Asynchronous Siding Mode Control of Two-dimensional Markov Jump Systems in Roesser Model

Abstract—**abstract**

Index Terms—**Markov jump systems, 2D systems, Siding mode control, Hidden Markov model**

I. Introduction

This part is introduciton.

II. Preliminaries

In this paper, we consider the following two-dimensional Markov jump systems in Roesser model:

$$\begin{cases} \boldsymbol{x(i,\, j)} = A_{r(i,j)}x(i,j) + E_{r(i,j)}w(i,j) \\ \qquad + B_{r(i,j)}[(u(i,j) + f(x(i,j), r(i,j))] \\ y(i,j) = C_{r(i,j)}x(i,j) + D_{r(i,j)}w(i,j) \end{cases} \tag{1}$$

where

$$\boldsymbol{x(i,\, j)} = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix},\ x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$$

$x^h(i,h) \in \mathbb{R}^{n_h}$ and $x^v(i,h) \in \mathbb{R}^{n_v}$ represent horizontal and vertical states respectively, $u(i,j) \in \mathbb{R}^{n_u}$ and $y(i,j) \in \mathbb{R}^{n_y}$ represent the controlled input and output respectively, and $w(i,j) \in \mathbb{R}^{n_w}$ represents the exogenous disturbance which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$. $A_{r(i,j)}$,$B_{r(i,j)}$,$C_{r(i,j)}$,$D_{r(i,j)}$ and $E_{r(i,j)}$ represent the time-varying system matrices, all of which are real known constant matrices with appropriate dimensions. Besides, we assume that the matrix $B_{r(i,j)}$ is full column rank for each $r(i,j) \in \mathcal{N}_1$, that is, rank($B_{r(i,j)}$)$= n_u$. The nonlinear function $f(x(i,j), r(i,j))$ satisfying the following property:

$$\|f(x(i,j), r(i,j)\| \le \delta_{r(i,j)}\|x(i,j)\| \tag{2}$$

where $\delta_{r(i,j)}$ is a known scalar, $\|\cdot\|$ denotes the Euclidean norm of a vector. The parameter $r(i,j)$ takes values in a finite set $\mathcal{N}_1 = \{1, 2..., N_1\}$ with transition probability matrix $\Lambda = \{\lambda_{k\tau}\}$, and the related transition probability from mode k to mode τ is given by

$$\begin{aligned} &\Pr\{r(i+1,j) = \tau | r(i,j) = k\} \\ =&\Pr\{r(i,j+1) = \tau | r(i,j) = k\} = \lambda_{k\tau},\ \forall k, \tau \in \mathcal{N}_1 \end{aligned} \tag{3}$$

where $\lambda_{k\tau} \in [0,1]$, for all $k, \tau \in \mathcal{N}_1$, and $\sum_{\tau=1}^{N_1} \lambda_{k\tau} = 1$ for every mode k.

We define the boundary condition (X_0, Γ_0) of system (1), as follows:

$$\begin{cases} X_0 = \{x^h(0,j), x^v(i,0)|\ i,j = 0,1,2...\} \\ \Gamma_0 = \{r(0,j), r(i,0)|\ i,j = 0,1,2...\} \end{cases} \tag{4}$$

And the corresponding zero boundary condition is assumed as $x^h(0,j) = 0, x^v(i,0) = 0, i,j = 0,1,2....$ Besides, we further impose following assumption on X_0.

Assumption 1. *The boundary condition X_0 satisfies:*

$$\lim_{L\to\infty} \mathbb{E}\Big\{\sum_{\ell=1}^{L}(\|x^h(0,\ell)\|^2 + \|x^v(\ell,0)\|^2)\Big\} < \infty \tag{5}$$

where $\mathbb{E}\{\cdot\}$ stands for mathematical expectation.

In practical applications, the complete information of $r(i,j)$ can not always be available to the controller. Hence, in this paper, the hidden Markov model $(r(i,j), \sigma(i,j), \Lambda, \Psi)$ as in [refto] is introduced to characterize the asynchronous phenomenon between the controller and the system. The parameter $\sigma(i,j)$, refers to controller mode, takes values in another finite set $\mathcal{N}_2 = \{1, 2...N_2\}$, and satisfies the conditional probability matrix $\Psi = \{\mu_{ks}\}$ with conditional mode transition probabilities

$$\Pr\{\sigma(i,j) = s | r(i,j) = k\} = \mu_{ks} \tag{6}$$

where $\mu_{ks} \in [0,1]$ for all $k \in \mathcal{N}_1, s \in \mathcal{N}_2$, and $\sum_{s=1}^{N2} \mu_{ks} = 1$ for any mode k.

