

# Asynchronous Siding Mode Control of Two-dimensional Markov Jump Systems in Roesser Model

**Abstract—abstract**

**Index Terms—Markov jump systems, 2D systems, Siding mode control, Hidden Markov model**

## I. INTRODUCTION

This part is introduction.

## II. PRELIMINARIES

In this paper, we consider the following two-dimensional Markov jump systems in Roesser model:

$$\begin{cases} \mathbf{x}(i, j) = A_{r(i,j)}\mathbf{x}(i, j) + E_{r(i,j)}w(i, j) \\ \quad + B_{r(i,j)}[u(i, j) + f(x(i, j), r(i, j))] \\ y(i, j) = C_{r(i,j)}\mathbf{x}(i, j) + D_{r(i,j)}w(i, j) \end{cases} \quad (1)$$

where

$$\mathbf{x}(i, j) = \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix}, \quad x(i, j) = \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}$$

$x^h(i, h) \in \mathbb{R}^{n_h}$  and  $x^v(i, h) \in \mathbb{R}^{n_v}$  represent horizontal and vertical states respectively,  $u(i, j) \in \mathbb{R}^{n_u}$  and  $y(i, j) \in \mathbb{R}^{n_y}$  represent the controlled input and output respectively, and  $w(i, j) \in \mathbb{R}^{n_w}$  represents the exogenous disturbance which belongs to  $\ell_2\{[0, \infty), [0, \infty)\}$ .  $A_{r(i,j)}, B_{r(i,j)}, C_{r(i,j)}, D_{r(i,j)}$  and  $E_{r(i,j)}$  represent the time-varying system matrices, all of which are real known constant matrices with appropriate dimensions. Besides, we assume that the matrix  $B_{r(i,j)}$  is full column rank for each  $r(i, j) \in \mathcal{N}_1$ , that is,  $\text{rank}(B_{r(i,j)}) = n_u$ . The nonlinear function  $f(x(i, j), r(i, j))$  satisfying the following property:

$$\|f(x(i, j), r(i, j))\| \leq \delta_{r(i,j)} \|x(i, j)\| \quad (2)$$

where  $\delta_{r(i,j)}$  is a known scalar,  $\|\cdot\|$  denotes the Euclidean norm of a vector. The parameter  $r(i, j)$  takes values in a finite set  $\mathcal{N}_1 = \{1, 2, \dots, N_1\}$  with transition probability matrix  $\Lambda = \{\lambda_{k\tau}\}$ , and the related transition probability from mode  $k$  to mode  $\tau$  is given by

$$\begin{aligned} \Pr\{r(i+1, j) = \tau | r(i, j) = k\} \\ = \Pr\{r(i, j+1) = \tau | r(i, j) = k\} = \lambda_{k\tau}, \quad \forall k, \tau \in \mathcal{N}_1 \end{aligned} \quad (3)$$

where  $\lambda_{k\tau} \in [0, 1]$ , for all  $k, \tau \in \mathcal{N}_1$ , and  $\sum_{\tau=1}^{N_1} \lambda_{k\tau} = 1$  for every mode  $k$ .

We define the boundary condition  $(X_0, \Gamma_0)$  of system (1), as follows:

$$\begin{cases} X_0 = \{x^h(0, j), x^v(i, 0) | i, j = 0, 1, 2, \dots\} \\ \Gamma_0 = \{r(0, j), r(i, 0) | i, j = 0, 1, 2, \dots\} \end{cases} \quad (4)$$

And the corresponding zero boundary condition is assumed as  $x^h(0, j) = 0, x^v(i, 0) = 0, i, j = 0, 1, 2, \dots$ . Besides, we further impose following assumption on  $X_0$ .

**Assumption 1.** The boundary condition  $X_0$  satisfies:

$$\lim_{L \rightarrow \infty} \mathbb{E} \left\{ \sum_{\ell=1}^L (\|x^h(0, \ell)\|^2 + \|x^v(\ell, 0)\|^2) \right\} < \infty \quad (5)$$

where  $\mathbb{E}\{\cdot\}$  stands for mathematical expectation.

In practical applications, the complete information of  $r(i, j)$  can not always be available to the controller. Hence, in this paper, the hidden Markov model  $(r(i, j), \sigma(i, j), \Lambda, \Psi)$  as in [refto] is introduced to characterize the asynchronous phenomenon between the controller and the system. The parameter  $\sigma(i, j)$ , refers to controller mode, takes values in another finite set  $\mathcal{N}_2 = \{1, 2, \dots, N_2\}$ , and satisfies the conditional probability matrix  $\Psi = \{\mu_{ks}\}$  with conditional mode transition probabilities

$$\Pr\{\sigma(i, j) = s | r(i, j) = k\} = \mu_{ks} \quad (6)$$

where  $\mu_{ks} \in [0, 1]$  for all  $k \in \mathcal{N}_1, s \in \mathcal{N}_2$ , and  $\sum_{s=1}^{N_2} \mu_{ks} = 1$  for any mode  $k$ .

