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Brief paper

Asynchronous sliding mode control of Markovian jump systems with time-varying delays and partly accessible mode detection probabilities ☆

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ABSTRACT

In this work, the problem of *asynchronous* sliding mode control (SMC) is investigated for a class of uncertain Markovian jump systems (MJSs) with time-varying delays and stochastic perturbation. It is assumed that the system modes cannot be obtained synchronously by the controller, but instead there is a detector that provides estimated values of the system modes. This asynchronous phenomenon between the system modes and controller modes will be described in this work via a hidden Markov model with partly accessible mode detection probabilities. Based on a common sliding surface, an asynchronous SMC law depending on the detector mode is synthesized to ensure the mean square stability of the sliding mode dynamics and the reachability of the specified sliding surface simultaneously. Moreover, a design algorithm for obtaining the asynchronous SMC law is established. Finally, an application of the automotive electronic throttle system is provided to illustrate the effectiveness and advantages of the proposed asynchronous sliding mode control approach.

1. Introduction

Markovian jump systems (MJSs) have received considerable attention in the past decades, since many real-world systems, such as manufacturing systems, aerospace systems and networked control systems with abrupt variations in their structures can be effectively represented by MJSs, in which the abrupt variations may happen due to random failures or repairs of components, changing of subsystem interconnections, abrupt variations in the operating point, etc. (Costa, Fragoso, & Marques, 2005; Shi & Li, 2015; Shi & Yu, 2009). A variety of works have been published with respect to the stability and stabilization of MJSs, see Bolzern, Colaneri, and de Nicolao (2013), Feng, Lam, and Shu (2010), Fioravant, Gonçalves, and Geromel (2013), Zhang and Lam (2010) and the references therein. Especially, the MJSs subject to time delays has become an active research topic, since the time delays may happen inevitably in practical applications (Wei, Qiu, Karimi, & Wang, 2013; Zhang, Boukas, & Lam, 2008).

On the other hand, as an effective robust control approach for uncertain systems with parameter variations and external disturbances, significant progresses have been recently achieved on the sliding mode control (SMC) of MJSs (Basin, Panathula, & Shtessel, 2017; Basin & Rodríguez-Ramírez, 2014). Among them, the mode-dependent sliding surface was designed in Niu, Ho, and Wang (2007) to establish the connections among different sliding functions under Markovian jumping. A singular system approach in Ma and Boukas (2009) was proposed to design both mode-independent and mode-dependent sliding surfaces. Following these excellent results, the SMC problem for various MJSs has been widely studied including time delays (Karimi, 2012), unmeasured states (Wu, Gao, Liu, & Li, 2017; Yin, Yang, & Kaynak, 2017), incomplete transition information (Kao, Xie, Zhang, & Karimi, 2015; Zhang, Wang, & Shi, 2013), actuator degradation (Chen, Niu, & Zou, 2013), missing measurement (Chen, Niu, & Zou, 2014), etc.

However, it should be pointed out that, in all the aforementioned works, it was implicitly assumed that the information of system modes was fully accessible for the sliding mode controller all the time, in order to ensure the synchronization between the controller mode and the system mode. Thus, the *mode-dependent* sliding mode controller can be achieved. Unfortunately, the above

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ideal assumption is difficult to be satisfied in practical applications. For example, the mode information of plant cannot be completely accessible due to communication delays and missing measurement, which may bring the asynchronization phenomenon between sliding mode controller modes and system modes. Although the mode-independent SMC approach in Huang, Lin, and Lin (2012) may be feasible for the above case, it also brings more conservatism due to neglecting all the available modes information (Liu, Ho, & Sun, 2008). Hence, the importance of asynchronous control/filtering for MJSs has already begun receiving more attentions. For example, the l_2–l_∞ and H_2 asynchronous filtering for discrete-time MJSs were addressed in de Oliveira and Costa (2017) and Wu, Shi, Su, and Chu (2014), respectively. Based on the hidden Markov model, the H_2 and passive asynchronous control methods were further proposed in Costa, Fragoso, and Todorov (2015) and Wu, Shi, Shu, Su, and Lu (2017), respectively. However, to the authors' best knowledge, until now, the problem of *asynchronous* SMC for time-delay MJSs has not been investigated. Moreover, due to the special structure of SMC systems, the asynchronous characteristic between the controller modes and the system modes is also more complex and interesting, which motivates the present work.

In this work, we seek to investigate the asynchronous SMC problem for a class of discrete-time MJSs subjected to time-varying delays and stochastic perturbation. With a similar framework in Costa et al. (2015), it is assumed that the information of system modes to controller can be only estimated by a detector via a hidden Markov model with a mode detection probability matrix (MDPM). A key feature of this work is that the mode detection probabilities are further considered to be *partly accessible* to controller design, which shows more generalized than the existing works as in Costa et al. (2015), de Oliveira and Costa (2017) and Wu et al. (2017). Thus, in order to design asynchronous SMC under the above partly accessible assumption, we have to answer the following questions:

Q1: How to design a suitable sliding function and SMC law just by using the detected modes?
Q2: How to tackle the *partly* accessible mode detection probabilities in designing asynchronous SMC law?
Q3: How to guarantee the mean square stability of the sliding mode dynamics and the reachability of the specified sliding surface subjected to *partly* accessible mode detection probabilities, time-varying delays and stochastic perturbation?

This work will provide satisfying answers to the above three questions. *The main contributions of this work are highlighted as follows. (1) The asynchronous phenomenon between the system modes and the detector modes to controller is coped with a hidden Markov model with the partly accessible mode detection probabilities, which is more generalized than the one considered in* Costa et al. (2015), de Oliveira and Costa (2017) *and* Wu et al. (2017). *(2) A new common sliding function is constructed properly, and by just using the information of the detected modes, an asynchronous SMC law is designed to ensure the mean square stability of the sliding mode dynamics. The framework of the proposed asynchronous SMC covers the existing mode-dependent SMC approaches as in* Chen et al. (2014), Kao et al. (2015) *and* Karimi (2012) and Zhang et al. (2013) *and the mode-independent SMC approach as in* Huang et al. (2012) *as two special cases. (3) The reachability of a sliding region around the specified sliding surface is proven by using a stochastic Lyapunov method, and the design algorithm of the proposed asynchronous SMC law is derived by means of a necessary and sufficient condition.*

Notation. All matrices in this work are supposed to have compatible dimensions. $\mathbb{Z}_-$ denotes the set of negative integers. $\mathbf{E}\{\cdot\}$ denotes the expectation operator with respect to probability measure. For a real symmetric matrix M, $M > 0$ represents that M is a positive-definite matrix. The shorthand "diag{·}" denotes a block diagonal matrix. In symmetric block matrices, the symbol "$\star$" is used as an ellipsis for terms induced for symmetry. $\|\cdot\|$ denotes the Euclidean norm of a vector or its induced matrix norm.

2. Problem formulation

2.1. System description

Given the probability space $(\Omega, \mathcal{F}, \mathrm{Prob}\{\cdot\})$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of events, and $\mathrm{Prob}\{\cdot\}$ is the probability measure defined on $\mathcal{F}$. The parameter $\{r(k) = i,\ k \geq 0\}$ is a Markov chain taking values on a finite set $\mathcal{N} \triangleq \{1, 2, \ldots, N\}$ with transition probability matrix $\Pi \triangleq [\pi_{ij}]$, $i, j \in \mathcal{N}$, and having the following transition probability from mode i at sample time k to mode j at sample time $k + 1$:

$$\pi_{ij} = \mathrm{Prob}\{r(k+1) = j \mid r(k) = i\}, \forall i, j \in \mathcal{N}, \tag{1}$$

where $\pi_{ij} \in [0, 1]$, and $\sum_{j=1}^{N} \pi_{ij} = 1$. Now, we consider the following discrete-time MJSs with time-varying delays and stochastic perturbation:

$$\begin{cases} x(k+1) = [A(r(k)) + \Delta A(r(k), k)]\, x(k) \\ \quad + [A_d(r(k)) + \Delta A_d(r(k), k)]\, x(k - d(k)) \\ \quad + B(r(k))\, [u(k) + f(x(k), x(k-d(k)), r(k))] \\ \quad + [G(r(k))x(k) + G_d(r(k))x(k-d(k))]\, \varpi(k), \\ x(n) = \phi(n),\ \forall n \in \mathbb{Z}_-, \end{cases} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ is the control input, and $\varpi(k) \in \mathbb{R}$ is a scalar Wiener process satisfying $\mathbf{E}\{\varpi(k)\} = 0$, $\mathbf{E}\left\{\varpi^2(k)\right\} = \zeta$, $\mathbf{E}\{\varpi(k_1)\varpi(k_2)\} = 0$ for $k_1 \neq k_2$. $\phi(n)$ is a given initial condition, $\forall n \in \mathbb{Z}_-$. The discrete time-varying delay satisfies $d_m \leq d(k) \leq d_M$, where d_m and d_M are known positive integers representing the lower and upper bounds of the time delay, respectively.

