy to show* that the current option price is given by:

$$\boxed{\mathrm{V} = (1+R)^{-N} \sum_{j=0}^{N} \binom{N}{j} g^j (1-g)^{N-j}\, V(S\, u^j d^{N-j})}$$

- ❖This can be used as the (European) option pricing “analytic” formula, which is simpler to use than going backward the Binomial tree
- ❖For the European binary option Example (discussed above) one simply has: $g = \dfrac{(1+R)-d}{u-d} = 0.7273$

$$\mathrm{V} = (1+0.1)^{-3} \sum_{j} \binom{N}{j} g^j (1-g)^{N-j}\, \Theta(1 \times 1.2^j \times 1/1.2\ 3{-}^j - 1) = 1.1^{-3} \left(\binom{3}{3} g^3 \times 1 + \binom{3}{2} g^2 (1-g) \times 1 \right) = 0.614$$

“Analytic” formula for European options

$$\boxed{V = (1+R)^{-N}\sum_{j=0}^{N}\binom{N}{j} g^{j}(1-g)^{N-j}\, V(S\,u^{j}\,d^{\,N-j})}$$

- In particular, for the European call option ($V \equiv c$) one has:

KNOWN PAYOFF AT EXPIRY !

$c(N)_j = \max(S\,u^{j}\,d^{\,N-j} - X\,;\,0)$

$$c = (1+R)^{-N}\sum_{j=0}^{N}\binom{N}{j} g^{j}(1-g)^{N-j}\max\left(Su^{j}d^{N-j} - X;0\right) = \ldots$$

$$\ldots = (1+R)^{-N}\sum_{j=a}^{N}\binom{N}{j} g^{j}(1-g)^{N-j}\left(Su^{j}d^{N-j} - X\right) = \ldots$$

a: *the smallest integer such that:* $Su^{a}d^{N-a} > X$ ⇒ $a = \lceil \ln(X/(Sd^{N}))\,/\ln(u/d)\rceil$

$$\ldots = S\sum_{j=a}^{N}\binom{N}{j}\underbrace{\left(\frac{ug}{1+R}\right)}_{g\,'}{}^{j}\underbrace{\left(\frac{d(1-g)}{1+R}\right)}_{1-g\,'}{}^{N-j} - X(1+R)^{-N}\underbrace{\sum_{j=a}^{N}\binom{N}{j} g^{j}(1-g)^{N-j}}_{\Phi(N;a;g)}$$

