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Sliding mode control of two-dimensional systems in Roesser model

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Abstract: The study is concerned with the problem of sliding mode control of two-dimensional (2D) discrete systems. Given a 2D system in Roesser model, attention is focused on the design of sliding mode controllers, which guarantee the resultant closed-loop systems to be asymptotically stable. This problem is solved by using two different methods: model transformation method and Choi's 1997 method. In terms of linear matrix inequality, sufficient conditions are formulated for the existence of linear switching surfaces guaranteeing asymptotic stability of the reduced-order equivalent sliding mode dynamics. Based on this, the problem of controller synthesis is investigated, with two different controller design procedures proposed, which can be easily implemented by using standard numerical software. A numerical example is provided to illustrate the effectiveness of the proposed controller design methods.

1 Introduction

Since its appearance in the 1950s, sliding mode control (SMC) has been proven to be an effective robust control strategy for incompletely modelled or uncertain systems. In the past two decades, SMC has been successfully applied to a wide variety of practical engineering systems such as robot manipulators, aircrafts, underwater vehicles, spacecrafts, flexible space structures, electrical motors, power systems and automotive engines [1]. Generally, SMC utilises a discontinuous control to force the system state trajectories to some pre-defined switching surfaces on which the system has desired properties such as stability, disturbance rejection capability and tracking ability. Many important results have been reported for this kind of control strategy. To mention a few, SMC has been investigated for uncertain systems [2, 3], time-delay systems [4, 5], stochastic systems [6, 7], Markovian jump systems [8] and discrete-time systems [9, 10]. In addition, sliding mode output feedback control has also been studied in [11, 12]. It is worth noting, however, that all the aforementioned results are concerned with one-dimensional (1D) systems, and to the best of the authors' knowledge, little effort has been made towards the problem of sliding model control for two-dimensional (2D) or n-dimensional (nD) systems.

Two-dimensional system model represents a wide range of practical systems, such as those in image data processing and transmission, thermal processes, gas absorption and water stream heating etc. [13, 14]. Therefore in recent years, 2D discrete systems received much attention, and many important results are easily available in the literature. To mention a few, the stability of 2D systems was investigated in [15, 16], the controller and filter designs were studied in [17, 18], and the model reduction problem for 2-D systems was addressed in [19, 20].

In this paper, we further extend the results obtained for 1D systems, to investigate the problem of SMC for 2D discrete-time systems. Given a 2D system in Roesser model, attention is focused on the design of sliding mode controllers, which guarantee the resultant closed-loop systems to be asymptotically stable. This problem is solved by using two different methods: model transformation method and Choi's 1997 method. In terms of linear matrix inequality (LMI), sufficient conditions are formulated for the existence of linear switching surfaces guaranteeing asymptotic stability of the reduced-order equivalent sliding mode dynamics. Based on this, the problem of controller synthesis is investigated, with two different controller design procedures proposed, which can be easily implemented by using standard numerical software. A numerical example is provided to illustrate the effectiveness of the proposed controller design methods. The contribution of the paper is summarised as follows:

- *SMC of discrete-time systems by using LMI approach.* Since the work of Choi in [21], SMC design utilising LMI has experienced a major development, and many important results have been reported. For example, LMI-based SMC design methods for a class of uncertain continuous-time systems either with or without time-delay were studied in [1, 3, 22]; Output feedback SMC design was investigated in [23]. However, all the above results are obtained for continuous-time systems and to the best of the authors' knowledge, few LMI-based results have been reported for the SMC of discrete-time systems. This paper contributes to the development of LMI-based SMC in discrete time, in particular 2D discrete-time systems in Roesser model.

- *Two different methods for SMC design.* This paper proposes two different methods of SMC design for 2D discrete systems. The first method is called model transformation method, which has been used in [3] for 1D continuous-time systems. This method needs to perform a model transformation to the original system such that the transformed model has a so called 'regular' form by introducing a nonsingular transformation matrix. The second method is called Choi's method, which was first proposed by Choi in [21] and has been further developed in [1, 4]. It is worth noting that none of the above two methods has not been applied to 2D systems, and it is our purpose to provide a systematic framework for LMI-based SMC of discrete-time 2D systems, by utilising the above two SMC methods.

- *Extension and improvement of the reaching law method for 2D discrete-time systems.* Since the work of Gao and Hung in [24], SMC design via reaching law method has received much attention. The reaching law is a differential equation which specifies the dynamics of a switching function. In addition, by appropriately choosing the parameters in the differential equation, the dynamics quality of SMC systems in the reaching phase can be controlled. Extension of the reaching law method to discrete-time system was made in [25]. However, some problems still remain unsolved. For example, it cannot guarantee the system state trajectories eventually converge to origin; serious chattering exist along the switching surface. In this paper, we further extend this method to 2D discrete-time systems and more importantly, an improved reaching law for 2D discrete-time systems is designed, which guarantees the system state trajectories converge to origin at last.

The remainder of this paper is organised as follows. Section 2 gives the problem formulation and some necessary assumptions. Section 3 develops and improves the reaching law approach for 2D discrete-time systems. Sections 4 and 5 present our main results of SMC designs by using the aforementioned two methods, respectively. Section 6 provides an illustrative example and we conclude this paper in Section 7.

