Asynchronous Sliding Mode Control of Two-dimensional Markov Jump Systems in Roesser Model

***Abstract*—This paper is concerned with the problem of asynchronous sliding mode control (SMC) for two-dimensional (2D) discrete-time Markov jump systems, which is described by Roesser model. A hidden Markov model (HHM) is introduced to describe the asynchronization which appears between the designed sliding model controller and the original system. Based on the constructed 2D sliding surface, an asynchronous 2D-SMC law is designed. The Lyapunov function and Linear matrix inequality technique are utilized to establish sufficient conditions of the asymptotic mean square stability with a prescribed H_∞ disturbance attenuation performance for the closed-loop 2D system. Moreover, the reachability of the constructed 2D sliding surface is ensured simultaneously. Then, the asynchronous 2D-SMC law can be obtained by solving a convex optimization problem. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed asynchronous 2D-SMC law design method.**

***Index Terms*—Markov jump systems, 2D systems, Sliding mode control, Hidden Markov model**

I. Introduction

This part is introduciton.

II. Preliminaries

In this paper, we consider the following two-dimensional Markov jump systems in Roesser model:

$$\begin{cases} \boldsymbol{x(i,\,j)} = A_{r(i,j)}x(i,j) + E_{r(i,j)}w(i,j) \\ \qquad\qquad + B_{r(i,j)}[(u(i,j) + f(x(i,j),r(i,j))] \\ y(i,j) = C_{r(i,j)}x(i,j) + D_{r(i,j)}w(i,j) \end{cases} \tag{1}$$

where

$$\boldsymbol{x(i,\,j)} = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix},\ x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$$

$x^h(i,h) \in \mathbb{R}^{n_h}$ and $x^v(i,h) \in \mathbb{R}^{n_v}$ represent horizontal and vertical states respectively, $u(i,j) \in \mathbb{R}^{n_u}$ and $y(i,j) \in \mathbb{R}^{n_y}$ represent the controlled input and output respectively, and $w(i,j) \in \mathbb{R}^{n_w}$ represents the exogenous disturbance which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$. $A_{r(i,j)}$,$B_{r(i,j)}$,$C_{r(i,j)}$,$D_{r(i,j)}$ and $E_{r(i,j)}$ represent the time-varying system matrices, all of which are real known constant matrices with appropriate dimensions. Besides, we assume that the matrix $B_{r(i,j)}$ is full column rank for each $r(i,j) \in \mathcal{N}_1$, that is, rank($B_{r(i,j)}$)= n_u. The nonlinear function $f(x(i,j),r(i,j))$ satisfying the following property:

$$\|f(x(i,j),r(i,j)\| \le \delta_{r(i,j)}\|x(i,j)\| \tag{2}$$

where $\delta_{r(i,j)}$ is a known scalar, $\|\cdot\|$ denotes the Euclidean norm of a vector. The parameter $r(i,j)$ takes values in a finite set $\mathcal{N}_1 = \{1,2...,N_1\}$ with transition probability matrix $\Lambda = \{\lambda_{k\tau}\}$, and the related transition probability from mode k to mode τ is given by

$$\begin{aligned} &\Pr\{r(i+1,j) = \tau | r(i,j) = k\} \\ =&\Pr\{r(i,j+1) = \tau | r(i,j) = k\} = \lambda_{k\tau},\ \forall k,\tau \in \mathcal{N}_1 \end{aligned} \tag{3}$$

where $\lambda_{k\tau} \in [0,1]$, for all $k,\tau \in \mathcal{N}_1$, and $\sum_{\tau=1}^{N_1}\lambda_{k\tau} = 1$ for every mode k.

We define the boundary condition (X_0,Γ_0) of system (1), as follows:

$$\begin{cases} X_0 = \{x^h(0,j),x^v(i,0)|\ i,j = 0,1,2...\} \\ \Gamma_0 = \{r(0,j),r(i,0)|\ i,j = 0,1,2...\} \end{cases} \tag{4}$$

And the corresponding zero boundary condition is assumed as $x^h(0,j) = 0, x^v(i,0) = 0, i,j = 0,1,2....$ Besides, we further impose following assumption on X_0.

Assumption 1. *The boundary condition X_0 satisfies:*

$$\lim_{L\to\infty} \mathbb{E}\Big\{\sum_{\ell=1}^{L}(\|x^h(0,\ell)\|^2 + \|x^v(\ell,0)\|^2)\Big\} < \infty \tag{5}$$

where $\mathbb{E}\{\cdot\}$ stands for mathematical expectation.

In practical applications, the complete information of $r(i,j)$ can not always be available to the controller. Hence, in this paper, the hidden Markov model $(r(i,j),\sigma(i,j),\Lambda,\Psi)$ as in [refto] is introduced to characterize the asynchronous phenomenon between the controller and the system. The parameter $\sigma(i,j)$, refers to controller mode, takes values in another finite set $\mathcal{N}_2 = \{1,2...N_2\}$, and satisfies the conditional probability matrix $\Psi = \{\mu_{ks}\}$ with conditional mode transition probabilities

$$\Pr\{\sigma(i,j) = s | r(i,j) = k\} = \mu_{ks} \tag{6}$$

where $\mu_{ks} \in [0,1]$ for all $k \in \mathcal{N}_1, s \in \mathcal{N}_2$, and $\sum_{s=1}^{N2}\mu_{ks} = 1$ for any mode k.

Next, the definitions of asymptotically mean square stable and H_∞ performance for 2D systems will be given in Definition 1 and Definition 2, respectively.

Definition 1. *The 2D Markov jump system (1) with $w(i,j) \equiv 0$ is said to be asymptotically mean square stable if the following holds:*

$$\lim_{i+j\to\infty} \mathbb{E}\{\|x(i,j)\|^2\} = 0 \tag{7}$$

for any boundary condition X_0 with Assumption 1.

