Lecture 1

Time Value of Money
(Basic Financial Calculus)

Financial instruments and pricing

Fall 2019

Time Value of Money

Having money NOW is more valuable than having money LATER *

Value of 1$  at PRESENT > Value of 1$  in the FUTURE

Time value of money is the difference between a (nominal) amount of money in the present and that same amount of money in the future

* „Classical" financial paradigm (starts to change now)

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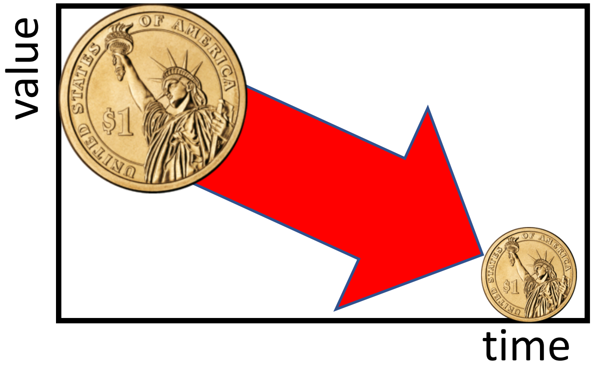
As time flows the value of money declines



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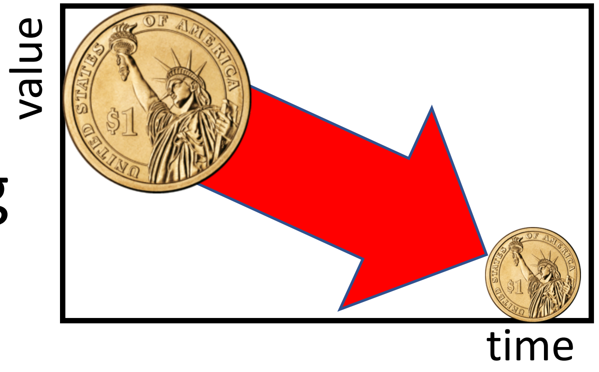

As time flows the value of money declines
This is due to the combined impact of the following

- Cost of money (opportunity cost): interest %
- Inflation (real vs nominal interest %)
- Risk (e.g. counterparty or transaction risk)

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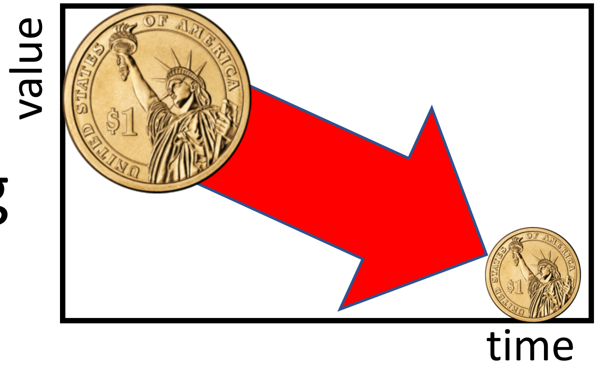

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Cost of Money: advanced spending

A dollar that you have today is worth more than the promise or expectation that you will receive a dollar in the future

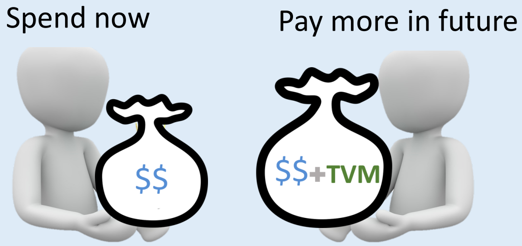
❖ Consumers are willing to pay interest (on their credit cards or bank loans) for the opportunity to receive cash now and advance spending

Example:

❑ To **spend** (pay) **$100 now** one borrows for one year, paying 5% interest, and will have to return (pay back) **$105 after one year**

❑ Therefore, **$100 paid now** *and* **$105 paid exactly one year later** *both* have the same value to a recipient who expects 5% interest

❑ Or alternatively: **$100 borrowed now** for one year at 5% interest has a ***future value*** **of $105**



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Cost of Money: opportunity cost

A dollar that you have today is worth more than the promise or expectation that you will receive a dollar in the future

❖ Investors are willing to forgo spending now if they expect a favorable return on their investment in the future (you invest your dollar today and earn interest)

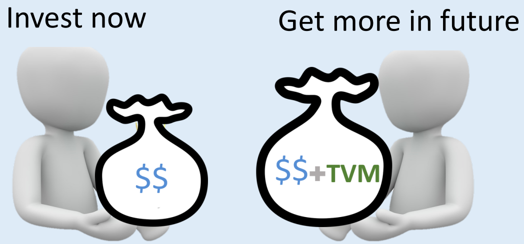
❖ „You cannot have your cake and eat it too"

Example:

❑ **$100 invested now** for one year, earning 5% interest, will be worth **$105 after one year**

❑ Therefore, **$100 paid now** *and* **$105 paid exactly one year later** *both* have the same value to a recipient who expects 5% interest

❑ Or alternatively: **$100 invested now** for one year at 5% interest has a ***future value* of $105**



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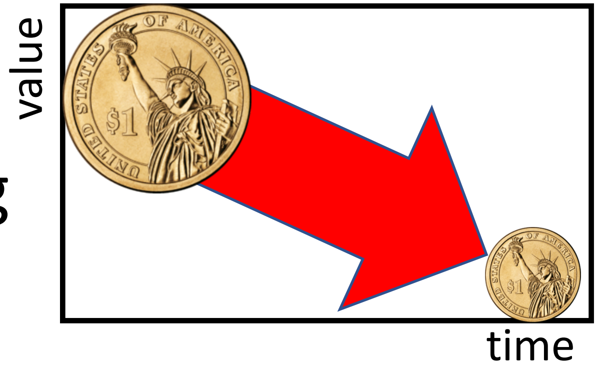

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Inflation

❖ Inflation erodes the value of money as it decreases future purchasing power

❖ All financial calculus can be done either in nominal or real values
(real % ≈ nominal % – inflation %)

❖ In the following we **will stick to the nominal scheme !** (it is simpler, more illustrative and commonly used as future inflation is a priori unknown)

Example: One invests **$100** and expects to get **$105** in one year.
Assume future (annual) **inflation of 2%**.
Due to inflation in a year one can buy less goods by a factor: 1 / (1+0.02) = 0.980

❑ Nominal calculus ("current prices"):
FV($100) = $100 (1 + 0.05) = $105 or PV($105) = $105 / (1 + 0.05) = $100
So the nominal interest % (rate of return) is 5% p.a.

❑ Real calculus ("constant prices"):
$105 in constant prices ("amount of goods") is worth only $105/1.02=$102.94
FV($100) = $100 (1.0294) = $102.94 or PV($102.94) = $102.94/1.0294 = $100

❑ Real % = (1 + 0.05) / (1 + 0.02) – 1 = 2.94 % (≈ 5% - 2% = 3%)

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- ❑ Delayed receipts of cash or financial transactions are risky (e.g. default / bankruptcy / credit risk)
- ❑ „a bird in the hand is worth two in the bush”
- ❑ It will be discussed in „Risk Management” Lectures (summer semester)
- ❑ Here we assume that various investments belong to the same credit risk class

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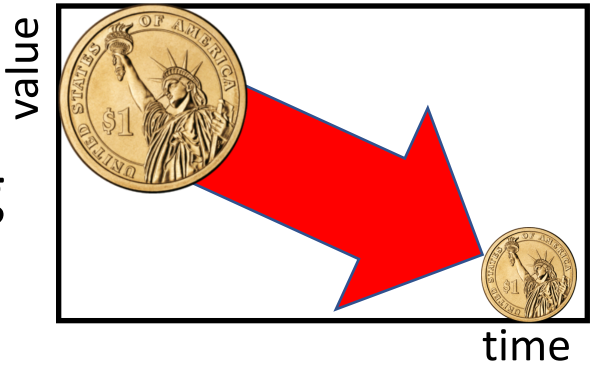

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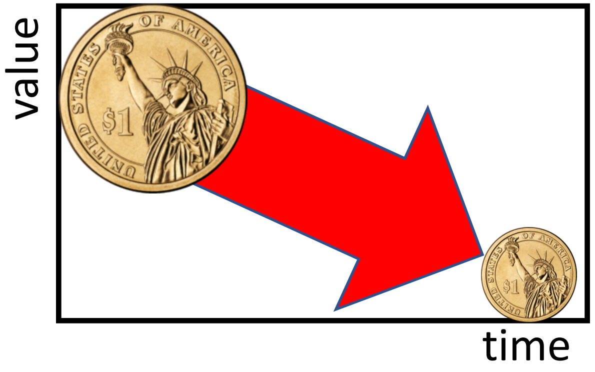
Cost of money = interest % > 0



The paradigm (*) starts to change now

- Should be at least true when one considers real interest % (real % ≈ nominal % – inflation % > 0) so that one could buy more goods in the future (in reward for postponed consumption). But economic data do not comply.

