List of problems for Descartes lecture "Random matrix theory, free random variables and applications"

- Use the prescription from Chapter 1 to generate ensemble of large matrices (e.g. 100 matrices of the size 50 by 50) coorsponding to GOE and GUE. Plot the result and compare to Wigner semicircle. Check normalization of the spectra.
- Generate 10^3 matrices of various sizes (2 by 2, 4 by 4, etc) for GOE. Plot the difference of the adjacent eigenvalues (staying far from the edges of the whole spectrum), and compare the results to the Wigner surmise. Why staying away from the edges improves the accuracy of the simulation?
- Using the GUE matrices generated in Problem 1, calculate first 10 spectral moments M_N. Compare these numbers with the first ten coefficients of expansion of the Green's function $G(z)$ for GUE at $z = \infty$. Check if the obtained series of non zero integers 1,2,... appears in Sloane's Online Encyclopedia of Integer Sequences (www.oeis.org).
- Using the listed explicit representations for the weight and for orthogonal polynomials in section 2.1.4, plot the average spectral function $\rho(\lambda)$ represented by (2.17) for $N = 1, 2, 5, 10$ and 20. Observe how the subsequent curves approximate better and better the Wigners semicircle. Check the normalization.
- Check numerically the "addition law" for two free Wigner semicircles with second cumulants κ_1 and κ_2.
- Generate several sums $M = H_1 + UH_2U$, where $H_{1,2}$ are diagonal matrices with randomly distributed equal number of values 1/2, and U is a Haar measure. Then, diagonalize an ensemble of matrices M, and plot the resulting spectral distribution, and compare it to analytic result (5.29), known as a free arcsine distribution. Hint. There are many ways to generate U. Most laborious comes from definition $UU^\dagger = 1$, i.e. generate random vector u_1 and normalize it. Then generate second vector u_2, make it orthogonal to the first one (e.g. by Gram-Schmidt procedure), and normalize, then draw the third one, made it orthogonal to both u_1, u_2, normalize etc. An easier way is to use the random eigenvectors corresponding to eigenvalues of the GUE ensemble, one only has to normalize them, since they are random and orthogonal by construction.
- (a) Check numerically Pastur equation in the case of $\lambda_i = a$. First perform the simulation. Note, that the eigenvectors of both ensembles are already maximally decorrelated, due to the Gaussian random character of GUE, so simply add diagonal matrix with randomly distributed values a to the GUE matrix, and perform averaging over such example for $a = 1/2, 1, 2, 10$ (b) Compare these results with analytic solution, i.e. solve analytically the

resulting cubic equation (Cardano) equation, identify the right solution for $G(z)$ from the three possible, and plot the spectral function $\rho(\lambda, a)$ as a 3d plot ρ, λ, a. At which value of parameter A the support of the spectrum splits from single interval into two intervals? (c) Optional: Do you see the way how to calculate the value of a corresponding to the split analytically?

- Check numerically the spectrum of the Ginibre ensemble, for real and complex cases. Do you see any differences?

- Check numerically Marchenko Pastur formula for few various values of rectangularity parameter $r = N/T$, where N is number of rows and T is number of columns. Consider cases $N << T, N < T, N = T, N > T$.

- (Optional) Confirm Tetilla law numerically.