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Brief paper

State estimation and sliding mode control for semi-Markovian jump systems with mismatched uncertainties☆

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ABSTRACT

This paper is concerned with the state estimation and sliding mode control problems for phase-type semi-Markovian jump systems. Using a supplementary variable technique and a plant transformation, a finite phase-type semi-Markov process has been transformed into a finite Markov chain, which is called its associated Markov chain. As a result, phase-type semi-Markovian jump systems can be equivalently expressed as its associated Markovian jump systems. A sliding surface is then constructed and a sliding mode controller is synthesized to ensure that the associated Markovian jump systems satisfy the reaching condition. Moreover, an observer-based sliding mode control problem is investigated. Sufficient conditions are established for the solvability of the desired observer. Two numerical examples are presented to show the effectiveness of the proposed design techniques.

1. Introduction

Markovian jump systems (MJS) are a special class of stochastic dynamic systems which are popular for modeling the random abrupt variations in their structures, since in practice many dynamical systems may subject to frequent unpredictable structural changes, such as random failures, repairs of sudden environment disturbances and abrupt variation of the operating point. Research into this class of systems and their applications have spanned several decades. For some representative work on this general topic, we refer to Costa and De Oliveira (2012), Gao, Fei, Lam, and Du (2011), Ji and Chizeck (1990), Mahmoud (2004), Shi, Boukas, and Agarwal (1999a,b), Shi and Yu (2009) and the references therein.

However, MJS have many limitations in applications, since the jump time of a Markov chain is, in general, exponentially distributed, and the results obtained for MJS are intrinsically conservative due to constant transition rates (Huang & Shi, 2013). Unlike the MJS, semi-Markovian jump systems (S-MJS) are characterized by a fixed matrix of transition probabilities and a matrix of sojourn time probability density functions (Hou, Luo, Shi, & Nguang, 2006). Due to their relaxed conditions on the probability distributions, S-MJS have much broader applications than the conventional MJS. Indeed, it is expected that most of the modeling, analysis and design results for MJS could be regarded as special cases of S-MJS. Hence, this area is significant not only in theory, but also in practice.

Sliding mode control (SMC) is an effective control approach due to its excellent advantage of strong robustness against model uncertainties, parameter variations and external disturbances. It is worthwhile to mention that the SMC strategy has been successfully applied to a variety of practical systems such as robot manipulators, aircraft navigation and control, and power system stabilizers. Consequently, the SMC design problem has received increasing research attention and there are a large number of significant results in the literature (see, for example, Basin, Ferreira, & Fridman, 2007; Basin & Rodriguez-Ramirez, 2011, 2012; Barambones, Alkorta, & de Durana, 2013; Niu, Ho, & Lam, 2005; Niu, Ho, & Wang, 2007; Soltanpour, Zolfaghari, Soltani, & Khooban, 2013; Wu & Shi, 2010; Wu &

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Zheng, 2009, and the references therein). Furthermore, the system states are not always available. Thus, sliding mode observer technique has been developed to deal with the state estimation problems for linear or nonlinear uncertain systems. Nevertheless, there are still a few results related to state estimation and SMC problems for S-MJS.

Motivated by the above discussion, in this paper, we investigate the state estimation and sliding mode control problems for semi-Markovian jump systems. This paper addresses three open questions: (1) how a phase-type semi-Markov process can be replaced by a Markov chain, which is equivalent to transforming a phase-type semi-Markovian jump system into its associated Markovian jump system; (2) how to design the appropriate sliding surface function to adjust the effect of the jumping phenomenon in the plant; and (3) how to perform the reachability analysis for the resulting sliding mode dynamics. Thus, sliding surface function design and reachability analysis of the resulting sliding mode dynamics are the main issues to be addressed in this paper. Numerical examples are given to illustrate the effectiveness of the proposed control scheme.

Notations. The notations used throughout the paper are standard. The superscripts 'T' and '-1' denote matrix transposition and matrix inverse, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The notation $X > 0$ ($\geqslant 0$) means that matrix X is positive definite (semi-definite); and $\lambda_{\min}(X)$ denotes the minimum eigenvalue of the symmetric matrix X. The notation $(\Omega, \mathcal{F}, \mathcal{P})$ represents the probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. In addition, $\mathbb{E}\{\cdot\}$ denotes the expectation operator. Symbol $\|\cdot\|$ stands for the Euclidean norm of a vector and its induced norm of a matrix. We use $\mathrm{diag}\{A_1, \ldots, A_n\}$ to denote the block-diagonal matrix with $A_1, \ldots, A_n$ on the diagonal. In symmetric block matrices, we use an asterisk $*$ to represent a term that is induced by symmetry.

2. Phase-type semi-Markov processes and Markovization

Consider a class of stochastic systems in the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ for $t \geqslant 0$

$$\dot{x}(t) = \left[\hat{A}(\hat{r}_t) + \Delta\hat{A}(\hat{r}_t, t)\right]x(t) + \hat{B}(\hat{r}_t)\left[u(t) + \varphi(t)\right],$$
$$y(t) = \hat{C}(\hat{r}_t)x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the system output, and $\varphi(t) \in \mathbb{R}^p$ is uncertainty disturbance. $\hat{A}(\hat{r}_t)$, $\hat{B}(\hat{r}_t)$ and $\hat{C}(\hat{r}_t)$ are matrix functions of the random process $\{\hat{r}_t, t \geqslant 0\}$; and $\Delta\hat{A}(\hat{r}_t, t)$ is system uncertainty. Let $\{\hat{r}_t, t \geqslant 0\}$ be a continuous time stochastic process on the state space $\{1, 2, \ldots, m+1\}$, where the states $1, 2, \ldots, m$ are transient and the state $m+1$ is absorbing. The infinitesimal generator is $Q = \begin{bmatrix} T & T^0 \\ 0_{1\times m} & 0 \end{bmatrix}$, where the matrix $T = (T_{ij})_{m\times m}$ satisfies $T_{ii} < 0, T_{ij} \geqslant 0, i \neq j$; and $T^0 = \begin{bmatrix} T_1^0 & T_2^0 & \cdots & T_m^0 \end{bmatrix}^T$ is a non-negative column vector such that $Te + T^0 = 0$, where e denotes an appropriately dimensioned column vector with all components equal to one. The initial distribution vector is $(\mathbf{a}, a_{m+1})$, where $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ satisfies $\mathbf{a}e + a_{m+1} = 1$. In addition, we have the following assumption.

Assumption 1. Assume that the absorbing state is reached with probability one for a finite time.

Definition 1 (*Neuts, 1975*)**.** A probability distribution is said to be of phase-type if it is the absorption time distribution of a finite Markov chain having an absorbing state and all the other states transient. This distribution is defined by the pair $(\mathbf{a}, T)$ and we say that the pair $(\mathbf{a}, T)$ is a representation of this distribution.

Definition 2 (*Hou et al., 2006*)**.** Let E be a finite set. A stochastic process $\hat{r}_t$ on the state space E is called a phase-type semi-Markov process, if the following conditions hold.