Next, the definitions of asymptotically mean square stable and H_∞ performance for 2D systems will be given in Definition 1 and Definition 2, respectively.

Definition 1. *The 2D Markov jump system* (1) *with $w(i,j) \equiv 0$ is said to be asymptotically mean square stable if the following holds:*

$$\lim_{i+j\to\infty} \mathbb{E}\{\|x(i,j)\|^2\} = 0 \tag{7}$$

for any boundary condition X_0 with Assumption 1.

Definition 2. *Given a scalar $\gamma > 0$, the 2D Markov jump system* (1) *is said to be asymptotically mean square stable with an H_∞ disturbance attenuation performance γ if the system satisfies* (7)*, and under zero boundary condition, the following holds:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|y(i,j)\|^2\} < \gamma^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|w(i,j)\|^2\} \tag{8}$$

for all $w(i,j) \in \ell_2\{[0,\infty),[0,\infty)\}$.

Now, we will make some notational simplification for convenience. The parameter $r(i,j)$ is represented by k, $r(i+1,j)$

and $r(i, j+1)$ are represented by τ, $\sigma(i, j)$ is represented by s.

The objective of this work is to devise an asynchronous SMC law $u(i, j)$, such that the 2D Markov jump system (1) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

III. Main Result

A. Sliding surface and sliding mode controller

In this paper, a novel Two-dimensional sliding surface function is constructed as follows:

$$s(i,j) = \begin{bmatrix} s^h(i,j) \\ s^v(i,j) \end{bmatrix} = Gx(i,j) \tag{9}$$

where $G = \sum_{k=1}^{N_1} \beta_k G_k^T$, and scalars β_k should be chosen such that GB_k is nonsingular for any $k \in \mathcal{N}_1$. Based on the the assumption that B_k is full column rank for any $k \in \mathcal{N}_1$, we can find that the above condition can be guaranteed easily with the properly selected parameter β_k.

An asynchronous 2D-SMC law is designed as follows:

$$u(i,j) = K_s x(i,j) - \rho(i,j)\frac{s(i,j)}{\|s(i,j)\|} \tag{10}$$

for any $s \in \mathcal{N}_2$, where the matrix $K_s \in \mathbb{R}^{n_u \times n_x}$ with $n_x = n_h + n_v$ will be determined later, and the parameter $\rho(i,j)$ is given as

$$\rho(i,j) = \varrho_1 \|x(i,j)\| + \varrho_2 \|w(i,j)\| \tag{11}$$

with $\varrho_1 = \max_{k \in \mathcal{N}_1}\{\delta_k\}$, $\varrho_2 = \max_{k \in \mathcal{N}_1}\{\|(GB_k)^{-1} G E_k\|\}$, and the parameter δ_k is given in (2).

Combining system (1) and the asynchronous 2D-SMC low (9), the closed-loop 2D markov jump system can be obtained easily as follows:

$$\boldsymbol{x(i,\, j)} = \bar{A}_{ks} x(i,j) + B_k \bar{\rho}_k(i,j) + E_k w(i,j) \tag{12}$$

where $\bar{A}_{ks} = A_k + B_k K_s$, and $\bar{\rho}_k(i,j)$ as follows

$$\bar{\rho}_k(i,j) = f_k(x(i,j)) - (\varrho_1\|x(i,j)\| + \varrho_2\|w(i,j)\|)\cdot \frac{s(i,j)}{\|s(i,j)\|}.$$

Then, based on the properties of norm, the following condition can be deduced easily

$$\|\bar{\rho}_k(i,j)\| \le (\varrho_1 + \delta_k)\|x(i,j)\| + \varrho_2\|w(i,j)\|. \tag{13}$$

B. Analysis of Stability and H_∞ attenuation performance

In this subsection, we focus on the stability and H_∞ attenuation performance analysis for the closed-loop 2D system (12). A sufficient condition will be derived to guarantee the considered system is asymptotically mean square stable with an H_∞ attenuation performance γ.