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Cost of money = interest % ≠ 0

	1Y rate %		
Switzerland	-0.6 %		
Eurozone	-0.3 %		

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	1Y rate %	Inflation %	Real %
Switzerland	-0.6 %	0.2 %	-0.8 %
Eurozone	-0.3 %	0.9 %	-1.2 %
Poland	1.8 %	2.8 %	-1.0 %

- Should be at least true when one considers real interest % (real % ≈ nominal % – inflation % > 0) so that one could buy more goods in the future (in reward for postponed consumption). But economic data do not comply.

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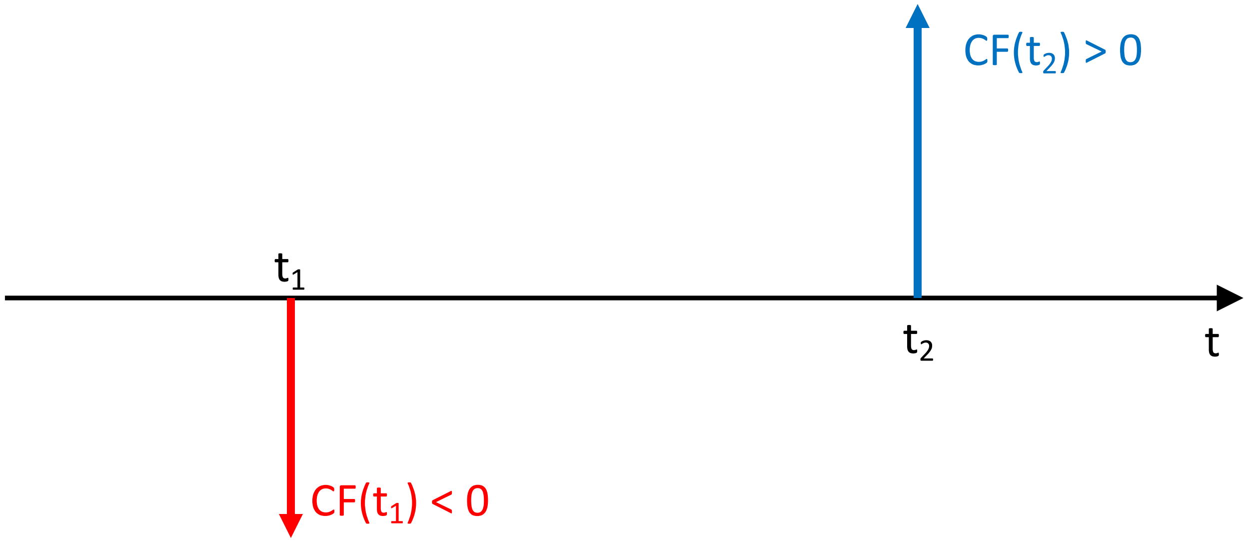
Cash Flows

Value of 1\$  at PRESENT ≠ Value of 1\$  in the FUTURE

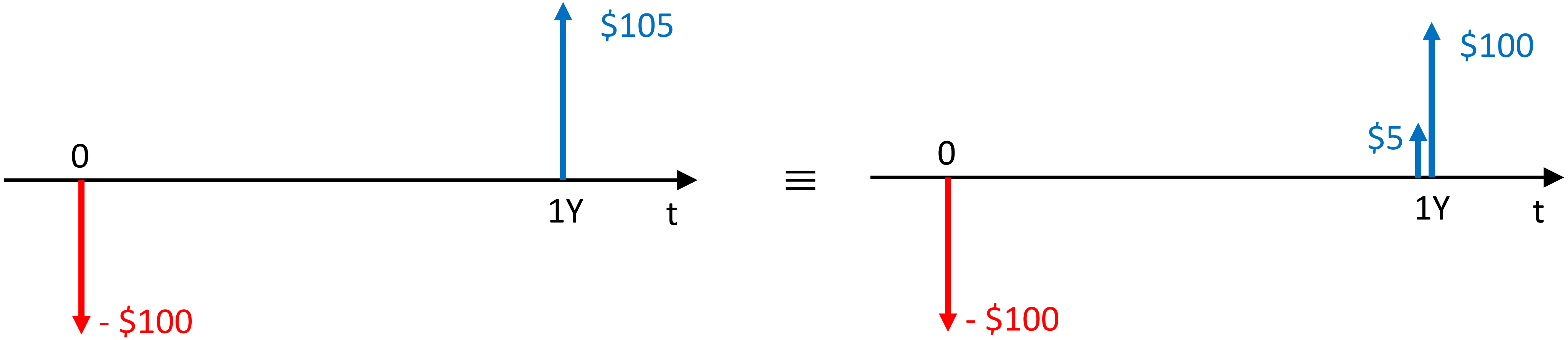
- ❖Cost of money = interest rate %
- ❖In finance interest rate % is used to "move" Cash Flows in time
- ❖Cash flow: CF(t) = amount of cash paid or received in time t
- ❖One usually sets now (present day): t = 0 and expresses [t] in years
- ❖Convention: received CF > 0 , paid CF < 0
- ❖Sign depends on the transaction side, e.g. initial CF is negative (-) for a buyer (he pays) and positive (+) for the seller (he receives money)

Cash Flows: visualizing CF (“arrow charts”)



Example:

Bank deposit of $100 for 1Y paying 5% interest (for the depositor)



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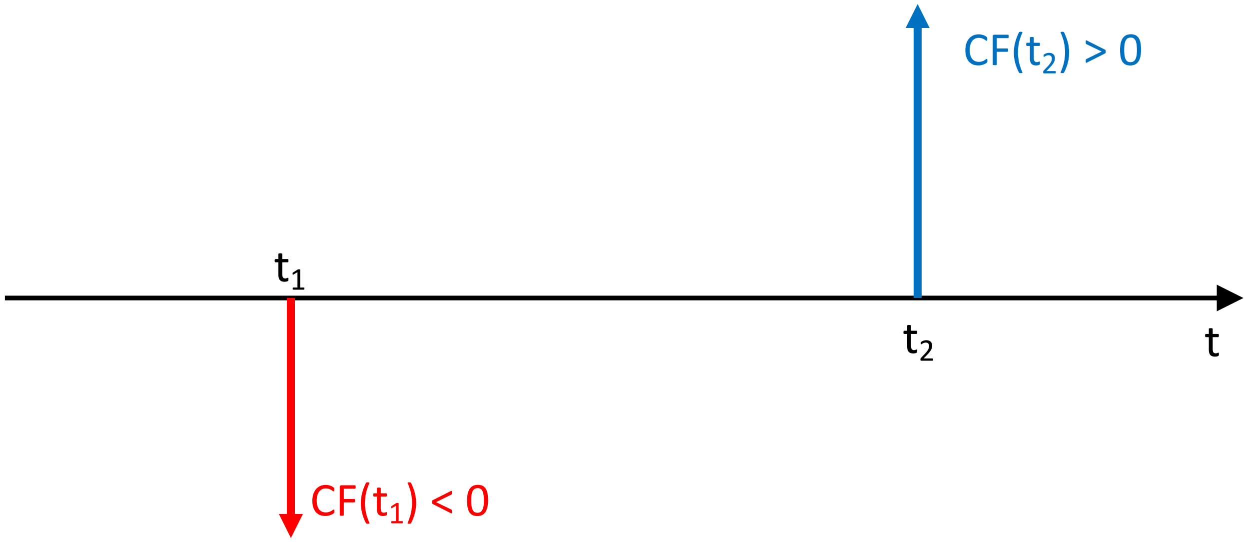
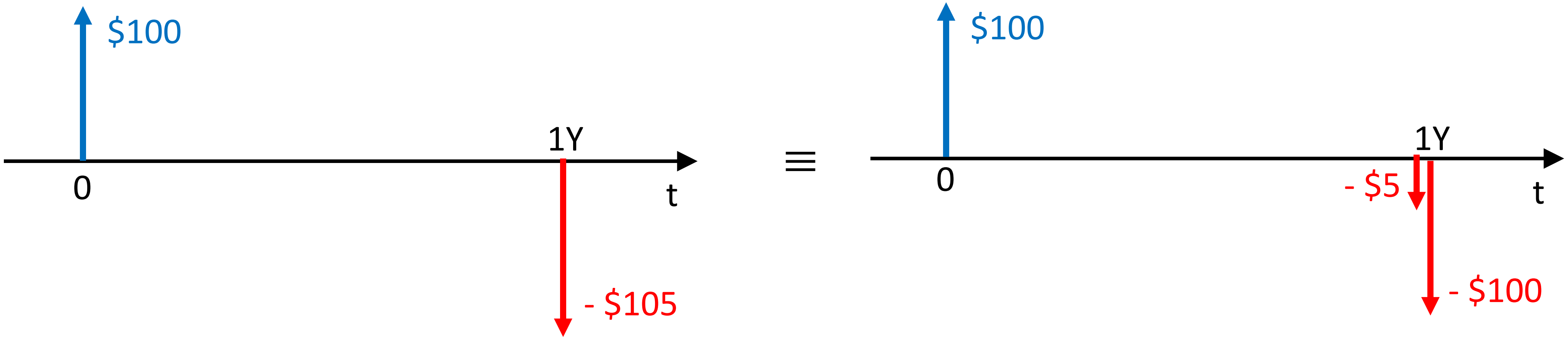
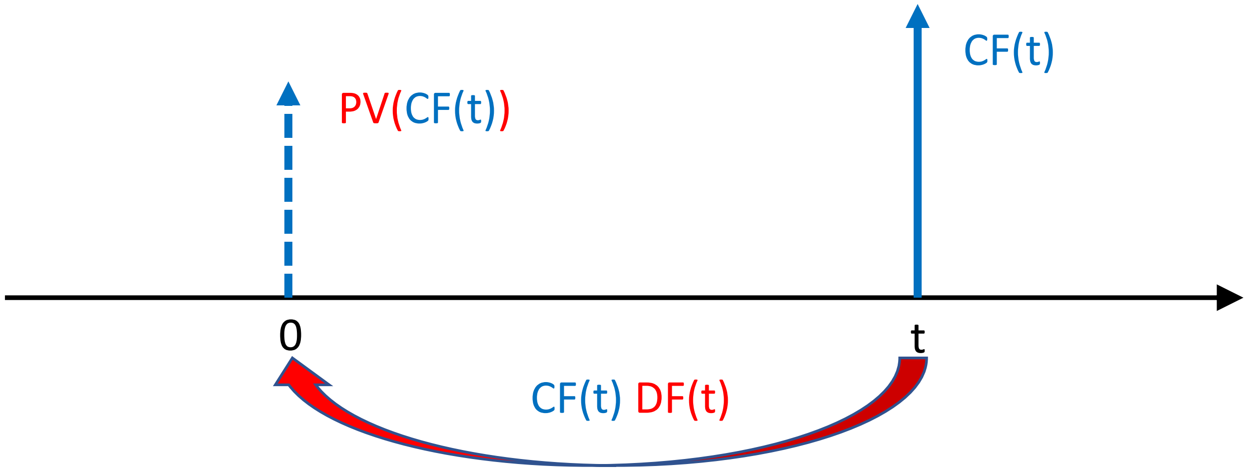


