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Stability of two-dimensional Roesser systems with time-varying delays via novel 2D finite-sum inequalities

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Abstract: This study considers the problem of stability analysis of discrete-time two-dimensional (2D) Roesser systems with interval time-varying delays. New 2D finite-sum inequalities, which provide a tighter lower bound than the existing ones based on 2D Jensen-type inequalities, are first developed. Based on an improved Lyapunov–Krasovskii functional, the newly derived inequalities are then utilised to establish delay-range-dependent linear matrix inequality-based stability conditions for a class of discrete time-delay 2D systems. The effectiveness of the obtained results is demonstrated by numerical examples.

1 Introduction

Two-dimensional (2D) systems, dated back to the mid 1970s [1–3], appear in a large number of practical and physical systems described by partial differential/difference equations, with examples in image processing, seismographic data processing, thermal processes, gas absorption or water stream heating [4–6]. During the past few decades, the study of 2D systems in the field of control and signal processing has attracted considerable attention due to their widespread applications, for example in repetitive processes, image data processing and transmission, multi-dimensional digital filtering or iterative learning control. To mention a few, with focus on applied studies, we refer the reader to recent works [7–13] and the references therein.

Among the available state-space models for 2D systems, Roesser model and Fornasini–Marchesini (FM) model are the two most representative ones. A number of methodologies and techniques have been developed for analysis and synthesis of 2D systems described by the Roesser and FM models including stability and H_∞ performance analysis [14–18], state feedback control [19–21], filtering design [22–24] and fault detection problems [25]. Particularly, in [15], definitions of asymptotic and exponential stability were introduced for discrete-time 2D systems and two Lyapunov-type theorems were proposed to provide stability conditions. The conventional Lyapunov stability theory for 1D systems was first extended to continuous-time 2D systems in [16]. The problems of state feedback H_∞ control and stabilisation for 2D sectorial nonlinear systems with intermittent measurements, 2D fuzzy systems and 2D switched systems were also considered in [18–21], respectively. Though the basic consideration of 2D systems theory can often be viewed as a generalisation from the conventional 1D systems theory, there exist deep and substantial differences between the 1D case and the 2D case. First, in the 2D context, the information propagation occurs in each of the two independent directions. Attempts to analyse these processes using traditional 1D systems theory generally fail because such an approach ignores their inherent 2D systems structure. Second, 2D framework has a more complex structure than 1D case and the available related 2D preliminaries are insufficient. In addition, unlike the case of 1D systems, the initial conditions of 2D systems are usually infinite sets which may be unbounded. These factors make the analysis and synthesis of 2D systems to be much more complicated and difficult than 1D systems [26].

On the other hand, it has been well recognised that time-delay frequently occurs in practical systems due to finite speed of information processing and data transmission among various parts of the system [27]. The existence of time-delay often degrades the system performance and even cause system instability. Therefore, the investigation of stability and control of time-delay systems plays an important role in applied models which has gained growing attention during the last two decades [28–34]. To derive stability conditions for time-delay systems, several approaches have been proposed in the literature of which the Lyapunov–Krasovskii functional (LKF) method combined with linear matrix inequality (LMI)-based techniques is the most commonly used approach [28, 31, 35]. According to the dependence of delay, the proposed stability conditions are classified into delay-dependent conditions and delay-independent conditions. Generally speaking, delay-dependent stability conditions, which utilise information on the size of time delays, are reasonable and less conservative than delay-independent ones. Therefore, much effort has been devoted to derive less conservative delay-dependent stability conditions for time-delay systems with the main concern is to enlarge the allowable delay bound. This task, however, relies heavily on how to choose an appropriate LKF and especially how to utilise inequality-based techniques in combination with model-transformations, delay-decomposition or the use of slack matrix variables [36, 37].

As mentioned above, 2D systems with delays appear in various engineering systems. While stability analysis and control of 1D systems with delays have been widely studied and developed, this problem for 2D systems has attracted increasing attention recently. For instance, in [38], some delay-independent robust stability conditions were presented for 2D systems in the FM models with constant delays. The obtained results were then applied to design a static state feedback stabilising controller for the corresponding control systems. Delay-dependent stability conditions based on the average dwell-time approach were derived in [39] for 2D switched systems with time-varying delays. In [40–44], delay-dependent stability conditions were proposed for 2D discrete systems with time-varying delay in the presence of saturation and quantisation nonlinearities. A similar problem for continuous-time 2D systems described by the Roesser model with constant delays was also investigated in [45, 46]. By employing the free-weighting matrix approach in the framework of the LKF method, delay-dependent stability and stabilisation conditions were derived in [47–49] for some classes of 2D systems in the second FM model with delays. The problems of H_∞ control and filtering for discrete- and continuous-time 2D systems with time-delay and disturbances were also considered in [50–54]. Looking at the literature, it can be realised that in most of the existing works which concern with stability analysis and control of 2D time-delay systems, stability conditions are derived based on an LMIzation process using Jensen-type inequalities and its variants. In addition, in order to deal with the relationship between time-varying delays and their upper and lower bounds, the free-weighting matrix method is utilised to improve feasibility of the derived conditions. However, the use of extra matrix variables combining with Jensen-type inequalities usually produces much conservatism in the derived stability conditions both in terms of feasibility and computational cost. Finding improved bounding techniques by improving integral/summation-based inequalities is always a relevant issue and an effective approach to derive less conservative stability conditions for time-delay systems. In the context of 1D systems, some improvements based on Wirtinger inequality or refined Jensen inequalities have recently been achieved (see, e.g. [35, 37, 55] and some discussions therein). However, similar techniques have not been developed for 2D systems which motivates our present study.

In this paper, we further study the problem of stability analysis of discrete-time 2D Roesser systems with interval time-varying delays. In comparison with existing results, the novel features of this paper rely on the following points:

- New 2D finite-sum inequalities are first proposed. It is shown that the newly derived 2D inequalities theoretically extend and provide a tighter lower bound than the ones based on 2D Jensen-type inequalities. Thus, it is expected that the derived inequalities in this paper can help to reduce conservativeness of the derived stability conditions.
- An improved LKF candidate which incorporates extra information of time delays into its augmented vectors is constructed.
- Together with our new 2D inequalities, the reciprocally convex combination technique is also utilised to consider the relationship between time-varying delays and their intervals, which is effective as it leads to less LMI decision variables.

On the basis of the above features, delay-range-dependent stability conditions are derived in terms of LMIs which can be solved by various computational tools.

2 Problem formulation and preliminaries

Notation: Throughout this paper, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and natural numbers, respectively. For $a, b \in \mathbb{Z}$, $a \le b$, $\mathbb{Z}[a, b]$ stands for $[a, b] \cap \mathbb{Z}$ and $\ell_{\mathbb{Z}}(a, b) = b - a + 1$. Likewise $\mathbb{Z}[a, b] \times [c, d] = \mathbb{Z}[a, b] \times \mathbb{Z}[c, d]$. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote the n-dimensional Euclidean space and the set of $n \times m$ real matrices whereas the set of symmetric positive definite matrices in $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{S}_n^+$.

Consider a 2D Roesser model with time-varying delays described by

$$\begin{bmatrix} x^{\mathrm{h}}(i+1, j) \\ x^{\mathrm{v}}(i, j+1) \end{bmatrix} = A \begin{bmatrix} x^{\mathrm{h}}(i, j) \\ x^{\mathrm{v}}(i, j) \end{bmatrix} + A_d \begin{bmatrix} x^{\mathrm{h}}(i - d_{\mathrm{h}}(i), j) \\ x^{\mathrm{v}}(i, j - d_{\mathrm{v}}(j)) \end{bmatrix} \quad (1)$$

where $x^{\mathrm{h}} \in \mathbb{R}^p$ and $x^{\mathrm{v}} \in \mathbb{R}^q$ are the horizontal state vector and the vertical state vector, respectively

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}$$

are the system matrices with compatible dimensions, $d_{\mathrm{h}}(i)$ and $d_{\mathrm{v}}(j)$ are time-varying delays along the horizontal and vertical directions, respectively, satisfying

$$0 \le d_1^{\mathrm{h}} \le d_{\mathrm{h}}(i) \le d_2^{\mathrm{h}}, \quad 0 \le d_1^{\mathrm{v}} \le d_{\mathrm{v}}(j) \le d_2^{\mathrm{v}}, \quad \forall i, j \in \mathbb{N} \quad (2)$$

where $d_1^{\mathrm{h}}, d_2^{\mathrm{h}}$, d_1^{v} and d_2^{v} are known integers representing the upper and the lower bounds of delays along the horizontal and vertical directions. The initial conditions are defined by sequences $\phi_{i,j}$ and $\psi_{i,j}$ as follows:

$$\begin{aligned} x^{\mathrm{h}}(i, j) &= \phi_{i,j}, \quad i \in \mathbb{Z}[-d_2^{\mathrm{h}}, 0], \ j \in \mathbb{N} \\ x^{\mathrm{v}}(i, j) &= \psi_{i,j}, \quad i \in \mathbb{N}, \ j \in \mathbb{Z}[-d_2^{\mathrm{v}}, 0] \end{aligned} \quad (3)$$

By adopting the concept of asymptotic stability proposed in [15] for 2D delay-free systems, we introduce the following definition.

Definition 1: System (1) is said to be asymptotically stable if the following two conditions hold

i. (ϵ-δ stability) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if $\hat{\phi}_j < \delta(\epsilon)$ and $\hat{\psi}_i < \delta(\epsilon)$ then $\| x(i, j) \| < \epsilon$ for all $i, j > 0$, where $x(i, j) = (x^{\mathrm{h}}(i, j), x^{\mathrm{v}}(i, j)) \in \mathbb{R}^{p+q}$ denotes the state vector of (1), $\hat{\phi}_j = \sup \{ \| \phi_{i,j} \| : i \in \mathbb{Z}[-d_2^{\mathrm{h}}, 0] \}$ and $\hat{\psi}_i = \sup \{ \| \psi_{i,j} \| : j \in \mathbb{Z}[-d_2^{\mathrm{v}}, 0] \}$.
ii. (Attractivity) $\lim_{i+j \to \infty} \| x(i, j) \| = 0$ when $\lim_{j \to \infty} \hat{\phi}_j = 0$ and $\lim_{i \to \infty} \hat{\psi}_i = 0$.