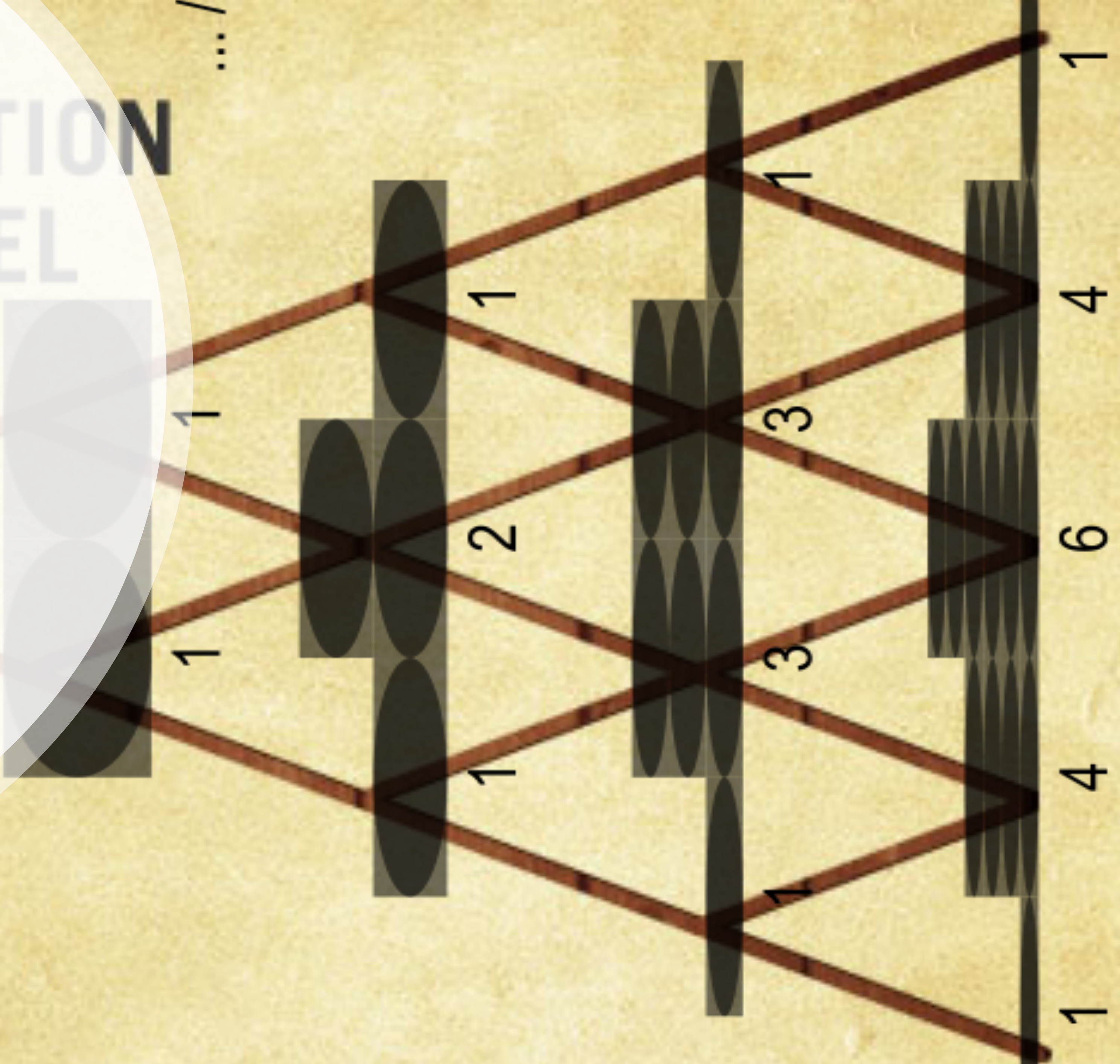
As g (and g’) can be interpreted as probability (see next section)