Notations. The notation used throughout the paper is standard. The superscript 'T' stands for matrix transposition; $\mathbb{R}^n$ denotes the nD Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$ and the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrix; $|\cdot|$ refers to the Euclidean vector norm; $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalues of a real symmetric matrix, respectively. In symmetric block matrices or long matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry and diag $(\ldots)$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2 Problem formulation

Consider the following 2D discrete-time system described in a state-space model

$$\mathcal{S}:\begin{cases} x^h(i+1,j) = \boldsymbol{A}_1 x^h(i,j) + \boldsymbol{A}_2 x^v(i,j) + \boldsymbol{B}_1 u^h(i,j) \\ x^v(i,j+1) = \boldsymbol{A}_3 x^h(i,j) + \boldsymbol{A}_4 x^v(i,j) + \boldsymbol{B}_2 u^v(i,j) \end{cases} \tag{1}$$

where $x^h(i,j) \in \mathbb{R}^{n_1}$, $x^v(i,j) \in \mathbb{R}^{n_2}$ represent the horizontal and vertical states, respectively; $u^h(i,j) \in \mathbb{R}^m$ and $u^h(i,j) \in \mathbb{R}^m$ are the control inputs. $\boldsymbol{A}_1 \in \mathbb{R}^{n_1\times n_1}$, $\boldsymbol{A}_2 \in \mathbb{R}^{n_1\times n_2}$, $\boldsymbol{A}_3 \in \mathbb{R}^{n_2\times n_1}$, $\boldsymbol{A}_4 \in \mathbb{R}^{n_2\times n_2}$, $\boldsymbol{B}_1 \in \mathbb{R}^{n_1\times m}$ and $\boldsymbol{B}_2 \in \mathbb{R}^{n_2\times m}$ are real valued system matrices.

Throughout the paper, we denote the system state as $x(i,j) = [x^{hT}(i,j)\ x^{vT}(i,j)]^T$. The boundary condition X_0 is defined as follows

$$X_0 \triangleq \begin{bmatrix} x^{hT}(0,0) & x^{hT}(0,1) & x^{hT}(0,2) & \cdots \\ x^{vT}(0,0) & x^{vT}(1,0) & x^{vT}(2,0) & \cdots \end{bmatrix}^T$$

Then, we make the following assumption on the boundary condition.

Assumption 1: The boundary condition is assumed to satisfy

$$\lim_{N\to\infty} \sum_{k=0}^{N} (|x^h(0,k)|^2 + |x^v(k,0)|^2) < \infty \tag{2}$$

Before presenting the main objective of this paper, we first introduce the following definitions for the 2D discrete-time system ($\mathcal{S}$) in (1), which will be essential for our derivation.

Definition 1: The 2D discrete-time system ($\mathcal{S}$) in (1) with $u(i, j) = 0$ is said to be asymptotically stable if

$$\lim_{i+j\to\infty} |x(i, j)|^2 = 0$$

for every boundary condition X_0 satisfying Assumption 1.

Our goal is to control the above 2D system in two steps. First, we design a linear switching surface as a function of the system states so that the system restricted to the switching surface has desirable properties, such as stability and tracking capability. The second step is to design a suitable relay-type controller to globally drive the system state trajectories to the predefined switching surface and maintain it there for all subsequent time.

3 Extension of reaching law approach

In this section, we further extend the reaching law method of designing SMC for 1D systems, to investigate the application of SMC to 2D systems. In 1995, Gao *et al.* in [25] gave the complete definition and detailed physical explanation for the quasi-sliding mode in discrete-time 1D system. They pointed out that the reaching condition should satisfy six conditions and gave a reaching condition in the form of an equation, that is, the exponential approximation law, rewritten as

$$s(k+1) = (1 - qT)s(k) - \varepsilon T\ \mathrm{sign}(s(k)),\ 1 - qT > 0 \tag{3}$$

where T, $\varepsilon > 0$, $q > 0$ are the sampling time, reaching rate and approximation rate, respectively. The approximation law described the movement mechanism of the discrete-time SMC system and overcame the shortcomings of those reaching conditions in the form of inequalities.

With the same principle as (3), we design the following reaching law, also named as exponential approximation law for 2D discrete-time systems based on the Roesser model

$$\begin{bmatrix} s^h(i+1, j) \\ s^v(i, j+1) \end{bmatrix} = \begin{bmatrix} 1 - q_1 T & 0 \\ 0 & 1 - q_2 T \end{bmatrix} \begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} - \begin{bmatrix} \varepsilon_1 T & 0 \\ 0 & \varepsilon_2 T \end{bmatrix} \begin{bmatrix} \mathrm{sign}(s^h(i, j)) \\ \mathrm{sign}(s^v(i, j)) \end{bmatrix} \tag{4}$$

where T is the sampling time, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $q_1 > 0$, $q_2 > 0$ are the reaching rates and approximation rates along the horizontal and vertical directions, respectively. A desirable reaching mode response can be achieved by a judicious choice of parameters ε_1, ε_2, q_1 and q_2.

Before proceeding further, we give the following definitions.

Definition 2: We call the quasi-sliding mode of a discrete SMC 2D system in the Δ_1 and Δ_2 vicinities of the sliding surfaces $s_h(i, j) = 0$ and $s_v(i, j) = 0$, respectively, such a motion of the system that

$$|s^h(i, j)| \leq \Delta_1, \quad |s^v(i, j)| \leq \Delta_2$$

where Δ_1 and Δ_2 are called the quasi-sliding mode band widthes in the horizontal and vertical directions, respectively.

Definition 3: The quasi-sliding mode becomes an ideal quasi-sliding mode when $\Delta = 0$.