Theorem 1. *Consider the Markov jump system (1) under the Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma > 0$, if there exist matrices $K_s \in \mathbb{R}^{n_u \times n_x}$, $R_k = \operatorname{diag}\{R_k^h, R_k^v\} > 0$, $Q_{ks} > 0$, $T_{ks} > 0$ and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, s \in \mathcal{N}_2$, such that the following inequalities hold:*

$$B_k^T \mathcal{R}_k B_k - \epsilon_k I \le 0 \tag{14}$$

,

$$\mathcal{A} + 2\Big(\sum_{s=0}^{N_2} \mu_{ks} \operatorname{diag}\{Q_{ks}, T_{ks}\}\Big) < 0 \tag{15}$$

,

$$\hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} - \operatorname{diag}\{Q_{ks}, T_{ks}\} < 0 \tag{16}$$

where

$$\mathcal{A} = \begin{bmatrix} \Pi_1 & \Pi_3 \\ * & \Pi_2 \end{bmatrix}$$

with

$$\begin{cases} \Pi_1 = -R_k + 4(\delta_k + \varrho_1)^2 \epsilon_k I + C_k^T C_k \\ \Pi_2 = -\gamma^2 I + D_k^T D_k + 4\varrho_2^2 \epsilon_k I \\ \Pi_3 = C_k^T D_k \end{cases}$$

and $\mathcal{R}_k = \sum_{\tau=1}^{N1} \lambda_{k\tau} R_\tau$, $\hat{A}_{ks} = \begin{bmatrix} \bar{A}_{ks} & E_k \end{bmatrix}$, then, the closed-loop system (12) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

Proof. Let's start the proof with the stability of system. We select the Lyapunov candidate as $V_1(i,j) = x^T(i,j) R_k x(i,j)$, then, define

$$\Delta V_1(i,j) = \boldsymbol{x(i,\, j)}^T R_\tau \boldsymbol{x(i,\, j)} - x^T(i,j) R_k x(i,j) \tag{17}$$

Based on the closed-loop system equation (12) with $w(i,j) = 0$, it is easy to find that

$$\begin{aligned} &\mathbb{E}\{\Delta V_1(i,j)\} \\ &= \sum_{s=0}^{N_2} \mu_{ks} \Big\{ \big[\bar{A}_{ks} x(i,j) + B_k \bar{\rho}_k(i,j)\big]^T \mathcal{R}_k \\ &\times \big[\bar{A}_{ks} x(i,j) + B_k \bar{\rho}_k(i,j)\big] \Big\} \\ &- x^T(i,j) R_k x(i,j) \\ &\le x^T(i,j) \Big\{ 2\Big(\sum_{s=1}^{N_2} \mu_{ks} \bar{A}_{ks}^T(i,j) \mathcal{R}_k \bar{A}_{ks}\Big) \Big\} x(i,j) \\ &+ 2\bar{\rho}_k^T(i,j) B_k^T \mathcal{R}_k B_k \bar{\rho}_k(i,j) \\ &- x^T(i,j) R_k x(i,j) \end{aligned} \tag{18}$$

Recalling the conditions given in (13) and (14), the following inequality can be further obtained

$$\mathbb{E}\{\Delta V_1(i,j)\} \le x^T(i,j) \mathcal{G}_{ks} x(i,j) \tag{19}$$

where $\mathcal{G}_{ks} = 2\big(\sum_{s=0}^{N_2} \mu_{ks} \bar{A}_{ks}^T \mathcal{R}_k \bar{A}_{ks}\big) + 2\epsilon_k(\delta_k + \varrho_1)^2 I - R_k$. The following inequality can be deduced from (15) based on the properties of matrix quadratic

$$2\Big(\sum_{s=1}^{N_2} \mu_{ks} Q_{ks}\Big) + 4\epsilon_k(\delta_k + \varrho_1)^2 I + C_k^T C_k - R_k < 0 \tag{20}$$

which will further deduce

$$2\Big(\sum_{s=1}^{N_2} \mu_{ks} Q_{ks}\Big) + 2\epsilon_k(\delta_k + \varrho_1)^2 I - R_k < 0 \tag{21}$$