Remark 1: Usually in the literature, the definition of asymptotic stability is set aside to retain only the attractivity condition (ii) in Definition 1 and the assumption that the initial conditions must go to zero at infinity is not required. In order to verify asymptotic stability of system (1), one needs to first check the ϵ-δ stability condition (i) and then to check the 2D attractivity condition. As mentioned in [15], unlike the 1D case where asymptotic stability implies that all trajectories go to zero at infinity, whatever the initial conditions are, in the 2D case, taking the initial conditions in the first quadrant, one cannot expect to have every trajectory approaching zero simply because it may not be the case for the initial conditions.

Assumption 1: The initial conditions are assumed to satisfy

$$\lim_{r \to \infty} \sum_{k=0}^{r} \left(\hat{\phi}_k^2 + \hat{\psi}_k^2 \right) < \infty \quad (4)$$

Remark 2: Condition (4) is obviously satisfied for all initial conditions $\phi_{i,j}, \psi_{i,j}$ with finite support, that is there exist positive integers K, L such that $\phi_{i,j} = 0, \forall j \ge K$, and $\psi_{i,j} = 0, \forall i \ge L$. In addition, (4) also implies that $\lim_{j \to \infty} \hat{\phi}_j = 0$ and $\lim_{i \to \infty} \hat{\psi}_i = 0$. Therefore, as mentioned in [21], to verify asymptotic stability of 2D linear systems (1), what we should do is to demonstrate that $\lim_{i+j \to \infty} \| x(i, j) \| = 0$.

The main objective of this paper is to derive new delay-dependent conditions ensuring asymptotic stability of system (1). To tackle this problem, we first propose new 2D finite-sum inequalities which theoretically encompass the existing 2D Jensen-type inequalities. We then construct an improved augmented 2D LKF candidate and utilise the newly derived 2D inequalities to derive less conservative stability conditions for system (1).

In the remaining of this section, let us introduce some 2D Jensen-type inequalities. Note that, for a matrix $R \in \mathbb{S}_n^+$, the function $q_R: \mathbb{R}^n \to \mathbb{R}$ defined by $q_R(x) = x^{\mathrm{T}} R x$ is a convex function. Thus, the inequality

$$q_R\left(\sum_{j=1}^{m} \lambda_j x_j \right) \le \sum_{j=1}^{m} \lambda_j q_R(x_j) \quad (5)$$

holds for all $x_j \in \mathbb{R}^n$, scalars $\lambda_j \ge 0$, $\sum_{j=1}^{m} \lambda_j = 1$, and positive integer m. From (5), we readily obtain the following 2D Jensen-type inequalities [44, 49].

Lemma 1: For a matrix $R \in \mathbb{S}_n^+$, integers $l_1 \le l_2$, $r_1 \le r_2$, and a function $\eta : \mathbb{Z}[l_1, l_2] \times [r_1, r_2] \to \mathbb{R}^n$, the following inequalities hold:

$$\ell_{\mathbb{Z}}(l_1, l_2) \sum_{i=l_1}^{l_2} \eta^{\mathrm{T}}(i,j) R \eta(i,j) \ge \Big(\sum_{i=l_1}^{l_2} \eta(i,j)\Big)^{\mathrm{T}} R \Big(\sum_{i=l_1}^{l_2} \eta(i,j)\Big) \tag{6a}$$

$$\ell_{\mathbb{Z}}(r_1, r_2) \sum_{j=r_1}^{r_2} \eta^{\mathrm{T}}(i,j) R \eta(i,j) \ge \Big(\sum_{j=r_1}^{r_2} \eta(i,j)\Big)^{\mathrm{T}} R \Big(\sum_{j=r_1}^{r_2} \eta(i,j)\Big) \tag{6b}$$

3 New 2D finite-sum inequalities

For simplicity, let us denote $J_1^g(\eta)$ and $J_2^g(\eta)$ as the gap of (6a) and (6b), respectively. Note that, if $l_1 = l_2$ then (6a) is reduced to an equality and thus $J_1^g(\eta) = 0$. The following result gives a tighter lower bound for $J_1^g(\eta)$.

Lemma 2: For a matrix $R \in \mathbb{S}_n^+$, integers $l_1 < l_2$, $r_1 \le r_2$, and a function $\eta : \mathbb{Z}[l_1, l_2] \times [r_1, r_2] \to \mathbb{R}^n$, the following inequality holds:

$$J_1^g(\eta) \ge \frac{3(\ell_{\mathbb{Z}}(l_1, l_2) + 1)}{\ell_{\mathbb{Z}}(l_1, l_2) - 1} \chi^{\mathrm{T}} R \chi \tag{7}$$

where $\chi = v_{11} - (2/(\ell_{\mathbb{Z}}(l_1, l_2) + 1)) v_{12}$, $v_{11} = \sum_{i=l_1}^{l_2} \eta(i,j)$ and $v_{12} = \sum_{i=l_1}^{l_2} \sum_{k=l_1}^{i} \eta(k,j)$.

Proof: Inspired by Hien and Trinh [37], we define an approximation function $\hat{\eta} : \mathbb{Z}[l_1, l_2] \times [r_1, r_2] \to \mathbb{R}^n$ as

$$\hat{\eta}(i,j) = \eta(i,j) - \frac{1}{\ell_{\mathbb{Z}}(l_1, l_2)} v_{11} + \alpha_i \hat{\chi} \tag{8}$$

where $\alpha_i = (i - l_1) + (1 - \ell_{\mathbb{Z}}(l_1, l_2))/2$, $i \in \mathbb{Z}[l_1, l_2]$, and $\hat{\chi} \in \mathbb{R}^n$ is a constant vector which will be defined later.
Throughout this proof let us denote $\ell = \ell_{\mathbb{Z}}(l_1, l_2)$. Since $\sum_{i=l_1}^{l_2} \alpha_i = 0$ and $\sum_{i=l_1}^{l_2} \alpha_i^2 = (\ell(\ell^2 - 1))/12$, it follows from (8) that:

$$\sum_{i=l_1}^{l_2} \hat{\eta}^{\mathrm{T}}(i,j) R \hat{\eta}(i,j) = \frac{1}{\ell} J_1^g(\eta) + 2\hat{\chi}^{\mathrm{T}} R \Big(\sum_{i=l_1}^{l_2} \alpha_i \eta(i,j)\Big) + \frac{\ell(\ell^2 - 1)}{12} \hat{\chi}^{\mathrm{T}} R \hat{\chi} \tag{9}$$

Now, we define $\eta_s(i,j) = \sum_{k=l_1}^{i-1} \eta(k,j)$ for $i > l_1$ and $\eta_s(l_1, j) = 0$, $j \in \mathbb{Z}[r_1, r_2]$, then $\eta(i,j) = \partial_1(\eta_s(i,j))$, where $\partial_1(.)$ denotes the difference operator with respect to the first variable. Consequently, $\alpha_i \eta(i,j) = \partial_1(\alpha_i \eta_s(i,j)) - \eta_s(i+1, j)$. Taking summation in i from l_1 to l_2 we obtain

$$\sum_{i=l_1}^{l_2} \alpha_i \eta(i,j) = \frac{\ell + 1}{2} v_{11} - v_{12} = \frac{\ell + 1}{2} \chi \tag{10}$$

From (9), (10) and the fact that $\sum_{i=l_1}^{l_2} \hat{\eta}^{\mathrm{T}}(i,j) R \hat{\eta}(i,j) \ge 0$, we have

$$J_1^g(\eta) + (\ell^2 + \ell) \hat{\chi}^{\mathrm{T}} R \chi + \frac{\ell^2(\ell^2 - 1)}{12} \hat{\chi}^{\mathrm{T}} R \hat{\chi} \ge 0 \tag{11}$$

Let $\hat{\chi} = -\lambda \chi$, where λ is a scalar, then (11) leads to

$$J_1^g(\eta) \ge (\ell^2 + \ell)\Big(\lambda - \frac{\ell(\ell - 1)}{12} \lambda^2\Big) \chi^{\mathrm{T}} R \chi \tag{12}$$

Since the function $\lambda - (\ell(\ell - 1)/12)\lambda^2$ attains its maximum $3/(\ell(\ell - 1))$ at $\lambda = 6/(\ell(\ell - 1))$ which yields $\hat{\chi} = (-6/(\ell(\ell - 1)))\chi$, from (11) we finally obtain

$$J_1^g(\eta) \ge \frac{3(\ell + 1)}{\ell - 1} \chi^{\mathrm{T}} R \chi$$

The proof is completed. □

Remark 3: Clearly, the inequality given in (7) provides a tighter lower bound than 2D Jensen-type inequality (6a) since a positive term is added into the right-hand side of (6a).

Remark 4: For a matrix $R \in \mathbb{S}_n^+$, integers $l_1 \le l_2$, $r_1 < r_2$, and a function $\eta : \mathbb{Z}[l_1, l_2] \times [r_1, r_2] \to \mathbb{R}^n$, by similar lines used in the proof of Lemma 2 we obtain the following estimate:

$$J_2^g(\eta) \ge \frac{3(\ell_{\mathbb{Z}}(r_1, r_2) + 1)}{\ell_{\mathbb{Z}}(r_1, r_2) - 1} \xi^{\mathrm{T}} R \xi \tag{13}$$

where $\xi = v_{21} - (2/(\ell_{\mathbb{Z}}(r_1, r_2) + 1)) v_{22}$, $v_{21} = \sum_{j=r_1}^{r_2} \eta(i,j)$ and $v_{22} = \sum_{j=r_1}^{r_2} \sum_{k=r_1}^{j} \eta(i,k)$.