Example:

Bank deposit of $100 for 1Y paying 5% interest (for the bank)



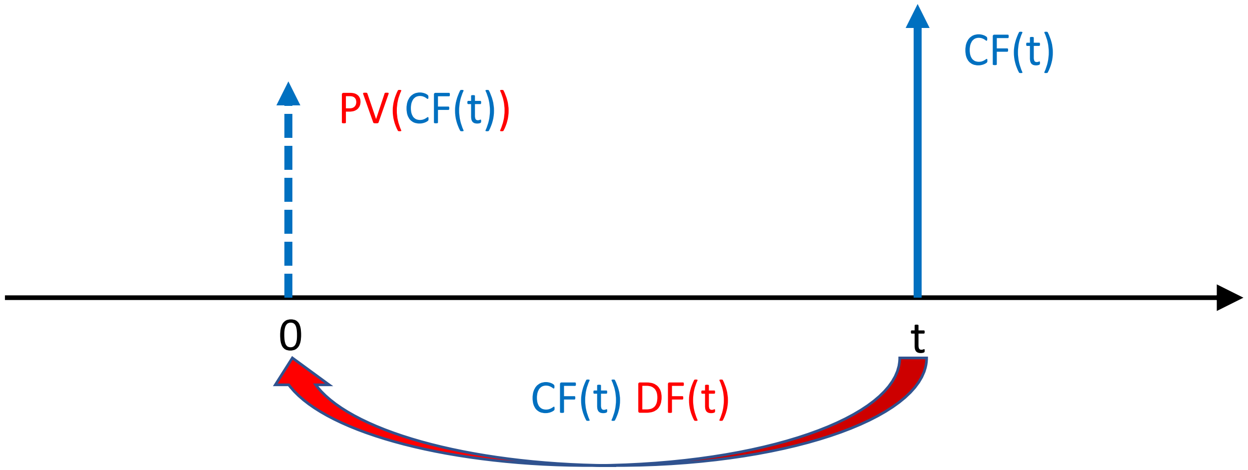
Moving CFs in t: Present Value (discounting)



- In order to compare CFs received/payed in different moments of time one needs to translate them into CFs in the same (universal) time
- One usually chooses t=0 (now): Present Value (PV)
- To „move CFs in time" one uses the concept of Time Value of Money, i.e. CF(0) is worth the same as $CF(t) = CF(0)\,(1 + R(t))$, where R(t) is the cost of money

  $PV(CF(t)) = CF(t) / (1 + R(t)) = CF(t)\,DF(t)$, where: $DF(t) = 1 / (1 + R(t))$
- DF(t) is called the Discount Factor

Moving CFs in t: Present Value (discounting)



$$PV(CF(t)) = CF(t)\,DF(t) \qquad DF(t) = 1 / (1 + R(t))$$

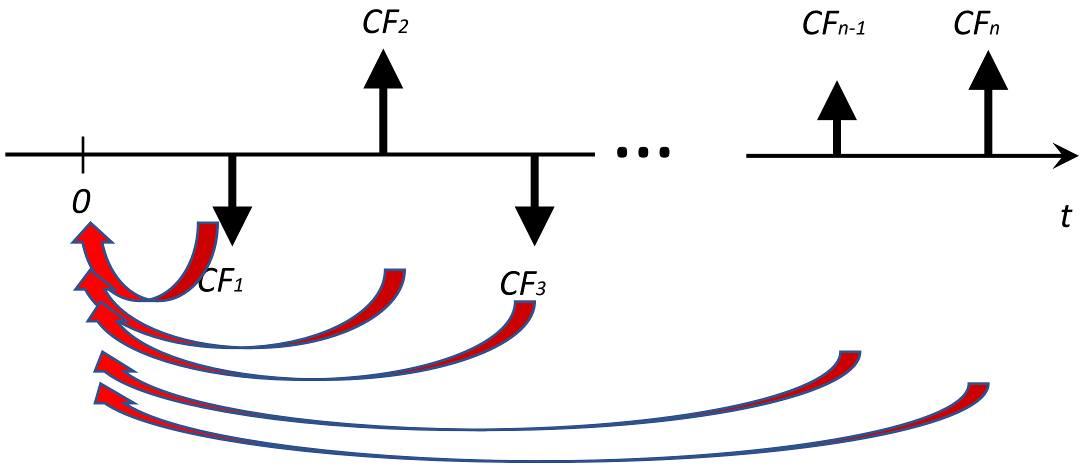
- Note that R(t) is the cost of money = interest rate from 0 to t, i.e. the unit is [%], so the unit varies for each t
- It is convenient to express DF(t) in terms of the interest rate y(t) having a universal unit of time, usually [% / 1 year] (”per annum”, p.a.)

$$DF(t) = 1 / (1 + y(t))^t$$

Moving CFs in t: Present Value (discounting)

$$PV(CF(t)) = CF(t)\ DF(t) \qquad\qquad DF(t) = 1 / (1 + y(t))^t$$

❖ In this case the (Net) Present Value (NPV) of the stream of future CFs (e.g. an investment, or an amortized loan) will be



$$NPV = \sum_{t>0} CF(t) DF(t) = \sum_{t>0} \frac{CF(t)}{(1 + y(t))^t}$$

Moving CFs in t: Present Value (discounting)

$$NPV = \sum_{t>0} CF(t)DF(t) = \sum_{t>0} \frac{CF(t)}{(1+y(t))^t}$$

- Note that for an investment (e.g. a deposit) NPV > 0 (one would invest only if PV of all future CFs > 0), and it is just equal the current value of investment which one has to pay now, i.e. it is (minus !) CF(0) < 0
- For a loan (for a borrower) NPV < 0 (one has to pay back or amortize the loan, so PV of all future CFs < 0), and it is just equal the current value of the loan which one receives now, i.e. it is (minus !) CF(0) > 0
- So in general one has:

$$-\mathrm{CF}(0) = NPV(\text{future CFs}) = \sum_{t>0} CF(t)DF(t) = \sum_{t>0} \frac{CF(t)}{\left(1+y(t)\right)^t}$$

Moving CFs in t: Present Value (discounting)

$$\sum_t CF(t)DF(t) = \sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \qquad (1)$$

- Where the initial CF(0) has been included in the sum !
- Expression (1) is very general, as will be shown during next lectures it can be used to evaluate most financial instruments
- Now we will discuss the simplest case where future CFs are known or can be computed in a straightforward way
- This is the case of bank deposits / loans, bonds and other simple fixed income instruments (assuming they do not hold bankruptcy / default / credit risk)
- For more complicated instruments, e.g. (some) derivatives one can use (1) but one has to "forecast" future CFs or their expectation values (will be discussed in future lectures)

Effective Interest Rate: *y*

$$\sum_t CF(t)DF(t) = \sum_t \frac{CF(t)}{(1+y(t))^t} = 0 \qquad (1)$$

- In general y(t) may vary from time to time (it is some, a priori unknown function of t)
- Having just one financial instrument (e.g. investment or a loan) with more than two CFs (i.e. more than CF(0) and CF(T)) one cannot solve unambiguously for y(t) *
- *This can be done when one has the whole collection of various instruments (will be discussed in future lectures)
- But one can make (a simplifying) assumption that y(t) = y = const. (so that we have the same interest rate for each period of time)