Definition 2. *Given a scalar $\gamma > 0$, the 2D Markov jump system (1) is said to be asymptotically mean square stable with an H_∞ disturbance attenuation performance γ if the system satisfies (7), and under zero boundary condition, the following holds:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|y(i,j)\|^2\} < \gamma^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|w(i,j)\|^2\} \tag{8}$$

for all $w(i,j) \in \ell_2\{[0,\infty),[0,\infty)\}$.

Now, we will make some notational simplification for convenience. The parameter $r(i,j)$ is represented by k, $r(i+1,j)$ and $r(i,j+1)$ are represented by τ, $\sigma(i,j)$ is represented by s.

The objective of this work is to devise an asynchronous SMC law $u(i,j)$, such that the 2D Markov jump system (1) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

III. Main Result

A. Sliding surface and sliding mode controller

In this paper, a novel Two-dimensional sliding surface function is constructed as follows:

$$s(i,j) = \begin{bmatrix} s^h(i,j) \\ s^v(i,j) \end{bmatrix} = Gx(i,j) \tag{9}$$

where $G = \sum_{k=1}^{N_1} \beta_k G_k^T$, and scalars β_k should be chosen such that GB_k is nonsingular for any $k \in \mathcal{N}_1$. Based on the the assumption that B_k is full column rank for any $k \in \mathcal{N}_1$, we can find that the above condition can be guaranteed easily with the properly selected parameter β_k.

An asynchronous 2D-SMC law is designed as follows:

$$u(i,j) = K_s x(i,j) - \rho(i,j)\frac{s(i,j)}{\|s(i,j)\|} \tag{10}$$

for any $s \in \mathcal{N}_2$, where the matrix $K_s \in \mathbb{R}^{n_u \times n_x}$ with $n_x = n_h + n_v$ will be determined later, and the parameter $\rho(i,j)$ is given as

$$\rho(i,j) = \varrho_1\|x(i,j)\| + \varrho_2\|w(i,j)\| \tag{11}$$

with $\varrho_1 = \max_{k\in\mathcal{N}_1}\{\delta_k\}$, $\varrho_2 = \max_{k\in\mathcal{N}_1}\{\|(GB_k)^{-1}GE_k\|\}$, and the parameter δ_k is given in (2).

Combining system (1) and the asynchronous 2D-SMC low (9), the closed-loop 2D markov jump system can be obtained easily as follows:

$$\boldsymbol{x(i,j)} = \bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_k w(i,j) \tag{12}$$

where $\bar{A}_{ks} = A_k + B_k K_s$, and $\bar{\rho}_k(i,j)$ as follows

$$\bar{\rho}_k(i,j) = f_k(x(i,j)) - (\varrho_1\|x(i,j)\| + \varrho_2\|w(i,j)\|)\cdot\frac{s(i,j)}{\|s(i,j)\|}.$$

Then, based on the properties of norm, the following condition can be deduced easily

$$\|\bar{\rho}_k(i,j)\| \le (\varrho_1 + \delta_k)\|x(i,j)\| + \varrho_2\|w(i,j)\|. \tag{13}$$

B. Analysis of Stability and H_∞ attenuation performance

In this subsection, we focus on the stability and H_∞ attenuation performance analysis for the closed-loop 2D system (12). A sufficient condition will be derived to guarantee the considered system is asymptotically mean square stable with an H_∞ attenuation performance γ.

Theorem 1. *Consider the Markov jump system (1) under the Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma > 0$, if there exist matrices $K_s \in \mathbb{R}^{n_u\times n_x}$, $R_k = \mathrm{diag}\{R_k^h, R_k^v\} > 0$, $Q_{ks} > 0$, $T_{ks} > 0$ and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, s \in \mathcal{N}_2$, such that the following inequalities hold:*

$$B_k^T \mathcal{R}_k B_k - \epsilon_k I \le 0 \tag{14}$$

,

$$\mathcal{A} + 2\Big(\sum_{s=0}^{N_2}\mu_{ks}\mathrm{diag}\{Q_{ks}, T_{ks}\}\Big) < 0 \tag{15}$$

,

$$\hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} - \mathrm{diag}\{Q_{ks}, T_{ks}\} < 0 \tag{16}$$

where

$$\mathcal{A} = \begin{bmatrix} \Pi_1 & \Pi_3 \\ * & \Pi_2 \end{bmatrix}$$

with

$$\begin{cases} \Pi_1 = -R_k + 4(\delta_k + \varrho_1)^2\epsilon_k I + C_k^T C_k \\ \Pi_2 = -\gamma^2 I + D_k^T D_k + 4\varrho_2^2\epsilon_k I \\ \Pi_3 = C_k^T D_k \end{cases}$$

and $\mathcal{R}_k = \sum_{\tau=1}^{N1}\lambda_{k\tau}R_\tau$, $\hat{A}_{ks} = \begin{bmatrix}\bar{A}_{ks} & E_k\end{bmatrix}$, then, the closed-loop system (12) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

Proof. Let's start the proof with the stability of system. We select the Lyapunov candidate as $V_1(i,j) = x^T(i,j)R_k x(i,j)$, then, define

$$\Delta V_1(i,j) = \boldsymbol{x(i,j)}^T R_\tau \boldsymbol{x(i,j)} - x^T(i,j)R_k x(i,j) \tag{17}$$

Based on the closed-loop system equation (12) with $w(i,j) = 0$, it is easy to find that

$$\begin{aligned} &\mathbb{E}\{\Delta V_1(i,j)\} \\ &= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j)\big]^T\mathcal{R}_k \\ &\times\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j)\big]\Big\} \\ &- x^T(i,j)R_k x(i,j) \\ &\le x^T(i,j)\Big\{2\Big(\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T(i,j)\mathcal{R}_k\bar{A}_{ks}\Big)\Big\}x(i,j) \\ &+ 2\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j) \\ &- x^T(i,j)R_k x(i,j) \end{aligned} \tag{18}$$