Now, we will analyse the band of quasi-sliding mode for 2-D systems with the reaching law described by (4). According to [25], we can obtain

$$\Delta_1 \triangleq \frac{\varepsilon_1 T}{1 - q_1 T}, \quad \Delta_2 \triangleq \frac{\varepsilon_2 T}{1 - q_2 T}$$

We can see from the above analysis that the values of $s^h(i, j)$ and $s^v(i, j)$ arbitrarily approach to $(\varepsilon_1 T)/(1 - q_1 T)$ and $(\varepsilon_2 T)/(1 - q_2 T)$, respectively. Once $s^h(i, j) \triangleq (\varepsilon_1 T)/(1 - q_1 T)$ and $s^v(i, j) \triangleq (\varepsilon_2 T)/(1 - q_2 T)$ hold, the system state trajectories enter two equiamplitude vibration movements. On the other hand, the values of $s^h(i, j)$ and $s^v(i, j)$ are determined by T, ε_1, ε_2, q_1 and q_2. $s^h(i, j) \to 0$ holds only if $\varepsilon_1 T \to 0$ and $s^v(i, j) \to 0$ holds only if $\varepsilon_2 T \to 0$. Therefore the quasi-sliding mode becomes an ideal quasi-sliding mode, that is $\Delta = 0$, only if both $\varepsilon_1 T \to 0$ and $\varepsilon_2 T \to 0$ hold. However, $\varepsilon_1 T$ and $\varepsilon_2 T$ are both nonzero, so the system state trajectories will never converge to origin.

To overcome this shortcoming, we make some improvement on (4), and a new reaching law is presented as

$$\begin{bmatrix} s^h(i+1, j) \\ s^v(i, j+1) \end{bmatrix} = \begin{bmatrix} 1 - q_1 T & 0 \\ 0 & 1 - q_2 T \end{bmatrix} \begin{bmatrix} s^h(i, j) \\ s^v(i, j) \end{bmatrix} - \begin{bmatrix} T(s^h(i, j))^2 \mathrm{sign}(s^h(i, j)) \\ T(s^v(i, j))^2 \mathrm{sign}(s^v(i, j)) \end{bmatrix} \tag{5}$$

We give the following result for the newly defined sliding surface function in

Theorem 1: The reaching law (5) can guarantee the system state trajectories eventually converge to origin.

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Proof: For simplicity, we only analyse the reaching course of the system state along the horizontal direction, and the vertical direction case can be shown in a similar way.

From the first equation of (5), we have

$$\begin{aligned} s^h(i+1,j) &= (1-q_1T)s^h(i,j) - T(s^h(i,j))^2\text{sign}(s^h(i,j)) \\ &= \left(1 - q_1T - \frac{T(s^h(i,j))^2}{\|s^h(i,j)\|}\right)s^h(i,j) \\ &\triangleq \gamma s^h(i,j) \end{aligned}$$

Obviously, only when $\|\gamma\| < 1$ (that is, $\|s^h(i,j)\| > T(s^h(i,j))^2/(2-q_1T)$, $\|s^h(i,j)\|$ is convergent and $\|s^h(i,j)\| = T(s^h(i,\ j)^2/(2-q_1T)$ is its lower bound. If $\|s^h(i,\ j)\| = T(s^h(i,j)^2/(2-q_1T)$, then $\gamma = -1$, that is, $s^h(\mathrm{i}+1,j) = -s^h(i,j)$, and the system state trajectories enter the equiamplitude vibration movement. Different from (4), $(s^h(i,j)^2$ is not a constant, but a function, hence the quasi-sliding mode band becomes small with $s^h(i,j) \to 0$. Therefore (5) can ensure the system trajectories eventually converge to origin. This completes the proof.

4 SMC design using model transformation method

4.1 Model transformation

The system ($\mathcal{S}$) in (1) can also be described as

$$\mathcal{P}:\quad \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix}\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_1 \\ 0_{n_2\times m} \end{bmatrix}u^h(i,j) + \begin{bmatrix} 0_{n_1\times m} \\ \boldsymbol{B}_2 \end{bmatrix}u^v(i,j) \tag{6}$$

We make the following assumptions.

Assumption 2: The matrix pairs ($\boldsymbol{A}_1$, $\boldsymbol{B}_1$) and ($\boldsymbol{A}_4$, $\boldsymbol{B}_2$) are stabilisable.

Assumption 3: Matrices $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ are of full column rank and satisfy $n_1 > m$, $n_2 > m$.

According to Assumptions 2 and 3, we know that there exist nonsingular matrices T_1 and T_2 such that

$$\boldsymbol{T}_1\boldsymbol{B}_1 = \begin{bmatrix} 0_{(n_1-m)\times m} \\ \bar{\boldsymbol{B}}_1 \end{bmatrix}, \quad \boldsymbol{T}_2\boldsymbol{B}_2 = \begin{bmatrix} 0_{(n_2-m)\times m} \\ \bar{\boldsymbol{B}}_2 \end{bmatrix}$$

where $\bar{\boldsymbol{B}}_1 \in \mathbb{R}^{m\times m}$, $\bar{\boldsymbol{B}}_2 \in \mathbb{R}^{m\times m}$ are nonsingular matrices. For convenience, let us choose

$$\boldsymbol{T}_1 \triangleq \begin{bmatrix} \boldsymbol{U}_2^{\mathrm{T}} \\ \boldsymbol{U}_1^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{T}_2 \triangleq \begin{bmatrix} \boldsymbol{V}_2^{\mathrm{T}} \\ \boldsymbol{V}_1^{\mathrm{T}} \end{bmatrix}$$