By simplifying (7) and (13), the following corollary is provided to make them easier to handle in applications to 2D discrete-time systems with delays.

Corollary 1: For a matrix $R \in \mathbb{S}_n^+$, positive integers l, r and a function $\eta : \mathbb{Z}[-l, 0] \times [-r, 0] \to \mathbb{R}^n$, the following inequalities hold:

$$\sum_{i=-l}^{-1} \eta_1^{\mathrm{T}}(i,j) R \eta_1(i,j) \ge \frac{1}{l} \Omega_1(j)^{\mathrm{T}} \hat{R} \Omega_1(j) \tag{14a}$$

$$\sum_{j=-r}^{-1} \eta_2^{\mathrm{T}}(i,j) R \eta(i,j) \ge \frac{1}{r} \Omega_2(i)^{\mathrm{T}} \hat{R} \Omega_2(i) \tag{14b}$$

where $\eta_1(i,j) = \eta(i+1, j) - \eta(i,j)$, $\eta_2(i,j) = \eta(i, j+1), - \eta(i,j)$

$\hat{R} = \mathrm{diag}\{R, 3R\}$ and (see equation below)

Proof: Clearly, $(l+1)/(l-1) > 1$ for all $l > 1$, it follows from (7) that:

$$l \sum_{i=-l}^{-1} \eta_1^{\mathrm{T}}(i,j) R \eta_1(i,j) \ge \tilde{v}_{11}^{\mathrm{T}} R \tilde{v}_{11} + 3\Big(\tilde{v}_{11} - \frac{2}{l+1} \tilde{v}_{12}\Big)^{\mathrm{T}} R \Big(\tilde{v}_{11} - \frac{2}{l+1} \tilde{v}_{12}\Big) \tag{15}$$

$$\Omega_1(j) = \mathrm{col}\Big\{\eta(0,j) - \eta(-l,j), \eta(0,j) + \eta(-l,j) - \frac{2}{l+1} \sum_{i=-l}^{0} \eta(i,j)\Big\}$$

$$\Omega_2(i) = \mathrm{col}\Big\{\eta(i,0) - \eta(i,-r), \eta(i,0) + \eta(i,-r) - \frac{2}{r+1} \sum_{j=-r}^{0} \eta(i,j)\Big\}$$

where $\tilde{v}_{11} = \sum_{i=-l}^{-1} \eta_1(i,j) = \eta(0,j) - \eta(-l,j)$ and $\tilde{v}_{12} = \sum_{i=-l}^{-1} \sum_{k=-l}^{i} \eta_1(k,j) = \sum_{i=-l}^{0} \eta(i,j) - (l+1)\eta(-l,j)$. From (15), we readily obtain (14a). The proof of (14b) is similar to (14a) and thus is omitted here. □

4 Stability conditions

In this section, 2D finite-sum inequalities proposed in the preceding section will be utilised to derive delay-dependent stability conditions for system (1). For this purpose, we construct an augmented LKF candidate as follows:

$$V(i,j) = \underbrace{\begin{bmatrix} \tilde{x}^{\mathrm{h}}(i,j) \\ \tilde{x}^{\mathrm{v}}(i,j) \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} \tilde{x}^{\mathrm{h}}(i,j) \\ \tilde{x}^{\mathrm{v}}(i,j) \end{bmatrix}}_{V_0(i,j)} + \sum_{k=1}^{4} V_k(i,j) \tag{16}$$

where (see equation below)

Remark 5: In the existing results, e.g. [39–43, 48, 51, 52], the functional $V_0(i,j)$ is usually constructed in the form

$$\begin{bmatrix} x^{\mathrm{h}}(i,j) \\ x^{\mathrm{v}}(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P^{\mathrm{h}} & 0 \\ 0 & P^{\mathrm{v}} \end{bmatrix} \begin{bmatrix} x^{\mathrm{h}}(i,j) \\ x^{\mathrm{v}}(i,j) \end{bmatrix}$$

which only uses the state vector. Different from those in the literature, the LKF candidate presented in (16) contains the state vector and its summations which allows one to incorporate extra information on time-delay into its augmented vectors. Based on this functional and by utilising our new 2D finite-sum inequalities, less conservative stability conditions for system (1) can be derived.

For simplicity, we let e_i $(i = 1, 2, \ldots, 7)$ denote the block-row vectors of the identity matrix I_{7n}, where $n = p + q$ is the dimension of system (1), $\mathcal{A} = (A - I_n)e_1 + A_d e_3$. We also denote (see equation below) and the following matrices: (see equation below) We now have the following result.

Theorem 1: System (1) with time-varying delays $d_{\mathrm{h}}(i) \in [d_1^{\mathrm{h}}, d_2^{\mathrm{h}}]$, $d_{\mathrm{v}}(j) \in [d_1^{\mathrm{v}}, d_2^{\mathrm{v}}]$, is asymptotically stable if there exist matrices $P^{\mathrm{h}} \in \mathbb{S}_{3p}^{+}$, $P^{\mathrm{v}} \in \mathbb{S}_{3q}^{+}$, $Q_\nu = \mathrm{diag}\{Q_\nu^{\mathrm{h}}, Q_\nu^{\mathrm{v}}\} \in \mathbb{S}_n^{+}$, $R_\nu = \mathrm{diag}\{R_\nu^{\mathrm{h}}, R_\nu^{\mathrm{v}}\} \in \mathbb{S}_n^{+}$ $(\nu = 1, 2)$ and matrices $X \in \mathbb{R}^{2p \times 2p}$, $Y \in \mathbb{R}^{2q \times 2q}$ such that $\Psi_1 \geq 0$, $\Psi_2 \geq 0$, and the following LMIs hold for $d^{\mathrm{h}} \in \{d_1^{\mathrm{h}}, d_2^{\mathrm{h}}\}$ and $d^{\mathrm{v}} \in \{d_1^{\mathrm{v}}, d_2^{\mathrm{v}}\}$:

$$\Pi(d_{\mathrm{h}}, d_{\mathrm{v}}) = \Pi_0(d_{\mathrm{h}}, d_{\mathrm{v}}) + \sum_{\nu=1}^{2} \Pi_\nu - \sum_{\nu=3}^{4} \Pi_\nu < 0 \tag{17}$$

Proof: We write the LKF (16), where $P = \mathrm{diag}\{P^{\mathrm{h}}, P^{\mathrm{v}}\}$, in the form $V(i,j) = V_{\mathrm{h}}(x^{\mathrm{h}}(i,j)) + V_{\mathrm{v}}(x^{\mathrm{v}}(i,j))$ and then the difference of $V(i,j)$ is defined by (see (18)) . Since

$$V_0(i,j) = \underbrace{\tilde{x}^{\mathrm{hT}}(i,j) P^{\mathrm{h}} \tilde{x}^{\mathrm{h}}(i,j)}_{V_{\mathrm{h}}^0(x^{\mathrm{h}}(i,j))} + \underbrace{\tilde{x}^{\mathrm{vT}}(i,j) P^{\mathrm{v}} \tilde{x}^{\mathrm{v}}(i,j)}_{V_{\mathrm{v}}^0(x^{\mathrm{v}}(i,j))}$$

and (see equation below) we have (see equation below). Therefore

$$\Delta V_0(i,j) = \chi^{\mathrm{T}}(i,j) \Pi_0(d_{\mathrm{h}}, d_{\mathrm{v}}) \chi(i,j) \tag{19}$$

Next, the difference $\Delta V_1(i,j)$ is given by

$$\tilde{x}^{\mathrm{h}}(i,j) = \mathrm{col}\{x^{\mathrm{h}}(i,j), \quad \sum_{k=i-d_1^{\mathrm{h}}}^{i-1} x^{\mathrm{h}}(k,j), \quad \sum_{k=i-d_2^{\mathrm{h}}}^{i-d_1^{\mathrm{h}}-1} x^{\mathrm{h}}(k,j)\}$$

$$\tilde{x}^{\mathrm{v}}(i,j) = \mathrm{col}\{x^{\mathrm{v}}(i,j), \quad \sum_{l=j-d_1^{\mathrm{v}}}^{j-1} x^{\mathrm{v}}(i,l), \quad \sum_{l=j-d_2^{\mathrm{v}}}^{j-d_1^{\mathrm{v}}-1} x^{\mathrm{v}}(i,l)\}$$

$$V_1(i,j) = \sum_{k=i-d_1^{\mathrm{h}}}^{i-1} x^{\mathrm{hT}}(k,j) Q_1^{\mathrm{h}} x^{\mathrm{h}}(k,j) + \sum_{l=j-d_1^{\mathrm{v}}}^{j-1} x^{\mathrm{vT}}(i,l) Q_1^{\mathrm{v}} x^{\mathrm{v}}(i,l)$$

$$V_2(i,j) = \sum_{k=i-d_2^{\mathrm{h}}}^{i-d_1^{\mathrm{h}}-1} x^{\mathrm{hT}}(k,j) Q_2^{\mathrm{h}} x^{\mathrm{h}}(k,j) + \sum_{l=j-d_2^{\mathrm{v}}}^{j-d_1^{\mathrm{v}}-1} x^{\mathrm{vT}}(i,l) Q_2^{\mathrm{v}} x^{\mathrm{v}}(i,l)$$

$$V_3(i,j) = d_1^{\mathrm{h}} \sum_{k=-d_1^{\mathrm{h}}}^{-1} \sum_{l=i+k}^{i-1} z_1^{\mathrm{T}}(l,j) R_1^{\mathrm{h}} z_1(l,j) + d_1^{\mathrm{v}} \sum_{k=-d_1^{\mathrm{v}}}^{-1} \sum_{l=j+k}^{j-1} z_2^{\mathrm{T}}(i,l) R_1^{\mathrm{v}} z_2(i,l)$$