Recalling the conditions given in (13) and (14), the following inequality can be further obtained

$$\mathbb{E}\{\Delta V_1(i,j)\} \le x^T(i,j)\mathcal{G}_{ks}x(i,j) \tag{19}$$

where $\mathcal{G}_{ks} = 2\big(\sum_{s=0}^{N_2} \mu_{ks}\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks}\big) + 2\epsilon_k(\delta_k+\varrho_1)^2 I - R_k$. The following inequality can be deduced from (15) based on the properties of matrix quadratic

$$2\big(\sum_{s=1}^{N_2}\mu_{ks}Q_{ks}\big) + 4\epsilon_k(\delta_k+\varrho_1)^2 I + C_k^T C_k - R_k < 0 \quad (20)$$

which will further deduce

$$2\big(\sum_{s=1}^{N_2}\mu_{ks}Q_{ks}\big) + 2\epsilon_k(\delta_k+\varrho_1)^2 I - R_k < 0 \quad (21)$$

The following inequality can be inferred directly from condition (16)

$$\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks} - Q_{ks} < 0 \quad (22)$$

Combine (21) and (22), we can infer that $\mathcal{G}_{ks} < 0$, which is equivalent to

$$\mathcal{G}_{ks} \le -\alpha I \quad (23)$$

with scalar $\alpha > 0$. Recalling (19), we can further infer that

$$\mathbb{E}\{\varDelta V_1(i,j)\} \le -\alpha\mathbb{E}\{\|x(i,j)\|^2\} \quad (24)$$

Summing up on the both side of (24), we have

$$\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\|x(i,j)\|^2\Big\} \le -\frac{1}{\alpha}\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\Big\} \quad (25)$$

where parameters κ_1, κ_2 are any positive integers. By substituting $\varDelta V_1$ and R_k with (17) and $R_k = \mathrm{diag}\{R_k^{\mathrm{h}}, R_k^{\mathrm{v}}\}$ respectively, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\\
&= \sum_{i=0}^{\kappa_1}\{V_1^v(i,\kappa_2+1) - V_1^v(i,0)\}\\
&- \sum_{j=0}^{\kappa_2}\{V_1^h(\kappa_1+1,j) - V_1^h(0,j)\}\\
&\le -\big(\sum_{i=0}^{\kappa_1}V_1^v(i,0) + \sum_{j=0}^{\kappa_2}V_1^h(0,j)\big)
\end{aligned} \quad (26)$$

where $V_1^h(i,j)$ and $V_1^v(i,j)$ are defined as

$$\begin{cases} V_1^h(i,j) = x^{hT}(i,j)R^h_{r(i,j)}x^h(i,j)\\ V_1^v(i,j) = x^{vT}(i,j)R^v_{r(i,j)}x^v(i,j)\end{cases}$$

Recalling the boundary condition in Assumption 1, and let κ_1, κ_2 tend to infinity, it follows from (25) and (26) that

$$\begin{aligned}
&\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\|x(i,j)\|^2\Big\}\\
&\le -\frac{\beta}{\alpha}\sum_{\ell=0}^{\infty}\big(\|x^v(\ell,0)\|^2 + \|x^h(0,\ell)\|^2\big)\\
&< \infty
\end{aligned} \quad (27)$$

where β is the maximum eigenvalue of $R^h(0,\ell)$ and $R^v(\ell,0)$, for any $\ell = 0,1,2...$, which implies that (7) holds. Thus, the asymptotically mean square stable of the considered system is proved.

Next, let's focus on the H_∞ attenuation performance under zero boundary condition. Based on the closed-loop system equation (12), it is easy to find that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j)\}\\
&= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_p w(i,j)\big]^T\\
&\times \mathcal{R}_k\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_p w(i,j)\big]\Big\}\\
&- x^T(i,j)R_k x(i,j)\\
&\le \hat{x}^T(i,j)\Big\{2\big(\sum_{s=1}^{N_2}\mu_{ks}\hat{A}_{ks}^T(i,j)\mathcal{R}_k\hat{A}_{ks}\big)\Big\}\hat{x}(i,j)\\
&+ 2\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&- x^T(i,j)R_k x(i,j)
\end{aligned} \quad (28)$$

where

$$\hat{x}(i,j) = \begin{bmatrix} x(i,j)\\ w(i,j)\end{bmatrix},\ \hat{A}_{ks}(i,j) = \begin{bmatrix}\bar{A}_{ks} & E_k\end{bmatrix}$$

Notice that from (13) and (14), we have

$$\begin{aligned}
&\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&\le 2\epsilon_k\big((\delta_k+\varrho_1)^2\|x(i,j)\|^2 + \varrho_2^2\|w(i,j)\|^2\big)
\end{aligned} \quad (29)$$

The following condition can be deduced easily from (15) and (16)

$$\varXi_{ks} < 0 \quad (30)$$

where $\varXi_{ks} \equiv \mathcal{A} + 2\sum_{s=1}^{N_2}\mu_{ks}\hat{A}_{ks}^T\mathcal{R}_k\hat{A}_{ks}$. Recalling the system (1), and substituting (29) into (28) yields

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j) + \|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&\le \hat{x}^T(i,j)\varXi_{ks}\hat{x}(i,j) < 0
\end{aligned} \quad (31)$$

Noting (26) with the zero boundary condition, we can infer that

$$\begin{aligned}
&\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\\
&= \sum_{i=0}^{\kappa_1}V_1^v(i,\kappa_2+1) + \sum_{j=0}^{\kappa_2}V_1^h(\kappa_1+1,j)\\
&\ge 0 \qquad \forall\kappa_1,\kappa_2 = 1,2,3...
\end{aligned} \quad (32)$$

Then, we can further deduce from (31) and (32) that

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\{\|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\{\varDelta V_1(i,j) + \|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&< 0
\end{aligned} \quad (33)$$

which implies (8) holds. And this completes the proof of Theorem 1.