(i) The sample paths of $\hat{r}_t$ are right-continuous functions and have left-hand limits with probability one;
(ii) Denote the sth jump point of the process $\hat{r}_t$ by τ_s, where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_s < \cdots$, and τ_s ($s = 1, 2, 3, \ldots$) are Markovian of the process $\hat{r}_t$;
(iii) $F_{ij}(t) \triangleq \mathcal{P}\left(\tau_{s+1} - \tau_s \leqslant t \mid \hat{r}_{\tau_s} = i, \hat{r}_{\tau_{s+1}} = j\right) = F_i(t), i, j \in E, t \geqslant 0$ do not depend on j and s; and
(iv) $F_i(t), i \in E$ is a phase-type distribution.

Remark 1. We considered the times between transitions are phase-type (PH) distributions. It is worth noting that the PH distribution is a generalization of the exponential distribution while still preserving much of its analytic tractability, and has been used in a wide range of stochastic modeling applications such as reliability theory, queueing theory and biostatistics. Furthermore, the family of PH distribution is dense in all the families of distributions on $[0, +\infty)$. So, for every probability distribution on $[0, +\infty)$, we may choose a PH distribution to approximate the original distribution in any accuracy (Neuts, 1975).

Let $(\mathbf{a}^{(i)}, T^{(i)})$, $i \in E$ denote the $m^{(i)}$ order representation of $F_i(t)$, and $E^{(i)}$ be the corresponding all transient states set (the number of the elements in $E^{(i)}$ is $m^{(i)}$), where

$$\mathbf{a}^{(i)} \triangleq \left(a_1^{(i)}, a_2^{(i)}, \ldots, a_{m^{(i)}}^{(i)}\right),$$
$$T^{(i)} \triangleq \left(T_{jk}^{(i)},\ j, k \in E^{(i)}\right).$$

Also, let

$$p_{ij} \triangleq \Pr\left(\hat{r}_{s+1} = j \mid \hat{r}_s = i\right), \quad i, j \in E,$$
$$P \triangleq (p_{ij}), \quad i, j \in E$$
$$(\mathbf{a}, T) \triangleq (\mathbf{a}^{(i)}, T^{(i)}), \quad i \in E.$$

It is easy to see that the probability distribution of $\hat{r}_t$ can be determined only by $\{P, (\mathbf{a}, T)\}$. For every $s (s = 1, 2, \ldots)$, $\tau_s \leqslant t \leqslant \tau_{s+1}$, define

$$J(t) \triangleq \text{the phase of } F_{\hat{r}(t)}(\cdot) \quad \text{at time } t - \tau_s. \tag{2}$$

Also, for any $i \in E$, define

$$T_j^{(i,0)} \triangleq -\sum_{k=1}^{m^{(i)}} T_{jk}^{(i)}, \quad j = 1, 2, \ldots, m^{(i)}, \tag{3}$$

$$G \triangleq \left\{(i, k^{(i)}) \mid i \in E,\ k^{(i)} = 1, 2, \ldots, m^{(i)}\right\}. \tag{4}$$

Lemma 1 (*Hou et al., 2006*)**.** *$Z(t) = (\hat{r}_t, J(t))$ is a Markov chain with state-space G. The infinitesimal generator of $Z(t)$ given by $Q = (q_{\mu\nu})$, $\mu, \nu \in G$ is determined only by the pair of $(\hat{r}_t, J(t))$ given by $\{P, (\mathbf{a}, T)\}$ as follows:*

$$\begin{cases} q_{(i,k^{(i)})(i,k^{(i)})} = T_{k^{(i)}k^{(i)}}^{(i)}, & (i, k^{(i)}) \in G \\ q_{(i,k^{(i)})(i,\bar{k}^{(i)})} = T_{k^{(i)}\bar{k}^{(i)}}^{(i)}, & k^{(i)} \neq \bar{k}^{(i)}, (i, k^{(i)}) \in G, \\ & (i, \bar{k}^{(i)}) \in G \\ q_{(i,k^{(i)})(j,k^{(j)})} = p_{ij} T_{k^{(i)}}^{(i,0)} a_{k^{(j)}}^{(j)}, & i \neq j, (i, k^{(i)}) \in G, \\ & (j, k^{(j)}) \in G. \end{cases}$$

From (4), we obtain that G has $N = \sum_{i \in E} m^{(i)}$ elements, so the state space of $Z(t)$ has N elements. For convenience, we further denote the number of (i, k) by $\sum_{r=1}^{i-1} m^{(r)} + k$, $1 \leqslant k \leqslant m^{(i)}$; and denote a transformation by $\psi(\cdot)$, hence

$$\psi(i, k) = \sum_{r=1}^{i-1} m^{(r)} + k, \quad i \in E,\ 1 \leqslant k \leqslant m^{(i)}.$$

Moreover, we define

$$r_t \triangleq \psi(Z(t)) \quad \text{and} \quad \lambda_{\psi(i,k)\psi(i',k')} \triangleq q_{\psi(i,k)\psi(i',k')}.$$

Therefore, r_t is an associated Markov process of $\hat{r}_t$ with the state space $\mathcal{N} = \{1, 2, \ldots, N\}$ and the infinitesimal generator is $\Lambda = (\lambda_{ij})$, $1 \leqslant i, j \leqslant N$, so that

$$\begin{aligned}\mathcal{P}\{r_{t+h} = j | r_t = i\} &= \mathcal{P}\{\psi(Z(t) + h) = j | \psi(Z(t)) = i\} \\ &= \begin{cases}\lambda_{ij} h + o(h), & i \neq j, \\ 1 + \lambda_{ij} h + o(h), & i = j,\end{cases}\end{aligned}$$

where λ_{ij} is the transition rate from mode i at time t to mode j at time $t + h$ when $i \neq j$; $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$; $h > 0$ and $\lim_{h\to 0} o(h)/h = 0$.

From Lemma 1, we know that every finite PH semi-Markov process can be transformed into a finite Markov chain which is called its associated Markov chain. Then, we can construct the following associated Markovian jump systems which are equivalent to (1):

$$\begin{aligned}\dot{x}(t) &= \big[A(r_t) + \Delta A(r_t, t)\big]x(t) + B(r_t)\big[u(t) + \varphi(t)\big], \\ y(t) &= C(r_t)x(t),\end{aligned} \tag{5}$$

where r_t is the associated Markov chain of PH semi-Markov chain $\hat{r}_t$. For notional simplicity, when the system operates in the ith mode, $A(r_t)$, $\Delta A(r_t, t)$, $B(r_t)$ and $C(r_t)$ are denoted by $A(i)$, $\Delta A(i, t)$, $B(i)$ and $C(i)$, respectively.