where $\boldsymbol{U}_1 \in \mathbb{R}^{n_1\times m}$, $\boldsymbol{U}_2 \in \mathbb{R}^{n_1\times(n_1-m)}$ and $\boldsymbol{V}_1 \in \mathbb{R}^{n_2\times m}$ $\boldsymbol{V}_2 \in \mathbb{R}^{n_2\times(n_2-m)}$ are two sub-blocks of a unitary matrix resulting from the singular value decomposition of $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, respectively, that is,

$$\boldsymbol{B}_1 = [\,\boldsymbol{U}_1 \quad \boldsymbol{U}_2\,]\begin{bmatrix} \Sigma \\ 0_{(n_1-m)\times m} \end{bmatrix}\boldsymbol{W}_1^{\mathrm{T}},$$

$$\boldsymbol{B}_2 = [\,\boldsymbol{V}_1 \quad \boldsymbol{V}_2\,]\begin{bmatrix} \Pi \\ 0_{(n_2-m)\times m} \end{bmatrix}\boldsymbol{W}_2^{\mathrm{T}}$$

where $\Sigma \in \mathbb{R}^{m\times m}$, $\Pi \in \mathbb{R}^{m\times m}$ are diagonal positive-definite matrices and $\boldsymbol{W}_1 \in \mathbb{R}^{m\times m}$, $\boldsymbol{W}_2 \in \mathbb{R}^{m\times m}$ are unitary matrices. By the state transformation $z(i,j) = \text{diag}\,(T_1,\, T_2)\, x(i,j)$, the system ($\mathcal{P}$) in (6) has the following 'regular' form:

$$\begin{bmatrix} z^h(i+1,j) \\ z^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{A}_1\boldsymbol{T}_1^{-1} & \boldsymbol{T}_1\boldsymbol{A}_2\boldsymbol{T}_2^{-1} \\ \boldsymbol{T}_2\boldsymbol{A}_3\boldsymbol{T}_1^{-1} & \boldsymbol{T}_2\boldsymbol{A}_4\boldsymbol{T}_2^{-1} \end{bmatrix} \times \begin{bmatrix} z^h(i,j) \\ z^v(i,j) \end{bmatrix} + \begin{bmatrix} \boldsymbol{T}_1\boldsymbol{B}_1 \\ 0_{n_2\times m} \end{bmatrix}u^h(i,j) + \begin{bmatrix} 0_{n_1\times m} \\ \boldsymbol{T}_2\boldsymbol{B}_2 \end{bmatrix}u^v(i,j)$$

which can be rewritten as

$$\mathcal{T}:\quad \begin{bmatrix} z_1^h(i+1,j) \\ z_2^h(i+1,j) \\ z_1^v(i,j+1) \\ z_2^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{A}}_{111} & \bar{\boldsymbol{A}}_{112} & \bar{\boldsymbol{A}}_{211} & \bar{\boldsymbol{A}}_{212} \\ \bar{\boldsymbol{A}}_{121} & \bar{\boldsymbol{A}}_{122} & \bar{\boldsymbol{A}}_{221} & \bar{\boldsymbol{A}}_{222} \\ \bar{\boldsymbol{A}}_{311} & \bar{\boldsymbol{A}}_{312} & \bar{\boldsymbol{A}}_{411} & \bar{\boldsymbol{A}}_{412} \\ \bar{\boldsymbol{A}}_{321} & \bar{\boldsymbol{A}}_{322} & \bar{\boldsymbol{A}}_{421} & \bar{\boldsymbol{A}}_{422} \end{bmatrix} \times \begin{bmatrix} z_1^h(i,j) \\ z_2^h(i,j) \\ z_1^v(i,j) \\ z_2^v(i,j) \end{bmatrix} + \begin{bmatrix} 0_{(n_1-m)\times m} \\ \bar{\boldsymbol{B}}_1 \\ 0_{(n_2-m)\times m} \\ 0_{m\times m} \end{bmatrix}u^h(i,j) + \begin{bmatrix} 0_{(n_1-m)\times m} \\ 0_{m\times m} \\ 0_{(n_2-m)\times m} \\ \bar{\boldsymbol{B}}_2 \end{bmatrix}u^v(i,j) \tag{7}$$

where $z_1^h(i,j) \in \mathbb{R}^{n_1-m}$, $z_2^h(i,j) \in \mathbb{R}^m$, $z_1^v(i,j) \in \mathbb{R}^{n_2-m}$, $z_2^v(i,j) \in \mathbb{R}^m$, $\bar{\boldsymbol{A}}_{111} = \boldsymbol{U}_2^{\mathrm{T}} \quad \boldsymbol{A}_1\boldsymbol{U}_2$, $\bar{\boldsymbol{A}}_{112} = \boldsymbol{U}_2^{\mathrm{T}} \quad \boldsymbol{A}_1\boldsymbol{U}_1$,

$\bar{A}_{211} = U_2^T A_2 V_2$, $\bar{A}_{212} = U_2^T A_2 V_1$, $\bar{A}_{311} = V_2^T A_3 U_2$, $\bar{A}_{312} = V_2^T A_3 U_1$, $\bar{A}_{411} = V_2^T A_4 V_2$, $\bar{A}_{412} = V_2^T A_4 V_1$, $\bar{B}_1 = \Sigma W_1^T$, $\bar{B}_2 = \Pi W_2^T$.