For each $r(k) = i$, the matrices $A_i \triangleq A(r(k))$, $A_{di} \triangleq A_d(r(k))$, $B_i \triangleq B(r(k))$, $G_i \triangleq G(r(k))$, $G_{di} \triangleq G_d(r(k))$ are known constant matrices, and the admissible uncertainties $\Delta A_i(k) \triangleq \Delta A(r(k), k)$, $\Delta A_{di}(k) \triangleq \Delta A_d(r(k), k)$ satisfy $\begin{bmatrix} \Delta A_i(k) & \Delta A_{di}(k) \end{bmatrix} = H_i \Phi_i(k) \begin{bmatrix} E_i & E_{di} \end{bmatrix}$, with E_i, E_{di} and H_i known constant matrices, and $\Phi_i(k)$ an unknown time-varying matrix satisfying $\Phi_i^{\mathrm{T}}(k)\Phi_i(k) \leq I$. Besides, we assume that

A1. For any $r(k) = i \in \mathcal{N}$, the nonlinear function $f_i(x(k), x(k - d(k))) \triangleq f(x(k), x(k-d(k)), r(k))$ possesses the following property:

$$\|f_i(x(k), x(k - d(k)))\| \leq \epsilon_i \|x(k)\| + \epsilon_{di} \|x(k - d(k))\|,$$

where $\epsilon_i > 0$ and $\epsilon_{di} > 0$ are two known scalars.

A2. The input matrix B_i is full column rank, that is, $\mathrm{rank}(B_i) = m$.

Definition 1 (*Costa et al., 2005*)**.** Stochastic MJSs (2) with $u(k) \equiv 0$ is said to be mean square stable if, for any initial conditions $\{x(0),\ \phi(n)\} \in \mathbb{R}^n$, $r(0) \in \mathcal{N}$, the following condition holds:

$$\lim_{k \to \infty} \mathbf{E}\left\{\|x(k)\|^2\right\} \big|_{x(0), \phi(n), r(0)} = 0. \tag{3}$$

2.2. Partly accessible mode detection probabilities

In practical applications, it is not always possible to directly measure the information of system modes $r(k)$, but instead there is a detector $\sigma(k)$ that provides estimated values of $r(k)$ with some probability (Costa et al., 2015). In this case, the emitted signal $\sigma(k)$ from detector to controller does not synchronize with the system mode $r(k)$. The hidden Markov model $\big(r(k), \sigma(k)\big)$ as in Costa et al. (2015), de Oliveira and Costa (2017) and Wu et al. (2017) is introduced to characterize the above asynchronous phenomenon as follows:

$$\mu_{i\nu} = \mathrm{Prob}\{\sigma(k) = \nu \mid r(k) = i\}, \tag{4}$$

where $i \in \mathcal{N}$, $\nu \in \mathcal{V}$, $\mathcal{V} \triangleq \{1, 2, \ldots, V\}$, $\mu_{i\nu}$ is the mode detection probability belonging to [0, 1]. For any $i \in \mathcal{N}$, it has $\sum_{\nu=1}^{V}\mu_{i\nu} = 1$. The MDPM Ω is defined as $\Omega \triangleq [\mu_{i\nu}]$.

It is clear that the hidden Markov model (4) covers the mode-dependent ($\mathcal{V} = \mathcal{N}$, $\mu_{i\nu} = 1$ for $\nu = i$) and mode-independent ($\mathcal{V} = \{1\}$) cases. However, the mode detection probabilities in Costa et al. (2015), de Oliveira and Costa (2017) and Wu et al. (2017) were supposed to be *completely* accessible. Apparently, this implicit assumption may be difficultly or costly attained in practical application due to possibly unobservable system modes or prohibitive costs in measuring mode information. A key advantage of this work is that some mode detection probabilities $\mu_{i\nu}$ in MDPM are allowed to be unknown, that is, Ω is *partly* accessible to controller design. For technical analysis, we denote the finite set $\ell = \ell_{\mathcal{K}}^{\nu} \cup \ell_{\mathcal{UK}}^{\nu}, \forall \nu \in \mathcal{V}$ with

$$\ell_{\mathcal{K}}^{\nu} \triangleq \{\nu \mid \mu_{i\nu} \text{ is known}\}, \ \ell_{\mathcal{UK}}^{\nu} \triangleq \{\nu \mid \mu_{i\nu} \text{ is unknown}\}. \tag{5}$$

Remark 1. Different from the well results on partially known transition probability matrix Π as in Fioravant et al. (2013), Kao et al. (2015), Wei et al. (2013), Zhang and Lam (2010), Zhang et al. (2008) and Zhang et al. (2013), the present work is focused on the partly accessible MDPM Ω associated with the hidden Markov model (4). It is easily seen that the partial accessibility to MDPM in this work covers the cases in previous works (Costa et al., 2015; de Oliveira & Costa, 2017; Wu et al., 2017) by letting $\ell_{\mathcal{UK}}^{\nu} = \emptyset$ in (5).

Now, our objective is to design an asynchronous SMC law $u(k)$, *depending only on the estimated mode* $\sigma(k)$, such that the resultant closed-loop system is mean square stable in spite of *partly* accessible mode detection probabilities, time-varying delays, stochastic perturbation and parameter uncertainties.

3. Asynchronous sliding mode control

3.1. Sliding surface and sliding mode controller

There are many results have been reported on SMC of discrete-time MJSs as in Chen et al. (2014) and Zhang et al. (2013), where the *mode-dependent* sliding surfaces were widely employed. However, due to the switching frequently from one mode to another, the reachability of sliding surfaces in these works may not always be attained actually. In order to avoid the above problem, a *common* sliding function will be utilized as follows:

$$s(k) = Zx(k), \tag{6}$$

where $Z \triangleq \sum_{i=1}^{N}\beta_i B_i^{\mathrm{T}}$, and scalars β_i ($i \in \mathcal{N}$) should be chosen such that $X_i \triangleq ZB_i$ is nonsingular for any $i \in \mathcal{N}$. As discussed in Liu, Niu, Lam, and Zhang (2014), based on the assumption that $\mathrm{rank}(B_i) = m$ for any $i \in \mathcal{N}$, the above nonsingularity conditions can be guaranteed easily by selecting the parameters β_i properly.

Notice that the system mode $r(k)$ can be only estimated via a detector $\sigma(k)$ with a *partly* accessible MDPM. That means that the controller design for the system (2) can just utilize the mode information emitting from the detector. To this end, we design an appropriate detected-mode-dependent SMC law as follows:

$$\begin{aligned} u(k) &= K_\nu x(k) + F_\nu x(k-d(k)) \\ &\quad - \rho(x(k), x(k-d(k)))\frac{s(k)}{\|s(k)\|}, \ \nu \in \mathcal{V}, \end{aligned} \tag{7}$$

where the matrices $K_\nu \in \mathbb{R}^{m\times n}$, $F_\nu \in \mathbb{R}^{m\times n}$ ($\nu \in \mathcal{V}$) will be determined later, and the robust term $\rho(x(k), x(k-d(k)))$ is given as

$$\rho(x(k), x(k-d(k))) \triangleq \varrho_1\|x(k)\| + \varrho_2\|x(k-d(k))\|, \tag{8}$$

with $\varrho_1 \triangleq \max_{i\in\mathcal{N}}\{\delta_{1i}\}$, $\varrho_2 \triangleq \max_{i\in\mathcal{N}}\{\delta_{2i}\}$, $\delta_{1i} \triangleq \|X_i^{-1}ZH_i\| \cdot \|E_i\| + \epsilon_i$, and $\delta_{2i} \triangleq \|X_i^{-1}ZH_i\| \cdot \|E_{di}\| + \epsilon_{di}$. Clearly, the asynchronous SMC law (7) is implementable in practice, since it just uses the detected mode $\nu \in \mathcal{V}$, which is nonsynchronous with the system mode $i \in \mathcal{N}$ via the hidden Markov model (4).

Remark 2. It is noted that the effect of stochastic jumping from one detected mode to another is reflected in the asynchronous SMC law (7) only via matrices K_ν and F_ν, which are the solutions of some coupled matrix inequalities concerning the transition probabilities π_{ij} and the mode detection probabilities $\mu_{i\nu}$ (see Theorem 3 later). This key feature gives the response to the first question (**Q1**). As mentioned before, the *synchronous* mode-dependent SMC approaches as in Chen et al. (2014), Kao et al. (2015), Karimi (2012) and Zhang et al. (2013) and the mode-independent SMC approach as in Huang et al. (2012) are the two kinds of special cases of the proposed asynchronous SMC law (7) in this work.