$$V_4(i,j) = d_{12}^{\mathrm{h}} \sum_{k=-d_2^{\mathrm{h}}}^{-d_1^{\mathrm{h}}-1} \sum_{l=i+k}^{i-1} z_1^{\mathrm{T}}(l,j) R_2^{\mathrm{h}} z_1(l,j) + d_{12}^{\mathrm{v}} \sum_{k=-d_2^{\mathrm{v}}}^{-d_1^{\mathrm{v}}-1} \sum_{l=j+k}^{j-1} z_2^{\mathrm{T}}(i,l) R_2^{\mathrm{v}} z_2(i,l)$$

$$z_1(i,j) = \partial_1(x^{\mathrm{h}}(i,j)), \quad z_2(i,j) = \partial_2(x^{\mathrm{v}}(i,j)), \quad d_{12}^{\mathrm{h}} = d_2^{\mathrm{h}} - d_1^{\mathrm{h}}, \quad d_{12}^{\mathrm{v}} = d_2^{\mathrm{v}} - d_1^{\mathrm{v}}$$

$$\chi(i,j) = \mathrm{col}\{x(i,j), x_l(i,j), x_d(i,j), x_u(i,j), x_l^s(i,j), x_d^s(i,j), x_u^s(i,j)\}$$

$$x(i,j) = \begin{bmatrix} x^{\mathrm{h}}(i,j) \\ x^{\mathrm{v}}(i,j) \end{bmatrix}, \quad x_l(i,j) = \begin{bmatrix} x^{\mathrm{h}}(i-d_1^{\mathrm{h}}, j) \\ x^{\mathrm{v}}(i, j-d_1^{\mathrm{v}}) \end{bmatrix}, \quad x_d(i,j) = \begin{bmatrix} x^{\mathrm{h}}(i-d_{\mathrm{h}}(i), j) \\ x^{\mathrm{v}}(i, j-d_{\mathrm{v}}(j)) \end{bmatrix}$$

$$x_u(i,j) = \begin{bmatrix} x^{\mathrm{h}}(i-d_2^{\mathrm{h}}, j) \\ x^{\mathrm{v}}(i, j-d_2^{\mathrm{v}}) \end{bmatrix}, \quad x_l^s(i,j) = \begin{bmatrix} \xi_{11}(i,j) \\ \xi_{21}(i,j) \end{bmatrix}, \quad x_d^s(i,j) = \begin{bmatrix} \xi_{12}(i,j) \\ \xi_{22}(i,j) \end{bmatrix}, \quad x_u^s(i,j) = \begin{bmatrix} \xi_{13}(i,j) \\ \xi_{23}(i,j) \end{bmatrix},$$

$$\xi_{11}(i,j) = \frac{1}{1+d_1^{\mathrm{h}}} \sum_{k=i-d_1^{\mathrm{h}}}^{i} x^{\mathrm{h}}(k,j), \quad \xi_{12}(i,j) = \frac{1}{1+d_{\mathrm{h}}(i)-d_1^{\mathrm{h}}} \sum_{k=i-d_{\mathrm{h}}(i)}^{i-d_1^{\mathrm{h}}} x^{\mathrm{h}}(k,j)$$

$$\xi_{13}(i,j) = \frac{1}{1+d_2^{\mathrm{h}}-d_{\mathrm{h}}(i)} \sum_{k=i-d_2^{\mathrm{h}}}^{i-d_{\mathrm{h}}(i)} x^{\mathrm{h}}(k,j), \quad \xi_{21}(i,j) = \frac{1}{1+d_1^{\mathrm{v}}} \sum_{l=j-d_1^{\mathrm{v}}}^{j} x^{\mathrm{v}}(i,l)$$

$$\xi_{22}(i,j) = \frac{1}{1+d_{\mathrm{v}}(j)-d_1^{\mathrm{v}}} \sum_{l=j-d_{\mathrm{v}}(j)}^{j-d_1^{\mathrm{v}}} x^{\mathrm{v}}(i,l), \quad \xi_{23}(i,j) = \frac{1}{1+d_2^{\mathrm{v}}-d_{\mathrm{v}}(i)} \sum_{l=j-d_2^{\mathrm{v}}}^{j-d_{\mathrm{v}}(j)} x^{\mathrm{v}}(i,l)$$

$$\Delta V_1(i,j) = x^{hT}(i,j)Q_1^h x^h(i,j) - x^{hT}(i-d_1^h,j)Q_1^h x^h(i-d_1^h,j) + x^{vT}(i,j)Q_1^v x^v(i,j) - x^{vT}(i,j-d_1^v)Q_1^v x^v(i,j-d_1^v) = \chi^T(i,j)(e_1^T Q_1 e_1 - e_2^T Q_1 e_2)\chi(i,j) \tag{20}$$

Similarly, we have (see (21)) . Combining (20) and (21) gives

$$\Delta V_1(i,j) + \Delta V_2(i,j) = \chi^T(i,j)\Pi_1\chi(i,j) \tag{22}$$

Now, we take the difference $\Delta V_3(i,j)$ and utilise the inequalities given in (7) and (13) to estimate $\Delta V_3(i,j)$. Note that

$$\Delta V_3(i,j) = (d_1^h)^2 z_1^T(i,j)R_1^h z_1(i,j) - d_1^h \sum_{k=i-d_1^h}^{i-1} z_1^T(k,j)R_1^h z_1(k,j) + (d_1^v)^2 z_2^T(i,j)R_1^v z_2(i,j) - d_1^v \sum_{l=j-d_1^v}^{j-1} z_2^T(i,l)R_1^v z_2(i,l) \tag{23}$$

and

$$\begin{bmatrix} z_1(i,j) \\ z_2(i,j) \end{bmatrix} = [(A-I_n)e_1 + A_d e_3]\chi(i,j) = \mathcal{A}\chi(i,j) \tag{24}$$

Therefore, (23) and (24) give

$$\Delta V_3(i,j) = \chi^T(i,j)\mathcal{A}^T\hat{R}_1\mathcal{A}\chi(i,j) - d_1^h \sum_{k=i-d_1^h}^{i-1} z_1^T(k,j)R_1^h z_1(k,j) - d_1^v \sum_{l=j-d_1^v}^{j-1} z_2^T(i,l)R_1^v z_2(i,l) \tag{25}$$

By utilising (7) and (14), respectively, we obtain (see (26) and (27)) and (see (27))

From (25)–(27), we have

$$\Delta V_3(i,j) \le \chi^T(i,j)(\mathcal{A}^T\hat{R}_1\mathcal{A} - \Pi_3)\chi(i,j) \tag{28}$$

$$\begin{aligned}
E(d_h) &= \begin{bmatrix} e_1^T & (1+d_1^h)e_5^T & \ell_{\mathbb{Z}}(d_1^h,d_h)e_6^T + \ell_{\mathbb{Z}}(d_h,d_2^h)e_7^T \end{bmatrix}^T \\
G(d_v) &= \begin{bmatrix} e_1^T & (1+d_1^v)e_5^T & \ell_{\mathbb{Z}}(d_1^v,d_v)e_6^T + \ell_{\mathbb{Z}}(d_v,d_2^v)e_7^T \end{bmatrix}^T \\
F_0 &= \begin{bmatrix} \mathcal{A}^T & (e_1-e_2)^T & (e_2-e_4)^T \end{bmatrix}^T, \quad F_1 = \begin{bmatrix} 0 & e_1^T & (e_2+e_3)^T \end{bmatrix}^T \\
F_2 &= \begin{bmatrix} -\mathcal{A}^T & e_2^T & (e_3+e_4)^T \end{bmatrix}^T, \quad \Gamma_1 = \begin{bmatrix} (e_1-e_2)^T & (e_1+e_2-2e_5)^T \end{bmatrix}^T \\
\Gamma_2 &= \begin{bmatrix} (e_2-e_3)^T & (e_2+e_3-2e_6)^T \end{bmatrix}^T, \quad \Gamma_3 = \begin{bmatrix} (e_3-e_4)^T & (e_3+e_4-2e_7)^T \end{bmatrix}^T \\
\Pi_0(d_h,d_v) &= \mathrm{Sym}\left(E(d_h)^T\overline{P^h}F_0 + G(d_v)^T\overline{P^v}F_0\right) + F_2^T\left(\overline{P^h}+\overline{P^v}\right)F_2 - F_1^T\left(\overline{P^h}+\overline{P^v}\right)F_1 \\
\overline{P^h} &= J_p^{3T}P^h J_p^3, \quad \overline{P^v} = J_q^{3T}P^v J_q^3, \quad \Pi_1 = e_1^T Q_1 e_1 - e_2^T Q_1 e_2 + e_2^T Q_2 e_2 - e_4^T Q_2 e_4 \\
\Pi_2 &= \mathcal{A}^T\left(\hat{R}_1+\hat{R}_2\right)\mathcal{A}, \quad \Pi_3 = \Gamma_1^T\Phi_1\Gamma_1, \quad \Pi_4 = \begin{bmatrix} \Gamma_2 \\ \Gamma_3 \end{bmatrix}^T \left\{ J_p^{4T}\Psi_1 J_p^4 + J_q^{4T}\Psi_2 J_q^4 \right\} \begin{bmatrix} \Gamma_2 \\ \Gamma_3 \end{bmatrix} \\
\hat{R}_1 &= \mathrm{diag}\left\{(d_1^h)^2 R_1^h, (d_1^v)^2 R_1^v\right\}, \quad \hat{R}_2 = \mathrm{diag}\left\{(d_{12}^h)^2 R_2^h, (d_{12}^v)^2 R_2^v\right\} \\
\Psi_1 &= \begin{bmatrix} \widetilde{R_2^h} & X \\ X^T & \widetilde{R_2^h} \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} \widetilde{R_2^v} & Y \\ Y^T & \widetilde{R_2^v} \end{bmatrix}, \quad \widetilde{R_2^h} = \mathrm{diag}\{R_2^h, 3R_2^h\}, \quad \widetilde{R_2^v} = \mathrm{diag}\{R_2^v, 3R_2^v\} \\
J_p^k &= \underbrace{\mathrm{diag}\{[I_p \quad 0_{p\times q}], \ldots, [I_p \quad 0_{p\times q}]\}}_{k\text{-blocks}}, \quad J_q^k = \underbrace{\mathrm{diag}\{[0_{q\times p} \quad I_q], \ldots, [0_{q\times p} \quad I_q]\}}_{k\text{-blocks}} \\
\Phi_1 &= \mathrm{diag}\left\{R_1, \frac{3(d_1^h+1)}{d_1^h-1}R_1^h, \frac{3(d_1^v+1)}{d_1^v-1}R_1^v\right\}
\end{aligned}$$