4.2 Switching surface design

In this subsection, we design the following linear switching functions along the horizontal and vertical directions, respectively

$$\begin{aligned} s^h(i,j) &\triangleq \begin{bmatrix} C_1 & I \end{bmatrix} \begin{bmatrix} z_1^h(i,j) \\ z_2^h(i,j) \end{bmatrix}, \\ s^v(i,j) &\triangleq \begin{bmatrix} C_2 & I \end{bmatrix} \begin{bmatrix} z_1^v(i,j) \\ z_2^v(i,j) \end{bmatrix} \end{aligned} \tag{8}$$

where $C_1 \in \mathbb{R}^{m\times(n_1-m)}$, $C_2 \in \mathbb{R}^{m\times(n_2-m)}$ are real valued matrices to be defined.

When the system states are driven onto the switching surfaces, we have $s^h(i,j) = 0$ and $s^v(i,j) = 0$. Therefore by substituting $z_2^h(i,j) = -C_1 z_1^h(i,j)$ and $z_2^v(i,j) = -C_2 z_1^v(i,j)$ into the first and third equations in (7), we obtain the following reduced-order sliding mode dynamics

$$\begin{aligned} \mathcal{S}-1\text{:} \quad & \begin{bmatrix} z_1^h(i+1,j) \\ z_1^v(i,j+1) \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_{111} - \bar{A}_{112}C_1 & \bar{A}_{211} - \bar{A}_{212}C_2 \\ \bar{A}_{311} - \bar{A}_{312}C_1 & \bar{A}_{411} - \bar{A}_{412}C_2 \end{bmatrix} \begin{bmatrix} z_1^h(i,j) \\ z_1^v(i,j) \end{bmatrix} \\ &= \left(\begin{bmatrix} \bar{A}_{111} & \bar{A}_{211} \\ \bar{A}_{311} & \bar{A}_{411} \end{bmatrix} - \begin{bmatrix} \bar{A}_{112} & \bar{A}_{212} \\ \bar{A}_{312} & \bar{A}_{412} \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \right) \\ &\quad \times \begin{bmatrix} z_1^h(i,j) \\ z_1^v(i,j) \end{bmatrix} \\ &\triangleq (\tilde{A}_{11} - \tilde{A}_{12}C) \begin{bmatrix} z_1^h(i,j) \\ z_1^v(i,j) \end{bmatrix} \end{aligned} \tag{9}$$

Remark 1: It is worthing pointing out that the designed switching function in (8) is, in fact, a double-plane function along the horizontal and vertical directions, respectively. Therefore we say that the system states reach onto the switching surface means both the horizontal states and the vertical states reach onto the corresponding horizontal surface $s^h(i,j) = 0$ and vertical surface $s^v(i,j) = 0$, respectively, at last. This design method plays a key role in deriving the sliding mode dynamics equation of $(\mathcal{S}-1)$ in (9) and in synthesising sliding mode controllers subsequently.

4.3 Analysis of sliding mode dynamics

In this subsection, we analyse the stability of the resultant sliding mode dynamics $(\mathcal{S}-1)$ in (9). The following theorem gives a sufficient condition for the existence of the sliding mode dynamics in terms of LMI, and by solving this condition we can obtain the parameters of the switching functions in (8).

Theorem 2: The sliding mode dynamics $(\mathcal{S}-1)$ in (9) is asymptotically stable if there exist matrices $Y = \text{diag}(Y^h, Y^v) > 0$, $X = \text{diag}(X^h, X^v)$ such that the following LMI holds

$$\begin{bmatrix} -Y & (\tilde{A}_{11}Y - \tilde{A}_{12}X)^T \\ * & -Y \end{bmatrix} < 0 \tag{10}$$

Moreover, the switching function parameter C can be written as $C = XY^{-1}$, that is, $C_1 = X^h(Y^h)_{-1}$ and $C_2 = X^v(Y^v)^{-1}$.

Proof: From (10) and $C = XY^{-1}$ we can see that the sliding mode dynamics is asymptotically stable if there exist matrix $Y \triangleq \text{diag}(Y^h, Y^v) > 0$ satisfying

$$\begin{bmatrix} -Y & Y(\tilde{A}_{11} - \tilde{A}_{12}C)^T \\ * & -Y \end{bmatrix} < 0 \tag{11}$$

Define $P \triangleq \text{diag}(P^h, P^v) = Y^{-1}$. Then, by performing a congruence transformation to (11) by diag (Y^{-1}, I), (11) is equivalent to

$$\begin{bmatrix} -P & (\tilde{A}_{11} - \tilde{A}_{12}C)^T \\ * & -P^{-1} \end{bmatrix} < 0 \tag{12}$$

By Schur complement [26], (12) is equivalent to

$$\Upsilon \triangleq (\tilde{A}_{11} - \tilde{A}_{12}C)^T P (\tilde{A}_{11} - \tilde{A}_{12}C) - P < 0 \tag{13}$$

Now consider the following index

$$\begin{aligned} \mathcal{I} \triangleq\ & z^{hT}(i+1,j) P^h z^h(i+1,j) \\ & + z^{vT}(i,j+1) P^v z^v(i,j+1) \\ & - z^T(i,j) P z(i,j) \end{aligned} \tag{14}$$

Then, along the solution of the sliding mode dynamics $(\mathcal{S}-1)$ in (9), we have

$$\begin{aligned} \mathcal{I} &= z^T(i,j)(\tilde{A}_{11} - \tilde{A}_{12}C)^T P (\tilde{A}_{11} - \tilde{A}_{12}C) z(i,j) \\ &\quad - z^T(i,j) P z(i,j) \\ &= z^T(i,j)\left[\left(\tilde{A}_{11} - \tilde{A}_{12}C\right)^T P \left(\tilde{A}_{11} - \tilde{A}_{12}C\right) - P\right] \\ &\quad \times z(i,j) \\ &\triangleq z^T(i,j) \Upsilon z(i,j) \end{aligned}$$