By substituting (7) into (2), the closed-loop system is obtained as

$$\begin{aligned} x(k+1) &= \tilde{A}_{i\nu}x(k) + \tilde{A}_{di\nu}x(k-d(k)) \\ &+ \left[G_i x(k) + G_{di}x(k-d(k))\right]\varpi(k) + B_i\rho_{Di}(k), \end{aligned} \tag{9}$$

with $\tilde{A}_{i\nu} \triangleq A_i + \tilde{Z}_i\Delta A_i(k) + B_iK_\nu$, $\tilde{A}_{di\nu} \triangleq A_{di} + \tilde{Z}_i\Delta A_{di}(k) + B_iF_\nu$, $\tilde{Z}_i \triangleq I - B_iX_i^{-1}Z$, $\rho_{Di}(k) \triangleq D_i(k) - \rho(x(k), x(k-d(k)))\frac{s(k)}{\|s(k)\|}$, and $D_i(k) \triangleq X_i^{-1}Z\Delta A_i(k)x(k) + X_i^{-1}Z\Delta A_{di}(k)x(k-d(k)) + f_i(x(k), x(k-d(k)))$. It is clear that

$$\|\rho_{Di}(k)\| \le (\delta_{1i} + \varrho_1)\|x(k)\| + (\delta_{2i} + \varrho_2)\|x(k-d(k))\|. \tag{10}$$

3.2. Analysis of sliding mode dynamics

In this subsection, the sufficient condition for mean square stability of the closed-loop system (9) will be derived.

Theorem 1. *Given a partly accessible MDPM Ω, the closed-loop system (9) is mean square stable if, for any $i \in \mathcal{N}$ and $\nu \in \mathcal{V}$, there exist matrices $K_\nu \in \mathbb{R}^{m\times n}$, $P_i > 0$, $Q > 0$, and scalars $\varsigma_i > 0$ such that the following inequalities hold:*

$$B_i^{\mathrm{T}}\mathcal{P}_iB_i - \varsigma_iI \le 0, \tag{11}$$

$$\begin{aligned} &\tilde{\Xi}_i + \sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu} \\ &\quad + \left(1 - \sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\right)\sum_{\nu\in\ell_{\mathcal{UK}}^{\nu}}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu} < 0, \end{aligned} \tag{12}$$

where $\mathcal{P}_i \triangleq \sum_{j=1}^{N}\pi_{ij}P_j$, $\tilde{\Xi}_i \triangleq \begin{bmatrix} \tilde{\mathcal{Z}}_i & \zeta G_i^{\mathrm{T}}\mathcal{P}_iG_{di} \\ \star & \tilde{\mathcal{X}}_i \end{bmatrix}$, $\tilde{\mathcal{A}}_{i\nu} \triangleq \begin{bmatrix}\sqrt{2}\tilde{A}_{i\nu} & \sqrt{2}\tilde{A}_{di\nu}\end{bmatrix}$, $\tilde{\mathcal{X}}_i \triangleq -Q + \zeta G_{di}^{\mathrm{T}}\mathcal{P}_iG_{di} + 4\varsigma_i(\delta_{2i} + \varrho_2)^2I$, *and* $\tilde{\mathcal{Z}}_i \triangleq -P_i + (d_M - d_m + 1)Q + \zeta G_i^{\mathrm{T}}\mathcal{P}_iG_i + 4\varsigma_i(\delta_{1i} + \varrho_1)^2I$.

Proof. For any system mode $r(k) = i \in \mathcal{N}$, we consider the following Lyapunov functional candidate for sliding mode dynamics (9):

$$V(x(k), i) \triangleq V_1(x(k), i) + V_2(x(k)) + V_3(x(k)), \tag{13}$$

where $V_1(x(k), i) \triangleq x^{\mathrm{T}}(k)P_ix(k)$, $V_2(x(k)) \triangleq \sum_{l=k-d(k)}^{k-1}x^{\mathrm{T}}(l)Qx(l)$ and $V_3(x(k)) \triangleq \sum_{z=-d_M+1}^{-d_m}\sum_{t=k+z}^{k-1}x^{\mathrm{T}}(t)Qx(t)$.

Along the state trajectories of (9), it has

$$\begin{aligned}
&\mathbf{E}\{\Delta V_1(x(k),i)\}\\
&=\mathbf{E}\{x^{\mathrm{T}}(k+1)\mathcal{P}_i x(k+1)\}-x^{\mathrm{T}}(k)P_i x(k)\\
&=\sum_{\nu=1}^{V}\mu_{i\nu}\Big\{\Big[\tilde{A}_{i\nu}x(k)+\tilde{A}_{di\nu}x(k-d(k))+B_i\rho_{Di}(k)\Big]^{\mathrm{T}}\\
&\times\mathcal{P}_i\Big[\tilde{A}_{i\nu}x(k)+\tilde{A}_{di\nu}x(k-d(k))+B_i\rho_{Di}(k)\Big]\Big\}\\
&+\zeta[G_ix(k)+G_{di}x(k-d(k))]^{\mathrm{T}}\mathcal{P}_i\\
&\times[G_ix(k)+G_{di}x(k-d(k))]-x^{\mathrm{T}}(k)P_ix(k).
\end{aligned}\tag{14}$$

Notice that from (10) and (11), we have

$$\begin{aligned}
&\rho_{Di}^{\mathrm{T}}(k)B_i^{\mathrm{T}}\mathcal{P}_iB_i\rho_{Di}(k)\\
&\le 2\varsigma_i\Big[(\delta_{1i}+\varrho_1)^2x^{\mathrm{T}}(k)x(k)\\
&\quad+(\delta_{2i}+\varrho_2)^2x^{\mathrm{T}}(k-d(k))x(k-d(k))\Big],
\end{aligned}\tag{15}$$

$$\begin{aligned}
&2\Big[\tilde{A}_{i\nu}x(k)+\tilde{A}_{di\nu}x(k-d(k))\Big]^{\mathrm{T}}\mathcal{P}_iB_i\rho_{Di}(k)\\
&\le\Big[\tilde{A}_{i\nu}x(k)+\tilde{A}_{di\nu}x(k-d(k))\Big]^{\mathrm{T}}\mathcal{P}_i\\
&\quad\times\Big[\tilde{A}_{i\nu}x(k)+\tilde{A}_{di\nu}x(k-d(k))\Big]\\
&\quad+2\varsigma_i\Big[(\delta_{1i}+\varrho_1)^2x^{\mathrm{T}}(k)x(k)\\
&\quad+(\delta_{2i}+\varrho_2)^2x^{\mathrm{T}}(k-d(k))x(k-d(k))\Big].
\end{aligned}\tag{16}$$

Substituting (15)–(16) into (14) yields

$$\begin{aligned}
&\mathbf{E}\{\Delta V_1(x(k),i)\}\\
&\le 2\sum_{\nu=1}^{V}\mu_{i\nu}\Big[\tilde{A}_{i\nu}x(k)+\tilde{A}_{di\nu}x(k-d(k))\Big]^{\mathrm{T}}\mathcal{P}_i\\
&\times\Big[\tilde{A}_{i\nu}x(k)+\tilde{A}_{di\nu}x(k-d(k))\Big]\\
&+4\varsigma_i\Big[(\delta_{1i}+\varrho_1)^2x^{\mathrm{T}}(k)x(k)\\
&+(\delta_{2i}+\varrho_2)^2x^{\mathrm{T}}(k-d(k))x(k-d(k))\Big]\\
&+\zeta[G_ix(k)+G_{di}x(k-d(k))]^{\mathrm{T}}\mathcal{P}_i\\
&\times[G_ix(k)+G_{di}x(k-d(k))]-x^{\mathrm{T}}(k)P_ix(k).
\end{aligned}\tag{17}$$

On the other hand, one has

$$\begin{aligned}
&\Delta V_2(x(k))=x^{\mathrm{T}}(k)Qx(k)-x^{\mathrm{T}}(k-d(k))Qx(k-d(k))\\
&+\sum_{l=k+1-d(k+1)}^{k-1}x^{\mathrm{T}}(l)Qx(l)-\sum_{l=k+1-d(k)}^{k-1}x^{\mathrm{T}}(l)Qx(l),
\end{aligned}\tag{18}$$

$$\begin{aligned}
&\Delta V_3(x(k))=(d_M-d_m)\,x^{\mathrm{T}}(k)Qx(k)\\
&\qquad-\sum_{l=k-d_M+1}^{k-d_m}x^{\mathrm{T}}(l)Qx(l).
\end{aligned}\tag{19}$$

Then, it follows from (17)–(19) that

$$\begin{aligned}
&\mathbf{E}\{\Delta V(x(k),i)\}\\
&=\mathbf{E}\{\Delta V_1(x(k),i)\}+(d_M-d_m+1)\,x^{\mathrm{T}}(k)Qx(k)\\
&\quad-x^{\mathrm{T}}(k-d(k))Qx(k-d(k))\\
&\quad+\sum_{l=k+1-d_m}^{k-1}x^{\mathrm{T}}(l)Qx(l)-\sum_{l=k+1-d(k)}^{k-1}x^{\mathrm{T}}(l)Qx(l)\\
&\quad+\sum_{l=k+1-d(k+1)}^{k-d_m}x^{\mathrm{T}}(l)Qx(l)-\sum_{l=k+1-d_M}^{k-d_m}x^{\mathrm{T}}(l)Qx(l)\\
&\le\eta^{\mathrm{T}}(k)\left[\sum_{\nu=1}^{V}\mu_{i\nu}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu}+\tilde{\Xi}_i\right]\eta(k),
\end{aligned}\tag{20}$$

where $\eta(k)\triangleq\big[x^{\mathrm{T}}(k)\quad x^{\mathrm{T}}(k-d(k))\big]^{\mathrm{T}}$.