Next, the definitions of asymptotically mean square stable and  $H_\infty$  performance for 2D systems will be given in Definition 1 and Definition 2, respectively.

**Definition 1.** The 2D Markov jump system (1) with  $w(i, j) \equiv 0$  is said to be asymptotically mean square stable if the following holds:

$$\lim_{i+j \rightarrow \infty} \mathbb{E}\{\|x(i, j)\|^2\} = 0 \quad (7)$$

for any boundary condition  $X_0$  with Assumption 1.

**Definition 2.** Given a scalar  $\gamma > 0$ , the 2D Markov jump system (1) is said to be asymptotically mean square stable with an  $H_\infty$  disturbance attenuation performance  $\gamma$  if the system satisfies (7), and under zero boundary condition, the following holds:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \{\|y(i, j)\|^2\} < \gamma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \{\|w(i, j)\|^2\} \quad (8)$$

for all  $w(i, j) \in \ell_2\{[0, \infty), [0, \infty)\}$ .

Now, we will make some notational simplification for convenience. The parameter  $r(i, j)$  is represented by  $k$ ,  $r(i+1, j)$

and  $r(i, j + 1)$  are represented by  $\tau$ ,  $\sigma(i, j)$  is represented by  $s$ .

The objective of this work is to devise an asynchronous SMC law  $u(i, j)$ , such that the 2D Markov jump system (1) is asymptotically mean square stable with an  $H_\infty$  disturbance attenuation performance  $\gamma$ .

### III. MAIN RESULT

#### A. Sliding surface and sliding mode controller

In this paper, a novel Two-dimensional sliding surface function is constructed as follows:

$$s(i, j) = \begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} = Gx(i, j) \quad (9)$$

where  $G = \sum_{k=1}^{N_1} \beta_k G_k^T$ , and scalars  $\beta_k$  should be chosen such that  $GB_k$  is nonsingular for any  $k \in \mathcal{N}_1$ . Based on the the assumption that  $B_k$  is full column rank for any  $k \in \mathcal{N}_1$ , we can find that the above condition can be guaranteed easily with the properly selected parameter  $\beta_k$ .

An asynchronous 2D-SMC law is designed as follows:

$$u(i, j) = K_s x(i, j) - \rho(i, j) \frac{s(i, j)}{\|s(i, j)\|} \quad (10)$$

for any  $s \in \mathcal{N}_2$ , where the matrix  $K_s \in \mathbb{R}^{n_u \times n_x}$  with  $n_x = n_h + n_v$  will be determined later, and the parameter  $\rho(i, j)$  is given as

$$\rho(i, j) = \varrho_1 \|x(i, j)\| + \varrho_2 \|w(i, j)\| \quad (11)$$

with  $\varrho_1 = \max_{k \in \mathcal{N}_1} \{\delta_k\}$ ,  $\varrho_2 = \max_{k \in \mathcal{N}_1} \{\|(GB_k)^{-1}GE_k\|\}$ , and the parameter  $\delta_k$  is given in (2).

Combining system (1) and the asynchronous 2D-SMC law (9), the closed-loop 2D markov jump system can be obtained easily as follows:

$$\mathbf{x}(i, j) = \bar{A}_{ks} x(i, j) + B_k \bar{\rho}_k(i, j) + E_k w(i, j) \quad (12)$$

where  $\bar{A}_{ks} = A_k + B_k K_s$ , and  $\bar{\rho}_k(i, j)$  as follows

$$\bar{\rho}_k(i, j) = f_k(x(i, j)) - (\varrho_1 \|x(i, j)\| + \varrho_2 \|w(i, j)\|) \cdot \frac{s(i, j)}{\|s(i, j)\|}.$$

Then, based on the properties of norm, the following condition can be deduced easily

$$\|\bar{\rho}_k(i, j)\| \leq (\varrho_1 + \delta_k) \|x(i, j)\| + \varrho_2 \|w(i, j)\|. \quad (13)$$

#### B. Analysis of Stability and $H_\infty$ attenuation performance

In this subsection, we focus on the stability and  $H_\infty$  attenuation performance analysis for the closed-loop 2D system (12). A sufficient condition will be derived to guarantee the considered system is asymptotically mean square stable with an  $H_\infty$  attenuation performance  $\gamma$ .