The following inequality can be inferred directly from condition (16)

$$\bar{A}_{ks}^T \mathcal{R}_k \bar{A}_{ks} - Q_{ks} < 0 \tag{22}$$

Combine (21) and (22), we can infer that $\mathcal{G}_{ks} < 0$, which is equivalent to

$$\mathcal{G}_{ks} \leq -\alpha I \tag{23}$$

with scalar $\alpha > 0$. Recalling (19), we can further infer that

$$\mathbb{E}\{\varDelta V_1(i,j)\} \leq -\alpha \mathbb{E}\{\|x(i,j)\|^2\} \tag{24}$$

Summing up on the both side of (24), we have

$$\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\|x(i,j)\|^2\Big\} \leq -\frac{1}{\alpha}\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\Big\} \tag{25}$$

where parameters κ_1, κ_2 are any positive integers. By substituting $\varDelta V_1$ and R_k with (17) and $R_k = \mathrm{diag}\{R_k^{\mathrm{h}}, R_k^{\mathrm{v}}\}$ respectively, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\\
&= \sum_{i=0}^{\kappa_1}\left\{V_1^v(i,\kappa_2+1) - V_1^v(i,0)\right\}\\
&- \sum_{j=0}^{\kappa_2}\left\{V_1^h(\kappa_1+1,j) - V_1^h(0,j)\right\}\\
&\leq -\Big(\sum_{i=0}^{\kappa_1}V_1^v(i,0) + \sum_{j=0}^{\kappa_2}V_1^h(0,j)\Big)
\end{aligned} \tag{26}$$

where $V_1^h(i,j)$ and $V_1^v(i,j)$ are defined as

$$\begin{cases} V_1^h(i,j) = x^{hT}(i,j)R^h_{r(i,j)}x^h(i,j)\\ V_1^v(i,j) = x^{vT}(i,j)R^v_{r(i,j)}x^v(i,j)\end{cases}$$

Recalling the boundary condition in Assumption 1, and let κ_1, κ_2 tend to infinity, it follows from (25) and (26) that

$$\begin{aligned}
&\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\|x(i,j)\|^2\Big\}\\
&\leq -\frac{\beta}{\alpha}\sum_{\ell=0}^{\infty}\left(\|x^v(\ell,0)\|^2 + \|x^h(0,\ell)\|^2\right)\\
&< \infty
\end{aligned} \tag{27}$$

where β is the maximum eigenvalue of $R^h(0,\ell)$ and $R^v(\ell,0)$, for any $\ell = 0, 1, 2...$, which implies that (7) holds. Thus, the asymptotically mean square stable of the considered system is proved.

Next, let's focus on the H_∞ attenuation performance under zero boundary condition. Based on the closed-loop system equation (12), it is easy to find that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j)\}\\
&= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\left[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_p w(i,j)\right]^T\\
&\times \mathcal{R}_k\left[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_p w(i,j)\right]\Big\}\\
&- x^T(i,j)R_k x(i,j)\\
&\leq \hat{x}^T(i,j)\Big\{2\Big(\sum_{s=1}^{N_2}\mu_{ks}\hat{A}_{ks}^T(i,j)\mathcal{R}_k\hat{A}_{ks}\Big)\Big\}\hat{x}(i,j)\\
&+ 2\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&- x^T(i,j)R_k x(i,j)
\end{aligned} \tag{28}$$

where

$$\hat{x}(i,j) = \begin{bmatrix} x(i,j)\\ w(i,j)\end{bmatrix},\ \hat{A}_{ks}(i,j) = \begin{bmatrix}\bar{A}_{ks} & E_k\end{bmatrix}$$

Notice that from (13) and (14), we have

$$\begin{aligned}
&\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&\leq 2\epsilon_k\left((\delta_k + \varrho_1)^2\|x(i,j)\|^2 + \varrho_2^2\|w(i,j)\|^2\right)
\end{aligned} \tag{29}$$

The following condition can be deduced easily from (15) and (16)

$$\varXi_{ks} < 0 \tag{30}$$

where $\varXi_{ks} \equiv \mathcal{A} + 2\sum_{s=1}^{N_2}\mu_{ks}\hat{A}_{ks}^T\mathcal{R}_k\hat{A}_{ks}$. Recalling the system (1), and substituting (29) into (28) yields

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j) + \|z(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&\leq \hat{x}^T(i,j)\varXi_{ks}\hat{x}(i,j) < 0
\end{aligned} \tag{31}$$