In this work, some mode detection probabilities $\mu_{i\nu}$, $\nu\in\mathcal{V}$, are unknown to controller design. By noting the fact that $\sum_{\nu=1}^{V}\mu_{i\nu}=1$, we have the following relationships for $0\le\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}<1$:

$$\begin{aligned}
&\sum_{\nu=1}^{V}\mu_{i\nu}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu}\\
&=\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu}+\Big(1-\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\Big)\\
&\times\sum_{\nu\in\ell_{\mathcal{UK}}^{\nu}}\frac{\mu_{i\nu}}{1-\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu}\\
&\le\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu}+\Big(1-\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\Big)\sum_{\nu\in\ell_{\mathcal{UK}}^{\nu}}\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu}.
\end{aligned}\tag{21}$$

Obviously, the inequality (21) also holds for $\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}=1$, i.e., $\ell_{\mathcal{UK}}^{\nu}=\emptyset$, MDPM Ω is *fully* accessible.

Therefore, it follows from (20) and (21) that the condition (12) ensures $\mathbf{E}\{\Delta V(x(k),i)\}<0$ for any $i\in\mathcal{N}$, which means that the mean square stability of the sliding mode dynamics (9) is attained. ■

Remark 3. One can see from the derivations in (21) that the partly known problem of the elements in MDPM has been tackled effectively. Thus, the mean square stability of sliding mode dynamics (9) is guaranteed despite *partly* accessible mode detection probabilities. That answers the second question (**Q2**). On the other hand, it is shown that the results in Theorem 1 are available for the *completely unaccessible* MDPM case ($\ell_{\mathcal{K}}^{\nu}=\emptyset$) meaning that the detector $\sigma(k)$ cannot provide any reliable information regarding $r(k)$. Actually, one can treat the worst case as the mode-independent one by simply letting $\mathcal{V}=\{1\}$ in (4).

3.3. Analysis of reachability

This subsection carries out the reachability analysis by using a stochastic Lyapunov method. It will be confirmed that the designed asynchronous SMC law (7) can force the state trajectories of the closed-loop system (9) into a time-varying sliding region around the specified sliding surface (6) *in mean square*.

Theorem 2. *Consider the stochastic MJS (2) with a* partly *accessible MDPM Ω and the detected-mode-dependent SMC law (7). For any $i\in\mathcal{N}$ and $\nu\in\mathcal{V}$, if there exist matrices $K_\nu\in\mathbb{R}^{m\times n}$, $F_\nu\in\mathbb{R}^{m\times n}$, $P_i>0$, $W_i>0$, $Q>0$, and scalar $\varsigma_i>0$ satisfying the condition (11) and the following matrix inequality:*

$$\begin{aligned}
&\vec{\Xi}_i+\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu}\\
&\quad+\Big(1-\sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\Big)\sum_{\nu\in\ell_{\mathcal{UK}}^{\nu}}\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu}<0,
\end{aligned}\tag{22}$$

where $\vec{\Xi}_i\triangleq\tilde{\Xi}_i+\begin{bmatrix}\zeta G_i^{\mathrm{T}}Z^{\mathrm{T}}\mathcal{W}_iZG_i & \zeta G_i^{\mathrm{T}}Z^{\mathrm{T}}\mathcal{W}_iZG_{di}\\ \star & \zeta G_{di}^{\mathrm{T}}Z^{\mathrm{T}}\mathcal{W}_iZG_{di}\end{bmatrix}$, $\mathbf{A}_{i\nu}\triangleq\big[\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\quad\tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\big]^{\mathrm{T}}$, $\mathbf{Y}_i\triangleq\mathrm{diag}\{\mathcal{P}_i,\mathcal{W}_i\}$, $\vec{\mathcal{A}}_{i\nu}\triangleq\big[\sqrt{2}\bar{A}_{i\nu}\quad\sqrt{2}\bar{A}_{di\nu}\big]$, $\mathcal{W}_i\triangleq\sum_{j=1}^{N}\pi_{ij}W_j$,

$\vec{A}_{i\nu} \triangleq ZA_i + X_iK_\nu$, $\vec{A}_{di\nu} \triangleq ZA_{di} + X_iF_\nu$, and other matrices are defined in Theorem 1, then the state trajectories of the closed-loop system (9) will be driven into the following sliding region $\mathcal{O}$ in mean square around the specified sliding surface (6):

$$\mathcal{O} \triangleq \left\{ s(k) \,\middle|\, \|s(k)\| \le \rho^*(k) \right\} \tag{23}$$

where $\rho^*(k) \triangleq \max_{i\in\mathcal{N}} \left\{ \sqrt{\frac{2\lambda_{\max}\left(X_i^{\mathrm{T}}\mathcal{W}_iX_i\right)}{\lambda_{\min}(W_i)}} \right\} \cdot \vec{\rho}(k)$ with $\vec{\rho}(k) \triangleq 2\varrho_1\|x(k)\| + 2\varrho_2\|x(k-d(k))\|$.

Proof. Combining (6) and (9) yields

$$\begin{aligned} s(k+1) &= \vec{A}_{i\nu}x(k) + \vec{A}_{di\nu}x(k-d(k)) + X_i\rho_{Di}(k) \\ &\quad + \left[ZG_ix(k) + ZG_{di}x(k-d(k))\right]\varpi(k). \end{aligned} \tag{24}$$

Select the Lyapunov functional as $\tilde{V}(k,i) \triangleq V(x(k),i) + s^{\mathrm{T}}(k)W_is(k)$. From (24), we have

$$\begin{aligned} &\mathbf{E}\left\{s^{\mathrm{T}}(k+1)\mathcal{W}_is(k+1)\right\} \\ &\le 2\sum_{\nu=1}^{V}\mu_{i\nu}\left[\vec{A}_{i\nu}x(k) + \vec{A}_{di\nu}x(k-d(k))\right]^{\mathrm{T}}\mathcal{W}_i \\ &\quad\times \left[\vec{A}_{i\nu}x(k) + \vec{A}_{di\nu}x(k-d(k))\right] \\ &\quad+ \zeta\left[ZG_ix(k) + ZG_{di}x(k-d(k))\right]^{\mathrm{T}}\mathcal{W}_i \\ &\quad\times \left[ZG_ix(k) + ZG_{di}x(k-d(k))\right] \\ &\quad+ 2\rho_{Di}^{\mathrm{T}}(k)X_i^{\mathrm{T}}\mathcal{W}_iX_i\rho_{Di}(k). \end{aligned} \tag{25}$$

Keeping (20) and (25) in mind, we obtain

$$\begin{aligned} &\mathbf{E}\left\{\Delta\tilde{V}(k,i)\right\} \\ &= \mathbf{E}\{\Delta V(k,i)\} + \mathbf{E}\left\{s^{\mathrm{T}}(k+1)\mathcal{W}_is(k+1)\right\} \\ &\quad - s^{\mathrm{T}}(k)W_is(k) \\ &\le \eta^{\mathrm{T}}(k)\left[\sum_{\nu=1}^{V}\mu_{i\nu}\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu} + \vec{\Xi}_i\right]\eta(k) \\ &\quad - \left[\lambda_{\min}(W_i)\|s(k)\|^2 - 2\lambda_{\max}\left(X_i^{\mathrm{T}}\mathcal{W}_iX_i\right)\|\rho_{Di}(k)\|^2\right]. \end{aligned} \tag{26}$$

By a similar line to (21), it is concluded that condition (22) ensures $\sum_{\nu=1}^{V}\mu_{i\nu}\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu} + \vec{\Xi}_i < 0$. Since outside the region $\mathcal{O}$, we have $\|s(k)\| > \rho^*(k) \ge \sqrt{\frac{2\lambda_{\max}\left(X_i^{\mathrm{T}}\mathcal{W}_iX_i\right)}{\lambda_{\min}(W_i)}}\|\rho_{Di}(k)\|$, then it is resulted from (26) that

$$\mathbf{E}\left\{\Delta\tilde{V}(k,i)\right\} < \eta^{\mathrm{T}}(k)\left[\sum_{\nu=1}^{V}\mu_{i\nu}\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu} + \vec{\Xi}_i\right]\eta(k) < 0, \tag{27}$$

which implies that the state trajectories of the closed-loop system (9) are strictly decreasing (*with mean square*) outside the region $\mathcal{O}$ defined in (23). This completes the proof. ■

Remark 4. Assuming that the conditions in Theorems 1 and 2 hold simultaneously, we conclude that the proposed asynchronous SMC law (7) now can guarantee the mean square stability of the closed-loop system (9) and the reachability of the sliding region $\mathcal{O}$ in spite of *partly* accessible mode detection probabilities, time-varying delays and stochastic perturbation, which just is the answer of the third question (**Q3**).