$$\begin{aligned}
\Delta V(i,j) &= \partial_1(V_h(x^h(i,j))) + \partial_2(V_v(x^v(i,j))) \\
&= V_h(x^h(i+1,j)) - V_h(x^h(i,j)) + V_v(x^v(i,j+1)) - V^v(x^v(i,j))
\end{aligned} \tag{18}$$

$$\begin{aligned}
\tilde{x}^h(i,j) &= J_p^3(E(d_h)-F_1)\chi(i,j), \quad \tilde{x}^v(i,j) = J_q^3(G(d_v)-F_1)\chi(i,j) \\
\tilde{x}^h(i+1,j) &= J_p^3(E(d_h)-F_2)\chi(i,j), \quad \tilde{x}^v(i,j+1) = J_q^3(G(d_v)-F_2)\chi(i,j)
\end{aligned}$$

$$\begin{aligned}
\partial_1(V_h^0(x^h(i,j))) &= \chi^T(i,j)\left[(E(d_h)-F_2)^T\overline{P^h}(E(d_h)-F_2) - (E(d_h)-F_1)^T\overline{P^h}(E(d_h)-F_1)\right]\chi(i,j) \\
&= \chi^T(i,j)\left[\mathrm{Sym}\left(E(d_h)^T\overline{P^h}F_0\right) + F_2^T\overline{P^h}F_2 - F_1^T\overline{P^h}F_1\right]\chi(i,j) \\
\partial_2(V_v^0(x^v(i,j))) &= \chi^T(i,j)\left[(G(d_v)-F_2)^T\overline{P^v}(G(d_v)-F_2) - (G(d_v)-F_1)^T\overline{P^v}(G(d_v)-F_1)\right]\chi(i,j) \\
&= \chi^T(i,j)\left[\mathrm{Sym}\left(G(d_v)^T\overline{P^v}F_0\right) + F_2^T\overline{P^v}F_2 - F_1^T\overline{P^v}F_1\right]\chi(i,j)
\end{aligned}$$

$$\begin{aligned}
\Delta V_2(i,j) &= x^{hT}(i-d_1^h,j)Q_2^h x^h(i-d_1^h,j) - x^{hT}(i-d_2^h,j)Q_2^h x^h(i-d_2^h,j) \\
&\quad + x^{vT}(i,j-d_1^v)Q_2^v x^v(i,j-d_1^v) - x^{vT}(i,j-d_2^v)Q_2^v x^v(i,j-d_2^v) \\
&= \chi^T(i,j)(e_2^T Q_2 e_2 - e_4^T Q_2 e_4)\chi(i,j)
\end{aligned} \tag{21}$$

Similar to (25), the difference $\Delta V_4(i,j)$ is given by

$$\Delta V_4(i,j) = \chi^{\mathrm{T}}(i,j)\mathcal{A}^{\mathrm{T}}\hat{R}_2\mathcal{A}\chi(i,j) - d_{12}^{\mathrm{h}}\sum_{k=i-d_2^{\mathrm{h}}}^{i-d_1^{\mathrm{h}}-1}\eta_1^{\mathrm{T}}(k,j)R_2^{\mathrm{h}}\eta_1(k,j) - d_{12}^{\mathrm{v}}\sum_{l=j-d_2^{\mathrm{v}}}^{j-d_1^{\mathrm{v}}-1}\eta_2^{\mathrm{T}}(i,l)R_2^{\mathrm{v}}\eta_2(i,l) \tag{29}$$

The following fact will be used in bounding $\Delta V_4(i,j)$. For scalars $\alpha \geq 0$, $\gamma \geq 0$, the inequality $\alpha/\sigma + \gamma/(1-\sigma) \geq (\sqrt{\alpha}+\sqrt{\gamma})^2$ holds for all $\sigma \in (0,1)$ which implies $\alpha/\sigma + \gamma/1-\sigma) \geq \alpha + 2\beta + \gamma$ provided that

$$\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix} \geq 0.$$

By utilising (14a), we have (see (30)) where $\tilde{\Omega}_1 = \mathrm{col}\{x^{\mathrm{h}}(i-d_1^{\mathrm{h}},j) - x^{\mathrm{h}}(i-d_{\mathrm{h}}(i),j),$ $x^{\mathrm{h}}(i-d_1^{\mathrm{h}},j) + x^{\mathrm{h}}(i-d_{\mathrm{h}}(i),j) - 2\xi_{12}\}$ and $\tilde{\Omega}_2 = \mathrm{col}\{x^{\mathrm{h}}(i-d_{\mathrm{h}}(i),j) - x^{\mathrm{h}}(i-d_2^{\mathrm{h}},j),$ $x^{\mathrm{h}}(i-d_{\mathrm{h}}(i),j) + x^{\mathrm{h}}(i-d_2^{\mathrm{h}},j) - 2\xi_{13}\}$. Note that, when $d_{\mathrm{h}}(i) = d_1^{\mathrm{h}}$ or $d_{\mathrm{h}}(i) = d_2^{\mathrm{h}}$, $\Omega_1 = 0$ and $\Omega_2 = 0$, respectively, and the above inequality is still valid.

Similar to (30), we have

$$-d_{12}^{\mathrm{v}}\sum_{l=j-d_2^{\mathrm{v}}}^{j-d_1^{\mathrm{v}}-1}\eta_2^{\mathrm{T}}(i,l)R_2^{\mathrm{v}}\eta_2(i,l) \leq -\begin{bmatrix}\Theta_1\\ \Theta_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\widetilde{R_2^{\mathrm{v}}} & Y\\ Y^{\mathrm{T}} & \widetilde{R_2^{\mathrm{v}}}\end{bmatrix}\begin{bmatrix}\Theta_1\\ \Theta_2\end{bmatrix} \tag{31}$$

where $\Theta_1 = \mathrm{col}\{x^{\mathrm{v}}(i,j-d_1^{\mathrm{v}}) - x^{\mathrm{v}}(i,j-d_{\mathrm{v}}(j)),$ $x^{\mathrm{v}}(i,j-d_1^{\mathrm{v}}) + x^{\mathrm{v}}(i,j-d_{\mathrm{v}}(j)) - 2\xi_{22}\}$ and $\Theta_2 = \mathrm{col}\{x^{\mathrm{v}}(i,j-d_{\mathrm{v}}(j)) - x^{\mathrm{v}}(i,j-d_2^{\mathrm{v}}),$ $x^{\mathrm{v}}(i,j-d_{\mathrm{v}}(j)) + x^{\mathrm{v}}(i,j-d_2^{\mathrm{v}}) - 2\xi_{23}\}$.

Since $\tilde{\Omega}_1 = J_p^2\Gamma_2\chi(i,j)$, $\tilde{\Omega}_2 = J_p^2\Gamma_3\chi(i,j)$, $\Theta_1 = J_q^2\Gamma_2\chi(i,j)$ and $\Theta_2 = J_q^2\Gamma_3(i,j)$, from (29)–(31), we have

$$\Delta V_4(i,j) \leq \chi^{\mathrm{T}}(i,j)\left(\mathcal{A}^{\mathrm{T}}\hat{R}_2\mathcal{A} - \Pi_4\right)\chi(i,j) \tag{32}$$

From (19), (22), (28) and (32), we finally obtain

$$\Delta V(i,j) \leq \chi^{\mathrm{T}}(i,j)\Big(\Pi_0(d_{\mathrm{h}},d_{\mathrm{v}}) + \sum_{\nu=1}^{2}\Pi_\nu - \sum_{\nu=3}^{4}\Pi_\nu\Big)\chi(i,j) \tag{33}$$

Since the matrix $\Pi(d_{\mathrm{h}},d_{\mathrm{v}}) = \Pi_0(d_{\mathrm{h}},d_{\mathrm{v}}) + \sum_{\nu=1}^2\Pi_\nu - \sum_{\nu=3}^4\Pi_\nu$ is affine with respect to delays $d_{\mathrm{h}}, d_{\mathrm{v}}$, $\Pi(d_{\mathrm{h}},d_{\mathrm{v}}) < 0$ for all $(d_{\mathrm{h}},d_{\mathrm{v}}) \in [d_1^{\mathrm{h}},d_2^{\mathrm{h}}] \times [d_1^{\mathrm{v}},d_2^{\mathrm{v}}]$ if and only if $\Pi(d_{\mathrm{h}},d_{\mathrm{v}}) < 0$ for $d_{\mathrm{h}} \in \{d_1^{\mathrm{h}},d_2^{\mathrm{h}}\}$, $d_{\mathrm{v}} \in \{d_1^{\mathrm{v}},d_2^{\mathrm{v}}\}$. Therefore, if (17) holds for $d_{\mathrm{h}} \in \{d_1^{\mathrm{h}},d_2^{\mathrm{h}}\}$ and $d_{\mathrm{v}} \in \{d_1^{\mathrm{v}},d^{\mathrm{v}}\}$ then, by (33), $\Delta V(i,j)$ is negative definite.