□

Remark 1. *Remark.*

C. Analysis of reachability

The reachability of the designed asynchronous 2D-SMC low for the closed-loop system (12) will be discussed in this subsection. By using a stochastic Lyapunov method, we provide a sufficient condition which will confirm that the designed asynchronous 2D-SMC law (10) can force the state trajectories of the closed-loop system (12) into a time-varying sliding region around the specified 2D sliding surface (9).

Theorem 2. *Consider the closed-loop 2D Markov jump system (12) with asynchronous 2D-SMC law (10). If there exists matrices $K_s \in \mathbb{R}^{n_u \times n_x}$, $R_k > 0$, $F_k > 0$, and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, s \in \mathcal{N}_2$, such that the condition (14) and the following inequality hold*

$$2\sum_{s=1}^{N_2} \bar{A}_{ks}^T\big(\mathcal{R}_k + G^T\mathcal{F}_k G\big)\bar{A}_{ks} - R_k < 0 \tag{34}$$

where $\mathcal{R}_k$ is defined in Theorem 1, and $\mathcal{F}_k = \sum_{\tau=1}^{N_1}\lambda_{k\tau}F_\tau$. Then, the state trajectories of the considered closed-loop system will be driven into the following sliding region $\mathcal{O}$, around the predefined sliding surface (9):

$$\mathcal{O} \equiv \Big\{\|s(i,j)\| \leq \ \rho^*(i,j)\Big\} \tag{35}$$

where $\rho^*(i,j) = \max_{k\in\mathcal{N}_1}\sqrt{\hat{\rho}_k(i,j)/\lambda_{\min}(F_k)}$ with

$$\begin{aligned}\hat{\rho}_k(i,j) = 4\big(&\|E_k^T\mathcal{R}_kE_k\| + \|E_k^TG^T\mathcal{R}_kGE_k\| \\ &+ 2\varrho_2^2(\|B_k^T\mathcal{F}_kB_k\| + \|B_k^TG^T\mathcal{F}_kGB_k\|)\big)\|w(i,j)\|^2 \\ &+ 8\big(\|B_k^T\mathcal{R}_TB_k\| + \|B_k^TG^T\mathcal{R}_TGB_k\|\big) \\ &\times(\varrho_1+\delta_k)^2\|x(i,j)\|^2.\end{aligned}$$

and $\lambda_{\min}(F_k)$ here denotes the minimum eigenvalue of F_k.

Proof. First, let's define $\boldsymbol{s(i,j)} = \begin{bmatrix} s^h(i+1,j) \\ s^v(i,j+1)\end{bmatrix}$, it is easy find that $\boldsymbol{s(i,j)} = G\boldsymbol{x(i,j)}$. Then, we select the Lyapunov candidate as

$$V(i,j) = V_1(i,j) + V_2(i,j) \tag{36}$$

where $V_1(i,j)$ is defined in Theorem 1, $V_2(i,j) = s^T(i,j)F_ks(i,j)$. Similar with the proof in Theorem 1, it is easy to find that

$$\begin{aligned}&\mathbb{E}\{\Delta V_1(i,j)\} \\ &= \mathbb{E}\Big\{\boldsymbol{x(i,j)}^TR_\tau\boldsymbol{x(i,j)} - x^T(i,j)R_kx(i,j)\Big\} \\ &= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T \\ &\times\mathcal{R}_k\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]\Big\} \\ &- x^T(i,j)R_kx(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks}x(i,j) \\ &+ 2\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T\mathcal{R}_k \\ &\times\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big] \\ &- x^T(i,j)R_kx(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks}x(i,j) \\ &+ \bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_kB_k\bar{\rho}_k(i,j) \\ &+ w^T(i,j)E_k^T\mathcal{R}_kE_kw(i,j) \\ &- x^T(i,j)R_kx(i,j)\end{aligned} \tag{37}$$

Along with the sliding function in (9), we have

$$\begin{aligned}&\mathbb{E}\{\Delta V_2(i,j)\} \\ &= \mathbb{E}\Big\{\boldsymbol{s(i,j)}^TF_\tau\boldsymbol{s(i,j)} - s^T(i,j)F_ks(i,j)\Big\} \\ &= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T \\ &\times G^T\mathcal{F}_kG\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]\Big\} \\ &- s^T(i,j)F_ks(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^TG^T\mathcal{F}_kG\bar{A}_{ks}x(i,j) \\ &+ 2\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^TG^T\mathcal{F}_k \\ &\times G\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big] \\ &- s^T(i,j)F_ks(i,j) \\ &\leq 2x^T(i,j)\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^TG^T\mathcal{F}_kG\bar{A}_{ks}x(i,j) \\ &+ \bar{\rho}_k^T(i,j)B_k^TG^T\mathcal{F}_kGB_k\bar{\rho}_k(i,j) \\ &+ w^T(i,j)E_k^TG^T\mathcal{F}_kGE_kw(i,j) \\ &- s^T(i,j)F_ks(i,j)\end{aligned} \tag{38}$$

Combing (37) and (38), we can infer that

$$\begin{aligned}&\mathbb{E}\{\Delta V(i,j)\} \\ &= \mathbb{E}\Big\{\Delta_1(i,j) + \Delta V_2(i,j)\Big\} \\ &\leq x^T(i,j)\Big\{2\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\big(\mathcal{R}_k + G^T\mathcal{F}_kG\big)\bar{A}_{ks}\Big\}x(i,j) \\ &+ \vec{\rho}_k(i,j) - x^T(i,j)R_kx(i,j) - \lambda_{\min}(F_k)\|s(i,j)\|^2\end{aligned} \tag{39}$$

where

$$\begin{aligned}&\vec{\rho}_k(i,j)\\&=4\big(\|B_k^T\mathcal{F}_kB_k\|+\|B_k^TG^T\mathcal{F}_kGB_k\|\big)\|\bar{\rho}_k(i,j)\|^2\\&+4\big(\|E_k^T\mathcal{R}_kE_k\|+\|E_k^TG^T\mathcal{R}_kGE_k\|\big)\|w(i,j)\|^2\end{aligned}$$