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This means that for all $z(i, j) \neq 0$, we have

$$\frac{\begin{array}{c} z^{h\mathrm{T}}(i+1, j)\boldsymbol{P}^h z^h(i+1, j) \\ +z^{v\mathrm{T}}(i, j+1)\boldsymbol{P}^v z^v(i, j+1) - z^{\mathrm{T}}(i, j)\boldsymbol{P}z(i, j) \end{array}}{z^{\mathrm{T}}(i, j)\boldsymbol{P}z(i, j)}$$
$$= -\frac{z^{\mathrm{T}}(i, j)(-\boldsymbol{Y})z(i, j)}{z^{\mathrm{T}}(i, j)\boldsymbol{P}z(i, j)} \leq -\frac{\lambda_{\min}(-\boldsymbol{Y})}{\lambda_{\max}(\boldsymbol{P})} = \alpha - 1$$

where $\alpha \triangleq 1 - \min\ (\lambda_{\min}(-\boldsymbol{Y})/\lambda_{\max}(\boldsymbol{P}))$. Since $\min\ (\lambda_{\min}(-\boldsymbol{Y})/\lambda_{\max}(\boldsymbol{P})) > 0$, we have $\alpha < 1$. Obviously,

$$\alpha \geq \frac{\begin{array}{c} z^{h\mathrm{T}}(i+1, j)\boldsymbol{P}^h z^h(i+1, j) \\ +z^{v\mathrm{T}}(i, j+1)\boldsymbol{P}^v z^v(i, j+1) \end{array}}{z^{\mathrm{T}}(i, j)\boldsymbol{P}z(i, j)} > 0$$

That is, α belongs to (0, 1) and is independent of $z(i, j)$. Therefore we have

$$\begin{aligned} &z^{h\mathrm{T}}(i+1, j)\boldsymbol{P}^h z^h(i+1, j) + z^{v\mathrm{T}}(i, j+1)\boldsymbol{P}^v z^v(i, j+1) \\ &\quad \leq \alpha z^{\mathrm{T}}(i, j)\boldsymbol{P}z(i, j) \end{aligned}$$

That is,

$$\begin{aligned} &z^{h\mathrm{T}}(i+1, j)\boldsymbol{P}^h z^h(i+1, j) + z^{v\mathrm{T}}(i, j+1)\boldsymbol{P}^v z^v(i, j+1) \\ &\quad \leq \alpha\{z^{h\mathrm{T}}(i, j)\boldsymbol{P}^h z^h(i, j) + z^{v\mathrm{T}}(i, j)\boldsymbol{P}^v z^v(i, j)\} \end{aligned} \tag{15}$$

Using relationship (15), it can be established that

$$\begin{aligned} &z^{h\mathrm{T}}(0, k+1)\boldsymbol{P}^h z^h(0, k+1) = z^{h\mathrm{T}}(0, k+1) \\ &\quad \times \boldsymbol{P}^h z^h(0, k+1) \\ &z^{h\mathrm{T}}(1, k)\boldsymbol{P}^h z^h(1, k) + z^{v\mathrm{T}}(0, k+1)\boldsymbol{P}^v z^v(0, k+1) \\ &\quad \leq \alpha\{z^{h\mathrm{T}}(0, k)\boldsymbol{P}^h z^h(0, k) + z^{v\mathrm{T}}(0, k)\boldsymbol{P}^v z^v(0, k)\} \\ &z^{h\mathrm{T}}(2, k-1)\boldsymbol{P}^h z^h(2, k-1) + z^{v\mathrm{T}}(1, k)\boldsymbol{P}^v z^v(1, k) \\ &\quad \leq \alpha\{z^{h\mathrm{T}}(1, k-1)\boldsymbol{P}^h z^h(1, k-1) \\ &\quad + z^{v\mathrm{T}}(1, k-1)\boldsymbol{P}^v z^v(1, k-1)\} \\ &\quad \vdots \\ &z^{h\mathrm{T}}(k+1, 0)\boldsymbol{P}^h z^h(k+1, 0) + z^{v\mathrm{T}}(k, 1)\boldsymbol{P}^v z^v(k, 1) \\ &\quad \leq \alpha\{z^{h\mathrm{T}}(k, 0)\boldsymbol{P}^h z^h(k, 0) + z^{v\mathrm{T}}(k, 0)\boldsymbol{P}^v z^v(k, 0)\} \\ &z^{v\mathrm{T}}(k+1, 0)\boldsymbol{P}^v z^v(k+1, 0) = z^{v\mathrm{T}}(k+1, 0) \\ &\quad \times \boldsymbol{P}^v z^v(k+1, 0) \end{aligned}$$

which implies

$$\begin{aligned} &\sum_{j=0}^{k+1}[z^{h\mathrm{T}}(k+1-j, j)\boldsymbol{P}^h z^h(k+1-j, j) \\ &\quad + z^{v\mathrm{T}}(k+1-j, j)\boldsymbol{P}^v z^v(k+1-j, j)] \\ &\leq \alpha\sum_{j=0}^{k}[z^{h\mathrm{T}}(k-j, j)\boldsymbol{P}^h z^h(k-j, j) \\ &\quad + z^{v\mathrm{T}}(k-j, j)\boldsymbol{P}^v z^v(k-j, j)] \\ &\quad + z^{h\mathrm{T}}(0, k+1)\boldsymbol{P}^h z^h(0, k+1) \\ &\quad + z^{v\mathrm{T}}(k+1, 0)\boldsymbol{P}^v z^v(k+1, 0) \end{aligned}$$