Let $E(r) = \sum_{i+j=r}[V_{\mathrm{h}}(x^{\mathrm{h}}(i,j)) + V_{\mathrm{v}}(x^{\mathrm{v}}(i,j)]$ denote the energy stored in the diagonal line $\{(i,j) \in \mathbb{N}^2 : i+j=r\}$. Then, it follows from (16) and (33) that there exist positive scalars λ_1, λ_2 such that

$$\lambda_1\sum_{i+j=r}\|x(i,j)\|^2 \leq E(r) \leq \sum_{k=1}^{r}\left[V_{\mathrm{h}}(x^{\mathrm{h}}(0,k)) + V_{\mathrm{v}}(x^{\mathrm{v}}(k,0))\right] \leq \lambda_2\sum_{k=1}^{r}\left(\hat{\phi}_k^2 + \hat{\psi}_k^2\right) \tag{34}$$

From (34), we readily obtain

$$\lim_{r\to\infty}\sum_{i+j=r}\|x(i,j)\|^2 \leq \frac{\lambda_2}{\lambda_1}\lim_{r\to\infty}\sum_{k=1}^{r}\left(\hat{\phi}_k^2 + \hat{\psi}_k^2\right) < \infty$$

which ensures the asymptotic stability of system (1). The proof is completed.□

Remark 6: The result of Theorem 1 provides a delay-dependent stability criterion for system (1) with proper interval time-varying delays, that is the lower bounds of delays $d_1^{\mathrm{h}}, d_1^{\mathrm{v}}$ are positive integers. When $d_1^{\mathrm{h}} = 0$ and $d_1^{\mathrm{v}} = 0$, the terms with Q_1, R_1 of the

$$\begin{aligned}-d_1^{\mathrm{h}}\sum_{k=i-d_1^{\mathrm{h}}}^{i-1}z_1^{\mathrm{T}}(k,j)R_1^{\mathrm{h}}z_1(k,j) \leq\ & -[x^{\mathrm{h}}(i,j) - x^{\mathrm{h}}(i-d_1^{\mathrm{h}},j)]^{\mathrm{T}}R_1^{\mathrm{h}}[x^{\mathrm{h}}(i,j) - x^{\mathrm{h}}(i-d_1^{\mathrm{h}},j)]\\ & -\frac{3(d_1^{\mathrm{h}}+1)}{d_1^{\mathrm{h}}-1}[x^{\mathrm{h}}(i,j) + x^{\mathrm{h}}(i-d_1^{\mathrm{h}},j) - 2\xi_{11}(i,j)]^{\mathrm{T}}R_1^{\mathrm{h}}\\ & \times[x^{\mathrm{h}}(i,j) + x^{\mathrm{h}}(i-d_1^{\mathrm{h}},j) - 2\xi_{11}(i,j)]\end{aligned} \tag{26}$$

$$\begin{aligned}-d_1^{\mathrm{v}}\sum_{l=j-d_1^{\mathrm{v}}}^{j-1}z_2^{\mathrm{T}}(i,l)R_1^{\mathrm{v}}z_2(i,l) \leq\ & -[x^{\mathrm{v}}(i,j) - x^{\mathrm{v}}(i,j-d_1^{\mathrm{v}})]^{\mathrm{T}}R_1^{\mathrm{v}}[x^{\mathrm{v}}(i,j) - x^{\mathrm{v}}(i,j-d_1^{\mathrm{v}})]\\ & -\frac{3(d_1^{\mathrm{v}}+1)}{d_1^{\mathrm{v}}-1}[x^{\mathrm{v}}(i,j) + x^{\mathrm{v}}(i,j-d_1^{\mathrm{v}}) - 2\xi_{21}(i,j)]^{\mathrm{T}}R_1^{\mathrm{v}}\\ & \times[x^{\mathrm{v}}(i,j) + x^{\mathrm{v}}(i,j-d_1^{\mathrm{v}}) - 2\xi_{21}(i,j)]\end{aligned} \tag{27}$$

$$\begin{aligned}-d_{12}^{\mathrm{h}}\sum_{k=i-d_2^{\mathrm{h}}}^{i-d_1^{\mathrm{h}}-1}z_1^{\mathrm{T}}(k,j)R_2^{\mathrm{h}}z_1(k,j) &= -d_{12}^{\mathrm{h}}\sum_{k=i-d_2^{\mathrm{h}}}^{i-d_{\mathrm{h}}(i)-1}z_1^{\mathrm{T}}(k,j)R_2^{\mathrm{h}}z_1(k,j) - d_{12}^{\mathrm{h}}\sum_{k=i-d_{\mathrm{h}}(i)}^{i-d_1^{\mathrm{h}}-1}z_1^{\mathrm{T}}(k,j)R_2^{\mathrm{h}}z_1(k,j)\\ &\leq -\frac{d_{12}^{\mathrm{h}}}{d_{\mathrm{h}}(i)-d_1^{\mathrm{h}}}\tilde{\Omega}_1^{\mathrm{T}}\widetilde{R_2^{\mathrm{h}}}\tilde{\Omega}_1 - \frac{d_{12}^{\mathrm{h}}}{d_2^{\mathrm{h}}-d_{\mathrm{h}}(i)}\tilde{\Omega}_2^{\mathrm{T}}\widetilde{R_2^{\mathrm{h}}}\tilde{\Omega}_2\\ &\leq -\begin{bmatrix}\tilde{\Omega}_1\\ \tilde{\Omega}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\widetilde{R_2^{\mathrm{h}}} & X\\ X^{\mathrm{T}} & \widetilde{R_2^{\mathrm{h}}}\end{bmatrix}\begin{bmatrix}\tilde{\Omega}_1\\ \tilde{\Omega}_2\end{bmatrix}\end{aligned} \tag{30}$$

functional $V(i, j)$ given in (16) will be omitted and a similar result of Theorem 1 can be obtained.

Remark 7: If the time delays in system (1) are constants, i.e. $d_1^{\mathrm{h}} = d_2^{\mathrm{h}} = d_{\mathrm{h}}$ and $d_1^{\mathrm{v}} = d_2^{\mathrm{v}} = d_{\mathrm{v}}$, where $d_{\mathrm{h}}, d_{\mathrm{v}}$ are positive integers, then the proposed functional (16) is reduced to a simpler form (see (35)) where $\hat{x}^{\mathrm{h}}(i, j) = \mathrm{col}\{x^{\mathrm{h}}(i, j), \sum_{k=i-d_{\mathrm{h}}}^{i-1} x^{\mathrm{h}}(k, j)\}$ and $\hat{x}^{\mathrm{v}}(i, j) = \mathrm{col}\{x^{\mathrm{v}}(i, j), \sum_{l=j-d_1^{\mathrm{v}}}^{j-1} x^{\mathrm{v}}(i, l)\}$.

By some similar lines used in the proof of Theorem 1 we have the following result.

Corollary 2: System (1) with constant delays $d_{\mathrm{h}}, d_{\mathrm{v}}$ is asymptotically stable if there exist matrices $P^{\mathrm{h}} \in \mathbb{S}_{2p}^{+}$, $P^{\mathrm{v}} \in \mathbb{S}_{2q}^{+}$, $Q^{\mathrm{h}}, R^{\mathrm{h}} \in \mathbb{S}_{p}^{+}$, $Q^{\mathrm{v}}, R^{\mathrm{v}} \in \mathbb{S}_{q}^{+}$, such that the following LMI holds:

$$\Xi_1 + \Xi_2 + e_1^{\mathrm{T}} Q e_1 - e_2^{\mathrm{T}} Q e_2 + \hat{\mathcal{A}}^{\mathrm{T}} \hat{R} \hat{\mathcal{A}} - \hat{\Gamma}^{\mathrm{T}} \hat{\Phi} \hat{\Gamma} < 0 \tag{36}$$

where $e_i \in \mathbb{R}^{n \times 3n}$ $(i = 1, 2, 3)$ denote the block-row vectors of the identity matrix I_{3n}, $\hat{\mathcal{A}} = (A - I_n)e_1 + A_d e_2$, $Q = \mathrm{diag}\{Q^{\mathrm{h}}, Q^{\mathrm{v}}\}$, $R = \mathrm{diag}\{R^{\mathrm{h}}, R^{\mathrm{v}}\}$, $\hat{R} = \mathrm{diag}\{d_{\mathrm{h}}^2 R^{\mathrm{h}}, d_{\mathrm{v}}^2 R^{\mathrm{v}}\}$ and (see equation below)

Remark 8: In the proof of Theorem 1, we utilise our new 2D summation inequalities to estimate the difference of the LKF candidate (16). Meanwhile, the reciprocally convex combination technique is also utilised to consider the relationship between the time-varying delays and their intervals as presented in (30) and (31). Moreover, unlike the method of [47, 48], where free-weighting matrices are introduced to improve the feasibility of LMI conditions, all the matrix variables in our stability conditions are directly derived from the LKF candidate and the reciprocally convex combination estimate. Thus, this approach can reduce the conservativeness and the computational burden of the derived stability conditions.

Remark 9: Another commonly used state-space model for 2D systems is the so-called Fornasini–Marchesini local state-space model (FMLSS) which can be presented in the following extended form with delays [47]:

$$\begin{aligned} x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) + A_{1d} x(i \\ &- d_1(i), j+1) + A_{2d} x(i+1, j - d_2(j)) \end{aligned} \tag{37}$$

It should be pointed out that the Roesser model (1) and FMLSS model (37) are not completely independent, i.e. in some special cases models (1) and (37) can be mutually recasted. Specifically, if $A_{d12} = 0_{p \times q}$ and $A_{d21} = 0_{q \times p}$ in (1) then, by using the state vector $x(i, j) = [x^{\mathrm{hT}}(i, j) \quad x^{\mathrm{vT}}(i, j)]^{\mathrm{T}}$, system (1) is converted to the FMLSS model (37), where

$$\begin{aligned} &A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} A_{d11} & 0 \\ 0 & 0 \end{bmatrix}, \\ &A_{d2} = \begin{bmatrix} 0 & 0 \\ 0 & A_{d22} \end{bmatrix} \end{aligned} \tag{38}$$

Conversely, if the matrices A_1, A_2, A_{1d} and A_{2d} are in the form of (38) then the state transformation $x^{\mathrm{h}}(i, j) = x(i, j+1) - A_2 x(i, j)$, $x^{\mathrm{v}}(i, j) = x(i, j)$ converts (37) into (1). In this case, the asymptotic stability of (37) can be determined by the proposed conditions of Theorem (1). However, unlike the case of delay-free systems, we cannot reduce the general form of (37) to (1) and vice versa.