3.4. Solving algorithm

It is clear that, in order to achieve the reachability of the specified sliding function (6) and the mean square stability of the closed-loop system (9) simultaneously, the detected-mode-dependent matrices K_ν and F_ν ($\nu \in \mathcal{V}$) in the asynchronous SMC law (7) should be designed according to the conditions in Theorems 1 and 2 simultaneously. So we are in a position to propose an LMI formulation for synthesizing the asynchronous SMC law (7) depending only $\sigma(k)$. The following *necessary and sufficient* condition will be helpful in converting the matrix inequality (22) into a set of coupled LMIs.

Lemma 1. *Define $\mathbf{U}_{i\nu} \triangleq \mathrm{diag}\{R_{i\nu}, J_{i\nu}\}$, and $\mathbf{A}_{i\nu}$, $\mathbf{Y}_i$ as in Theorem 2. There exist matrices $K_\nu \in \mathbb{R}^{m\times n}$, $F_\nu \in \mathbb{R}^{m\times n}$, $R_{i\nu} > 0$, $J_{i\nu} > 0$, $P_i > 0$, $W_i > 0$ satisfying $\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu} \le \mathbf{U}_{i\nu}$, or $\begin{bmatrix} -\mathbf{U}_{i\nu} & \mathbf{A}_{i\nu}^{\mathrm{T}} \\ \mathbf{A}_{i\nu} & -\mathbf{Y}_i^{-1} \end{bmatrix} \le 0$, if and only if there exist matrices $K_\nu \in \mathbb{R}^{m\times n}$, $F_\nu \in \mathbb{R}^{m\times n}$, $R_{i\nu} > 0$, $J_{i\nu} > 0$, $P_i > 0$, $W_i > 0$, $T_i > 0$, $L_i > 0$ such that $\mathcal{P}_i < T_i$, $\mathcal{W}_i < L_i$ and $\begin{bmatrix} -\mathbf{U}_{i\nu} & \mathbf{A}_{i\nu}^{\mathrm{T}} \\ \mathbf{A}_{i\nu} & -\hat{\mathbf{Y}}_i^{-1} \end{bmatrix} \le 0$ with $\hat{\mathbf{Y}}_i \triangleq \mathrm{diag}\{T_i, L_i\}$.*

Proof. See Appendix. ■

We now give the following sufficient conditions to obtain the asynchronous SMC law (7).

Theorem 3. *Consider the following assertions:*

(i) *There exist matrices $K_\nu \in \mathbb{R}^{m\times n}$, $F_\nu \in \mathbb{R}^{m\times n}$, $P_i > 0$, $W_i > 0$, $Q > 0$ and scalars $\varsigma_i > 0$ such that the conditions (11)* and *(22) hold for any $i \in \mathcal{N}$ and $\nu \in \mathcal{V}$.*

(ii) *There exist matrices $K_\nu \in \mathbb{R}^{m\times n}$, $F_\nu \in \mathbb{R}^{m\times n}$, $R_{i\nu} > 0$, $J_{i\nu} > 0$, $P_i > 0$, $W_i > 0$, $Q > 0$ and scalars $\varsigma_i > 0$ such that the condition (11) and the following conditions hold for any $i \in \mathcal{N}$ and $\nu \in \mathcal{V}$:*

$$\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu} \le \mathbf{U}_{i\nu}, \tag{28}$$

$$\vec{\Xi}_i + \sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\mathbf{U}_{i\nu} + \left(1 - \sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}\right)\sum_{\nu\in\ell_{\mathcal{UK}}^{\nu}}\mathbf{U}_{i\nu} < 0, \tag{29}$$

where $\mathbf{U}_{i\nu} \triangleq \mathrm{diag}\{R_{i\nu}, J_{i\nu}\}$.

(iii) *There exist matrices $\bar{K}_\nu \in \mathbb{R}^{m\times n}$, $\bar{F}_\nu \in \mathbb{R}^{m\times n}$, $\bar{M}_\nu$, $\bar{D}_\nu$, $\bar{R}_{i\nu} > 0$, $\bar{J}_{i\nu} > 0$, $\bar{P}_i > 0$, $\bar{W}_i > 0$, $\bar{T}_i > 0$, $\bar{L}_i > 0$, $\bar{Q} > 0$ and scalars $\vartheta_i > 0$, $\kappa_i > 0$ such that the following coupled LMIs hold for any $i \in \mathcal{N}$ and $\nu \in \mathcal{V}$:*

$$\begin{bmatrix} -\vartheta_iI & \mathbb{S}_i \\ \mathbb{S}_i^{\mathrm{T}} & -\mathbb{P} \end{bmatrix} \le 0, \tag{30}$$

$$\begin{bmatrix} -\bar{T}_i & \mathbb{T}_i \\ \mathbb{T}_i^{\mathrm{T}} & -\mathbb{P} \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} -\bar{L}_i & \mathbb{L}_i \\ \mathbb{L}_i^{\mathrm{T}} & -\mathbb{W} \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} \mathbb{U}_{i\nu} & \mathbb{A}_{i\nu} & \mathbb{E}_{i\nu} \\ \mathbb{A}_{i\nu}^{\mathrm{T}} & -\mathbb{Y}_i & \mathbb{H}_i \\ \mathbb{E}_{i\nu}^{\mathrm{T}} & \mathbb{H}_i^{\mathrm{T}} & -\mathrm{diag}\{\kappa_iI, \kappa_iI\} \end{bmatrix} \le 0, \tag{33}$$

$$\begin{bmatrix} -\Lambda_i & \mathbb{I}_i & \mathbb{G}_i \\ \mathbb{I}_i^{\mathrm{T}} & -\Upsilon_i & 0 \\ \mathbb{G}_i^{\mathrm{T}} & 0 & -\Sigma_i \end{bmatrix} < 0, \tag{34}$$

where $\tilde{\mu}_{i\nu} \triangleq \begin{cases} \mu_{i\nu}, & \nu \in \ell_{\mathcal{K}}^{\nu} \\ 1 - \sum_{\nu\in\ell_{\mathcal{K}}^{\nu}}\mu_{i\nu}, & \nu \in \ell_{\mathcal{UK}}^{\nu} \end{cases}$, and

$$\begin{aligned} &\mathbb{P} \triangleq \mathrm{diag}\left\{\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_N\right\}, \ \mathbb{Y}_i \triangleq \mathrm{diag}\left\{\bar{T}_i, \bar{L}_i\right\}, \\ &\mathbb{W} \triangleq \mathrm{diag}\left\{\bar{W}_1, \bar{W}_2, \ldots, \bar{W}_N\right\}, \ \Lambda_i \triangleq \mathrm{diag}\left\{\bar{P}_i, \bar{Q}\right\}, \\ &\mathbb{S}_i \triangleq \left[\sqrt{\pi_{i1}}B_i^{\mathrm{T}}\vartheta_i \quad \cdots \quad \sqrt{\pi_{iN}}B_i^{\mathrm{T}}\vartheta_i\right], \\ &\mathbb{T}_i \triangleq \left[\sqrt{\pi_{i1}}\bar{T}_i \quad \cdots \quad \sqrt{\pi_{iN}}\bar{T}_i\right], \\ &\mathbb{L}_i \triangleq \left[\sqrt{\pi_{i1}}\bar{L}_i \quad \cdots \quad \sqrt{\pi_{iN}}\bar{L}_i\right], \end{aligned}$$