Effective Interest Rate: *y*

$$\sum_t CF(t)DF(t) = \sum_t \frac{CF(t)}{(1+y)^t} = 0 \qquad (2)$$

- The rate *y* computed this way is called the Effective Interest Rate (EIR)
- Other names: Yield (y), (Internal) Rate of Return (IRR), PL: Rzeczywista Roczna Stopa Oprocentowania (RRSO), …
- In general EIR *y* can be computed for any (or at least most) financial instruments
- But one should keep in mind that it has some disadvantages:
  - it may be negative (but that's OK as the classical financial paradigm has changed)
  - to compute y from (2) one has to solve the polynomial equation: there may be more than one real solution*
  - * fortunately a single *y* for "simple" instruments (if sign(CF(0) ≠ sign(future CFs))

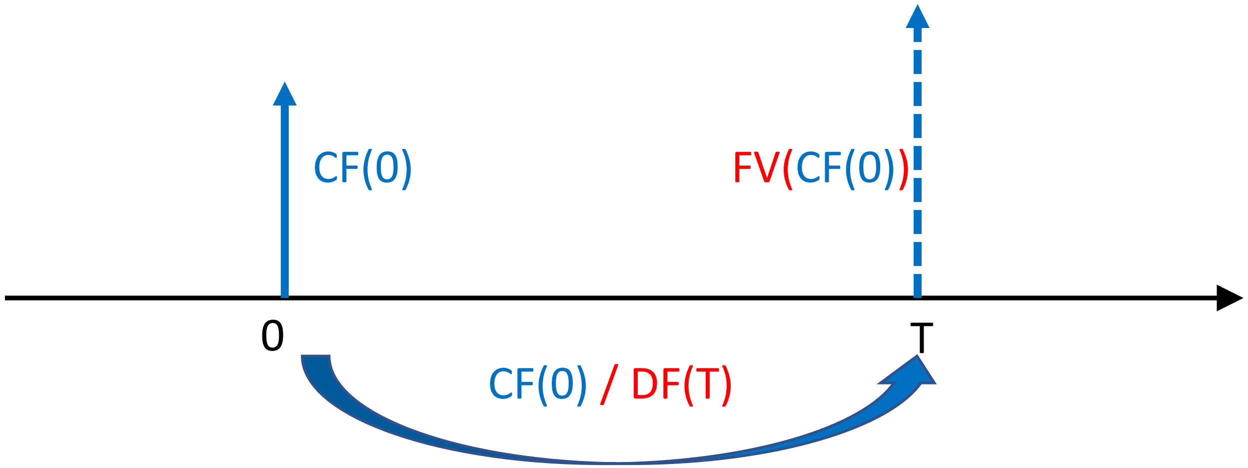
Effective Interest Rate: *y*

$$\sum_t CF(t)DF(t) = \sum_t \frac{CF(t)}{(1+y)^t} = 0 \qquad (2)$$

- Effective Interest Rate *y* does not have to be equal to the Nominal Interest Rate *r* *, i.e. the one stated in the (legal) contract to compute interest (e.g. deposit / loan agreement, coupons paid by a Bond, ...), it can be either higher or lower as will be shown in a minute
- Nominal rate is just used to compute future interest (CFs)
- Effective rate is the one important in finance (i.e. it is used to compare various financial instruments, e.g. deposits or loans)

* Note: "Nominal" has two meanings, another meaning of "Nominal" is Nominal vs Real ("real = nominal – inflation", as already discussed)

Moving CFs in time: Future Value (compounding)

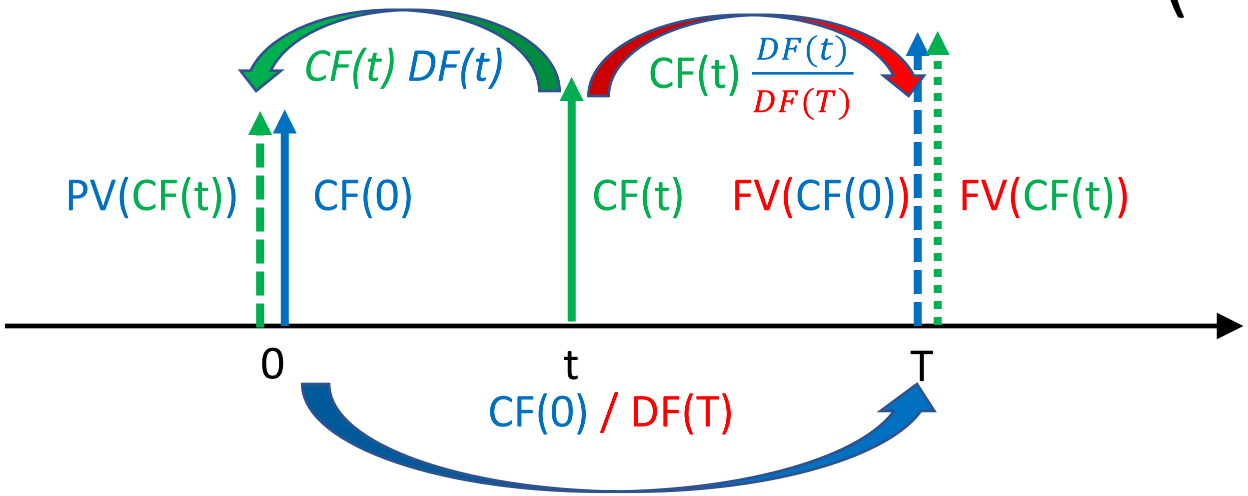


- In some cases it is convenient to use Future Value (FV), i.e the one computed for t=T (T: time of the last CF) *
- FV of the initial CF(0) is simply

$$FV(CF(0)) = CF(0)\,(1+y(T))^{T} = CF(0)/DF(T)$$

* One could as well choose any other moment of time t (but it is rarely used)

Moving CFs in time: Future Value (compounding)



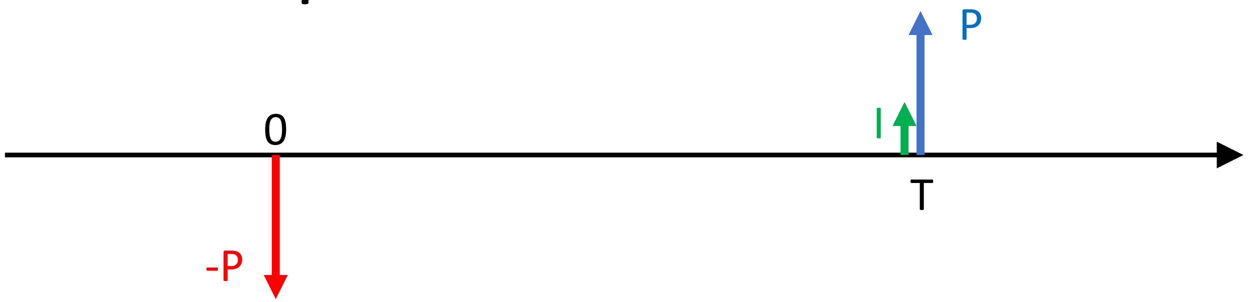
- It is also trivial to show that for any CF(t) one has

$$FV(CF(t)) = CF(t)\,\frac{(1+y(T))^T}{(1+y(t))^t} = CF(t)\,\frac{DF(t)}{DF(T)}$$

- One may think of it as moving CF(t) back in time to t=0 and then forward in time to t=T
- For a constant EIR y it simplifies to: $FV(CF(t)) = CF(t)\,(1+y)^{T-t}$
- Note that our general formula (1) or (2): $\sum_t CF(t)DF(t) = 0$ stays unchanged !!!

Example 1: Simple Interest

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



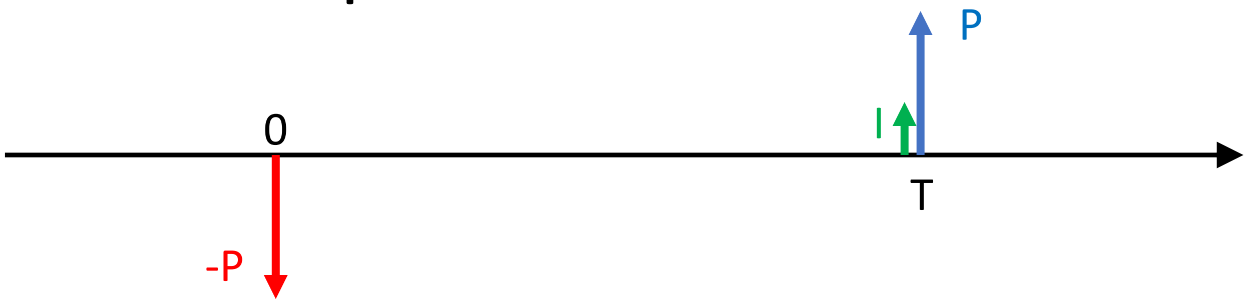
- A Bank offers a standard deposit P = 1000 PLN for T = 2 years with interest rate r = 3% p.a. (principal amount P and interest are paid in the end)
- Interest: I = P × r × T = 1000 PLN × 0.03 / year × 2 years = 60 PLN *
- Initial: CF(0) = -P = -1000 PLN
- Final: CF(T) = P + I = P (1 + r T) = 1060 PLN
- Effective Interest Rate: $CF(0) + CF(T) / (1+y)^T = 0 \Rightarrow y = 2.96\ \% \neq r$
- In general y < r if T > 1 and y > r if T < 1 (y=r if T=1)