Noting (26) with the zero boundary condition, we can infer that

$$\begin{aligned}
&\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\\
&= \sum_{i=0}^{\kappa_1}V_1^v(i,\kappa_2+1) + \sum_{j=0}^{\kappa_2}V_1^h(\kappa_1+1,j)\\
&\geq 0 \qquad \forall \kappa_1,\kappa_2 = 1,2,3...
\end{aligned} \tag{32}$$

Then, we can further deduce from (31) and (32) that

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\{\|z(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\{\varDelta V_1(i,j) + \|z(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&< 0
\end{aligned} \tag{33}$$

which implies (8) holds. And this completes the proof of Theorem 1.

□

Remark 1. *Remark.*

C. Analysis of reachability

The reachability of the designed asynchronous 2D-SMC low for the closed-loop system (12) will be discussed in this subsection. By using a stochastic Lyapunov method, we provide a sufficient condition which will confirm that the designed asynchronous 2D-SMC law (10) can force the state trajectories of the closed-loop system (12) into a time-varying sliding region around the specified 2D sliding surface (9).

Theorem 2. *Consider the closed-loop 2D Markov jump system (12) with asynchronous 2D-SMC law (10). If there exists matrices $K_s \in \mathbb{R}^{n_u \times n_x}$, $R_k > 0$, $F_k > 0$, and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, s \in \mathcal{N}_2$, such that the condition (14) and the following inequality hold*

$$2\sum_{s=1}^{N_2} \bar{A}_{ks}^T\big(\mathcal{R}_k + G^T\mathcal{F}_kG\big)\bar{A}_{ks} - R_k < 0 \tag{34}$$

where $\mathcal{R}_k$ is defined in Theorem 1, and $\mathcal{F}_k = \sum_{\tau=1}^{N_1}\lambda_{k\tau}F_\tau$. Then, the state trajectories of the considered closed-loop system will be driven into the following sliding region $\mathcal{O}$, around the predefined sliding surface (9):

$$\mathcal{O} \equiv \Big\{\|s(i,j)\| \leq \ \rho^*(i,j)\Big\} \tag{35}$$

where $\rho^*(i,j) = \max_{k\in\mathcal{N}_1}\sqrt{\hat{\rho}_k(i,j)/\lambda_{\min}(F_k)}$ with

$$\begin{aligned}\hat{\rho}_k(i,j) = {} & 4\big(\|E_k^T\mathcal{R}_kE_k\| + \|E_k^TG^T\mathcal{R}_kGE_k\| \\ & + 2\varrho_2^2(\|B_k^T\mathcal{F}_kB_k\| + \|B_k^TG^T\mathcal{F}_kGB_k\|)\big)\|w(i,j)\|^2 \\ & + 8\big(\|B_k^T\mathcal{R}_TB_k\| + \|B_k^TG^T\mathcal{R}_TGB_k\|\big) \\ & \times (\varrho_1+\delta_k)^2\|x(i,j)\|^2.\end{aligned}$$

and $\lambda_{\min}(F_k)$ here denotes the minimum eigenvalue of F_k.

Proof. First, let's define $\boldsymbol{s(i,\,j)} = \begin{bmatrix} s^h(i+1,j) \\ s^v(i,j+1)\end{bmatrix}$, it is easy find that $\boldsymbol{s(i,\,j)} = G\boldsymbol{x(i,\,j)}$. Then, we select the Lyapunov candidate as

$$V(i,j) = V_1(i,j) + V_2(i,j) \tag{36}$$

where $V_1(i,j)$ is defined in Theorem 1, $V_2(i,j) = s^T(i,j)F_ks(i,j)$. Similar with the proof in Theorem 1, it is easy to find that

$$\begin{aligned}&\mathbb{E}\{\Delta V_1(i,j)\} \\ &= \mathbb{E}\Big\{\boldsymbol{x(i,\,j)}^TR_\tau\boldsymbol{x(i,\,j)} - x^T(i,j)R_kx(i,j)\Big\} \\ &= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T \\ &\times \mathcal{R}_k\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]\Big\} \\ &- x^T(i,j)R_kx(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks}x(i,j) \\ &+ 2\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T\mathcal{R}_k \\ &\times \big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big] \\ &- x^T(i,j)R_kx(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks}x(i,j) \\ &+ \bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_kB_k\bar{\rho}_k(i,j) \\ &+ w^T(i,j)E_k^T\mathcal{R}_kE_kw(i,j) \\ &- x^T(i,j)R_kx(i,j)\end{aligned} \tag{37}$$