$\Phi(N;a;g) = 1 - CDF(Binom.(N,g))$

$$\boxed{c = S\Phi(N;a;g') - X(1+R)^{-N}\,\Phi(N;a;g)}$$

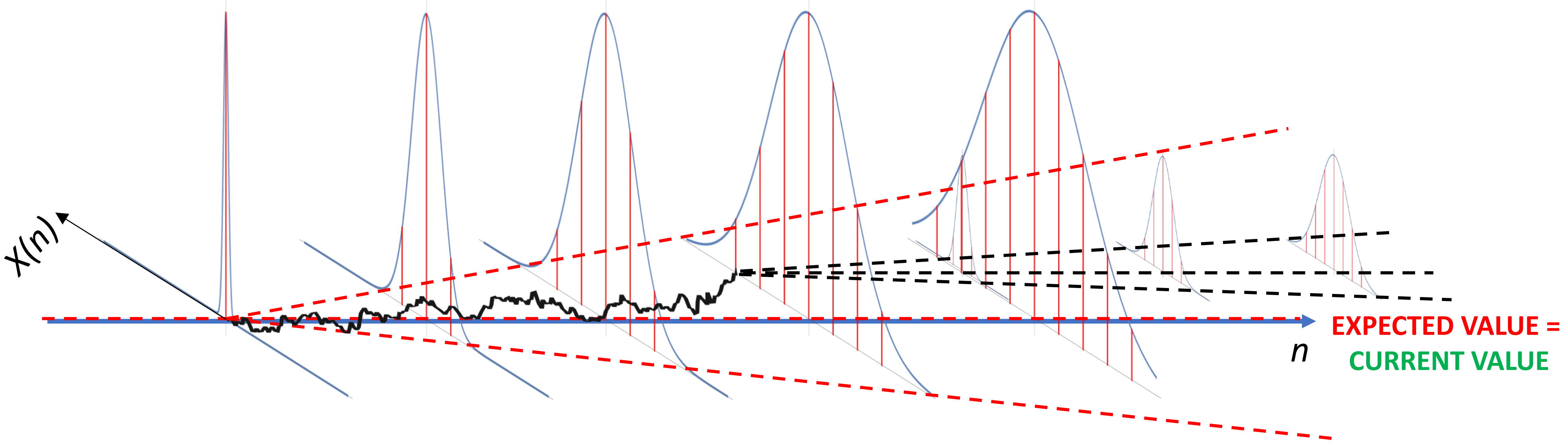
Option pricing in Binomial Model

- Stochastic processes
- One-step Binomial Model
- Multi-step Binomial Model
- "Analytic" formula for European options
- Risk-neutral / martingale pricing



Martingales (short introduction)

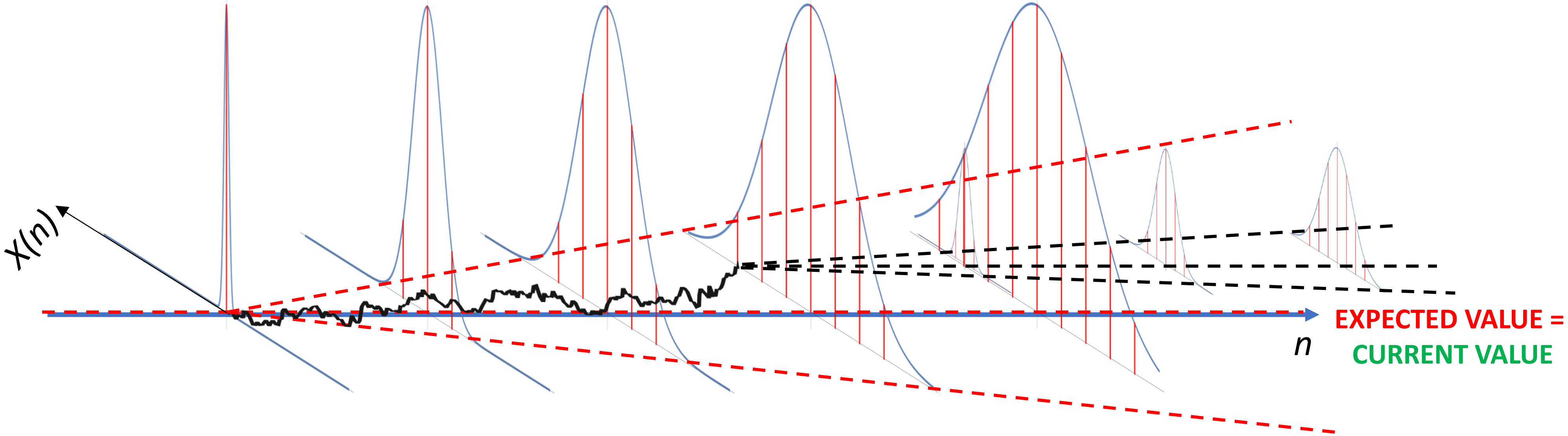
- Martingales are stochastic processes representing “fair games”, i.e. in each step the probability to win or loose is such that the expected player’s capital = current capital



- One can thus talk about conditional expectation of the next value (observation) $X(n+1)$ in a stochastic process sequence, given all prior values (observations): $X(0), X(1), \dots, X(n)$
- For a martingale one has: $E(X(n+1) \mid X(0), X(1), \dots, X(n)) = X(n)$ for any step n
- For a continuous-time martingale it is replaced by: $E(X(t') \mid \{X(\tau), \tau \leq t\}) = X(t)$: $\forall t \leq t'$