* Note: for short time deposits T is a fraction of a year, so T = days of deposit / 365 **

** This may depend on the, so called, day count convention (discussed in next lecture)

Example 2: Compound Interest

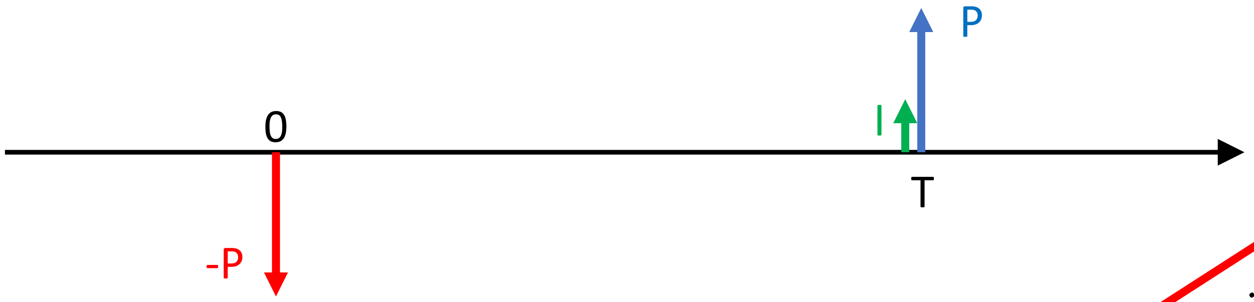
$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



- A Bank offers a savings account for P = 1000 PLN for T = 2 years with interest rate r = 3% p.a. (principal amount P and interest are paid in the end) but the bank adds (capitalizes) accrued interest n=2 times a year, so that interest in next period is computed on a higher balance of the account (principal + accumulated interest)
- Initial: CF(0) = -P = -1000 PLN
- Final: $CF(T) = P + I = P\,(1 + r\,/\,n)^{n\,T}$ = 1000 PLN $(1 + 0.03/2)^{4}$= 1061.36 PLN
- Interest: $I = CF(T) - P = P\,((1 + r\,/\,n)^{n\,T} - 1)$
- Effective Interest Rate: it is easy to show that: $y = (1 + r/n)^n - 1 \Rightarrow y = 3.02\ \% \neq r$

Example 3: Continuous Interest

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



$$FV(P) = P\,(1 + r\,/\,n)^{n\,T}$$

$$1 + y = (1 + r/n)^n$$

This explains why we have: $(1+y)^t$ EIR y is in fact anually compounded rate

- In general (for positive r): $y < r$ if $n < 1$ and $y > r$ if $n > 1$ ($y=r$ if $n=1$): compound interest is good for a depositor but it is bad for the borrower
- The higher n the higher y, in the limit $n \to \infty$ one obtains

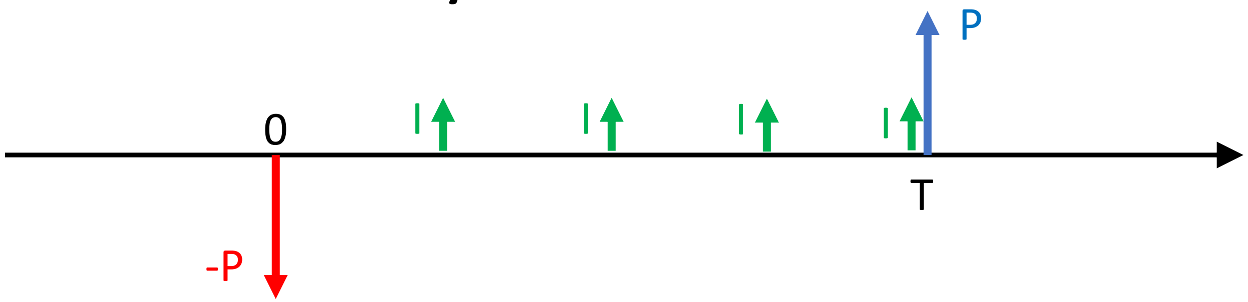
$$1 + y = \exp r$$

$$\sum_t \frac{CF(t)}{\exp(r_c\,t)} = 0$$

- The above r is called continuous(ly) (compounded) rate
- If interest is added continuously one has $dP(t) = P(t)\,r_c\,dt$ and thus $P(t) = P(0)\exp(r_c\,t)$
- Sometimes it is convenient to use effective cont. rate r_c (it has a dim. of % / year, p.a.)

Example 4: Annuity

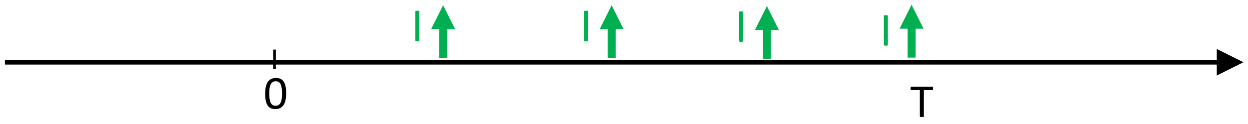
$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



- A Bank offers a deposit for P = 1000 PLN for T = 2 years with interest rate r = 3% p.a. This time interest is paid n = 2 times a year (principal amount is constant and paid in the end)
- Initial: CF(0) = -P = -1000 PLN
- Final: CF(T) = P = 1000 PLN
- Interest (payed every ½ year: CF(t=i/n), i = 1,…,nT) is now constant: I = P×r/n = 15 PLN
- The interest payments are an example of an annuity, i.e. a series of equal CFs (payments or receipts) that occur at evenly spaced intervals and the first payment takes place in the future (1st in t > 0, last in t = T)

Example 4: Annuity

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



❖One may show that:

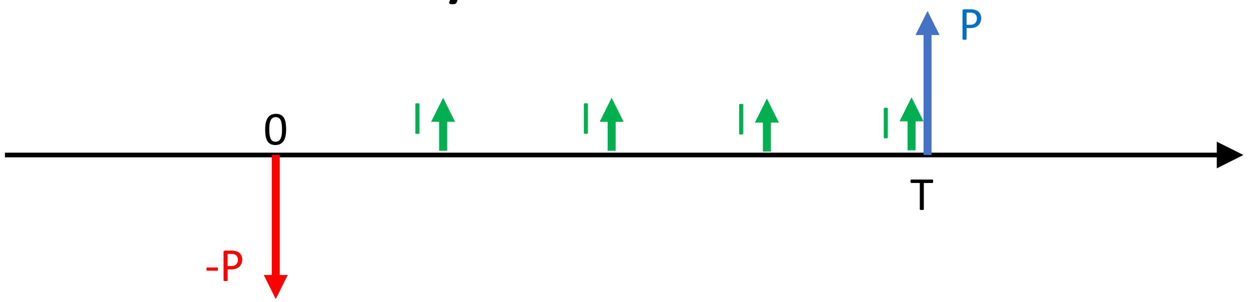
$$\text{PVA} = \sum_{i=1}^{nT} \frac{I}{(1+y)^{i/n}} = \frac{I}{(1+y)^{1/n-1}} \left(1 - \frac{1}{(1+y)^T}\right)$$

❖For n=1 (equal annual payments, since the name "annuity") PVA simplifies to:

$$\text{PVA} = \frac{I}{y}\left(1 - \frac{1}{(1+y)^T}\right)$$

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$

Example 4: Annuity



❖ One may show that:

$$\mathrm{PVA} = \sum_{i=1}^{nT} \frac{I}{(1+y)^{i/n}} = \frac{I}{(1+y)^{1/n}-1}\left(1-\frac{1}{(1+y)^T}\right)$$