Along with the sliding function in (9), we have

$$\begin{aligned}&\mathbb{E}\{\Delta V_2(i,j)\} \\ &= \mathbb{E}\Big\{\boldsymbol{s(i,\,j)}^TF_\tau\boldsymbol{s(i,\,j)} - s^T(i,j)F_ks(i,j)\Big\} \\ &= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T \\ &\times G^T\mathcal{F}_kG\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]\Big\} \\ &- s^T(i,j)F_ks(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^TG^T\mathcal{F}_kG\bar{A}_{ks}x(i,j) \\ &+ 2\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^TG^T\mathcal{F}_k \\ &\times G\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big] \\ &- s^T(i,j)F_ks(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^TG^T\mathcal{F}_kG\bar{A}_{ks}x(i,j) \\ &+ \bar{\rho}_k^T(i,j)B_k^TG^T\mathcal{F}_kGB_k\bar{\rho}_k(i,j) \\ &+ w^T(i,j)E_k^TG^T\mathcal{F}_kGE_kw(i,j) \\ &- s^T(i,j)F_ks(i,j)\end{aligned} \tag{38}$$

Combing (37) and (38), we can infer that

$$\begin{aligned}&\mathbb{E}\{\Delta V(i,j)\} \\ &= \mathbb{E}\Big\{\Delta_1(i,j) + \Delta V_2(i,j)\Big\} \\ &\leq x^T(i,j)\Big\{2\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\big(\mathcal{R}_k + G^T\mathcal{F}_kG\big)\bar{A}_{ks}\Big\}x(i,j) \\ &+ \vec{\rho}_k(i,j) - x^T(i,j)R_kx(i,j) - \lambda_{\min}(F_k)\|s(i,j)\|^2\end{aligned} \tag{39}$$

where

$$\begin{aligned}&\vec{\rho}_k(i,j)\\&=4\big(\|B_k^T\mathcal{F}_kB_k\|+\|B_k^TG^T\mathcal{F}_kGB_k\|\big)\|\bar{\rho}_k(i,j)\|^2\\&+4\big(\|E_k^T\mathcal{R}_kE_k\|+\|E_k^TG^T\mathcal{R}_kGE_k\|\big)\|w(i,j)\|^2\end{aligned}$$

Recalling the condition (13), we can get an inequality as follows

$$\|\bar{\rho}_k(i,j)\|^2\le 2(\varrho_1+\delta_k)^2\|x(i,j)\|^2+2\varrho_2^2\|w(i,j\|^2 \quad (40)$$

It is obvious that $\vec{\rho}_k(i,j)<\hat{\rho}_k(i,j)$ for any $k\in\mathcal{N}_1$ after substitute (40) into $\vec{\rho}_k(i,j)$. Then, based on the condition (15), when the state trajectories is out of the region $\mathcal{O}$ around the specified siding surface (9), we can infer that

$$-\lambda_{\min}(F_k)\|s(i,j)\|^2+\vec{\rho}_k(i,j)<0 \quad (41)$$

It yields from (34), (37), (39) and (41) that

$$\begin{aligned}&\mathbb{E}\{\varDelta V(i,j)\}\\&\le x^T(i,j)\Big\{2\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\big(\mathcal{R}_k+G^T\mathcal{F}_kG\big)\bar{A}_{ks}\\&-R_k\Big\}x(i,j)<0\end{aligned} \quad (42)$$

which means the state trajectories of the close-loop (12) are strictly decreasing (with mean square) outside the region $\mathcal{O}$ defined in (35). Now, the proof is complete.

□

Remark 2. *Remark.*

D. Synthesis of Asynchronous 2D-SMC Law

It is obvious that, if the theorem 1 and the theorem 2 hold simultaneously, then, the asymptotically mean square stability with an H_∞ disturbance attenuation performance γ of the closed-loop 2D system (12) and the reachability of the predefined sliding function (9) can be guaranteed simultaneously. That is, the to be determined matrix K_s in 2D-SMC law (10) should ensure that theorem 1, 2 are established at the same time. Now, in this subsection, we will continue our study with this idea.