3. Sliding mode control

This section presents the design results of the sliding surface and reaching motion controller. First, we will analyze the sliding mode dynamics. Since $B(r_t)$ is of full column rank by assumption, there exists the following singular value decomposition:

$$B(r_t) = \begin{bmatrix} U_1(r_t) & U_2(r_t)\end{bmatrix}\begin{bmatrix}\Sigma(r_t) \\ 0_{(n-p)\times p}\end{bmatrix} V^T(r_t),$$

where $U_1(r_t) \in \mathbb{R}^{n\times p}$ and $U_2(r_t) \in \mathbb{R}^{n\times(n-p)}$ are unitary matrices, $\Sigma(r_t) \in \mathbb{R}^{p\times p}$ is a diagonal positive-definite matrix, and $V(r_t) \in \mathbb{R}^{p\times p}$ is a unitary matrix. Let $T(r_t) \triangleq \begin{bmatrix} U_2(r_t) & U_1(r_t)\end{bmatrix}^T$. For each possible value $r_t = i$, $i \in \mathcal{N}$, we denote $T(i) \triangleq T(r_t = i)$, $i \in \mathcal{N}$. Then, by the state transformation $z(t) \triangleq T(i)x(t)$, system (5) has the regular form given by

$$\dot{z}(t) = \big[\bar{A}(i) + \Delta\bar{A}(i, t)\big]z(t) + \begin{bmatrix}0_{(n-p)\times p} \\ B_2(i)\end{bmatrix}\big[u(t) + \varphi(t)\big], \tag{6}$$

where

$$\begin{aligned}&\bar{A}(i) \triangleq T(i)A(i)T^{-1}(i), \qquad B_2(i) \triangleq \Sigma(i)V^T(i), \\ &\Delta\bar{A}(i, t) \triangleq T(i)\Delta A(i, t)T^{-1}(i).\end{aligned}$$

Let $z(t) \triangleq \begin{bmatrix} z_1^T(t) & z_2^T(t)\end{bmatrix}^T$, where $z_1(t) \in \mathbb{R}^{n-p}$ and $z_2(t) \in \mathbb{R}^p$, then system (6) can be transformed into

$$\begin{cases}\dot{z}_1(t) = \big[\bar{A}_{11}(i) + \Delta\bar{A}_{11}(i, t)\big]z_1(t) \\ \qquad\quad + \big[\bar{A}_{12}(i) + \Delta\bar{A}_{12}(i, t)\big]z_2(t), \\ \dot{z}_2(t) = \big[\bar{A}_{21}(i) + \Delta\bar{A}_{21}(i, t)\big]z_1(t) \\ \qquad\quad + \big[\bar{A}_{22}(i) + \Delta\bar{A}_{22}(i, t)\big]z_2(t) \\ \qquad\quad + B_2(i)\big[u(t) + \varphi(t)\big], \quad i \in \mathcal{N},\end{cases} \tag{7}$$

where

$$\begin{aligned}&\bar{A}_{11}(i) \triangleq U_2^T(i)A(i)U_2(i), \qquad \bar{A}_{12}(i) \triangleq U_2^T(i)A(i)U_1(i), \\ &\bar{A}_{21}(i) \triangleq U_1^T(i)A(i)U_2(i), \qquad \bar{A}_{22}(i) \triangleq U_1^T(i)A(i)U_1(i).\end{aligned}$$

The mismatched time-varying uncertainties $\Delta\bar{A}_{uv}(i, t)$, $u, v \in \{1, 2\}$ are assumed to carry the following structure

$$\begin{bmatrix}\Delta\bar{A}_{11}(i, t) & \Delta\bar{A}_{12}(i, t) \\ \Delta\bar{A}_{21}(i, t) & \Delta\bar{A}_{22}(i, t)\end{bmatrix} = \begin{bmatrix}E_1(i) \\ E_2(i)\end{bmatrix}\Delta F(i, t)\begin{bmatrix}H_1(i) & H_2(i)\end{bmatrix},$$

where $E_1(i)$, $E_2(i)$, $H_1(i)$ and $H_2(i)$, $i \in \mathcal{N}$ are known real-constant matrices, and $\Delta F(i, t)$ is the unknown time-varying matrix function satisfying

$$\Delta F^T(i, t)\Delta F(i, t) \leqslant I, \quad i \in \mathcal{N}.$$

It is obvious that (7) represents the sliding mode dynamics of system (6), and hence the corresponding sliding surface can be chosen as follows for each $i \in \mathcal{N}$,

$$s(t, i) = \begin{bmatrix}-\mathbb{C}(i) & I\end{bmatrix}z(t) = -\mathbb{C}(i)z_1(t) + z_2(t) = 0, \tag{8}$$

where $\mathbb{C}(i)$, $i \in \mathcal{N}$ are the parameters to be designed. When the system trajectories reach onto the sliding surface $s(t, i) = 0$, that is, $z_2(t) = \mathbb{C}(i)z_1(t)$, the sliding mode dynamics is attained. Substituting $z_2(t) = \mathbb{C}(i)z_1(t)$ into the first equation of system (7) gives the sliding mode dynamics:

$$\dot{z}_1(t) = \mathscr{A}(i)z_1(t), \quad i \in \mathcal{N}, \tag{9}$$

where

$$\mathscr{A}(i) \triangleq \bar{A}_{11}(i) + E_1(i)\Delta F_i H_1(i) + \big(\bar{A}_{12}(i) + E_1(i)\Delta F_i H_2(i)\big)\mathbb{C}(i).$$

Let us recall the definition of the exponential stable of system (9).

Definition 3. System (9) is said to be exponentially stable, if for any initial condition $z_1(0) \in \mathbb{R}^n$, and $r_0 \in \mathcal{N}$, there exist constants α_1 and α_2, such that

$$\mathbb{E}\left\{\|z_1(0, r_0)\|^2\right\} \leqslant \alpha_1\|z_1(0)\|^2 \exp(-\alpha_2 t).$$

Let $C^2(\mathbb{R}^2 \times \mathcal{N}; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(z_1, i)$ on $\mathbb{R}^n \times \mathcal{N}$ which are continuously twice differentiable in z_1. For $V \in C^2(\mathbb{R}^2 \times \mathcal{N}; \mathbb{R}_+)$, define an infinitesimal operator $\mathcal{L}V(z_1, i)$ as in Mao (1999, 2002). Then, we have the following lemma.

Lemma 2 (Mao, 1999, 2002)**.** *If there exist a function $V \in C^2(\mathbb{R}^2 \times \mathcal{N}; \mathbb{R}_+)$ and positive constants c_1, c_2 and c_3 such that*

$$c_1\|z_1\|^2 \leqslant V(z_1, i) \leqslant c_2\|z_1\|^2,$$

and $\mathcal{L}V(z_1, i) \leqslant -c_3\|z_1\|^2$, for all $(z_1, i) \in \mathbb{R}^n \times \mathcal{N}$, then system (9) is exponentially stable.

In the following, we consider the problem of sliding mode surface design.