Recalling the condition (13), we can get an inequality as follows

$$\|\bar{\rho}_k(i,j)\|^2\le 2(\varrho_1+\delta_k)^2\|x(i,j)\|^2+2\varrho_2^2\|w(i,j\|^2 \quad (40)$$

It is obvious that $\vec{\rho}_k(i,j)<\hat{\rho}_k(i,j)$ for any $k\in\mathcal{N}_1$ after substitute (40) into $\vec{\rho}_k(i,j)$. Then, based on the condition (15), when the state trajectories is out of the region $\mathcal{O}$ around the specified siding surface (9), we can infer that

$$-\lambda_{\min}(F_k)\|s(i,j)\|^2+\vec{\rho}_k(i,j)<0 \quad (41)$$

It yields from (34), (37), (39) and (41) that

$$\begin{aligned}&\mathbb{E}\{\mathit{\Delta} V(i,j)\}\\&\le x^T(i,j)\Big\{2\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T\big(\mathcal{R}_k+G^T\mathcal{F}_kG\big)\bar{A}_{ks}\\&-R_k\Big\}x(i,j)<0\end{aligned} \quad (42)$$

which means the state trajectories of the close-loop (12) are strictly decreasing (with mean square) outside the region $\mathcal{O}$ defined in (35). Now, the proof is complete.

□

Remark 2. *Remark.*

D. Synthesis of Asynchronous 2D-SMC Law

It is obvious that, if the theorem 1 and the theorem 2 hold simultaneously, then, the asymptotically mean square stability with an H_∞ disturbance attenuation performance γ of the closed-loop 2D system (12) and the reachability of the predefined sliding function (9) can be guaranteed simultaneously. That is, the to be determined matrix K_s in 2D-SMC law (10) should ensure that theorem 1, 2 are established at the same time. Now, in this subsection, we will continue our study with this idea.

Theorem 3. *Consider the Markov jump system (1) under the Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma>0$, if there exist matrices $\tilde{K}_s\in\mathbb{R}^{n_u\times n_x}$, $L_s\in\mathbb{R}^{n_x\times n_x}$, $\tilde{R}_k=\mathrm{diag}\{\tilde{R}_k^h,\tilde{R}_k^v\}>0$, $\tilde{F}_k>0$, $\tilde{Q}_{ks}>0$, $\tilde{T}_{ks}>0$ and scalars $\tilde{\epsilon}_k>0$, for any $k\in\mathcal{N}_1, s\in\mathcal{N}_2$, such that the following inequalities hold:*

$$\begin{bmatrix}-\tilde{\epsilon}_kI & \mathscr{B}_k\\ * & \mathscr{R}_k\end{bmatrix}<0 \quad (43)$$

$$\begin{bmatrix}\mathscr{L}_{ks} & \mathscr{A}_{ks} & \mathscr{G}_{ks}\\ * & \mathscr{R}_k & 0\\ * & * & \mathscr{F}_k\end{bmatrix}<0 \quad (44)$$

$$\begin{bmatrix}\mathscr{H}_k & \mathscr{D}_k & \mathscr{P}_{ks} & \mathscr{Y}_{ks}\\ * & \mathscr{I}_k & 0 & 0\\ * & * & \mathscr{Q}_{ks} & 0\\ * & * & * & \mathscr{T}_{ks}\end{bmatrix}<0 \quad (45)$$

where

$$\begin{aligned}\mathscr{B}_k&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{\epsilon}_kB_k^T & \sqrt{\lambda_{k2}}\tilde{\epsilon}_kB_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{\epsilon}_kB_k^T\end{bmatrix}\\ \mathscr{R}_k&=\mathrm{diag}\{-\tilde{R}_1,-\tilde{R}_2,\cdots,-\tilde{R}_{N_1}\}\\ \mathscr{F}_k&=\mathrm{diag}\{-\tilde{F}_1,-\tilde{F}_2,\cdots,-\tilde{F}_{N_1}\}\\ \mathscr{I}_k&=\mathrm{diag}\{-\tilde{\epsilon}_k,-I,-I,-\tilde{\epsilon}_k\}\\ \mathscr{Q}_{ks}&=\mathrm{diag}\{-\tilde{Q}_{k1},-\tilde{Q}_{k2},\cdots,-\tilde{Q}_{kN_2}\}\\ \mathscr{T}_{ks}&=\mathrm{diag}\{-\tilde{T}_{k1},-\tilde{T}_{k2},\cdots,-\tilde{T}_{kN_2}\}\\ \mathscr{L}_{ks}&=\mathrm{diag}\{\tilde{Q}_{ps}-L_s^T-L_s,-\tilde{T}_{ks}\}\\ \mathscr{H}_k&=\begin{bmatrix}-\tilde{R}_k & \tilde{R}C_k^TD_k^T\\ * & -\gamma^2I\end{bmatrix}\\ \mathscr{A}_{ks}&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{ks}^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{ks}^T\\ \sqrt{\lambda_{k1}}\tilde{T}_{ks}B_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{T}_{ks}B_k^T\end{bmatrix}\\ \mathscr{G}_{ks}&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{ks}^TG^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{ks}^TG^T\\ 0 & \cdots & 0\end{bmatrix}\\ \mathscr{D}_{ks}&=\begin{bmatrix}2(\varrho_1+\delta_k)\tilde{R}_k & \tilde{R}_kC_k^T & 0 & 0\\ 0 & 0 & D_k^T & 2\varrho_2\end{bmatrix}\\ \mathscr{P}_{ks}&=\begin{bmatrix}\sqrt{2\mu_{k1}}\tilde{R}_k & \cdots & \sqrt{2\mu_{kN_2}}\tilde{R}_k\\ 0 & \cdots & 0\end{bmatrix}\\ \mathscr{Y}_{ks}&=\begin{bmatrix}0 & \cdots & 0\\ \sqrt{2\mu_{k1}}I & \cdots & \sqrt{2\mu_{kN_2}}I\end{bmatrix}\end{aligned}$$