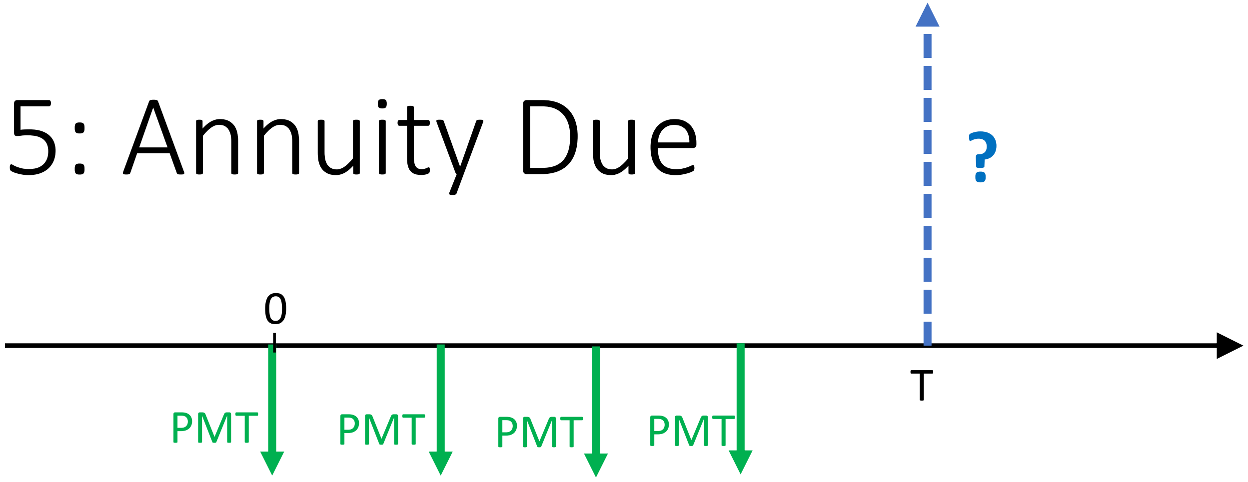
❖ Effective Interest Rate:

$$-P + PVA + P/(1+y)^T = 0 \;\Rightarrow\; PVA = P\left(1-\frac{1}{(1+y)^T}\right) \;\Rightarrow\; \frac{I}{(1+y)^{1/n}-1} = P$$

- ❑ using $I = P\,r\,/\,n$ one immediately gets: $y = (1 + r/n)^n - 1 = 3.02\ \% \neq r$
- ❑ this is exactly the same as for the compound interest !!!
- ❑ EIR assumes that one can reinvest CFs with rate of return y, so it does not matter if interest is paid off or capitalized (NOT ALWAYS TRUE ! => see Exercise 8 in Set 1)

Example 5: Annuity Due

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



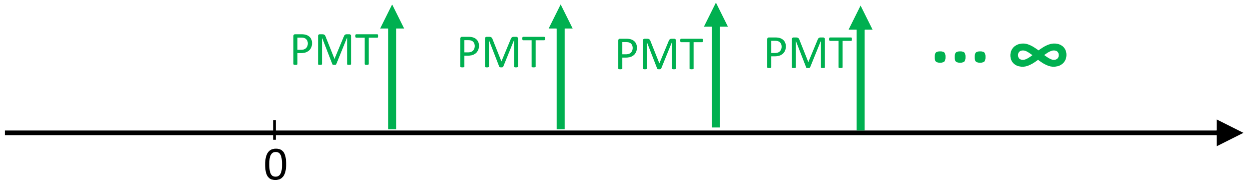
- A client plans to put aside PMT = 1000 PLN each year on its savings account (which offers annual capitalization of interest: n = 1). First CF is made now (in t = 0). What will be the balance of the account after T = 4 years if interest rate is r = 3% p.a. ?
- The series of PMTs is an example of annuity due, i.e. a series of equal CFs (payments or receipts) that occur at evenly spaced intervals and the first payment takes place immediately (1st in t = 0, last in t < T)
- In this case one can use (a simplified: n = 1) formula for PVA and transfer it to PVAD:

$$\mathrm{PVAD} = \sum_{i=0}^{T-1} \frac{PMT}{(1+y)^i} = \sum_{i=1}^{T} \frac{PMT}{(1+y)^i}(1+y) = PVA(1+y)$$

- And then transfer it to: $\mathrm{FVAD} = \mathrm{PVAD}(1+y)^T = \frac{PMT}{y}\left(1 - \frac{1}{(1+y)^T}\right)(1+y)^{T+1} = 4\ 309.14\ \mathrm{PLN}$

Example 6: Perpetuity

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



❖From mid 18th to early 20th century British and U.S. governments issued perpetual bonds*, called Consols, which gave right to the infinite series of fixed annual payments. Compute what would be the price of $r = 4\%$ p.a. consols with principal value $P = \$50$ if investors expect $y = 3\%$ p.a. effective rate of return ?

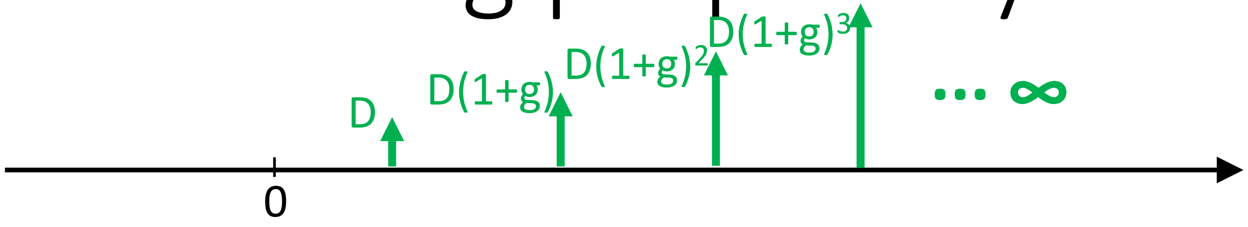
❖One can again use (a simplified: $n = 1$) formula for PVA and take $T \to \infty$ limit:

$$\mathrm{PVA} = \frac{PMT}{y}\left(1 - \frac{1}{(1+y)^T}\right) \xrightarrow{T\to\infty} \mathrm{PVP} = \frac{PMT}{y} = \frac{\$50 \times 0.04}{0.03} = \$\ 66.67$$

* In fact they were not really perpetual as could be redeemed at the option of the government (which happened)

Example 7: Growing perpetuity

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



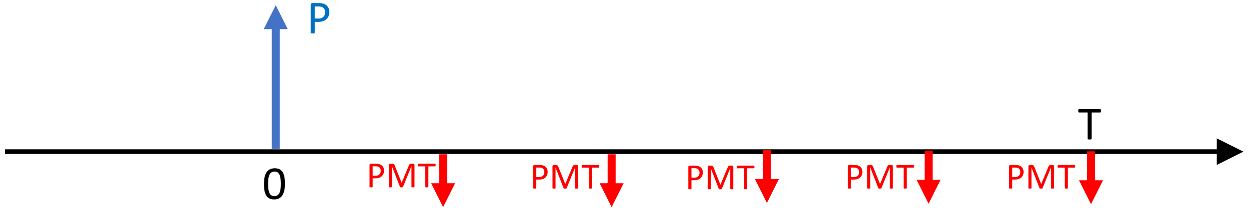
- Consider a dividend paying stock. The current dividend yield (i.e. dividend / price) is r = 4% p.a. Assume a simple pricing model based on perpetual dividends which rise each year by g = 2%. What is the effective rate of return y on the stock expected by investors ?
- It is straightforward to show:

$$\mathrm{PVGP} = \frac{D}{y-g}$$

- The dividend yield is: r = D / PVGP
- One immediately gets: y = r + g
- So the yield (expected rate of return) is simply dividend yield + growth rate

Example 8: Loan Amortization

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



❖A Bank offers a Loan for P = 10 000 PLN for T = 5 years with interest rate r = 5% p.a. The loan will be repaid using the Annuity Amortization Method* (i.e. each year the borrower pays the same amount PMT, out of which part goes to interest payment and part to principal payment: PMT = I + ΔP).

❖Compute the payment PMT done each year.

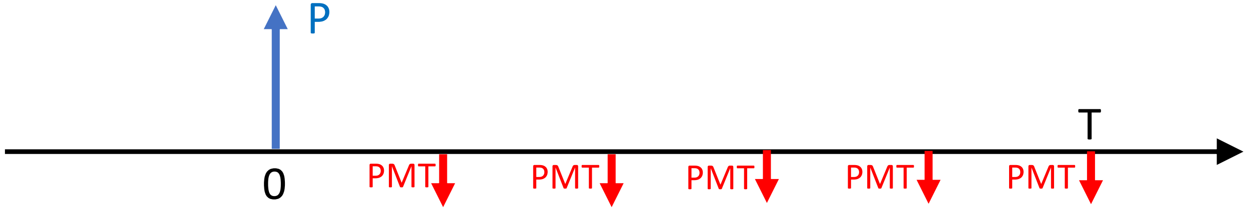
This amortization method is an annuity (and, as n=1 Cash Flow / year, y = r):

$$\mathrm{P} = \mathrm{PVA} = \sum_{i=1}^{T} \frac{PMT}{(1+r)^i} = \frac{PMT}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

*There are many possible loan amortization schemes. Another popular one is constant principal payments scheme (see Exercise 9 in Set 1)

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$

Example 8: Loan Amortization



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- Compute the payment PMT done each year.