**Theorem 1.** Consider the Markov jump system (1) under the Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar  $\gamma > 0$ , if there exist matrices  $K_s \in \mathbb{R}^{n_u \times n_x}$ ,  $R_k = \text{diag}\{R_k^h, R_k^v\} > 0$ ,  $Q_{ks} > 0$ ,  $T_{ks} > 0$  and scalars  $\epsilon_k > 0$ , for any  $k \in \mathcal{N}_1, s \in \mathcal{N}_2$ , such that the following inequalities hold:

$$B_k^T \mathcal{R}_k B_k - \epsilon_k I \leq 0 \quad (14)$$

$$\mathcal{A} + 2 \left( \sum_{s=0}^{N_2} \mu_{ks} \text{diag}\{Q_{ks}, T_{ks}\} \right) < 0 \quad (15)$$

$$\hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} - \text{diag}\{Q_{ks}, T_{ks}\} < 0 \quad (16)$$

where

$$\mathcal{A} = \begin{bmatrix} \Pi_1 & \Pi_3 \\ * & \Pi_2 \end{bmatrix}$$

with

$$\begin{cases} \Pi_1 = -R_k + 4(\delta_k + \varrho_1)^2 \epsilon_k I + C_k^T C_k \\ \Pi_2 = -\gamma^2 I + D_k^T D_k + 4\varrho_2^2 \epsilon_k I \\ \Pi_3 = C_k^T D_k \end{cases}$$

and  $\mathcal{R}_k = \sum_{\tau=1}^{N_1} \lambda_{k\tau} R_\tau$ ,  $\hat{A}_{ks} = [\bar{A}_{ks} \ E_k]$ , then, the closed-loop system (12) is asymptotically mean square stable with an  $H_\infty$  disturbance attenuation performance  $\gamma$ .

*Proof.* Let's start the proof with the stability of system. We select the Lyapunov candidate as  $V_1(i, j) = x^T(i, j) R_k x(i, j)$ , then, define

$$\Delta V_1(i, j) = \mathbf{x}(i, j)^T R_\tau \mathbf{x}(i, j) - x^T(i, j) R_k x(i, j) \quad (17)$$

Based on the closed-loop system equation (12) with  $w(i, j) = 0$ , it is easy to find that

$$\begin{aligned} \mathbb{E}\{\Delta V_1(i, j)\} &= \sum_{s=0}^{N_2} \mu_{ks} \left\{ [\bar{A}_{ks} x(i, j) + B_k \bar{\rho}_k(i, j)]^T \mathcal{R}_k \right. \\ &\quad \times [\bar{A}_{ks} x(i, j) + B_k \bar{\rho}_k(i, j)] \\ &\quad \left. - x^T(i, j) R_k x(i, j) \right\} \\ &\leq x^T(i, j) \left\{ 2 \left( \sum_{s=1}^{N_2} \mu_{ks} \bar{A}_{ks}^T \mathcal{R}_k \bar{A}_{ks} \right) \right\} x(i, j) \\ &\quad + 2 \bar{\rho}_k^T(i, j) B_k^T \mathcal{R}_k B_k \bar{\rho}_k(i, j) \\ &\quad - x^T(i, j) R_k x(i, j) \end{aligned} \quad (18)$$

Recalling the conditions given in (13) and (14), the following inequality can be further obtained

$$\mathbb{E}\{\Delta V_1(i, j)\} \leq x^T(i, j) \mathcal{G}_{ks} x(i, j) \quad (19)$$

where  $\mathcal{G}_{ks} = 2 \left( \sum_{s=0}^{N_2} \mu_{ks} \bar{A}_{ks}^T \mathcal{R}_k \bar{A}_{ks} \right) + 2\epsilon_k (\delta_k + \varrho_1)^2 I - R_k$ . The following inequality can be deduced from (15) based on the properties of matrix quadratic

$$2 \left( \sum_{s=1}^{N_2} \mu_{ks} Q_{ks} \right) + 4\epsilon_k (\delta_k + \varrho_1)^2 I + C_k^T C_k - R_k < 0 \quad (20)$$

which will further deduce

$$2 \left( \sum_{s=1}^{N_2} \mu_{ks} Q_{ks} \right) + 2\epsilon_k (\delta_k + \varrho_1)^2 I - R_k < 0 \quad (21)$$