$$\mathbb{U}_{i\nu} \triangleq \mathrm{diag}\left\{\bar{R}_{i\nu} - \bar{M}_\nu - \bar{M}_\nu^{\mathrm{T}}, \bar{J}_{i\nu} - \bar{D}_\nu - \bar{D}_\nu^{\mathrm{T}}\right\},$$

$$\mathbb{A}_{i\nu} \triangleq \begin{bmatrix} \sqrt{2}\left[A_i\bar{M}_\nu + B_i\bar{K}_\nu\right]^{\mathrm{T}} & \sqrt{2}\left(ZA_i\bar{M}_\nu + X_i\bar{K}_\nu\right)^{\mathrm{T}} \\ \sqrt{2}\left[A_{di}\bar{D}_\nu + B_i\bar{F}_\nu\right]^{\mathrm{T}} & \sqrt{2}\left(ZA_{di}\bar{D}_\nu + X_i\bar{F}_\nu\right)^{\mathrm{T}} \end{bmatrix},$$

$$\mathbb{E}_{i\nu} \triangleq \begin{bmatrix} \sqrt{2}\bar{M}_\nu^{\mathrm{T}}E_i^{\mathrm{T}} & 0 \\ \sqrt{2}\bar{D}_\nu^{\mathrm{T}}E_{di}^{\mathrm{T}} & 0 \end{bmatrix}, \mathbb{H}_i \triangleq \begin{bmatrix} 0 & \tilde{Z}_iH_i\kappa_i \\ 0 & 0 \end{bmatrix},$$

$$\mathbb{I}_i \triangleq \begin{bmatrix} \sqrt{\tilde{\mu}_{i1}}\bar{P}_i & \cdots & \sqrt{\tilde{\mu}_{iV}}\bar{P}_i & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sqrt{\tilde{\mu}_{i1}}\bar{Q} & \cdots & \sqrt{\tilde{\mu}_{iV}}\bar{Q} \end{bmatrix},$$

$$\mathbb{G}_i \triangleq \begin{bmatrix} \sqrt{\zeta}\bar{P}_iG_i^{\mathrm{T}} & \sqrt{\zeta}\bar{P}_iG_i^{\mathrm{T}}Z^{\mathrm{T}} & \sqrt{d_M - d_m + 1}\bar{P}_i & 2(\delta_{1i} + \varrho_1)\bar{P}_i & 0 \\ \sqrt{\zeta}\bar{Q}G_{di}^{\mathrm{T}} & \sqrt{\zeta}\bar{Q}G_{di}^{\mathrm{T}}Z^{\mathrm{T}} & 0 & 0 & 2(\delta_{2i} + \varrho_2)\bar{Q} \end{bmatrix},$$

$$\Upsilon_i \triangleq \mathrm{diag}\left\{\bar{R}_{i1}, \ldots, \bar{R}_{iV}, \bar{J}_{i1}, \ldots, \bar{J}_{iV}\right\},$$

$$\Sigma_i \triangleq \mathrm{diag}\left\{\bar{T}_i, \bar{L}_i, \bar{Q}, \vartheta_i I, \vartheta_i I\right\}.$$

We have that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Moreover, if the coupled LMIs (30)–(34) hold, then the mean square stability of the closed-loop system (9) and the reachability of the sliding region $\mathcal{O}$ defined in (23) around the sliding surface (6) can be guaranteed simultaneously by the asynchronous SMC law (7) with $K_\nu = \bar{K}_\nu\bar{M}_\nu^{-1}$ and $F_\nu = \bar{F}_\nu\bar{D}_\nu^{-1}$.

Proof. (*iii*) $\Rightarrow$ (*ii*). Denote $\bar{R}_{i\nu} = R_{i\nu}^{-1}$, $\bar{J}_{i\nu} = J_{i\nu}^{-1}$, $\bar{P}_i = P_i^{-1}$, $\bar{W}_i = W_i^{-1}$, $\bar{T}_i = T_i^{-1}$, $\bar{L}_i = L_i^{-1}$, $\bar{Q} = Q^{-1}$ and $\vartheta_i = \varsigma_i^{-1}$. Bearing in mind that $-\bar{M}_\nu^{\mathrm{T}}R_{i\nu}\bar{M}_\nu \le \bar{R}_{i\nu} - \bar{M}_\nu - \bar{M}_\nu^{\mathrm{T}}$ and $-\bar{D}_\nu^{\mathrm{T}}J_{i\nu}\bar{D}_\nu \le \bar{J}_{i\nu} - \bar{D}_\nu - \bar{D}_\nu^{\mathrm{T}}$, it is obtained from Lemma 1 and the Schur complement that the condition (28) is ensured by LMIs (31)–(33). On the other hand, it is easily shown that LMIs (30) and (34) are equivalent to conditions (11) and (29), respectively.

(*ii*) $\Rightarrow$ (*i*). It is noted that condition (22) is guaranteed by the conditions (28)–(29). Moreover, since $\mathbf{A}_{i\nu}^{\mathrm{T}}\mathbf{Y}_i\mathbf{A}_{i\nu} > \tilde{\mathcal{A}}_{i\nu}^{\mathrm{T}}\mathcal{P}_i\tilde{\mathcal{A}}_{i\nu}$ and $\bar{\Xi}_i > \tilde{\Xi}_i$, we conclude that conditions (11) and (22) ensure the mean square stability of the closed-loop system (9) and the reachability of the sliding region $\mathcal{O}$ in (23) simultaneously, which completes the proof. ■

According to Theorem 3, the design procedures of the asynchronous SMC law (7) can be summarized as Algorithm 1. About the computational complexity of the conditions (30)–(34) in Theorem 3, we need to solve $3NV+10N+1$ LMIs to get $2NV+6N+4V+1$ decision variables, or $2Vmn+2Vn^2+(2NV+2N+1)\frac{n(n+1)}{2}+Nm(m+1)+2N$ scalar variables.

Algorithm 1

- ***Step 1.*** Select the parameters β_i properly such that the matrix X_i is nonsingular for any $i \in \mathcal{N}$.
- ***Step 2.*** Compute the scalars δ_{1i}, δ_{2i}, ϱ_1 and ϱ_2, and obtain the robust term (8).
- ***Step 3.*** Get the matrices $\bar{K}_\nu, \bar{F}_\nu, \bar{M}_\nu$ and $\bar{D}_\nu$ by solving the coupled LMIs (30)–(34).
- ***Step 4.*** Produce the asynchronous SMC law (7) with matrices $K_\nu = \bar{K}_\nu\bar{M}_\nu^{-1}$ and $F_\nu = \bar{F}_\nu\bar{D}_\nu^{-1}$, and then apply it to the discrete-time MJSs (2) with time-varying delays and stochastic perturbation.

4. An illustrative example

We consider an application of the automotive electronic throttle body discussed in Vargas et al. (2016). Denote the system state as $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k)\end{bmatrix}^{\mathrm{T}}$, where $x_1(k)$, $x_2(k)$, and $x_3(k)$ stand for the position, velocity, and electrical current, respectively.

Table 1
Parameters in the electronic throttle system (Vargas et al., 2016).

Parameter	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$a_{12}^{(i)}$	0.010937	0.011968	0.0152	0.0111
$a_{21}^{(i)}$	−0.11649	−0.02536	−0.02978	−0.0229
$a_{22}^{(i)}$	−0.8072	0.6058	0.2737	0.7779
$a_{23}^{(i)}$	−1.5061	0.1254	−1.19207	−0.18986
$a_{32}^{(i)}$	−0.2285	1.1056	−0.3584	−0.6315
$a_{33}^{(i)}$	0.7967	0.6238	0.3835	0.41783
$b^{(i)}$	−0.09479	0.412	−0.07351	−0.2441

Now, the electronic throttle system with Markov-driven voltage failures can be modeled as the stochastic MJS (2) with

$$A_i = \begin{bmatrix} 1 & a_{12}^{(i)} & 0 \\ a_{21}^{(i)} & a_{22}^{(i)} & a_{23}^{(i)} \\ 0 & a_{32}^{(i)} & a_{33}^{(i)} \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ 0 \\ b^{(i)} \end{bmatrix}, i \in \{1, 2, 3, 4\},$$

and the parameters shown in Table 1. The modes $r(k) = i$ characterize the random failures programmed in the power amplifier, that is, normal ($i = 1$), soft failure ($i = 2$), intermediate failure ($i = 3$), and hard failure ($i = 4$). The transition probability matrix is given as $\Pi = \begin{bmatrix} 0.738 & 0.112 & 0.150 & 0 \\ 0.201 & 0.579 & 0.115 & 0.105 \\ 0.180 & 0.120 & 0.508 & 0.192 \\ 0.109 & 0.111 & 0.131 & 0.649 \end{bmatrix}$.