Solving for PMT one immediately gets:

$$PMT = P\, r \left(1 - \frac{1}{(1+r)^T}\right)^{-1} \implies PMT = 2\,309.75\ PLN$$

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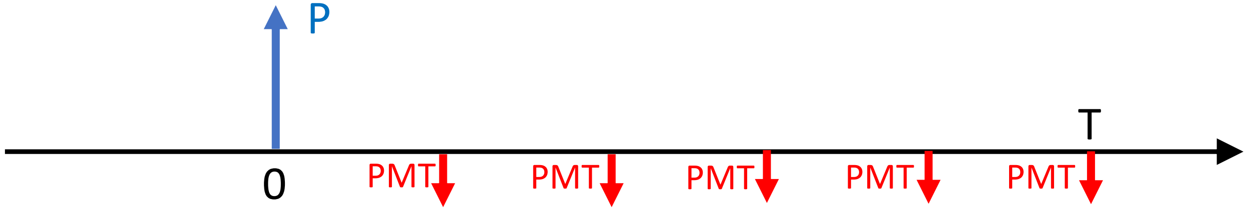
Example 8: Loan Amortization

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



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  Solving for PMT one immediately gets:

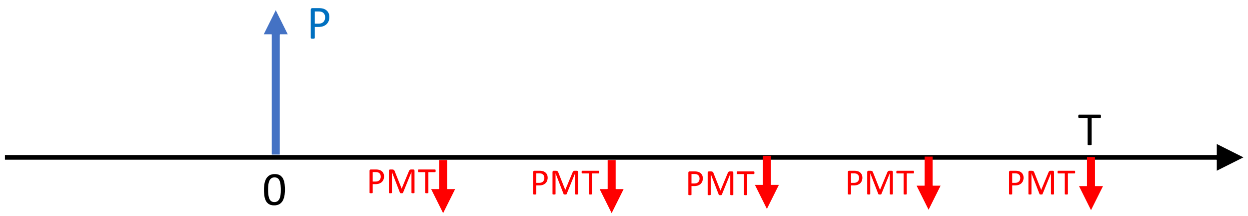
$$PMT = P\,r\left(1 - \frac{1}{(1+r)^T}\right)^{-1} \Rightarrow PMT = 2\,309.75\ PLN$$

- Prepare the Loan Amortization Schedule, i.e. a table showing how the loan is repaid

*There are many possible loan amortization schemes. Another popular one is constant principal payments scheme (see Exercise 9 in Set 1)

Example 8: Loan Amortization

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



❖A Bank offers a Loan for P = 10 000 PLN for T = 5 years with interest rate r = 5% p.a.

Loan Amortization Schedule (in PLN)

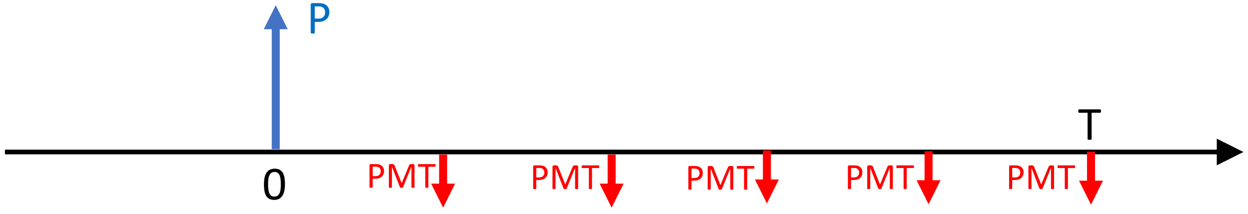
Period t	Initial Principal P(t-1)	Payment CF PMT(t)	Interest CF I(t)	Principal CF ΔP(t)	Final Principal P(t)
1	10 000,00	2 309,75	500,00	1 809,75	8 190,25
2	8 190,25	2 309,75	409,51	1 900,24	6 290,02
3	6 290,02	2 309,75	314,50	1 995,25	4 294,77
4	4 294,77	2 309,75	214,74	2 095,01	2 199,76
5	2 199,76	2 309,75	109,99	2 199,76	-

PMT = 2 309.75 **I(t) = P(t-1) r** **ΔP(t) = PMT – I(t)** **P(t) = P(t-1) -ΔP(t)**

PMT = I + ΔP

Example 8: Loan Amortization

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



- A Bank offers a Loan for P = 10 000 PLN for T = 5 years with interest rate r = 5% p.a.
- What is the outstanding balance P(t) after Period t = 3 (in the beg. of period t=4) ?

Period t	Initial Principal P(t-1)	Payment CF PMT(t)	Interest CF I(t)	Principal CF ΔP(t)	Final Principal P(t)
1	**10 000,00**	2 309,75	500,00	1 809,75	8 190,25
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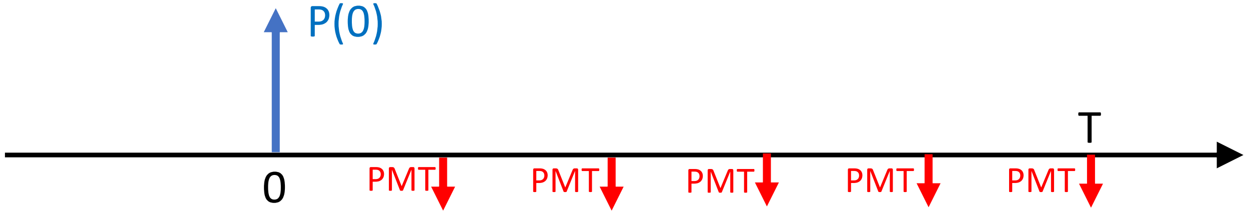


- A Bank offers a Loan for P = 10 000 PLN for T = 5 years with interest rate r = 5% p.a.
- What is the outstanding balance P(t) after Period t = 3 (in the beg. of period t=4) ?
  - Note that in the end of each period the principal balance decreases by:

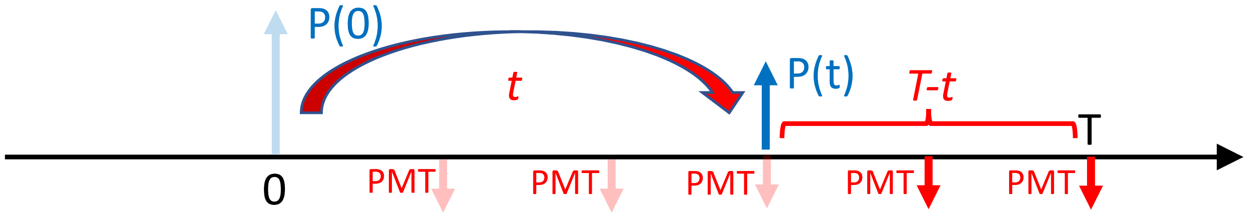
$$\Delta P(t) = P(t) - P(t+1) = PMT - I(t) = PMT - P(t)\,r \qquad (*)$$

  - For t = 0 one has (this way we computed PMT):

$$\mathrm{P(0)} = \mathrm{P} = \mathrm{PVA(T)} = \sum_{i=1}^{T} \frac{PMT}{(1+r)^i} = \frac{PMT}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

Example 8: Loan Amortization

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



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$$\Delta P(t) = P(t) - P(t+1) = PMT - I(t) = PMT - P(t)\,r \qquad (*)$$

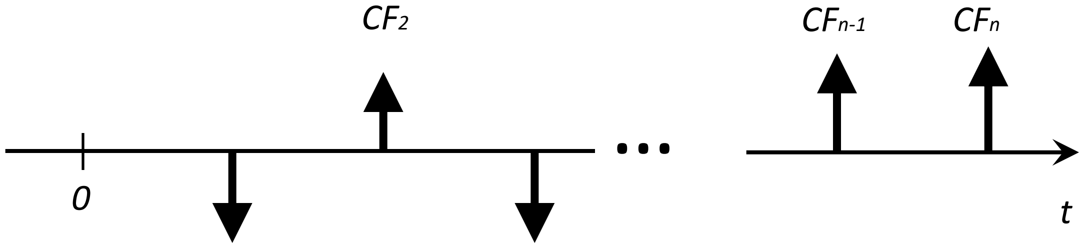
  - Assume it holds for any t (when only T-t future CFs = PMT are left):

$$\mathrm{P(t)} = \mathrm{PVA(T-t)} = \sum_{i=1}^{T-t} \frac{PMT}{(1+r)^i} = \frac{PMT}{r}\left(1 - \frac{1}{(1+r)^{T-t}}\right) \Rightarrow P(3) = 4294.77\ PLN$$

  - It is straightforward to show that (*) is fulfilled !
- The balance P(t) of an amortized loan is the PV (at time t) of all future CFs after time t
- Rule of thumb: this holds for any amortization scheme as long as r = const. !!!