Theorem 1. *Associated Markovian jump system (9) is exponentially stable, if there exist matrices $Q(i) > 0$ and general matrices $M(i)$, $i \in \mathcal{N}$ such that the following inequalities hold,*

$$\begin{bmatrix}\Theta_{11}(i) & \Theta_{12}(i) & \Theta_{13}(i) \\ * & \Theta_{22}(i) & 0 \\ * & * & \Theta_{33}(i)\end{bmatrix} < 0, \tag{10}$$

where

$$\begin{aligned}\Theta_{11}(i) &\triangleq \lambda_{ii}Q(i)+\bar{A}_{11}(i)Q(i)+Q(i)\bar{A}_{11}^T(i)\\ &\quad+\bar{A}_{12}(i)M(i)+M^T(i)\bar{A}_{12}^T(i),\\ \Theta_{12}(i) &\triangleq \left[E_1(i)\ \ \epsilon_1 Q_1^T(i)H_1(i)\ \ E_2(i)\ \ \epsilon_2 M^T(i)H_1(i)\right],\\ \Theta_{22}(i) &\triangleq \mathrm{diag}\{-\epsilon_1 I, -\epsilon_1 I, -\epsilon_2 I, -\epsilon_2 I\},\\ \Theta_{13}(i) &\triangleq \left[\sqrt{\lambda_{i1}}Q(i)\ \cdots\ \sqrt{\lambda_{iN}}Q(i)\right],\\ \Theta_{33}(i) &\triangleq \mathrm{diag}\{-Q(1),\ldots,-Q(N)\}.\end{aligned}$$

Moreover, the sliding surface of system (9) is

$$s(t,i)=-M(i)Q^{-1}(i)z_1(t)+z_2(t),\quad i\in\mathcal{N}. \tag{11}$$

Proof. To analyze the stability of the sliding motion (9), we choose the following Lyapunov function:

$$V\big(z_1(t),i\big)\triangleq z_1^T(t)P(i)z_1(t),$$

where $P(i)\triangleq P(r_t=i)>0, i\in\mathcal{N}$ are real matrices to be determined.

The infinitesimal generator $\mathcal{L}$ can be considered as a derivative of the Lyapunov function $V(z_1(t),i)$ along the trajectories of the associated Markov process $\{r_t, t\geqslant 0\}$. Then

$$\mathcal{L}V(z_1(t),i)=z_1^T(t)\Phi(t)z_1(t),$$

where

$$\begin{aligned}\Phi(t)&\triangleq\sum_{j=1}^{N}P(j)\lambda_{ij}+2P(i)[\bar{A}_{11}(i)+\bar{A}_{12}(i)\mathbb{C}(i)]\\ &\quad+2P(i)[E_1(i)\Delta F_i H_1(i)+E_1(i)\Delta F_i H_2(i)\mathbb{C}(i)].\end{aligned}$$

Obviously, if $\Phi(t)<0$, then $\mathcal{L}V(z_1(t),i)<0, \forall i\in\mathcal{N}$. Next, define $Q(i)=P^{-1}(i)$, then pre- and post-multiplying $\Phi(t)$ by $Q(i)$. Using $\lambda_{ii}=-\sum_{j=1,j\neq i}^{N}\lambda_{ij}$ and from the Schur complement, it gives that $\Phi(t)<0$, which is equivalent to (10). Thus, one obtains $\mathcal{L}V(z_1(t),i)<0$ for all $z_1(t)\neq 0$ and $i\in\mathcal{N}$. Then, $\mathcal{L}V(z_1(t),i)\leqslant -\lambda_1\|z_1(t)\|^2$, where $\lambda_1\triangleq\min_{i\in\mathcal{N}}\{-\lambda_{\min}(\Phi(t))\}>0$.

This result implies that the inequalities depend only on the global constant λ_1. By Lemma 2 and employing the same techniques as in Mao (2002), inequalities (10) ensure that system (9) is exponentially stable. □

Next, we synthesize a SMC law to drive the system trajectories onto the predefined sliding surface $s(t,i)=0$ in (11) in a finite time.

Theorem 2. *Suppose (10) has solutions $M(i)$ and $Q(i)$, $i\in\mathcal{N}$, and the linear sliding surface is given by (11). Then, the following controller (12) makes the sliding surface $s(t,i)=0, i\in\mathcal{N}$, stable and globally attractive in a finite time*

$$\begin{aligned}u(t)&=-B_2^{-1}(i)\Big[\mathcal{M}_i\big(\bar{A}(i)+\Delta\bar{A}(i,t)\big)z(t)\\ &\quad+\big(\varsigma(i)+\varrho(i)\big)\mathrm{sgn}(s(t,i))\Big],\end{aligned} \tag{12}$$

where

$$\mathcal{M}_i\triangleq\left[-M(i)Q^{-1}(i)\ \ I\right],\quad \varsigma(i)>0, i\in\mathcal{N},$$

are constants and $\varrho(i)\triangleq\max_{i\in\mathcal{N}}(\|B_2(i)\varphi(t)\|)$.

Proof. We will complete the proof by showing that the control law (12) can not only make the system exponentially stable but also globally attractive in a finite time. For each $i\in\mathcal{N}$, from the sliding surface

$$s(t,i)=\left[-\mathbb{C}(i)\ \ I\right]z(t)\triangleq\mathcal{M}_i z(t),$$

select the following Lyapunov function:

$$V(i,s(t,i))\triangleq\frac{1}{2}s^T(t,i)s(t,i),\quad i\in\mathcal{N}.$$

According to (6) and (8), for each $i\in\mathcal{N}$, we obtain

$$\begin{aligned}\dot{s}(t,i)&=\mathcal{M}_i\big[\bar{A}(i)+\Delta\bar{A}(i,t)\big]z(t)\\ &\quad+\mathcal{M}_i\begin{bmatrix}0_{(n-p)\times p}\\ B_2(i)\end{bmatrix}\big[u(t)+\varphi(t)\big].\end{aligned} \tag{13}$$

Substituting (12) into (13) yields

$$\dot{s}(t,i)=-(\varsigma(i)+\varrho(i))\mathrm{sgn}(s(t,i))+B_2(i)\varphi(t),\quad i\in\mathcal{N}.$$

Thus, taking the derivative of $V(i,s(t,i))$ and considering $|s(t,i)|\geq\|s(t,i)\|$, we obtain that

$$\begin{aligned}\dot{V}(i,s(t,i))&=s^T(t,i)\dot{s}(t,i)\\ &=-s^T(t,i)(\varsigma(i)+\varrho(i))\mathrm{sgn}(s(t,i))+s^T(t,i)B_2(i)\varphi(t)\\ &\leqslant-\varsigma(i)\|s(t,i)\|\\ &=-\varsigma(i)\sqrt{2V(i,s(t,i))}.\end{aligned}$$

Then, by applying Assumption 1 we obtain that the state trajectories of the dynamics (9) arrive in the same subsystem within a finite time. Furthermore, the state trajectories will reach the sliding path in a finite time and will remain within it. This completes the proof. □

4. Observer-based sliding mode control

In this section, we will utilize a state observer to generate the estimate of unmeasured state components, and then synthesize a sliding mode control law based on the state estimates.

We design the following observer for systems (5):

$$\dot{\hat{x}}(t)=A(i)\hat{x}(t)+B(i)u(t)+L(i)\big[y(t)-C(i)\hat{x}(t)\big], \tag{14}$$

where $\hat{x}(t)\in\mathbb{R}^n$ represents the estimate of $x(t)$, and $L(i)\in\mathbb{R}^{n\times q}$, $i\in\mathcal{N}$, are the observer gains to be designed.