By considering the instability and performance degradation to design the real dynamic process, other parameters in (2) are set as follows:

$$f_i(x, x_d) = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_{d1}^2 + x_{d2}^2 + x_{d3}^2},$$
$$(x_{dl} \triangleq x_l(k - d(k)),\ l = 1, 2, 3),$$

$$A_{di} = \begin{bmatrix} -0.1 & 0.2 & 0.3 \\ 0.2 & 0.1 & 0.09 \\ -0.3 & 0 & 0.1 \end{bmatrix}, G_i = \begin{bmatrix} 0.2 & -0.1 & 0.1 \\ -0.5 & 0.3 & -0.1 \\ 0.1 & 0.2 & 0.4 \end{bmatrix},$$

$$G_{di} = \begin{bmatrix} 0.05 & 0.03 & 0.01 \\ 0.01 & -0.05 & 0.02 \\ 0.4 & 0.2 & 0.01 \end{bmatrix}, \ \forall i \in \{1, 2, 3, 4\};$$

$$\begin{bmatrix} H_1^{\mathrm{T}} \mid H_2^{\mathrm{T}} \mid H_3^{\mathrm{T}} \mid H_4^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 0.02\ -0.05\ 0.04 \mid -0.04\ -0.01\ 0.03 \mid -0.03\ -0.07\ 0.02 \mid 0.03\ 0.01\ -0.01 \end{bmatrix},$$

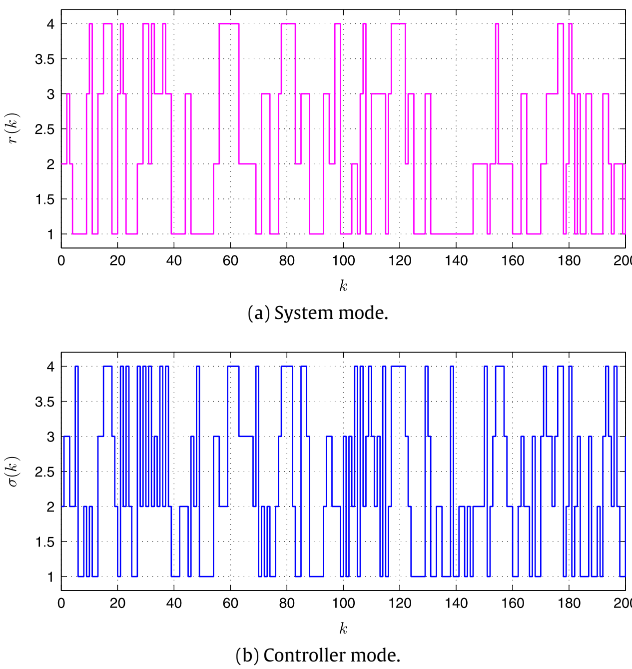
$$\begin{bmatrix} E_1 \mid E_2 \mid E_3 \mid E_4 \end{bmatrix} = \begin{bmatrix} 0.3\ 0.7\ 0.1 \mid 0.1\ 0.2\ 0.1 \mid -0.3\ -0.7\ 0.2 \mid 0.3\ 0.1\ -0.1 \end{bmatrix},$$

$$\begin{bmatrix} E_{d1} \mid E_{d2} \mid E_{d3} \mid E_{d4} \end{bmatrix} = \begin{bmatrix} -0.1\ -0.1\ 0.2 \mid -0.4\ 0.1\ -0.3 \mid 0.4\ 0.4\ -0.1 \mid -0.4\ -0.4\ 0.3 \end{bmatrix}, \Phi_1(k) = \sin(2k),$$

$$\Phi_2(k) = \cos(0.3k), \Phi_3(k) = \frac{0.8}{2 + k^2}, \Phi_4(k) = \sin^2(0.1k).$$

We assume that a detector is employed to determine which power failure is active in the power amplifier, and emit an estimated failure mode ν to the controller via a hidden Markov process satisfying (4) with the partly accessible MDPM $\Omega = \begin{bmatrix} 0.6 & 0.2 & ? & ? \\ 0.1 & 0.5 & 0.3 & 0.1 \\ ? & ? & 0.4 & 0.2 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix}$, where ? means that the detector cannot provide the estimation probability due to the possible detector degradation or the environment effects.

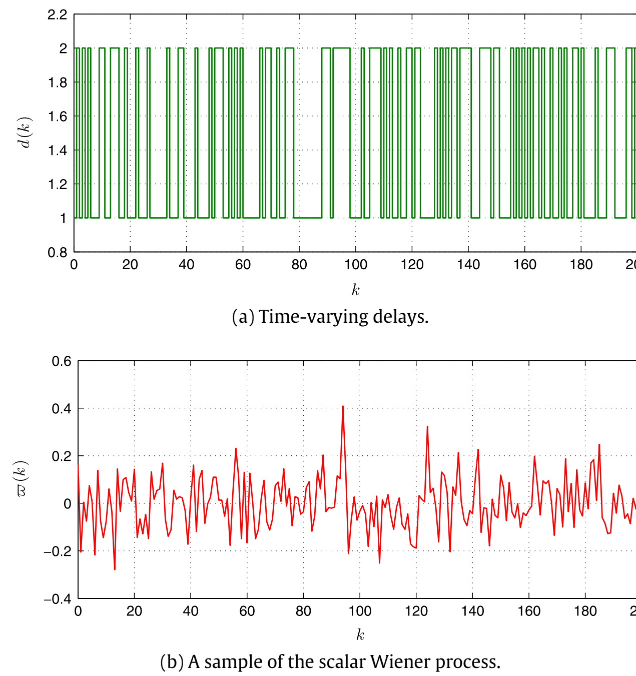
Based on the estimated failure mode ν, our purpose is to synthesize an asynchronous SMC law (7) to stochastically stabilize



(a) System mode.

(b) Controller mode.

Fig. 1. A possible sequence of system and controller modes.



(a) Time-varying delays.

(b) A sample of the scalar Wiener process.

Fig. 2. Evolutions of the time-varying delays $d(k)$ and a sample of the scalar Wiener process $\varpi(k)$ in simulation.

the electronic throttle system subjected to the time-varying delays satisfying $1 \le d(k) \le 2$ with $\mathrm{Prob}\{d(k) = 1\} = \mathrm{Prob}\{d(k) = 2\} = \frac{1}{2}$, the stochastic perturbation $\varpi(k)$ with the variance $\zeta = 0.1$, and the above partly accessible MDPM Ω. Letting $\beta_1 = \beta_3 = \beta_4 = \frac{1}{2}$ and $\beta_2 = \frac{1}{4}$ in (6), we have the sliding matrix $Z = \begin{bmatrix} 0 & 0 & -0.1032 \end{bmatrix}$. It is easily verified that the nonsingularity of the matrix $X_i = ZB_i$ can be ensured for every mode i. Now, by solving Theorem 3 with choosing $\epsilon_i = \epsilon_{di} = 1$, we obtain the following asynchronous SMC law:

$$u(k) = \begin{cases} \begin{bmatrix} -0.1134 & -3.6444 & -0.2239 \end{bmatrix} x(k) \\ + \begin{bmatrix} 0.6298 & 0.1318 & -0.0197 \end{bmatrix} x(k-d(k)) \\ - \rho(x(k), x(k-d(k))) \dfrac{s(k)}{\|s(k)\|}, & \nu = 1, \\ \begin{bmatrix} -0.2327 & -1.5742 & -1.2980 \end{bmatrix} x(k) \\ + \begin{bmatrix} 0.0240 & 0.0371 & 0.0946 \end{bmatrix} x(k-d(k)) \\ - \rho(x(k), x(k-d(k))) \dfrac{s(k)}{\|s(k)\|}, & \nu = 2, \\ \begin{bmatrix} -0.2560 & -1.6306 & -1.2819 \end{bmatrix} x(k) \\ + \begin{bmatrix} 0.0168 & 0.0716 & 0.1436 \end{bmatrix} x(k-d(k)) \\ - \rho(x(k), x(k-d(k))) \dfrac{s(k)}{\|s(k)\|}, & \nu = 3, \\ \begin{bmatrix} -0.1908 & -2.8251 & -0.7231 \end{bmatrix} x(k) \\ + \begin{bmatrix} 0.3622 & 0.1358 & 0.0927 \end{bmatrix} x(k-d(k)) \\ - \rho(x(k), x(k-d(k))) \dfrac{s(k)}{\|s(k)\|}, & \nu = 4, \end{cases} \tag{35}$$

where $\rho(x(k), x(k-d(k))) = 1.3241\|x(k)\| + 1.1034\|x(k-d(k))\|$.