Example 9: Company valuation

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$



❖We were asked to price a company. Financial advisors provide us with a forecast of EBITDA* (Earnings Before Interest, Taxes, Depreciation and Amortization), i.e. some proxy of global CFs that the company generates, see table below:

t (years)	1	2	3	4	5	6	7	8	9	10
EBITDA ($ mln)	-10	10	-5	15	25	30	40	45	55	65

❖As the forecast ends after 10 years, we assume that the last EBITDA will turn into a growing perpetuity with CAGR (Compound Annual Growth Rate) $g = 5\%$

❖The company's Balance Sheet shows $ 60 mln Equity and $ 40 mln Debt

❖We know that the cost of of Equity is 15%** and the cost of Debt is 5%

❖The WACC (Weighted Average Cost of Capital) is thus: $y = 0.6 \times 15\% + 0.4 \times 5\% = 11\%$

* Beyond scope of these lectures
**Will be discussed in „Risk Management"

Example 9: Company valuation

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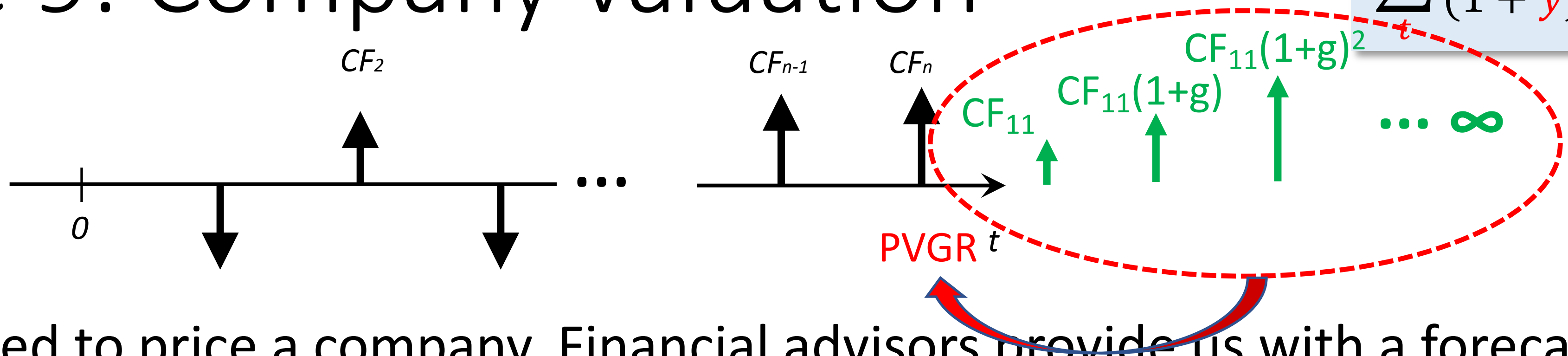

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$$\text{PVGP} = \frac{CF(11)}{y-g} = \frac{CF(10)(1+g)}{y-g}$$

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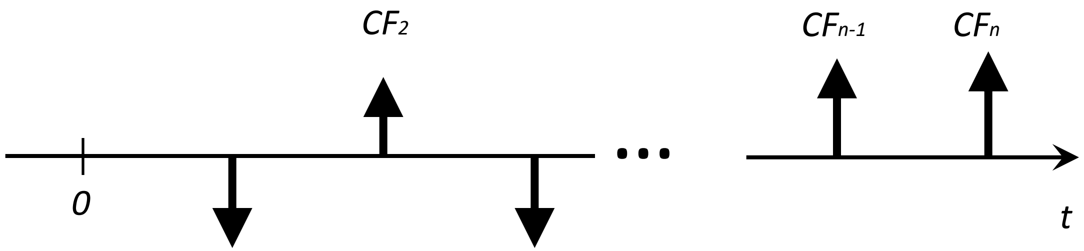

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DF(t)	0.90	0.81	0.73	0.66	0.59	0.53	0.48	0.43	0.39	0.35
PV(EBITDA)	-9.01	8.12	-3.66	13.17	14.84	16.04	19.27	19.52	21.50	423.50

$$\mathrm{DF}(t) = \frac{1}{(1+y)^t}$$

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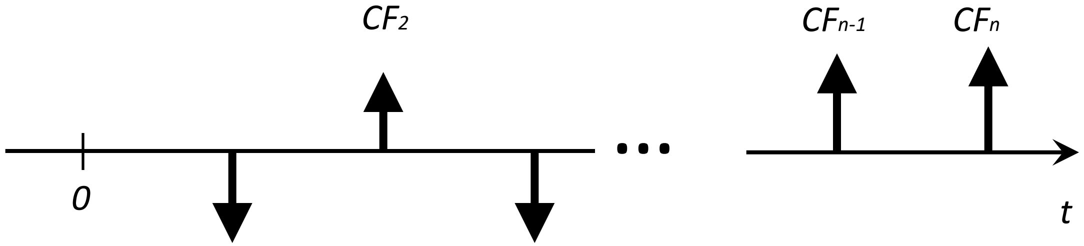

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TOTAL ~$ 523 mln

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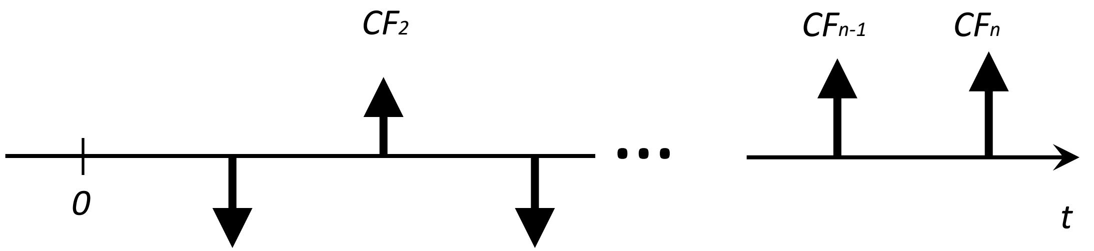

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TOTAL ~$ 523 mln

❖The company's Balance Sheet shows $ 60 mln Equity and $ 40 mln Debt

❖An investor can pay, i.e. the Equity is worth, $ 523 mln - $ 40 mln = $ 483 mln

❖The WACC (Weighted Average Cost of Capital) is thus: $y = 0.6 \times 15\% + 0.4 \times 5\% = 11\%$

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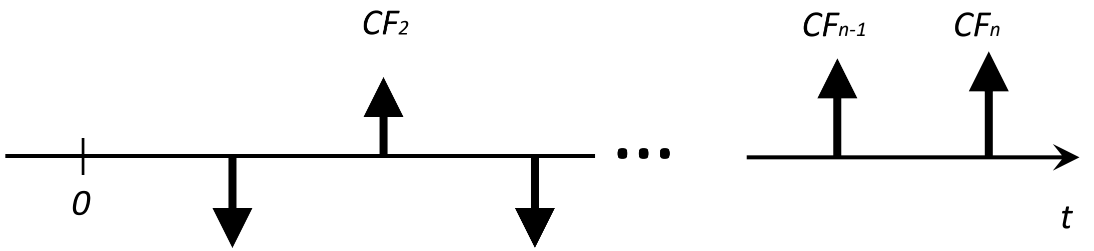

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❖Sensitivity analysis:

g \ y	10%	11%	12%
4%	525	**423**	347
5%	**618**	**483**	**389**
6%	756	**568**	444

growing perpetuity with CAGR (Compound Annual Growth Rate) g = 5%

❖The company's Balance Sheet shows \$ 60 mln Equity and \$ 40 mln Debt

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❖The WACC (Weighted Average Cost of Capital) is thus: y = 0.6 × 15% + 0.4 × 5% = 11%

Summary & Dictionary

$$\sum_t \underbrace{CF(t)\overbrace{\frac{1}{(1+y)^t}}^{DF(t)}}_{PV(CF(t))} = 0$$

✓Time Value of Money (PL: wartość pieniądza w czasie)

✓Interest / Interest Rate (PL: odsetki / stopa (o)procentowa(nia))

✓Nominal / Real interest rate (PL: nominalna / realna stopa %)

- r : rate from a contract, used to compute future CFs (interest)

✓Cash Flow (PL: przepływ pieniądza)

✓Present / Future Value (PL: wartość obecna / przyszła)

✓Discounting <> Compounding / Discount Factor (PL: dyskontowanie <> kapitalizacja / ???)

✓Effective Interest Rate / Yield / (Internal) Rate of Return / …
(PL: efektywna stopa procentowa / rentowność / (wewnętrzna) stopa zwrotu / RRSO …)

- y : the one important in finance !
- Rule of thumb: if the nominal rate r = const., then y depends only on the compounding frequency n and not on how principal (PL: kapitał) is amortized

Summary & Dictionary

$$\sum_t CF(t) \underbrace{\overbrace{\frac{1}{(1+y)^t}}^{DF(t)}}_{PV(CF(t))} = 0$$

✓ Simple Interest (PL: procent prosty)

- FV(t) = PV (1 + r t)

✓ Compound Interest (PL: procent złożony)

- FV(t) = PV $(1 + r / n)^{n\,t}$

✓ Continuous rate (PL: stopa o kapitalizacji ciągłej)

- FV(t) = PV exp(r t)

✓ Annuity / Annuity Due (PL: ???)

- a series of T equal CFs at evenly spaced intervals (1st CF now / in the future)
- PV $= \sum_{i=1}^{T} \frac{PMT}{(1+y)^i} = \frac{PMT}{y}\left(1 - \frac{1}{(1+y)^T}\right)$

Note: here y should be adjusted to frequency n (time intervals) of CFs !!!

$(1+y)^n = 1+y_{p.a.}$

✓ (Growing) Perpetuity (PL: ???)

- a (growing) infinite series of equal CFs at evenly spaced intervals (1st CF in the future)
- PV = PMT / (y – g)

Summary: what to remember ?

$$\sum_t \frac{CF(t)}{(1+y)^t} = 0$$

Time Value of Money
Present / Future Value
Effective Interest Rate
Simple interest
Compound interest
Continuous Rate
Annuity (Due)
Perpetuity

The rest is just simple algebra …