Let $\delta(t)\triangleq x(t)-\hat{x}(t)$ denote the estimation error. Consider (5) and (14), the estimate error dynamics can be obtained as

$$\begin{aligned}\dot{\delta}(t)&=\Big(A(i)-L(i)C(i)+\Delta A(i,t)\Big)\delta(t)\\ &\quad+\Delta A(i,t)\hat{x}(t)+B(i)\varphi(t),\end{aligned} \tag{15}$$

where the matched uncertainty disturbance $\varphi(t)$ is unknown but bounded as $\|\varphi(t)\|\leqslant\rho\|\delta(t)\|$, where ρ is a known scalar. Define the following integral sliding mode surface function:

$$\tilde{s}(t,i)=G(i)\hat{x}(t)-\int_0^t G(i)\Big(A(i)+B(i)K(i)\Big)\hat{x}(v)\mathrm{d}v, \tag{16}$$

where $G(i)$ and $K(i)$, $i\in\mathcal{N}$ are coefficient matrices. In addition, $K(i)$ is chosen such that $A(i)+B(i)K(i)$ is Hurwitz, and $G(i)$ is designed such that $G(i)B(i)$ is non-singular.

It follows from (16) that

$$\begin{aligned}\dot{\tilde{s}}(t,i)&=G(i)\dot{\hat{x}}(t)-G(i)\Big(A(i)+B(i)K(i)\Big)\hat{x}(t)\\ &=G(i)B(i)u(t)+G(i)L(i)C(i)\delta(t)-G(i)B(i)K(i)\hat{x}(t).\end{aligned} \tag{17}$$

Let $\dot{\tilde{s}}(t,i)=0$ for each $i\in\mathcal{N}$, we obtain the following equivalent control law:

$$u_{eq}(t)=K(i)\hat{x}(t)-\big(G(i)B(i)\big)^{-1}G(i)L(i)C(i)\delta(t). \tag{18}$$

Substituting (18) into (14), one can obtain the following sliding mode dynamics:

$$\dot{\hat{x}}(t) = \Big(A(i) + B(i)K(i)\Big)\hat{x}(t) + \left[I - B(i)\big(G(i)B(i)\big)^{-1}G(i)\right]L(i)C(i)\delta(t). \quad (19)$$

In the following theorem, a sufficient condition for the stability analysis is given for the overall closed-loop system composed of the estimation error dynamics (15) and the sliding mode dynamics (19).

Theorem 3. *Consider the associated Markovian jump system (5). Its unmeasured states are estimated by the observer (14). The sliding surface functions in the state estimation space and in the state estimation error space are chosen as (16). If there exist matrices $X(i) > 0$ and $\mathcal{L}(i)$ such that for all $i \in \mathcal{N}$,*

$$\begin{bmatrix} \Omega_1(i) & \Omega_2(i) \\ * & \Omega_3(i) \end{bmatrix} < 0, \quad (20)$$

where

$$\Omega_1(i) \triangleq \begin{bmatrix} \Omega_{11}(i) & \mathcal{L}(i)C(i) & X(i)B(i) \\ * & \Omega_{22}(i) & 0 \\ * & * & -B^T(i)X(i)B(i) \end{bmatrix},$$

$$\Omega_2(i) \triangleq \begin{bmatrix} 0 & 0 & 0 \\ C^T(i)\mathcal{L}^T(i) & X(i)E(i) & X(i)B(i) \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_3(i) \triangleq \mathrm{diag}\{-X(i), -\varepsilon_1 I, -\varepsilon_2 I\},$$

$$\Omega_{11}(i) \triangleq X(i)\Big(A(i) + B(i)K(i)\Big) + \varepsilon_1 H^T(i)H(i) + \Big(A(i) + B(i)K(i)\Big)^T X(i) + \sum_{j=1}^{N} X(j)\lambda_{ij},$$

$$\Omega_{22}(i) \triangleq X(i)A(i) + A^T(i)X(i) - \mathcal{L}(i)C(i) - C^T(i)\mathcal{L}^T(i) + \varepsilon_2\rho^2 I + \sum_{j=1}^{N} X(j)\lambda_{ij},$$

then the overall closed-loop systems composed of (15) and (19) are mean-square exponentially stable. Moreover, the observer gain is given by

$$L(i) = X^{-1}(i)\mathcal{L}(i), \quad i \in \mathcal{N}. \quad (21)$$

Proof. Select the following Lyapunov function:

$$\tilde{V}(\hat{x}, \delta, i) \triangleq \hat{x}^T(t)X(i)\hat{x}(t) + \delta^T(t)X(i)\delta(t). \quad (22)$$

Along the solution of systems (15) and (19), we have

$$\begin{aligned} \mathcal{L}(\tilde{V}(\hat{x}, \delta, i)) = {} & 2\hat{x}^T(t)X(i)\Big[\Big(A(i) + B(i)K(i)\Big)\hat{x}(t) \\ & + \Big(I - B(i)\big(G(i)B(i)\big)^{-1}G(i)\Big)L(i)C(i)\delta(t)\Big] \\ & + 2\delta^T(t)X(i)\Big[\Big(A(i) - L(i)C(i) \\ & + \Delta A(i,t)\Big)\delta(t) + \Delta A(i,t)\hat{x}(t) + B(i)\varphi(t)\Big] \\ & + \hat{x}^T(t)\left(\sum_{j=1}^{N} X(j)\lambda_{ij}\right)\hat{x}(t) \\ & + \delta^T(t)\left(\sum_{j=1}^{N} X(j)\lambda_{ij}\right)\delta(t). \end{aligned} \quad (23)$$

From $G(i) = B^T(i)X(i)$, for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we obtain

$$\begin{aligned} & -2\hat{x}^T(t)X(i)B(i)\big(B^T(i)X(i)B(i)\big)^{-1}G(i)L(i)C(i)\delta(t) \\ & \leqslant \hat{x}^T(t)X(i)B(i)\big(B^T(i)X(i)B(i)\big)^{-1}B^T(i)X(i)\hat{x}(t) \\ & \quad + \delta^T(t)C^T(i)L^T(i)X(i)L(i)C(i)\delta(t), \end{aligned} \quad (24)$$

$$\begin{aligned} & -2\delta^T(t)X(i)\Delta A(i,t)\hat{x}(t) \\ & \leqslant \varepsilon_1^{-1}\delta^T(t)X(i)E(i)E^T(i)X(i)\delta(t) \\ & \quad + \varepsilon_1\hat{x}^T(t)H^T(i)H(i)\hat{x}(t), \end{aligned} \quad (25)$$

and

$$-2\delta^T(t)X(i)B(i)\varphi(t) \leqslant \varepsilon_2^{-1}\delta^T(t)X(i)B(i)B^T(i)X(i)\delta(t) + \varepsilon_2\rho^2\delta^T(t)\delta(t). \quad (26)$$