Martingales (short introduction)

- **Martingales** are stochastic processes representing "fair games", i.e. in each step the probability to win or loose is such that the expected player's capital = current capital



- A stochastic process {X(n)} can be also a martingale wrt another process {Y(n)} if:

  E(X(n+1) | Y(0), Y(1), …, Y(n)) = X(n) for any step n

- All information available at a time n, i.e. Y(0), Y(1), …, Y(n) is often called a "filtration"
- A process is "adapted to filtration" if for every realization: X(n) is known at time n

Risk-neutral / Martingale pricing

$$\sum_t < PV\big(CF(t)\big) > = 0 \quad (4)$$

- In Lecture 5 we showed that one can use Eqn. (4) to price some derivative instruments (e.g. forwards) if one adjusts the stochastic process of the underlying asset (e.g. the share) price in such a way that its present value is a MARTINGALE, i.e.: $\langle \mathrm{PV}(S(T))\rangle = S(0)$ for each time T
- In this Lecture we have shown that in the one-step Binomial model one has:

$$\mathrm{c} \;=\; \frac{g\, c(1)_u + (1-g)c(1)_d}{(1+R)} \;\;(*)\,,\;\; \text{where: } \mathrm{g} = \frac{(1+R)-d}{u-d}$$

- NOTE, that due to (static) ARBITRAGE arguments: d ≤ 1+R ≤ u
  - If d > 1+R: one can take a LOAN for S @ R% and BUY 1 SHARE @ S (NET CF=0), after one step one SELLS the share getting (at least) Sd and REPAYS the LOAN paying S(1+R) (NET CF > 0) ⇒ ARBITRAGE
  - If u < 1+R: one SELLS 1 SHARE SHORT @ S and makes DEPOSIT for S @ R% (NET CF=0), after one step one BUYS the share back paying (at most) Su and from the DEPOSIT one gets S(1+R) (NET CF > 0) ⇒ ARBITRAGE
- Thus: 0 ≤ g ≤ 1 is a PROBABILITY, and Eqn. (*) becomes: $c = \langle \mathrm{PV}(c(1))\rangle_g$, so it loos exactly like Eqn. (4)

$$\sum_t < PV(CF(t)) > = 0 \qquad (4)$$

Risk-neutral / Martingale pricing

❖In Lecture 5 we showed that one can use Eqn. (4) to price some derivative instruments (e.g. forwards) if one adjusts the stochastic process of the underlying asset (e.g. the share) price in such a way that its present value is a MARTINGALE, i.e.: $\langle \mathrm{PV}(S(T)) \rangle = S(0)$ for each time T

❖The same applies for the N-step Binomial model where one has:

$$c(n)_j = \frac{g\,\overbrace{c(n+1)_{j+1}}^{c_u} + (1-g)\overbrace{c(n+1)_j}^{c_d}}{(1+R)}\ , \qquad g = \frac{(1+R)-d}{u-d}$$

KNOWN CF(N) AT EXPIRY !

$c(N)_j = \max(S\,u^{j}\,d^{N\text{-}j} - X\ ;\, 0)$

❖Resulting in (see p. 43): $c = \sum_{j=0}^{N} \underbrace{\binom{N}{j} g^j (1-g)^{N-j}}_{\tilde{\mathcal{P}}_g(N)_j} \underbrace{(1+R)^{-N} c(N)_j}_{PV(CF(N))} \quad (**)$

For $0 \le g \le 1$ this is a *probability*
*in **Binomial(N;g)** distribution*

❖So again Eqn. $(**)$ becomes: $c = \langle \mathrm{PV}(c(N)) \rangle_g$ in accordance with Eqn. (4)