Using this relationship iteratively, we obtain

$$\begin{aligned} &\sum_{j=0}^{k+1}[z^{h\mathrm{T}}(k+1-j, j)\boldsymbol{P}^h z^h(k+1-j, j) \\ &\quad + z^{v\mathrm{T}}(k+1-j, j)\boldsymbol{P}^v z^v(k+1-j, j)] \\ &\leq \alpha^{k+1}\{z^{h\mathrm{T}}(0, 0)\boldsymbol{P}^h z^h(0, 0) \\ &\quad + z^{v\boldsymbol{T}}(0, 0)\boldsymbol{P}^v z^v(0, 0)\} \\ &\quad + \sum_{j=0}^{k}\alpha^j[z^{h\mathrm{T}}(0, k+1-j)\boldsymbol{P}^h z^h(0, k+1-j) \\ &\quad + z^{v\mathrm{T}}(k+1-j, 0)\boldsymbol{P}^v z^v(k+1-j, 0)] \\ &= \sum_{j=0}^{k+1}\alpha^j[z^{h\mathrm{T}}(0, k+1-j)\boldsymbol{P}^h z^h(0, k+1-j) \\ &\quad + z^{v\mathrm{T}}(k+1-j, 0)\boldsymbol{P}^v z^v(k+1-j, 0)] \end{aligned}$$

Therefore we have

$$\begin{aligned} &\sum_{j=0}^{k+1}[|z^h(k+1-j, j)|^2 + |z^v(k+1-j, j)|^2] \\ &\quad \leq \beta\sum_{j=0}^{k+1}\alpha^j\{|z^h(0, k+1-j)|^2 + |z^v(k+1-j, 0)|^2\} \end{aligned} \tag{16}$$

where $\beta \triangleq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$. Now, denote $\mathcal{X}_k \triangleq \sum_{j=0}^{k}[|z^h(k-j, j)|^2 + |z^v(k-j, j)|^2]$. Upon the inequality in (16) we have

$$\begin{aligned} &\mathcal{X}_0 \leq \beta\{|z^h(0, 0)|^2 + |z^v(0, 0)|^2\} \\ &\mathcal{X}_1 \leq \beta[\alpha\{|z^h(0, 0)|^2 + |z^v(0, 0)|^2\} + \{|z^h(0, 1)|^2 \\ &\qquad + |z^v(1, 0)|^2\}] \\ &\mathcal{X}_2 \leq \beta[\alpha^2\{|z^h(0, 0)|^2 + |z^v(0, 0)|^2\} + \alpha\{|z^h(0, 1)|^2 \\ &\qquad + |z^v(1, 0)|^2\} + \{|z^h(0, 2)|^2 + |z^v(2, 0)|^2\}] \\ &\quad \vdots \end{aligned}$$

$$\mathcal{X}_N \le \beta[\alpha^N\{|z^h(0,0)|^2 + |z^v(0,0)|^2\} + \alpha^{N-1}\{|z^h(0,1)|^2 + |z^v(1,0)|^2\} + \cdots + \{|z^h(0,N)|^2 + |z^v(N,0)|^2\}]$$

Adding both sides of the above inequality system yields

$$\begin{aligned}\sum_{k=0}^{N}\mathcal{X}_k &\le \beta(1+\alpha+\cdots+\alpha^N)\{|z^h(0,0)|^2 \\ &\quad + |z^v(0,0)|^2\} + \beta(1+\alpha+\cdots+\alpha^{N-1}) \\ &\quad \times \{|z^h(0,1)|^2 + |z^v(1,0)|^2\} + \cdots \\ &\quad + \beta\{|z^h(0,N)|^2 + |z^v(N,0)|^2\} \\ &\le \beta(1+\alpha+\cdots+\alpha^N)\{|z^h(0,0)|^2 + |z^v(0,0)|^2\} \\ &\quad + \beta(1+\alpha+\cdots+\alpha^N) \\ &\quad \times \{|z^h(0,1)|^2 + |z^v(1,0)|^2\} + \cdots \\ &\quad + \beta(1+\alpha+\cdots+\alpha^N) \\ &\quad \times \{|z^h(0,N)|^2 + |z^v(N,0)|^2\} \\ &= \beta\frac{1-\alpha^N}{1-\alpha}\sum_{k=0}^{N}\{|z^h(0,k)|^2 + |z^v(k,0)|^2\}\end{aligned}$$

Then, under Assumption 1, the right side of the above inequality is bounded, which means $\lim_{k\to\infty}\mathcal{X}_k = 0$, that is, $|z(i,j)|^2 \to 0$ as $i+j\to\infty$. Then, by Definition 1 the sliding mode dynamics $(\mathcal{S}-1)$ in (9) is asymptotically stable. This completes the proof. □

Remark 2: Theorem 2 provides a sufficient condition for the existence of sliding mode dynamics in terms of LMI, which can be readily solved by utilising standard numerical software [26].