5 Numerical examples

Example 1: In this example, we apply the obtained results to a control system of the thermal process in chemical reactors, heat exchanger or pipe furnaces shown in Fig. 1, which is described by the following partial differential equation [4]:

$$\begin{cases} \dfrac{\partial T(x, t)}{\partial x} + \dfrac{\partial T(x, t)}{\partial t} = -a_0 T(x, t) - a_1 T(x - d_x, t) + b u(x, t) \\ y(x, t) = c T(x, t) \end{cases} \tag{39}$$

where $T(x, t)$ is usually the temperature at $x \in [0, x_f]$ (space) and $t \in [0, \infty)$ (time), $u(x, t)$ is the control input, $y(x, t)$ is the measured output, d_x is the space delay and a_0, a_1, b, c are real constants.
Taking (see equation below) then (39) can be written in the form

$$\begin{aligned} T(i, j+1) = {} & \Big(1 - \frac{\Delta t}{\Delta x} - a_0 \Delta t\Big) T(i, j) + \frac{\Delta t}{\Delta x} T(i-1, j) \\ & - a_1 \Delta t T(i - 1 - d_1, j) + b \Delta t u(i, j) \end{aligned} \tag{40}$$

where $d_1 = \lfloor d_x / \Delta x \rfloor$ and $\lfloor . \rfloor$ denotes the floor function.
As discussed in the literature, e.g. in [56], due to practical reasons such as the finite speed of the data processing through a low-rate communication channel or sensor technology, time-delay associated with the output naturally arises in a variety of engineering applications to which we must consider the impact of time delay. In the model given in Fig. 1, we assume the communication delay occurs in the signal processing is τ_{sc}. Generally, this delay is unknown or time-varying. Thus, a static delayed output feedback for system (40) is designed in the form $u(i, j) = k y(i, j - d_2(j)) = k c T(i, j - d_2(j))$. Let $x^{\mathrm{h}}(i, j) = T(i-1, j)$, $x^{\mathrm{v}}(i, j) = T(i, j)$ then the closed-loop system of (40) is converted to system (1) where

$$\begin{aligned} \hat{V}(i, j) = {} & \begin{bmatrix} \hat{x}^{\mathrm{h}}(i, j) \\ \hat{x}^{\mathrm{v}}(i, j) \end{bmatrix}^{\mathrm{T}} P \begin{bmatrix} \hat{x}^{\mathrm{h}}(i, j) \\ \hat{x}^{\mathrm{v}}(i, j) \end{bmatrix} + \sum_{k=i-d_{\mathrm{h}}}^{i-1} x^{\mathrm{hT}}(k, j) Q^{\mathrm{h}} x^{\mathrm{h}}(k, j) + \sum_{l=j-d_{\mathrm{v}}}^{j-1} x^{\mathrm{vT}}(i, l) Q^{\mathrm{v}} x^{\mathrm{v}}(i, l) \\ & + d_{\mathrm{h}} \sum_{k=-d_{\mathrm{h}}}^{-1} \sum_{l=i+k}^{i-1} \eta_1^{\mathrm{T}}(l, j) R^{\mathrm{h}} \eta_1(l, j) + d_{\mathrm{v}} \sum_{k=-d_{\mathrm{v}}}^{-1} \sum_{l=j+k}^{j-1} \eta_2^{\mathrm{T}}(i, l) R^{\mathrm{v}} \eta_2(i, l) \end{aligned} \tag{35}$$

$$\begin{aligned} \Xi_1 &= \begin{bmatrix} \hat{\mathcal{A}} + e_1 \\ (1 + d_{\mathrm{h}}) e_3 - e_2 \end{bmatrix}^{\mathrm{T}} \hat{P}^{\mathrm{h}} \begin{bmatrix} \hat{\mathcal{A}} + e_1 \\ (1 + d_{\mathrm{h}}) e_3 - e_2 \end{bmatrix} - \begin{bmatrix} e_1 \\ (1 + d_{\mathrm{h}}) e_3 - e_1 \end{bmatrix}^{\mathrm{T}} \hat{P}^{\mathrm{h}} \begin{bmatrix} e_1 \\ (1 + d_{\mathrm{h}}) e_3 - e_1 \end{bmatrix} \\ \Xi_2 &= \begin{bmatrix} \hat{\mathcal{A}} + e_1 \\ (1 + d_{\mathrm{v}}) e_3 - e_2 \end{bmatrix}^{\mathrm{T}} \hat{P}^{\mathrm{v}} \begin{bmatrix} \hat{\mathcal{A}} + e_1 \\ (1 + d_{\mathrm{v}}) e_3 - e_2 \end{bmatrix} - \begin{bmatrix} e_1 \\ (1 + d_{\mathrm{v}}) e_3 - e_1 \end{bmatrix}^{\mathrm{T}} \hat{P}^{\mathrm{v}} \begin{bmatrix} e_1 \\ (1 + d_{\mathrm{v}}) e_3 - e_1 \end{bmatrix} \\ \hat{\Gamma} &= \big[(e_1 - e_2)^{\mathrm{T}} \quad (e_1 + e_2 - 2e_3)^{\mathrm{T}}\big]^{\mathrm{T}}, \quad \hat{\Phi} = \mathrm{diag}\Big\{R, \frac{3(d_{\mathrm{h}} + 1)}{d_{\mathrm{h}} - 1} R^{\mathrm{h}}, \frac{3(d_{\mathrm{v}} + 1)}{d_{\mathrm{v}} - 1} R^{\mathrm{v}}\Big\} \\ \hat{P}^{\mathrm{h}} &= J_p^{2T} P^{\mathrm{h}} J_p^2, \quad \hat{P}^{\mathrm{v}} = J_q^{2T} P^{\mathrm{v}} J_q^2 \end{aligned}$$

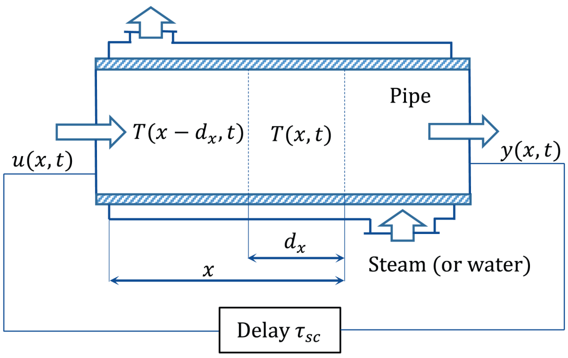

Fig. 1 *Heat exchanger control*

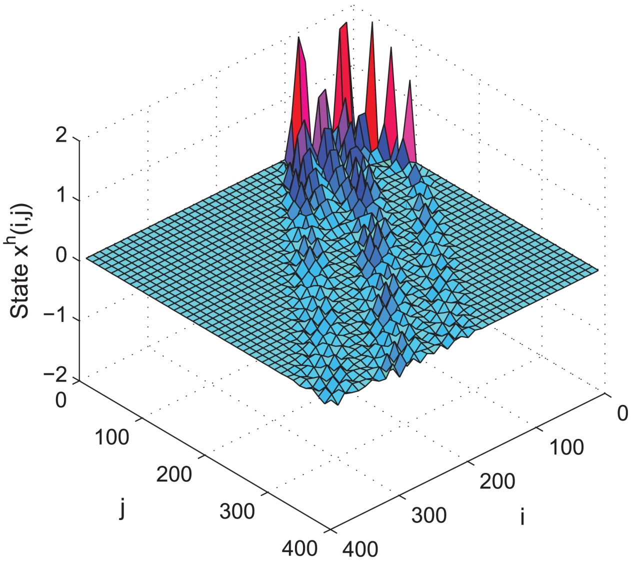

Fig. 2 *State x^h of the open-loop system*

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{\Delta t}{\Delta x} & 1 - \dfrac{\Delta t}{\Delta x} - a_0 \Delta t \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ -a_1 \Delta t & bkc\Delta t \end{bmatrix}$$

Let $\Delta t = 0.1$, $\Delta x = 0.2$, $a_0 = 1, a_1 = 1.25$. For $d_1 = 6$, the open-loop system of (40) is unstable. Simulation results given in Figs. 2 and 3 are taken with finite support initial conditions $x^h(i,j) = 2\sin(0.1j)$, $i \in \mathbb{Z}[-6,0], j \le 100$, $x^v(i,0) = 2\cos(0.1i)$, $i \le 100$, and $x^h(0,j) = 0, x^v(i,0) = 0$ elsewhere. It can be seen that both state trajectories $x^h(i,j)$ and $x^v(i,j)$ are divergent as $i + j \to \infty$. Now, for illustrative purpose, we let $b = 1$, $c = 0.4$ and $k = -0.5$. The time-varying delay $d_2(j) \in [2,4]$. By Matlab LMI toolbox, it is found that the derived stability conditions of Theorem 1 are satisfied with (see equation below) By Theorem 1, the closed-loop system of (40) is asymptotically stable. The state trajectories x^h, x^v of the closed-loop system with $d_2(j) = 2 + 2|\sin(j\pi/4)|$ are presented in Figs. 4 and 5.