The following inequality can be inferred directly from condition (16)

$$\bar{A}_{ks}^T \mathcal{R}_k \bar{A}_{ks} - Q_{ks} < 0 \quad (22)$$

Combine (21) and (22), we can infer that  $\mathcal{G}_{ks} < 0$ , which is equivalent to

$$\mathcal{G}_{ks} \leq -\alpha I \quad (23)$$

with scalar  $\alpha > 0$ . Recalling (19), we can further infer that

$$\mathbb{E}\{\Delta V_1(i, j)\} \leq -\alpha \mathbb{E}\{\|x(i, j)\|^2\} \quad (24)$$

Summing up on the both side of (24), we have

$$\mathbb{E}\left\{\sum_{i=0}^{\kappa_1} \sum_{j=0}^{\kappa_2} \|x(i, j)\|^2\right\} \leq -\frac{1}{\alpha} \mathbb{E}\left\{\sum_{i=0}^{\kappa_1} \sum_{j=0}^{\kappa_2} \Delta V_1(i, j)\right\} \quad (25)$$

where parameters  $\kappa_1, \kappa_2$  are any positive integers. By substituting  $\Delta V_1$  and  $R_k$  with (17) and  $R_k = \text{diag}\{R_k^h, R_k^v\}$  respectively, we obtain

$$\begin{aligned} & \sum_{i=0}^{\kappa_1} \sum_{j=0}^{\kappa_2} \Delta V_1(i, j) \\ &= \sum_{i=0}^{\kappa_1} \{V_1^v(i, \kappa_2 + 1) - V_1^v(i, 0)\} \\ & \quad - \sum_{j=0}^{\kappa_2} \{V_1^h(\kappa_1 + 1, j) - V_1^h(0, j)\} \\ & \leq -\left(\sum_{i=0}^{\kappa_1} V_1^v(i, 0) + \sum_{j=0}^{\kappa_2} V_1^h(0, j)\right) \end{aligned} \quad (26)$$

where  $V_1^h(i, j)$  and  $V_1^v(i, j)$  are defined as

$$\begin{cases} V_1^h(i, j) = x^{hT}(i, j) R_{r(i, j)}^h x^h(i, j) \\ V_1^v(i, j) = x^{vT}(i, j) R_{r(i, j)}^v x^v(i, j) \end{cases}$$

Recalling the boundary condition in Assumption 1, and let  $\kappa_1, \kappa_2$  tend to infinity, it follows from (25) and (26) that

$$\begin{aligned} & \mathbb{E}\left\{\sum_{i=0}^{\kappa_1} \sum_{j=0}^{\kappa_2} \|x(i, j)\|^2\right\} \\ & \leq -\frac{\beta}{\alpha} \sum_{\ell=0}^{\infty} (\|x^v(\ell, 0)\|^2 + \|x^h(0, \ell)\|^2) \\ & < \infty \end{aligned} \quad (27)$$

where  $\beta$  is the maximum eigenvalue of  $R^h(0, \ell)$  and  $R^v(\ell, 0)$ , for any  $\ell = 0, 1, 2, \dots$ , which implies that (7) holds. Thus, the asymptotically mean square stable of the considered system is proved.

Next, let's focus on the  $H_\infty$  attenuation performance under zero boundary condition. Based on the closed-loop system

equation (12), it is easy to find that

$$\begin{aligned} & \mathbb{E}\{\Delta V_1(i, j)\} \\ &= \sum_{s=0}^{N_2} \mu_{ks} \left\{ [\bar{A}_{ks} x(i, j) + B_k \bar{\rho}_k(i, j) + E_p w(i, j)]^T \right. \\ & \quad \times \mathcal{R}_k [\bar{A}_{ks} x(i, j) + B_k \bar{\rho}_k(i, j) + E_p w(i, j)] \left. \right\} \\ & \quad - x^T(i, j) R_k x(i, j) \\ & \leq \hat{x}^T(i, j) \left\{ 2 \left( \sum_{s=1}^{N_2} \mu_{ks} \hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks} \right) \hat{x}(i, j) \right. \\ & \quad + 2 \bar{\rho}_k^T(i, j) B_k^T \mathcal{R}_k B_k \bar{\rho}_k(i, j) \\ & \quad \left. - x^T(i, j) R_k x(i, j) \right\} \end{aligned} \quad (28)$$

where

$$\hat{x}(i, j) = \begin{bmatrix} x(i, j) \\ w(i, j) \end{bmatrix}, \quad \hat{A}_{ks}(i, j) = \begin{bmatrix} \bar{A}_{ks} & E_k \end{bmatrix}$$

Notice that from (13) and (14), we have

$$\begin{aligned} & \bar{\rho}_k^T(i, j) B_k^T \mathcal{R}_k B_k \bar{\rho}_k(i, j) \\ & \leq 2\epsilon_k ((\delta_k + \varrho_1)^2 \|x(i, j)\|^2 + \varrho_2^2 \|w(i, j)\|^2) \end{aligned} \quad (29)$$