Substituting (24)–(26) into (23), we have

$$\mathcal{L}(\tilde{V}(\hat{x}, \delta, i)) \leqslant \xi^T(t)\Theta(i)\xi(t), \quad (27)$$

where $\xi(t) \triangleq \left[\hat{x}^T(t) \;\; \delta^T(t)\right]^T$ and

$$\Theta(i) \triangleq \begin{bmatrix} \Theta_{11}(i) & \mathcal{L}(i)C(i) \\ * & \Theta_{22}(i) \end{bmatrix},$$

$$\begin{aligned} \Theta_{11}(i) \triangleq {} & X(i)\Big(A(i) + B(i)K(i)\Big) + \sum_{j=1}^{N} X(j)\lambda_{ij} \\ & + X(i)B(i)\big(B^T(i)X(i)B(i)\big)^{-1}B^T(i)X(i) \\ & + \Big(A(i) + B(i)K(i)\Big)^T X(i) + \varepsilon_1 H^T(i)H(i), \end{aligned}$$

$$\begin{aligned} \Theta_{22}(i) \triangleq {} & X(i)A(i) + A^T(i)X(i) - \mathcal{L}(i)C(i) \\ & + C^T(i)L^T(i)X(i)L(i)C(i) - C^T(i)\mathcal{L}^T(i) \\ & + \varepsilon_1^{-1}X(i)E(i)E^T(i)X(i) + \varepsilon_2\rho^2 I \\ & + \varepsilon_2^{-1}X(i)B(i)B^T(i)X(i) + \sum_{j=1}^{N} X(j)\lambda_{ij}. \end{aligned}$$

Then by the Schur complement, it follows that (20) implies $\Theta(i) < 0$. Thus,

$$\mathcal{L}(\tilde{V}(\hat{x}, \delta, i)) < 0, \quad i \in \mathcal{N}.$$

Employing the same techniques as in Theorem 1 and (Mao, 2002), we know that the overall system composed of the estimation error dynamics (15) and the sliding mode dynamics in the state estimation space (19) is mean-square exponentially stable. This completes the proof. □

In the following, we synthesize a SMC law, by which the sliding motion can be driven onto the pre-specified sliding surface $\tilde{s}(t, i) = 0$ in a finite time and then are maintained there for all subsequent time.

Theorem 4. *The trajectories of systems (14) can be driven onto the sliding surface $\tilde{s}(t, i) = 0$ in a finite time by the following observer-based SMC*

$$u(t) = -\varsigma\tilde{s}(t, i) + K(i)\hat{x}(t) - \chi(i, t)\mathrm{sgn}(\tilde{s}(t, i)), \quad (28)$$

where $\varsigma > 0$ is a small constant, and $\chi(i, t)$ is given by

$$\chi(i, t) = \max_{i \in \mathcal{N}}\Big\{\|B^T(i)X(i)L(i)\| \, \|y(t)\| + \|B^T(i)X(i)C(i)\| \, \|\hat{x}(t)\|\Big\}. \quad (29)$$

Proof. Select the following Lyapunov function:

$$\mathcal{V}(\tilde{s}(t, i), i) = \frac{1}{2}\tilde{s}^T(t, i)\left[B^T(i)X(i)B(i)\right]^{-1}\tilde{s}(t, i).$$

From $\|\tilde{s}(t,i)\| \leqslant |\tilde{s}(t,i)|$ and $\tilde{s}^T(t,i)\mathrm{sgn}(\tilde{s}(t,i)) \leqslant |\tilde{s}(t,i)|$, for any $i \in \mathcal{N}$, we obtain

$$\begin{aligned}
\dot{\mathcal{V}}(\tilde{s}(t,i),i) &= \tilde{s}^T(t,i)\left(B^T(i)X(i)B(i)\right)^{-1}\dot{\tilde{s}}(t,i) \\
&= \tilde{s}^T(t,i)\left(B^T(i)X(i)B(i)\right)^{-1}B^T(i)X(i)\Big[B(i)u(t) \\
&\quad + L(i)(y(t)-C(i)\hat{x}(t)) - B(i)K(i)\hat{x}(t)\Big] \\
&= \tilde{s}^T(t,i)(B^T(i)X(i)B(i))^{-1}B^T(i)X(i)\Big[-\varsigma B(i)\tilde{s}(t,i) \\
&\quad - B(i)\chi(i,t)\mathrm{sgn}(\tilde{s}(t,i)) + L(i)(y(t)-C(i)\hat{x}(t))\Big] \\
&\leqslant -\varsigma\|\tilde{s}(t,i)\| - \|\tilde{s}(t,i)\|\,\|(B^T(i)X(i)B(i))^{-1}\| \\
&\quad \times \|B^T(i)X(i)\|\,\|B(i)\chi(i,t)\mathrm{sgn}(\tilde{s}(t,i))\| \\
&\quad + x0s\|\tilde{s}(t,i)\|\,\|(B^T(i)X(i)B(i))^{-1}\| \\
&\quad \times \|B^T(i)X(i)\|\,\|L(i)(y(t)-C(i)\hat{x}(t))\| \\
&\leqslant -\varsigma\|\tilde{s}(t,i)\| \leqslant -\vartheta_i\mathcal{V}^{\frac{1}{2}}(t),
\end{aligned}$$

where $\vartheta_i \triangleq \varsigma\sqrt{\frac{2}{\lambda_{\min}(B^T(i)X(i)B(i))}} > 0, i \in \mathcal{N}$.

Therefore, by applying Assumption 1 we can conclude that the state trajectories of the observer dynamics (14) can be driven onto the sliding surface $\tilde{s}(t,i) = 0$ by the observer-based SMC (28) in a finite time. This completes the proof. □

Remark 2. As we know, the major drawback of SMC is that it is discontinuous across sliding surfaces. The discontinuity leads to control chattering in practice, and involves high frequency dynamics. How to reduce chattering will be a research topic in future studies.

5. Numerical examples

In this section, we present two examples to show the effectiveness of the control schemes proposed in this paper.

Example 1 (*SMC Problem*)**.** Consider the semi-Markovian jump system in (1) with two operating modes and the following parameters:

$$A(1) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B(1) = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \epsilon_1 = 0.3,$$

$$A(2) = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad B(2) = \begin{bmatrix} -1 & 0 \\ -0.1 & -1 \end{bmatrix}, \quad \epsilon_2 = 0.1,$$

$$E(1) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad H(1) = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},$$

$$H(2) = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}, \quad \Delta F(1,t) = \Delta F(2,t) = 0.1\sin(t),$$

$$\varphi(t) = 0.5\exp(-t)\sqrt{x_1^2 + x_2^2}.$$

Let $\hat{r}_t$ be a PH semi-Markov process taking values in $\{1,2\}$. The sojourn time in the first state is a random variable distributed according to a negative exponential distribution with parameter λ_1. The sojourn time in the second state is a random variable distributed according to a two-order Erlang distribution. From Section 2 of this paper, we know that if we want to investigate a PH semi-Markovian switching system, we can investigate its associated Markovian switching system. The key is to look for the associated Markov chain and its infinitesimal generator, and define the proper function.