Theorem 3. *Consider the Markov jump system (1) under the Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma>0$, if there exist matrices $\tilde{K}_s\in\mathbb{R}^{n_u\times n_x}$, $L_s\in\mathbb{R}^{n_x\times n_x}$, $\tilde{R}_k=\operatorname{diag}\{\tilde{R}_k^h,\tilde{R}_k^v\}>0$, $\tilde{F}_k>0$, $\tilde{Q}_{ks}>0$, $\tilde{T}_{ks}>0$ and scalars $\tilde{\epsilon}_k>0$, for any $k\in\mathcal{N}_1, s\in\mathcal{N}_2$, such that the following inequalities hold:*

$$\begin{bmatrix}-\tilde{\epsilon}_kI & \mathscr{B}_k\\ * & \mathscr{R}_k\end{bmatrix}<0 \quad (43)$$

$$\begin{bmatrix}\mathscr{L}_{ks} & \mathscr{A}_{ks} & \mathscr{G}_{ks}\\ * & \mathscr{R}_k & 0\\ * & * & \mathscr{F}_k\end{bmatrix}<0 \quad (44)$$

$$\begin{bmatrix}\mathscr{H}_k & \mathscr{D}_k & \mathscr{P}_{ks} & \mathscr{Y}_{ks}\\ * & \mathscr{I}_k & 0 & 0\\ * & * & \mathscr{Q}_{ks} & 0\\ * & * & * & \mathscr{T}_{ks}\end{bmatrix}<0 \quad (45)$$

where

$$\begin{aligned}\mathscr{B}_k&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{\epsilon}_kB_k^T & \sqrt{\lambda_{k2}}\tilde{\epsilon}_kB_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{\epsilon}_kB_k^T\end{bmatrix}\\ \mathscr{R}_k&=\operatorname{diag}\{-\tilde{R}_1,-\tilde{R}_2,\cdots,-\tilde{R}_{N_1}\}\\ \mathscr{F}_k&=\operatorname{diag}\{-\tilde{F}_1,-\tilde{F}_2,\cdots,-\tilde{F}_{N_1}\}\\ \mathscr{I}_k&=\operatorname{diag}\{-\tilde{\epsilon}_k,-I,-I,-\tilde{\epsilon}_k\}\\ \mathscr{Q}_{ks}&=\operatorname{diag}\{-\tilde{Q}_{k1},-\tilde{Q}_{k2},\cdots,-\tilde{Q}_{kN_2}\}\\ \mathscr{T}_{ks}&=\operatorname{diag}\{-\tilde{T}_{k1},-\tilde{T}_{k2},\cdots,-\tilde{T}_{kN_2}\}\\ \mathscr{L}_{ks}&=\operatorname{diag}\{\tilde{Q}_{ps}-L_s^T-L_s,-\tilde{T}_{ks}\}\\ \mathscr{H}_k&=\begin{bmatrix}-\tilde{R}_k & \tilde{R}C_k^TD_k^T\\ * & -\gamma^2I\end{bmatrix}\\ \mathscr{A}_{ks}&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{ks}^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{ks}^T\\ \sqrt{\lambda_{k1}}\tilde{T}_{ks}B_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{T}_{ks}B_k^T\end{bmatrix}\\ \mathscr{G}_{ks}&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{ks}^TG^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{ks}^TG^T\\ 0 & \cdots & 0\end{bmatrix}\\ \mathscr{D}_{ks}&=\begin{bmatrix}2(\varrho_1+\delta_k)\tilde{R}_k & \tilde{R}_kC_k^T & 0 & 0\\ 0 & 0 & D_k^T & 2\varrho_2\end{bmatrix}\\ \mathscr{P}_{ks}&=\begin{bmatrix}\sqrt{2\mu_{k1}}\tilde{R}_k & \cdots & \sqrt{2\mu_{kN_2}}\tilde{R}_k\\ 0 & \cdots & 0\end{bmatrix}\\ \mathscr{Y}_{ks}&=\begin{bmatrix}0 & \cdots & 0\\ \sqrt{2\mu_{k1}}I & \cdots & \sqrt{2\mu_{kN_2}}I\end{bmatrix}\end{aligned}$$

and $\tilde{A}_{ks}=A_kL_s+B_k\tilde{K}_s$. Then, the closed-loop system (12) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ, and the state trajectories of the considered closed-loop system will be driven into a sliding region $\mathcal{O}$, around the predefined sliding surface (9). Moreover, the to be determined matrix K_s in 2D-SMC law (10) can be chosen as

$$K_s=\tilde{K}_sL_s^{-1} \quad (46)$$

if the LMIs (43), (44) and (45) have feasible solutions.