Risk-neutral / Martingale pricing

$$\sum_t < PV(CF(t)) > = 0 \quad (4)$$

- In Lecture 5 we showed that one can use Eqn. (4) to price some derivative instruments (e.g. forwards) if one adjusts the stochastic process of the underlying asset (e.g. the share) price in such a way that its present value is a MARTINGALE, i.e.: $\langle \mathrm{PV}(S(T))\rangle = S(0)$ for each time T
- If, in the Binomial model, the (true) share price evolution probability q is replaces by a (fictitious) probability $\mathrm{g} = \frac{(1+R)-d}{u-d}$, then one can explicitly check that:

$$\langle \mathrm{PV}(S(n+1))\rangle_{\mathrm{g}} = \mathrm{g}\,\frac{\mathrm{S(n)\ u}}{(1+R)^{n+1}} + (1-\mathrm{g})\,\frac{\mathrm{S(n)\ d}}{(1+R)^{n+1}} = \frac{\mathrm{S(n)}}{(1+R)^{n}} = PV(S(n))$$

- So: $\mathrm{E(PV(S(n{+}1))\ |\ S(0), PV(S(1)), \ldots, PV(S(n)))} = \mathrm{PV(S(n))}$

and the (discounted) share price process becomes a MARTINGALE* !

*This also automatically implies: $\langle \mathrm{PV}(S(n))\rangle_g = \sum_j \tilde{\mathcal{P}}_g(n)_j \frac{S(n)_j}{(1+R)^n} = S(0)$ for any n, where: $\tilde{\mathcal{P}}_g(n)_j = \binom{n}{j} g^j (1-g)^{n-j}$
so if the share price evolution is driven by martingale probability g ⇒ expected rate of return from shares = risk-free rate

"RISK-NEUTRAL" PRICING !!!

Risk-neutral / Martingale pricing

$$\sum_t < PV(CF(t)) > = 0 \quad (4)$$

- In Lecture 5 we showed that one can use Eqn. (4) to price some derivative instruments (e.g. forwards) if one adjusts the stochastic process of the underlying asset (e.g. the share) price in such a way that its present value is a MARTINGALE, i.e.: $\langle \mathrm{PV}(S(T)) \rangle = S(0)$ for each time T
- The option is a derivative instrument, whose price (stochastic process) depends on the share price (stochastic process)
    - if we artificially adjust the (discounted) share price process $\{PV(S(t))\}_t$ by changing probability q → probability g such that it becomes a martingale and thus Eqn. (4) is fulfilled by S(t), or in general:

      $$E(PV(S(t')) \mid \{PV(S(\tau)), \tau \le t\}) = PV(S(t)): \ \forall t \le t'$$
    - then also the option price process $\{c(t)\}_t$ becomes a martingale wrt (filtration of) the share price process, and thus Eqn. (4) is fulfilled also by c(t), or in general:

      $$E(PV(c(t')) \mid \{PV(S(\tau)), \tau \le t\}) = PV(S(t)): \ \forall t \le t'$$
    - this is TRUE ONLY IF the derivative (e.g. option) MARKET IS (DYNAMICALLY) COMPLETE, i.e. if options can be replicated by a portfolio of shares and risk-free instruments (loans/deposits)
- Summing up* (in the Binomial model): $\langle \mathrm{PV}(S(N)) \rangle_g = S(0) \ \Rightarrow \ c(0) = \langle \mathrm{PV}(c(S(N))) \rangle_g$

*The expectation value $\langle \quad \rangle_g$ is computed with probability measure $\tilde{\mathcal{P}}_g(n)_j = \binom{n}{j} g^j (1-g)^{n-j}$, where: $g = \frac{(1+R)-d}{u-d}$

$$\sum_t < PV\big(CF(t)\big) > = 0 \quad (4)$$

Radon-Nikodým derivative

- We have discussed that if the market is (dynamically) complete then options (and in general derivative instruments) can be priced using Eqn. (4) but one has to change the real probability measure $\mathcal{P}_q$ into the martingale probability measure $\tilde{\mathcal{P}}_g$ when computing $\langle\quad\rangle$.
- In order to do so one can use the, so called, (discrete) Radon-Nikodým derivative of $\tilde{\mathcal{P}}_g$ wrt $\mathcal{P}_q$:

$$\mathrm{Z} \equiv \frac{\tilde{\mathcal{P}}_g}{\mathcal{P}_q}$$

- For a continuous prob. distrib. one defines a (continuous) Radon-Nikodým derivative: $\mathrm{Z} \equiv \frac{\mathrm{d}\tilde{\mathcal{P}}_g}{\mathrm{d}\mathcal{P}_q}$
- Z is a random variable under probability measure $\mathcal{P}_q$ with the following properties:
  - ○ Pr(Z > 0) = 1
  - ○ $\langle\ Z\ \rangle_{\mathrm{q}} = 1$
  - ○ For any random variable V: $\langle\ V\ \rangle_{\mathrm{g}} = \langle\ V\ Z \rangle_{\mathrm{q}}$ ⟸ This enables to compute the expectation value in (4)* using prob. $\mathcal{P}_q$ instead of $\tilde{\mathcal{P}}_g$ if one knows Z

*E.g. for a continuous prob. distrib. one has: $\langle V\rangle_{\mathrm{g}} = \int V\,\mathrm{d}\tilde{\mathcal{P}}_g = \int V \frac{\mathrm{d}\tilde{\mathcal{P}}_g}{\mathrm{d}\mathcal{P}_q}\mathrm{d}\mathcal{P}_q = \int V\,Z\,\mathrm{d}\mathcal{P}_q = \langle V\ Z\rangle_{\mathrm{q}}$

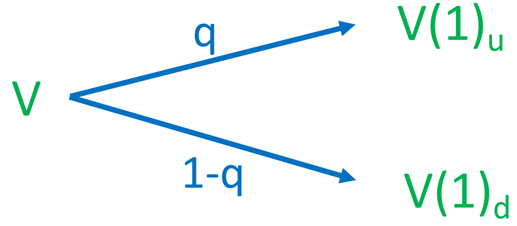
Radon-Nikodým derivative

$\langle V \rangle_g = \langle V \, Z \rangle_q$

$\sum_t < PV(CF(t)) > = 0 \quad (4)$

- We have discussed that if the market is (dynamically) comlete then options (and in general derivative instruments) can be priced using Eqn. (4) but one has to change the real probability measure $\mathcal{P}_q$ into the martingale probability measure $\tilde{\mathcal{P}}_g$ when computing $\langle \quad \rangle$.
- In order to do so one can use the, so called, (discrete) Radon-Nikodým derivative of $\tilde{\mathcal{P}}_g$ wrt $\mathcal{P}_q$:

$$Z \equiv \frac{\tilde{\mathcal{P}}_g}{\mathcal{P}_q}$$



- For a (discrete) one-step Binomial process one simply has:
  - $Z(1)_u = \frac{g}{q}$ with probability q and $Z(1)_d = \frac{(1-g)}{(1-q)}$ with probability 1-q
  - So trivially: $\langle V \, Z \rangle_q = V(1)_u \, Z(1)_u \, q + V(1)_d \, Z(1)_d \, (1-q) = V(1)_u \, g + V(1)_d \, (1-g) = \langle V \rangle_g$
- For a multi-step Binomial process for each n:
  - $Z(n)_j = \frac{\tilde{\mathcal{P}}_g(n)_j}{\mathcal{P}_q(n)_j} = \left(\frac{g}{q}\right)^j \left(\frac{1-g}{1-q}\right)^{n-j}$ with probability $\mathcal{P}_q(n)_j$, j= 0,…, n
  - One can show that the Radon-Nikodým derivative process $\{Z(n)\}_n$ is a martingale under measure $\mathcal{P}_q$

$$E(Z(n+1) \mid Z(0), \ldots, Z(n)_j) = Z(n)_j$$

Summary

THE END ;)