In fact, the sojourn time in the second part can be divided into two parts. The sojourn time in the first (respectively second) subdivision is a random variable that is negative exponentially distributed with parameter λ_2, (respectively λ_3). More specifically, if the process $\hat{r}_t$ enters state 2, it must stay at the first subdivision for some time, then enter the second subdivision, and finally returns to state 1 again. We know that $p_{12} = p_{21} = 1$. Obviously,

$$\mathbf{a}^{(1)} = (a_1^{(1)}) = 1, \qquad \mathbf{a}^{(2)} = (a_1^{(2)}, a_2^{(2)}) = (1,0),$$

$$T^{(1)} = (T_{11}^{(1)}) = (-\lambda_1),$$

$$T^{(2)} = \begin{bmatrix} T_{11}^{(2)} & T_{12}^{(2)} \\ T_{21}^{(2)} & T_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} -\lambda_2 & \lambda_2 \\ 0 & -\lambda_3 \end{bmatrix}.$$

It is easy to see the state space of $Z(t) = (\hat{r}_t, J(t))$ is $G = \big((1,1), (2,1), (2,2)\big)$. We number the elements of G as $\varphi\big((1,1)\big) = 1$, $\varphi\big((2,1)\big) = 2$, and $\varphi\big((2,2)\big) = 3$. Hence, the infinitesimal generator of $\varphi(Z(t))$ is

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ -\lambda_3 & 0 & \lambda_3 \end{bmatrix}.$$

Now, let $r_t = \varphi(Z(t))$. It is obvious that r_t is the associated Markov chain of $\hat{r}_t$ with state space $\{1,2,3\}$. The infinitesimal generator of r_t is given by Q.

Therefore, our aim is to design a SMC such that the closed-loop system with associated Markov chain r_t is stable. To check the stability of (5), we solve (10) in Theorem 1, and obtain

$$\mathbb{C}(1) = -0.580, \qquad \mathbb{C}(2) = 1.084.$$

Thus, the sliding surface function in (8) can be computed as

$$s(t,i) = \begin{cases} s(t,1) = \begin{bmatrix} 0.580 & 1 \end{bmatrix} z(t), & i = 1, \\ s(t,2) = \begin{bmatrix} -1.084 & 1 \end{bmatrix} z(t), & i = 2. \end{cases}$$

Now, we will design the SMC of (12) in Theorem 2. By computation, we have

$$\mathcal{M}_1 = \begin{bmatrix} -0.580 & 1 \end{bmatrix}, \qquad \mathcal{M}_2 = \begin{bmatrix} 1.084 & 1 \end{bmatrix},$$

$$\varrho(1) = 0.6548, \qquad \varrho(2) = 1.2538,$$

and set $\varsigma(1) = \varsigma(2) = 0.15$. Thus, the SMC in (26) can be computed as

$$u(t) = \begin{cases} \begin{bmatrix} -0.58 & 1 \end{bmatrix} z(t) + 0.804\mathrm{sgn}(s(t,1)), & i = 1, \\ \begin{bmatrix} 1.084 & 1 \end{bmatrix} z(t) + 1.403\mathrm{sgn}(s(t,2)), & i = 2. \end{cases} \tag{30}$$

To prevent the control signals from chattering, we replace $\mathrm{sgn}(s(t,i))$ with $\frac{s(t,i)}{0.1+\|s(t,i)\|}$, $i = \{1,2\}$. For a given initial condition of $x(0) = [-0.8\ 1.0]^T$, the simulation results are given in Figs. 1–4. A switching signal is displayed in Fig. 1; here, '1' and '2' correspond to the first and second modes, respectively. Fig. 2 shows the state response of the closed-loop system with control input (30). The SMC input and sliding function are shown in Figs. 3 and 4, respectively.

Example 2 (*Observer-based SMC Problem*)**.** Consider the semi-Markovian jump system in (1) with two operating modes and the following parameters:

$$A(1) = \begin{bmatrix} -2.5 & -1 \\ 1 & -1.5 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E(1) = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} -1 & 0.15 \\ 2 & -1.5 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E(2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

$$C(1) = \begin{bmatrix} -2.5 & -2 \\ -2 & -1 \end{bmatrix}, \quad C(2) = \begin{bmatrix} -2.5 & -2 \\ -2 & -1 \end{bmatrix}, \quad \rho = 0.1,$$

$$H(1) = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, \quad \Delta F(1,t) = \Delta F(2,t) = 0.2\sin(t),$$

$$H(2) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad \varphi(t) = \exp(-t)\sqrt{x_1^2 + x_2^2},$$

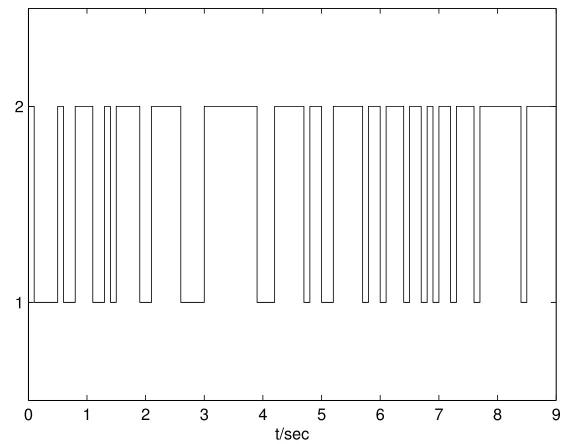

Fig. 1. Switching signal with two modes.

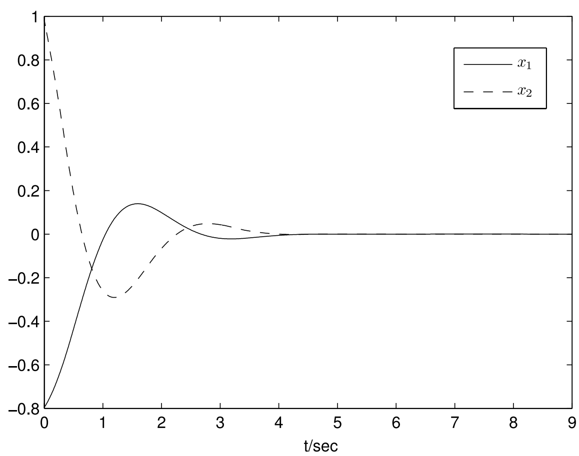

Fig. 2. State response of the closed-loop system.

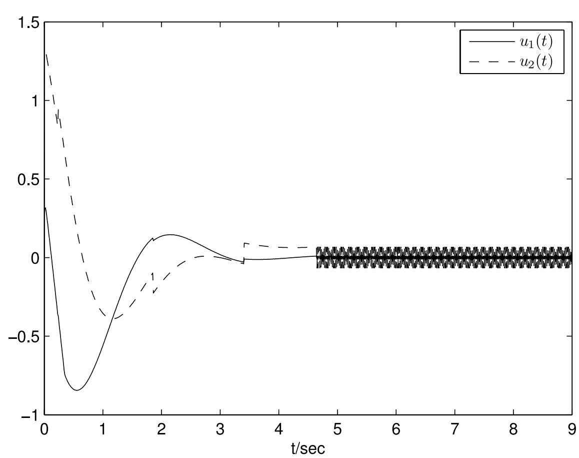

Fig. 3. Control input $u(t)$.