The following condition can be deduced easily from (15) and (16)

$$\Xi_{ks} < 0 \quad (30)$$

where  $\Xi_{ks} \equiv \mathcal{A} + 2 \sum_{s=1}^{N_2} \mu_{ks} \hat{A}_{ks}^T \mathcal{R}_k \hat{A}_{ks}$ . Recalling the system (1), and substituting (29) into (28) yields

$$\begin{aligned} & \mathbb{E}\{\Delta V_1(i, j) + \|z(i, j)\|^2 - \gamma^2 \|w(i, j)\|^2\} \\ & \leq \hat{x}^T(i, j) \Xi_{ks} \hat{x}(i, j) < 0 \end{aligned} \quad (31)$$

Noting (26) with the zero boundary condition, we can infer that

$$\begin{aligned} & \sum_{i=0}^{\kappa_1} \sum_{j=0}^{\kappa_2} \Delta V_1(i, j) \\ &= \sum_{i=0}^{\kappa_1} V_1^v(i, \kappa_2 + 1) + \sum_{j=0}^{\kappa_2} V_1^h(\kappa_1 + 1, j) \\ & \geq 0 \quad \forall \kappa_1, \kappa_2 = 1, 2, 3, \dots \end{aligned} \quad (32)$$

Then, we can further deduce from (31) and (32) that

$$\begin{aligned} & \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{E}\{\|z(i, j)\|^2 - \gamma^2 \|w(i, j)\|^2\} \\ & \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{E}\{\Delta V_1(i, j) + \|z(i, j)\|^2 - \gamma^2 \|w(i, j)\|^2\} \\ & < 0 \end{aligned} \quad (33)$$

which implies (8) holds. And this completes the proof of Theorem 1.  $\square$

**Remark 1.** Remark.

### C. Analysis of reachability

The reachability of the designed asynchronous 2D-SMC law for the closed-loop system (12) will be discussed in this subsection. By using a stochastic Lyapunov method, we provide a sufficient condition which will confirm that the designed asynchronous 2D-SMC law (10) can force the state trajectories of the closed-loop system (12) into a time-varying sliding region around the specified 2D sliding surface (9).

**Theorem 2.** Consider the closed-loop 2D Markov jump system (12) with asynchronous 2D-SMC law (10). If there exists matrices  $K_s \in \mathbb{R}^{n_u \times n_x}$ ,  $R_k > 0$ ,  $F_k > 0$ , and scalars  $\epsilon_k > 0$ , for any  $k \in \mathcal{N}_1$ ,  $s \in \mathcal{N}_2$ , such that the condition (14) and the following inequality hold

$$2 \sum_{s=1}^{N_2} (\bar{A}_{ks}^T \mathcal{R}_k \bar{A}_{ks} + \bar{A}_{ks}^T G^T \mathcal{F}_k G \bar{A}_{ks}) - R_k < 0 \quad (34)$$

where  $\mathcal{R}_k$  is defined in Theorem 1, and  $\mathcal{F}_k = \sum_{\tau=1}^{N_1} \lambda_{k\tau} F_\tau$ . Then, the state trajectories of the considered closed-loop system will be driven into the following sliding region  $\mathcal{O}$ , around the specified sliding surface (9):

$$\mathcal{O} \equiv \left\{ \|s(i, j)\| \leq \rho^*(i, j) \right\} \quad (35)$$

where  $\rho^*(i, j) = \max_{k \in \mathcal{N}_1} \sqrt{\hat{\rho}_k(i, j) / \lambda_{\min}(F_k)}$  with

$$\begin{aligned} \hat{\rho}_k(i, j) = & 4(\|E_k^T \mathcal{R}_k E_k\| + \|E_k^T G^T \mathcal{R}_k G E_k\| \\ & + 2\varrho_2^2(\|B_k^T \mathcal{F}_k B_k\| + \|B_k^T G^T \mathcal{F}_k G B_k\|)) \|w(i, j)\|^2 \\ & + 8(\|B_k^T \mathcal{R}_T B_k\| + \|B_k^T G^T \mathcal{R}_T G B_k\|) \\ & \times (\varrho_1 + \delta_k)^2 \|x(i, j)\|^2. \end{aligned}$$

and  $\lambda_{\min}(F_k)$  here denotes the minimum eigenvalue of  $F_k$ .

*Proof.* Proof of theorem.  $\square$

**Remark 2.** Remark.

## IV. NUMERICAL EXAMPLE

In this section, we provide an example to verify the validity of the proposed method.

## V. CONCLUSIONS

### REFERENCES

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