In the simulation, the initial conditions are $x(0) = \begin{bmatrix} 1 & -1 & 0.5 \end{bmatrix}^{\mathrm{T}}$ and $\phi(n) = \mathbf{0}_{3\times 1}$ $(n \in \{-2, -1\})$. The simulation results are shown in Figs. 1–6, where Fig. 1 depicts a possible sequence of system and controller modes. Under the time-varying delays $d(k)$ and a sample of the scalar Wiener process $\varpi(k)$ as plotted in Fig. 2, it can be seen clearly from Fig. 3 that the electronic throttle system in open-loop case is unstable. However, under the same scenarios, the electronic throttle system can be stochastically stabilized by the asynchronous SMC law (35) as shown in Fig. 4. The responses

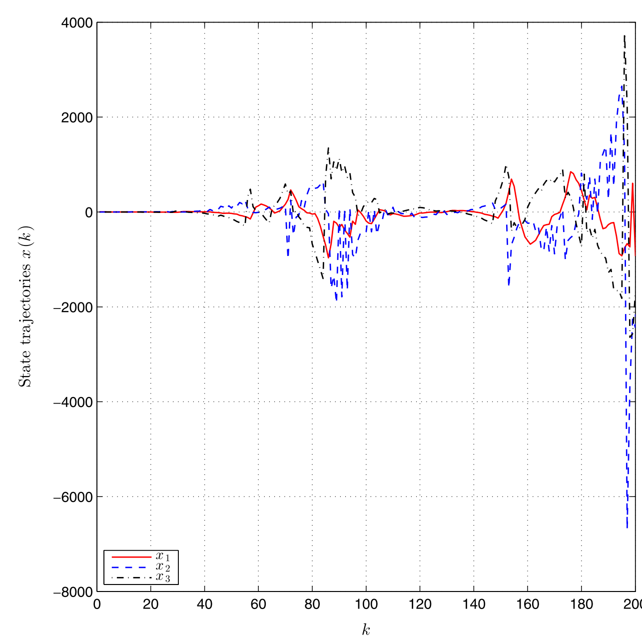


Fig. 3. Evolution of state trajectories $x(k)$ in open-loop case.

of the sliding variable $s(k)$ and the asynchronous SMC input $u(k)$ are depicted in Figs. 5 and 6, respectively. It is worth pointing out that all of the existing approaches as in Chen et al. (2014), Costa et al. (2015), Huang et al. (2012), Kao et al. (2015), Wu et al. (2017) and Zhang et al. (2013) cannot be effectively employed to design a stabilizing controller for the electronic throttle system under the partly accessible mode detection probabilities, from which the advantages of the proposed asynchronous SMC approach have been confirmed.

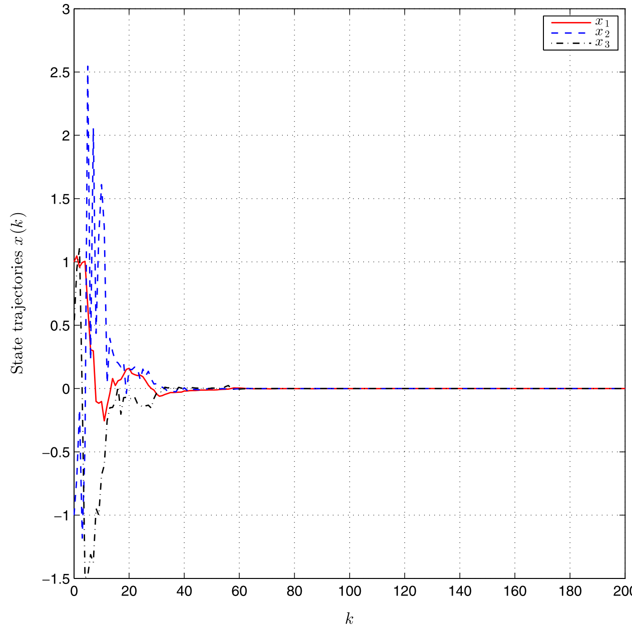


Fig. 4. Evolution of state trajectories $x(k)$ in closed-loop case.

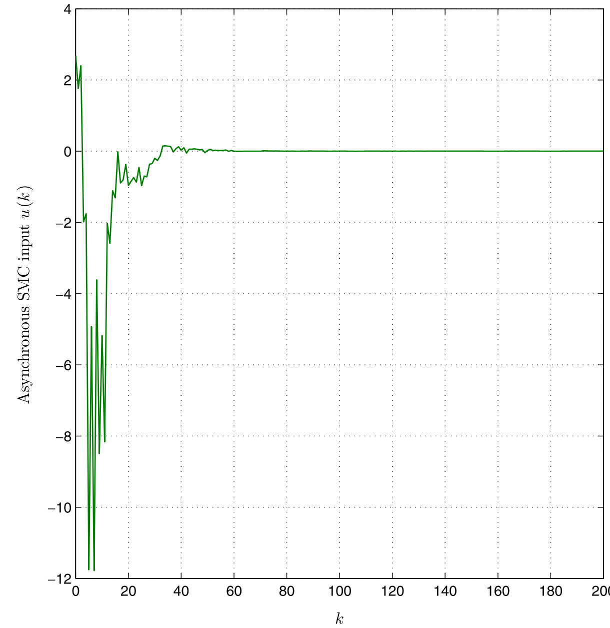


Fig. 6. Evolution of the asynchronous SMC input $u(k)$.

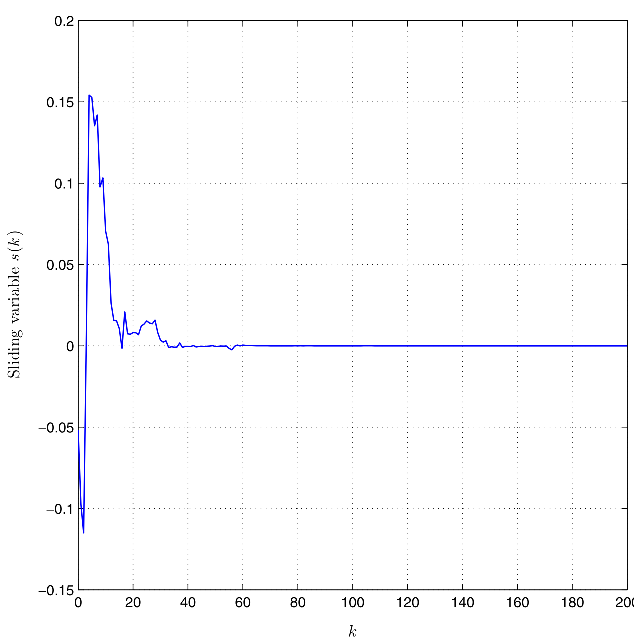


Fig. 5. Evolution of the sliding variable $s(k)$.

5. Conclusion

This work has investigated the asynchronous SMC issue for a class of discrete-time Markovian jump systems with time-varying delays and stochastic perturbation. The information of system modes to controller is considered to be only estimated via a detector with a *partly accessible* MDPM. Sufficient conditions for the mean square stability of the sliding mode dynamics and the reachability of a sliding region around the specified sliding surface have been derived by using hidden Markov approach. The proposed asynchronous SMC law exhibits the following distinct features: (1) a new *common* sliding function is employed to avoid the drawbacks of the mode-dependent ones as in Chen et al. (2014) and Zhang et al. (2013); (2) the designed SMC law just depends on the detected modes, which are nonsynchronous with the system modes via a hidden Markov model with partly accessible mode detection probabilities. Our future research work on the asynchronous SMC would be focused on tackling continuous-time MJSs and semi-Markovian jump systems (Li, Wu, Shi, & Lim, 2015; Shi, Li, Wu, & Lim, 2017).

Appendix. Proof of Lemma 1

Sufficiency is obvious. Next, we show the necessity. From $\begin{bmatrix} -\mathbf{U}_{i\nu} & \mathbf{A}_{i\nu}^{\mathrm{T}} \\ \mathbf{A}_{i\nu} & -\mathbf{Y}_i^{-1} \end{bmatrix} \leq 0$, we can find a sufficiently small scalar $\alpha > 0$ such that $\begin{bmatrix} -\mathbf{U}_{i\nu} & \mathbf{A}_{i\nu}^{\mathrm{T}} \\ \mathbf{A}_{i\nu} & -\mathbf{Y}_i^{-1} + \alpha I \end{bmatrix} \leq 0$. Setting $T_i^{-1} = \mathcal{P}_i^{-1} - \alpha I$ and $L_i^{-1} = \mathcal{W}_i^{-1} - \alpha I$, we get that $\mathcal{P}_i < T_i, \mathcal{W}_i < L_i$ and $\begin{bmatrix} -\mathbf{U}_{i\nu} & \mathbf{A}_{i\nu}^{\mathrm{T}} \\ \mathbf{A}_{i\nu} & -\hat{\mathbf{Y}}_i^{-1} \end{bmatrix} \leq 0$.

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