For constant delay d_2, by the result of Theorem 1 in [39] (with $\alpha = 1$) and that of Corollary 2 in this paper, the closed system of (40) is asymptotically stable with delays d_1, d_2 in the domains $\mathcal{S}_1 = \{(1,l),(2,l),(3,3):\ l \in \mathbb{N}\}$ and $\mathcal{S}_2 = \{(k,l),(l,k):\ k,l \in \mathbb{N}, k \le 5\}$, respectively. Clearly, our conditions give better results than the ones proposed in [39]. In addition, the number of LMI decision variables (NoDv) of Corollary 2 and of Theorem 1 in [39] are $2(p^2+q^2) + 3(p+q)$ and $5.5(p^2+q^2) + 2.5(p+q)$, respectively. This shows the effectiveness of our method in reducing the conservativeness of the derived stability conditions.

Example 2: Consider a 2D system in the form of (1), where

$$A = \left[\begin{array}{cc:c} 0.8 & -0.6 & 0.12 \\ 0.15 & 0.1 & 0 \\ \hdashline 0.15 & 0.1 & 0.25 \end{array}\right], \quad A_d = \left[\begin{array}{cc:c} 0.15 & -0.1 & 0 \\ 0.12 & -0.2 & -0.1 \\ \hdashline 0.05 & -0.06 & 0.11 \end{array}\right]$$

When the delays $d_h(i)$ and $d_v(j)$ are time-varying and belonging to intervals $[d_1^h, d_2^h]$, $[d_1^v, d_2^v]$, where $d_1^h < d_2^h$ and $d_1^v < d_2^v$, the results of [39] are not applicable. Alternatively, Theorem 1 in [53] and our result of Theorem 1 in this paper can be applied to this case. For illustrative purpose, we let $d_1^h = 2, d_1^v = 2$, the upper bound of d_h for various d_v obtained by Theorem 1 in this paper and Theorem 1 in [53] is presented in Table 1. Clearly, our method leads to better results than the one based on the 2D Jensen-type inequalities [53]. Fig. 6 depicts the norm-trajectory $\| [x^h(i,j), x^v(i,j)] \|$ of the system with time-varying delays $d_h(i) = 2 + 8|\sin(i\pi/2)|$ and $d^v(j) = 2 + 4|\cos(j\pi/2)|$ which validates our theoretical results.

6 Concluding remarks

In this paper, new 2D finite-sum inequalities have been proposed. By constructing an improved LKF and utilising the newly derived inequalities, delay-dependent LMI-based stability conditions have been derived for a class of discrete-time 2D Roesser systems with interval time-varying delays. At the same time, the technique of reciprocally convex combination has also been utilised to consider the relationship between the time-varying delays and their intervals, which is effective to reduce the conservatism of the derived stability conditions. Two numerical examples have been provided to illustrate the effectiveness and the merits of the proposed method.

The approach in this paper can also be effectively applied to various problems in the field of stability analysis and control of discrete 2D time-delay systems (see, e.g. Chapter 14 in [22]) such as H_∞ performance analysis [19, 39, 52], state/output feedback controller design [18, 20, 47], filtering design [10, 22] or state estimation [43]. In particular, the proposed method in this paper can be extended to general discrete time-delay 2D switched systems [21, 53], which comprise a number of subsystems with a deterministic (average dwell time) or stochastic (Markovian jump

$$T(i,j) = T(i\Delta x, j\Delta t), \quad u(i,j) = u(i\Delta x, j\Delta t)$$
$$\frac{\partial T(x,t)}{\partial x} \simeq \frac{T(i,j) - T(i-1,j)}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \simeq \frac{T(i,j+1) - T(i,j)}{\Delta t}$$

$$P^h = \begin{bmatrix} 13.1863 & -0.0956 & 0.0891 \\ -0.0956 & 0.0014 & -0.0149 \\ 0.0891 & -0.0149 & 4.7744 \end{bmatrix}, \quad P^v = \begin{bmatrix} 22.2257 & -0.6020 & 0.0161 \\ -0.6020 & 0.1801 & -0.0654 \\ 0.0161 & -0.0654 & 0.0383 \end{bmatrix}$$
$$Q_1 = \begin{bmatrix} 2.4925 & 0 \\ 0 & 0.0379 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1.4253 & 0 \\ 0 & 0.0086 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 7.0 & 0 \\ 0 & 9.0 \end{bmatrix}$$
$$R_2 = \begin{bmatrix} 8.0 & 0 \\ 0 & 10.0 \end{bmatrix}, \quad X = \begin{bmatrix} -20.9594 & 0.0877 \\ -6.0660 & -13.9732 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.1090 & 0.0167 \\ -0.0260 & 0.0318 \end{bmatrix}$$

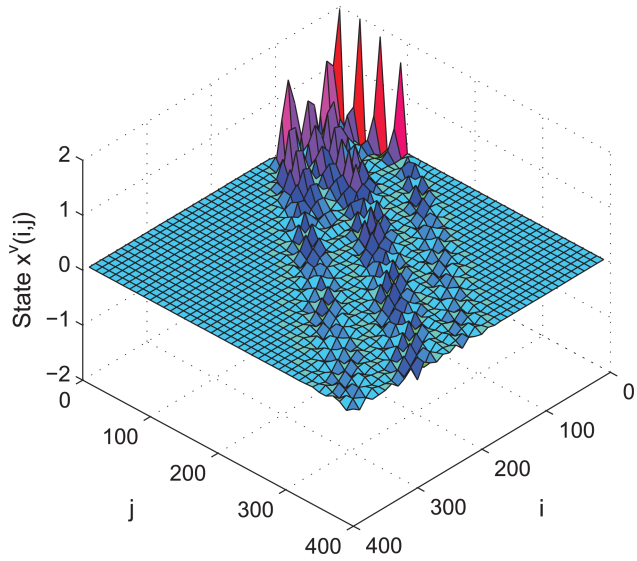

Fig. 3 *State x^v of the open-loop system*

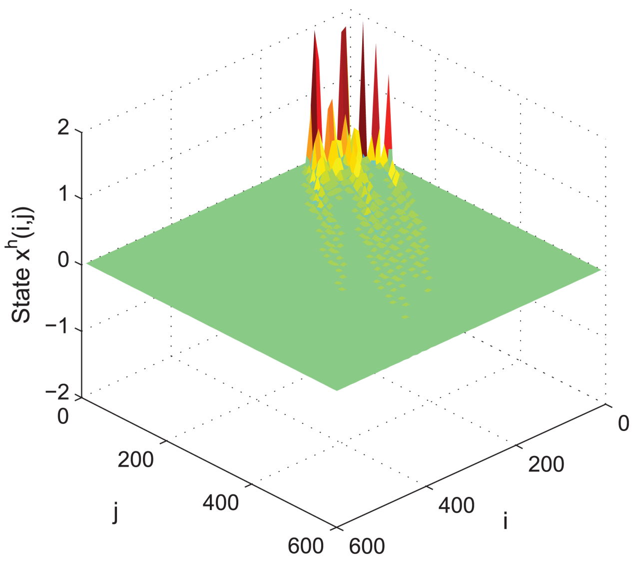

Fig. 4 *State x^h of the closed-loop system*

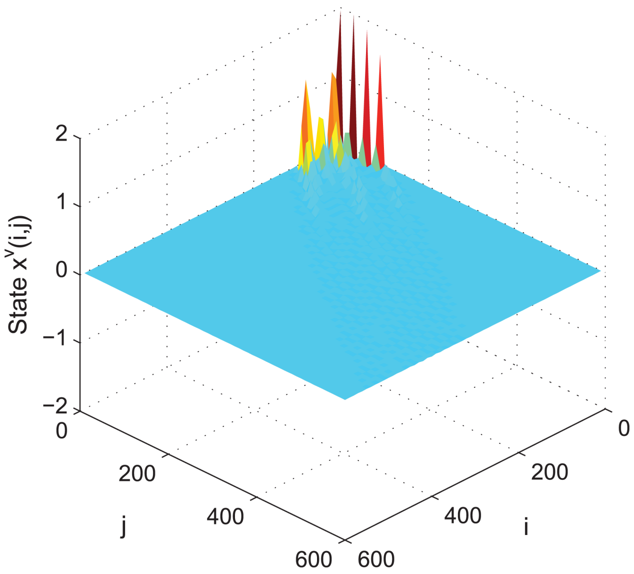

Fig. 5 *State x^v of the closed-loop system*

process) switching rule among them. However, specific applications to such problems especially for useful classes of

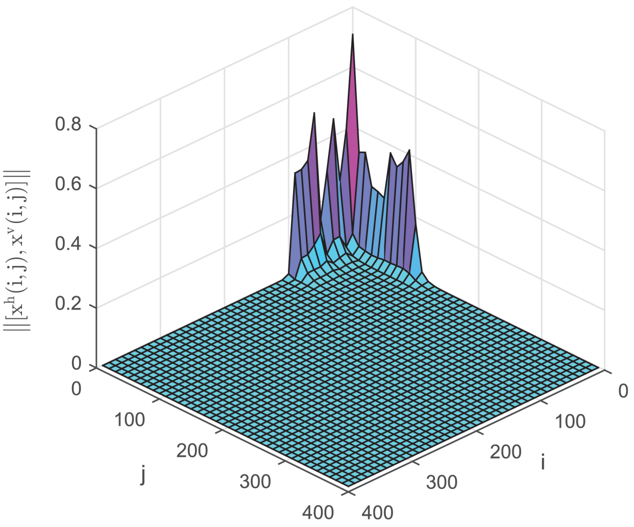

Fig. 6 *State $\| [x^h, x^v] \|$ of the system in Example 2*

Table 1 Upper bound of d_h for various d_v

d_2^v	3	5	7	9	10
Huang and Xiang [53]	7	6	5	4	3
Theorem 1	10	10	9	8	6

realistic 2D systems require further investigations. This will be addressed in the future works.

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