and $\tilde{A}_{ks}=A_kL_s+B_k\tilde{K}_s$. Then, the closed-loop system (12) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ, and the state trajectories of the considered closed-loop system will be driven into a sliding region $\mathcal{O}$, around the predefined sliding surface (9). Moreover, the to be determined matrix K_s in 2D-SMC law (10) can be chosen as

$$K_s=\tilde{K}_sL_s^{-1} \quad (46)$$

if the LMIs (43), (44) and (45) have feasible solutions.

Proof. As we discussed above, the objective is to testify that, the conditions (14), (15), (16) in Theorem 1 and (34) in Theorem 2 can be guaranteed simultaneously by (43), (44), (45). Before that, let's make some notations as $\tilde{K}_s=K_sL_s$, $\tilde{R}_k=R_k^{-1}$, $\tilde{F}_k=F_k^{-1}$, $\tilde{Q}_{ks}=Q_{ks}^{-1}$, $\tilde{T}_{ks}=T_{ks}^{-1}$ and $\tilde{\epsilon}=\epsilon_k^{-1}$. Firstly, we will prove that (14) and (43) are equivalent. Pre- and post- multiplying the inequalities given in (43) by $\mathrm{diag}\{\epsilon_kI,I,I,\cdots,I\}$, respectively, and applying Schur complement after that, then, we can see (14) satisfied. Next, we will verity that (44) and (45) are sufficient to ensure (15), (16) and (34) hold simultaneously. Using $\mathrm{diag}\{R_k,I,I,\cdots,I\}$ to pre- and post-multiply the inequality given in (45), and applying Schur complement after that, then we will have (15) satisfied. It follows from (44) that $\tilde{Q}_{ps}-L_s^T-L_s<0$, that is $L_s^T+L_s$ is positive definite, which guarantees that the matrix L_s is invertible. We can infer the following formulation based on $Q_{ps}>0$

$$(\tilde{Q}_{ps}-L_s)^T\tilde{Q}_{ks}^{-1}(\tilde{Q}_{ps}-L_s)\ge 0 \quad (47)$$

which means

$$-L_s^T\tilde{Q}_{ks}L_s\le\tilde{Q}_{ps}-L_s^T-L_s \quad (48)$$

Recalling the condition give in (45), then we can infer that

$$\begin{bmatrix} \tilde{\mathscr{L}}_{ks} & \mathscr{A}_{ks} & \mathscr{G}_{ks} \\ * & \mathscr{R}_k & 0 \\ * & * & \mathscr{F}_k \end{bmatrix} < 0 \tag{49}$$

where $\tilde{\mathscr{L}}_{ks} = \mathrm{diag}\{-L_s^T \tilde{Q}_{ks} L_s, -\tilde{T}_{ks}\}$.
Noting that the slack matrix L_s is invertible, we denote $h_{ks} = \mathrm{diag}\{L_s^{-1}, T_{ks}, I, I, \cdots, I\}$. Using h_{ks} to pre- and post-multiply the inequality given in (49), and applying Schur complement after that, then the following inequality will be obtained

$$\hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} + \check{A}_{ks}^T \mathcal{F}_k \check{A}_{ks} - \mathrm{diag}\{Q_{ks}, T_{ks}\} < 0 \tag{50}$$

where $\check{A}_{ks} = \begin{bmatrix} \bar{A}_{ks} & 0 \end{bmatrix}$. Combing (15) and (50), we will have

$$\mathcal{A} + 2\sum_{s=0}^{N_2} \mu_{ks} \left\{ \hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} + \check{A}_{ks}^T \mathcal{F}_k \check{A}_{ks} \right\} < 0 \tag{51}$$

which further implies (34) holds based on the property of positive definite matrix. It is clear that, the gain matrix K_s can not obtained directly from LMIs in Theorem 3 while $\tilde{K}_s$ is obtained. Thanks to the matrix L_s is invertible, K_s can be calculated indirectly with $K_s = \tilde{K}_s L_s^{-1}$. Now, the proof is finished. □

It is clear that, the conditions in Theorem 3 are presented in the form of Linear matrix inequality, which can be easily solved with the help of Matlab LMI toolbox. Next, we are going to propose an algorithm to obtain the asynchronous 2D-SMC law with the minimum disturbance attenuation performance γ^*. The design procedures of the algorithm are summarized as follows.

1. Select the parameters β_k properly and compute matrix G, such that the matrix GB_k is nonsingular for any $k \in \mathcal{N}_1$.
2. Compute the scalars ϱ_1 and ϱ_2 in (11).
3. Get the matrices $\tilde{K}_s$ and L_s by solving the following optimization problem

$$\min\ \tilde{\sigma} \text{ subject to (43)-(45) with } \tilde{\sigma} = \gamma^2\ . \tag{52}$$

4. Then, the sliding mode controller matrix K_s can be obtained with $K_s = \tilde{K}_s L_s^{-1}$.

IV. Numerical Example

In this section, we provide an example to verify the effectiveness of the proposed asynchronous 2D-SMC law design method. It is known that some dynamical processes, for instance, gas absorption, air drying and water stream heating can be described by the Darboux equation [3]. It is generally described in the form of partial differential equation, and can be easily converted into a 2D Markov jump system in Roesser model with a similar approach applied in [2]. The considered 2D Markov jump system and the asynchronous 2D-SMC law both consist of two operation modes, that is, $\mathcal{N}_1 = \{1, 2\}$, $\mathcal{N}_2 = \{1, 2\}$, and the corresponding system matrices are listed as follows:

Mode 1:

$$A_1 = \begin{bmatrix} -1.0 & 0.4 \\ 0.2 & -1.0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & -0.3 \end{bmatrix},\ D_1 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.1 \end{bmatrix},\ E_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

Mode 2:

$$A_2 = \begin{bmatrix} -0.8 & 0.6 \\ 0.2 & -1.2 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.2 & -0.2 \\ -0.1 & -0.2 \end{bmatrix},\ D_2 = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} -0.2 & -0.1 \\ 0.1 & 0.2 \end{bmatrix},\ E_2 = E_1$$

The nonlinear function $f(x(i,j), r(i,j))$ is set as

$$f((x(i,j), r(i,j)) = \begin{cases} 0.2 \sin \|x(i,j)\|, & r(i,j) = 1 \\ 0.3 \sin \|x(i,j)\|, & r(i,j) = 2 \end{cases}$$

Thus, the parameter δ_k can be selected as first mode with $\delta_1 = 0.2$ and second mode with $\delta_2 = 0.3$.
The mode jumps of the considered 2D system and 2D-SMC law are governed by the transition probability matrix Λ and Ψ, respectively, which are set as follows:

$$\Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix},\ \Psi = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

A possible time sequences with two different directions of the system modes and the asynchronous 2D-SMC law modes are depicted in Fig.1 and Fig.2, respectively. By comparison, it is clear that the designed 2D-SMC law run asynchronously with the original 2D system.

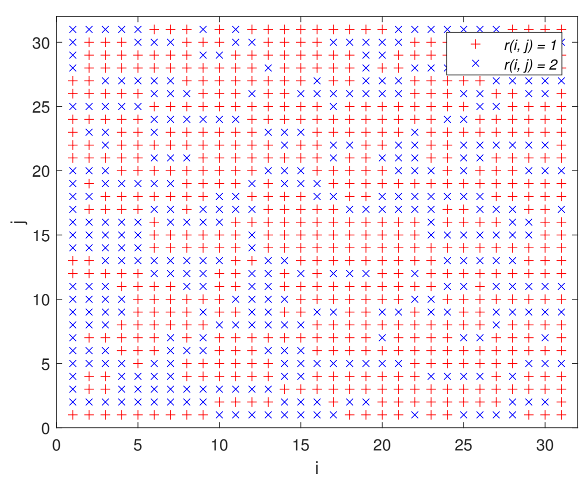

Fig. 1. Modes of the considered 2D system

Based on the corresponding assumptions, the boundary con-

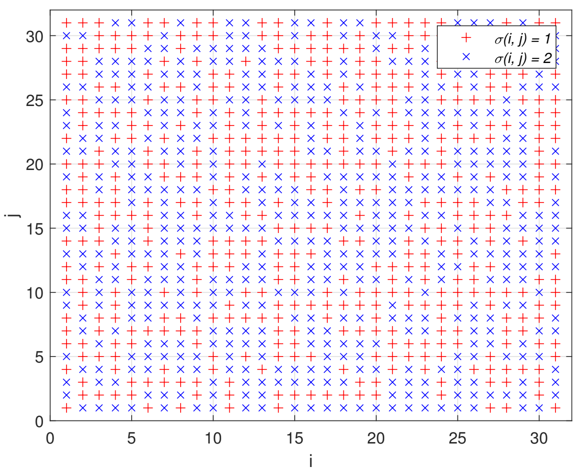

Fig. 2. Modes of the 2D-SMC law

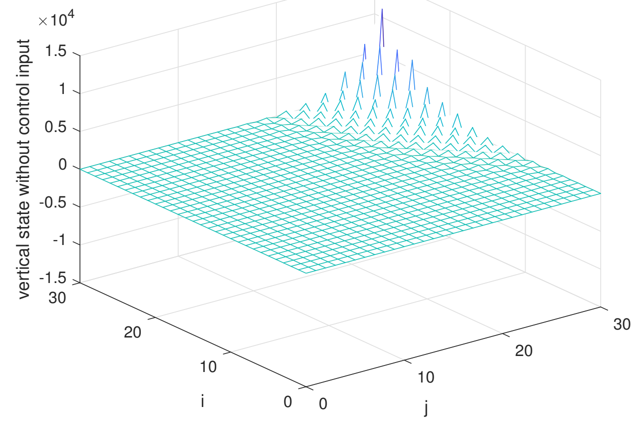

Fig. 4. Modes of the 2D-SMC law

dition X_0 and the exogenous disturbance $w(i,j)$ are assumed to be

$$x^h(0,j) = \begin{cases} 0.1, & 0 \leq j \leq 10 \\ 0, & elsewhere \end{cases}$$

$$x^v(i,0) = \begin{cases} 0.1, & 0 \leq i \leq 10 \\ 0, & elsewhere \end{cases}$$

$$w(i,j) = \begin{cases} 0.2, & 0 \leq i,j \leq 10 \\ 0, & elsewhere \end{cases}$$