and $\hat{r}_t$ is chosen as in Example 1. In this example, we consider the sliding mode observer design when some system states are not available. According to Section 4, we first design a sliding mode observer in the form of (14) to estimate the system states, and synthesize an observer-based SMC as in (28). We select matrices $K(1)$ and $K(2)$ as follows:

$$K(1) = K(2) = \begin{bmatrix} -0.1 & -0.3 \end{bmatrix}.$$

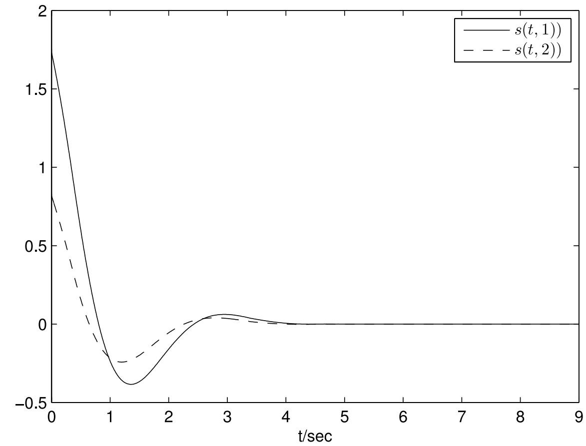

Fig. 4. Sliding surface function $s(t, i)$.

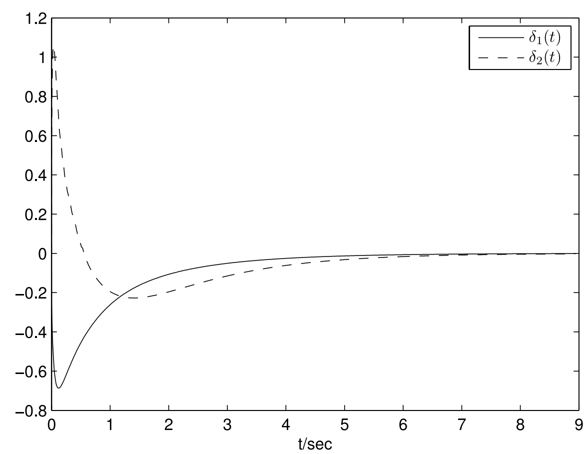

Fig. 5. State estimation error $\delta(t)$.

Solving the conditions (20) and (21) in Theorem 3, we have

$$L(1) = \begin{bmatrix} -0.230 & 0.372 \\ -0.768 & 0.926 \end{bmatrix}, \qquad L(2) = \begin{bmatrix} 0.137 & -0.365 \\ -0.092 & -0.191 \end{bmatrix}.$$

According to (16)–(17), we have

$$\tilde{s}(t, i) = \begin{cases} \tilde{s}(t, 1) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix} \hat{x}(t) + \eta_1(t), & i = 1, \\ \tilde{s}(t, 2) = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix} \hat{x}(t) + \eta_2(t), & i = 2, \end{cases}$$

with

$$\dot{\eta}_1(t) = \begin{bmatrix} -0.07 & -0.46 \end{bmatrix} \hat{x}(t),$$
$$\dot{\eta}_2(t) = \begin{bmatrix} 0.28 & -0.34 \end{bmatrix} \hat{x}(t).$$

The state observer-based SMC is designed in (28) with $\varsigma = 0.1$, then the sliding mode controller designed in (28) can be obtained as

$$u(t) = \begin{cases} u_1(t) = -0.1\tilde{s}(t, 1) + \begin{bmatrix} -0.1 & -0.3 \end{bmatrix} \hat{x}(t) \\ \qquad - \chi(i, t)\mathrm{sgn}(\tilde{s}(t, 1)), & i = 1, \\ u_2(t) = -0.2\tilde{s}(t, 2) + \begin{bmatrix} 0.2 & -0.42 \end{bmatrix} \hat{x}(t) \\ \qquad - \chi(i, t)\mathrm{sgn}(\tilde{s}(t, 2)), & i = 2, \end{cases} \tag{31}$$

with $\chi(i, t) = 0.0057\|y(t)\| + 0.2142\|\hat{x}(t)\|$.

To avoid the control signals from chattering, we replace $\mathrm{sgn}(\tilde{s}(t, i))$ with $\frac{\tilde{s}(t,i)}{0.1+\|\tilde{s}(t,i)\|}$, $i = \{1, 2\}$. For a given initial condition of $x(0) = [1.1\ -0.4]^T$, and with $\hat{x}(0) = 0$, the results from the simulation are given in Figs. 5–7. Fig. 5 shows the state response of the

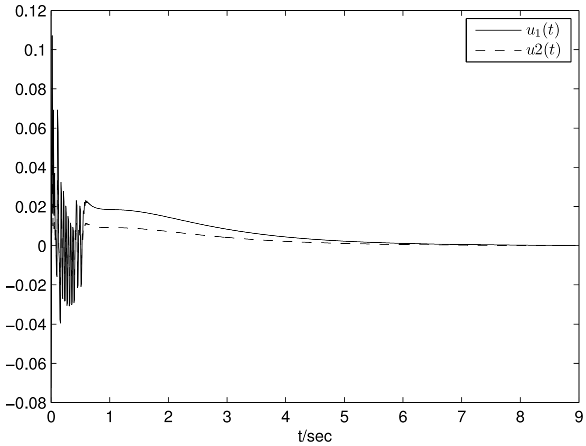

Fig. 6. Control input $u(t)$.

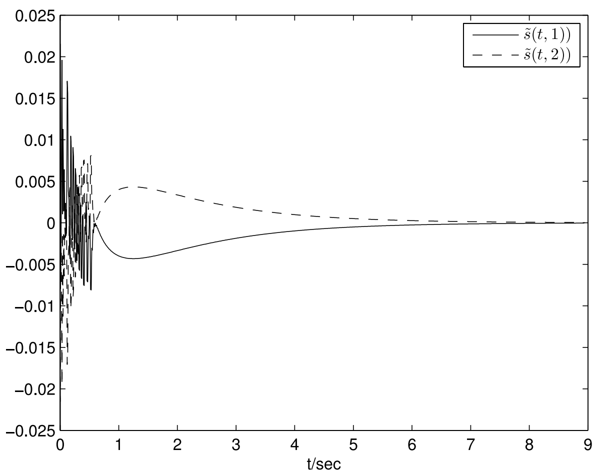

Fig. 7. Sliding surface function $\tilde{s}(t, i)$.

error system with control input (31). The control variables and the sliding surface functions are shown in Figs. 6 and 7, respectively.

6. Conclusion

In this paper, the state estimation and sliding mode control problems have been addressed for semi-Markovian jump systems with mismatched uncertainties. Sufficient conditions for the existence of sliding mode dynamics have been established, and an explicit parametrization for the desired sliding surface has also been given. Then, the sliding mode controller for reaching motion has been synthesized. Moreover, an observer-based sliding mode controller has been synthesized to guarantee the reachability of the system's trajectories to the predefined integral-type sliding surface. Finally, two numerical examples have been provided to illustrate the effectiveness of the proposed design schemes.

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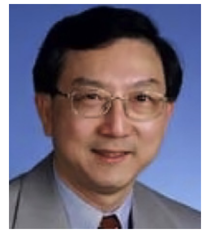

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