Proof. As we discussed above, the objective is to testify that, the conditions (14), (15), (16) in Theorem 1 and (34) in Theorem 2 can be guaranteed simultaneously by (43), (44), (45). Before that, let's make some notations as $\tilde{K}_s=K_sL_s$, $\tilde{R}_k=R_k^{-1}$, $\tilde{F}_k=F_k^{-1}$, $\tilde{Q}_{ks}=Q_{ks}^{-1}$, $\tilde{T}_{ks}=T_{ks}^{-1}$ and $\tilde{\epsilon}=\epsilon_k^{-1}$. Firstly, we will prove that (14) and (43) are equivalent. Pre- and post- multiplying the inequalities given in (43) by $\operatorname{diag}\{\epsilon_kI,I,I,\cdots,I\}$, respectively, and applying Schur complement after that, then, we can see (14) satisfied. Next, we will verity that (44) and (45) are sufficient to ensure (15), (16) and (34) hold. Using $\operatorname{diag}\{R_k,I,I,\cdots,I\}$ to pre- and post-multiply the inequality given in (45), and applying Schur complement after that, then we will have (15) satisfied. It follows from (44) that $\tilde{Q}_{ps}-L_s^T-L_s<0$, that is $L_s^T+L_s$ is positive definite, which guarantees that the matrix L_s is invertible. We can infer the following formulation based on $Q_{ps}>0$

$$(\tilde{Q}_{ps}-L_s)^T\tilde{Q}_{ks}^{-1}(\tilde{Q}_{ps}-L_s)\ge 0 \quad (47)$$

which means

$$-L_s^T\tilde{Q}_{ks}L_s\le\tilde{Q}_{ps}-L_s^T-L_s \quad (48)$$

Noting the condition give in (45), we can infer that

$$\begin{bmatrix} \tilde{\mathscr{L}}_{ks} & \mathscr{A}_{ks} & \mathscr{G}_{ks} \\ * & \mathscr{R}_k & 0 \\ * & * & \mathscr{F}_k \end{bmatrix} < 0 \tag{49}$$

where $\tilde{\mathscr{L}}_{ks} = \mathrm{diag}\{-L_s^T \tilde{Q}_{ks} L_s, -\tilde{T}_{ks}\}$.
Noting the slack matrix L_s is invertible, we denote $h_{ks} = \mathrm{diag}\{L_s^{-1}, T_{ks}, I, I, \cdots, I\}$. Using h_{ks} to pre- and post-multiply the inequality given in (49), and applying Schur complement after that, then the following inequality will be obtained

$$\hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} + \breve{A}_{ks}^T \mathcal{F}_k \breve{A}_{ks} - \mathrm{diag}\{Q_{ks}, T_{ks}\} < 0 \tag{50}$$

where $\breve{A}_{ks} = \begin{bmatrix} \bar{A}_{ks} & 0 \end{bmatrix}$. Combing (15) and (50) we have

$$\mathcal{A} + 2\sum_{s=0}^{N_2} \mu_{ks} \Big\{ \hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} + \breve{A}_{ks}^T \mathcal{F}_k \breve{A}_{ks} \Big\} < 0 \tag{51}$$

which further implies (34) holds based on the property of positive definite matrix. It is clear that, the gain matrix K_s can not obtained directly from LMIs in Theorem 3 while $\tilde{K}_s$ is obtained. Thanks to the matrix L_s is invertible, K_s can be calculated indirectly with $K_s = \tilde{K}_s L_s^{-1}$. Now, the proof is finished. □

IV. Numerical Example

In this section, we provide an example to verify the validity of the proposed method.

V. Conclusions

References

[1] Zhang, Guangming, et al. "Finite-time H static output control of Markov jump systems with an auxiliary approach." Applied Mathematics & Computation 273.C(2016):553-561.