The horizontal and vertical state responses of the open-loop

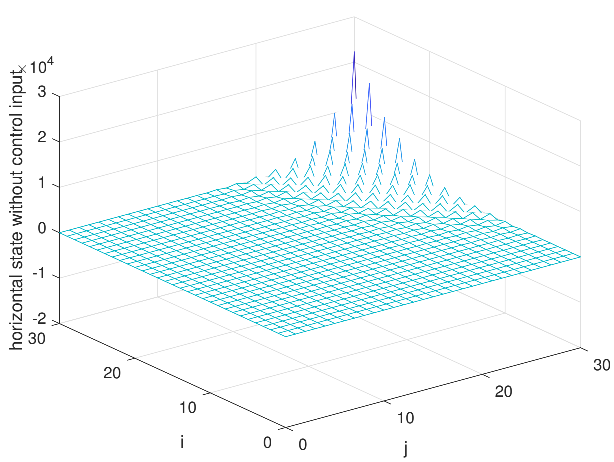

Fig. 3. Modes of the considered 2D system

system in (1) under $u(i,j) = 0$ are depicted in Fig.3 and Fig.4, respectively. The obtained result implies the considered 2D system is unstable under zero control input. Next, let's follow the procedures of asynchronous 2D-SMC law design algorithm, and obtain the sliding model controller gain. Letting $\beta_1 = 0.6$ and $\beta_2 = 0.4$, and the matrix G can be calculated as follows:

$$G = \begin{bmatrix} 0.2 & -0.02 \\ 0.16 & -0.26 \end{bmatrix}$$

It's easy to verify that the non-singularity of the matrix GB_k is satisfied for any $k \in \mathcal{N}_1$. The parameters ϱ_1 and ϱ_2 can be calculated as $\varrho_1 = 0.3$ and $\varrho_2 = 0.6872$, respectively. By solving optimization problem (52), we can obtain the following sliding mode controller gains

Mode1

$$K_1 = \begin{bmatrix} 4.4148 & -4.3508 \\ -0.6827 & -2.1887 \end{bmatrix}$$

Mode2

$$K_2 = \begin{bmatrix} 4.3707 & -3.8450 \\ -0.6826 & -2.3252 \end{bmatrix}$$

with the minimum H_∞ disturbance attenuation performance $\gamma^* = 0.3164$.

Figs.7-8 plot the state responses of the resultant closed-loop system,

with the minimum H_∞ disturbance attenuation performance $\gamma^* = 0.3164$.

V. Conclusions

This is conclusion.

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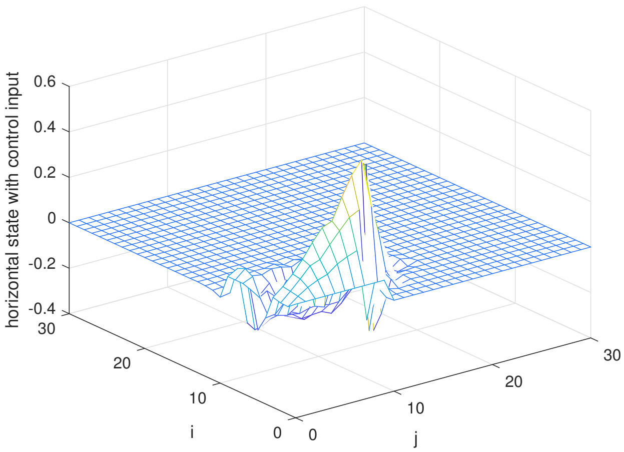

Fig. 5. Modes of the considered 2D system

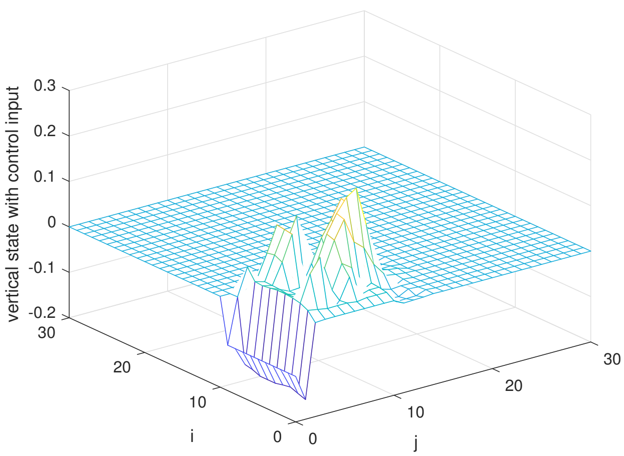

Fig. 6. Modes of the 2D-SMC law

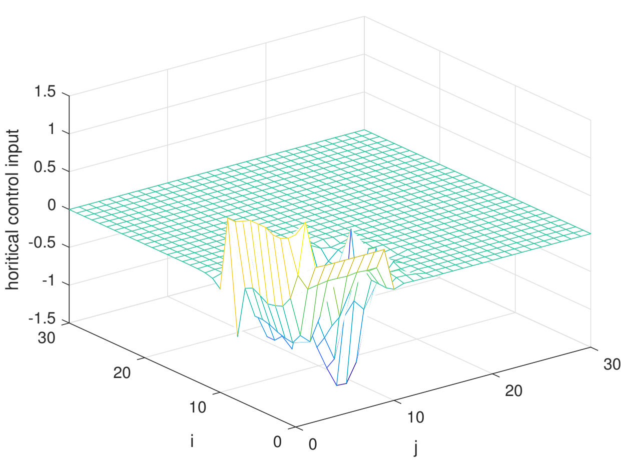

Fig. 7. Modes of the considered 2D system

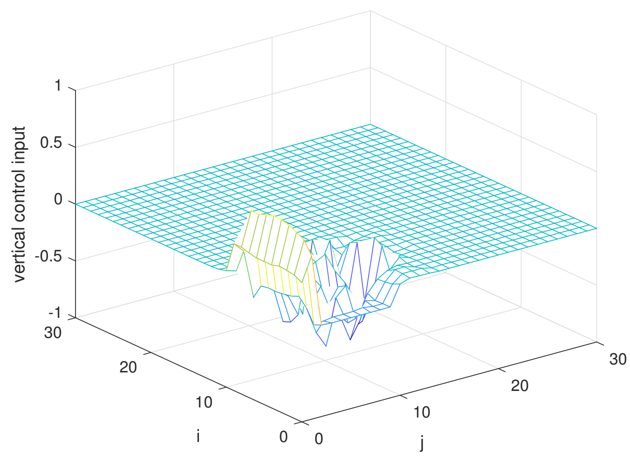

Fig. 8. Modes of the 2D-SMC law