4.4 Synthesis of sliding mode controller

In this subsection, we synthesise a sliding mode controller by using the reaching law method developed in Section 3. According to (5), (7) and (8), we have

$$\begin{aligned}s^h(i+1,j) &= [\,\boldsymbol{C}_1 \quad \boldsymbol{I}\,]\left\{\begin{bmatrix}\bar{\boldsymbol{A}}_{111} & \bar{\boldsymbol{A}}_{112} & \bar{\boldsymbol{A}}_{211} & \bar{\boldsymbol{A}}_{212}\\ \bar{\boldsymbol{A}}_{121} & \bar{\boldsymbol{A}}_{122} & \bar{\boldsymbol{A}}_{221} & \bar{\boldsymbol{A}}_{222}\end{bmatrix}\right. \\ &\quad \left.\times\begin{bmatrix}z^h(i,j)\\ z^v(i,j)\end{bmatrix} + \begin{bmatrix}0\\ \bar{\boldsymbol{B}}_1\end{bmatrix}u^h(i,j)\right\} \\ &= (1-q_1T)s^h(i,j) - T(s^h(i,j))^2\mathrm{sign}(s^h(i,j))\end{aligned} \tag{17}$$

$$\begin{aligned}s^v(i,j+1) &= [\,\boldsymbol{C}_2 \quad \boldsymbol{I}\,]\left\{\begin{bmatrix}\bar{\boldsymbol{A}}_{311} & \bar{\boldsymbol{A}}_{312} & \bar{\boldsymbol{A}}_{411} & \bar{\boldsymbol{A}}_{412}\\ \bar{\boldsymbol{A}}_{321} & \bar{\boldsymbol{A}}_{322} & \bar{\boldsymbol{A}}_{421} & \bar{\boldsymbol{A}}_{422}\end{bmatrix}\right. \\ &\quad \left.\times\begin{bmatrix}z^h(i,j)\\ z^v(i,j)\end{bmatrix} + \begin{bmatrix}0\\ \bar{\boldsymbol{B}}_2\end{bmatrix}u^v(i,j)\right\} \\ &= (1-q_2T)s^v(i,j) - T(s^v(i,j))^2\mathrm{sign}(s^v(i,j))\end{aligned} \tag{18}$$

From (17) and (18), we obtain the following two sliding mode controllers

$$\begin{aligned}u^h(i,j) &= -\bar{\boldsymbol{B}}_1^{-1}\{\boldsymbol{C}_1[\bar{\boldsymbol{A}}_{111}z_1^h(i,j) + \bar{\boldsymbol{A}}_{112}z_2^h(i,j) \\ &\quad + \bar{\boldsymbol{A}}_{211}z_1^v(i,j) + \bar{\boldsymbol{A}}_{212}z_2^v(i,j)] \\ &\quad + \bar{\boldsymbol{A}}_{121}z_1^h(i,j) + \bar{\boldsymbol{A}}_{122}z_2^h(i,j) \\ &\quad + \bar{\boldsymbol{A}}_{221}z_1^v(i,j) + \bar{\boldsymbol{A}}_{222}z_2^v(i,j) \\ &\quad - (1-q_1T)s^h(i,j) + T(s^h(i,j))^2\mathrm{sign}(s^h(i,j))\}\end{aligned} \tag{19}$$

$$\begin{aligned}u^v(i,j) &= -\bar{\boldsymbol{B}}_2^{-1}\{\boldsymbol{C}_2[\bar{\boldsymbol{A}}_{311}z_1^h(i,j) \\ &\quad + \bar{\boldsymbol{A}}_{312}z_2^h(i,j) + \bar{\boldsymbol{A}}_{411}z_1^v(i,j) + \bar{\boldsymbol{A}}_{412}z_2^v(i,j)] \\ &\quad + \bar{\boldsymbol{A}}_{321}z_1^h(i,j) \\ &\quad + \bar{\boldsymbol{A}}_{322}z_2^h(i,j) + \bar{\boldsymbol{A}}_{421}z_1^v(i,j) + \bar{\boldsymbol{A}}_{422}z_2^v(i,j) \\ &\quad - (1-q_2T)s^v(i,j) + T(s^v(i,j))^2\mathrm{sign}(s^v(i,j))\}\end{aligned} \tag{20}$$

Theorem 3: Suppose (10) has a set of feasible solutions $\boldsymbol{Y} = \mathrm{diag}(\boldsymbol{Y}^h, \boldsymbol{Y}^v) > 0$, $\boldsymbol{X} = \mathrm{diag}(\boldsymbol{X}^h, \boldsymbol{X}^v)$ and the parameters of linear switching functions are given by $\boldsymbol{C}_1 = \boldsymbol{X}^h(\boldsymbol{Y}^h)^{-1}$ and $\boldsymbol{C}_2 = \boldsymbol{X}^v(\boldsymbol{Y}^v)^{-1}$. Then, by the controller given in (19) and (20), the state trajectories of the closed-loop system (7) can be driven onto the switching surfaces $s^h(i,j) = 0$ and $s^v(i,j) = 0$ in finite time and be maintained there all the time.

Proof: Since the sliding mode controllers (19) and (20) are designed by using the reaching law method, the following relationships hold

$$\begin{aligned}&s^h(i+1,j) = (1-q_1T)s^h(i,j) - T(s^h(i,j))^2\mathrm{sign}(s^h(i,j)) \\ &\Downarrow \\ &\Delta s^h(i,j) = s^h(i+1,j) - s^h(i,j) \\ &\qquad = -q_1Ts^h(i,j) - T(s^h(i,j))^2\mathrm{sign}(s^h(i,j)) \\ &\Downarrow \\ &\lim_{s^h(i,j)\to 0^+}\Delta s^h(i,j) < 0, \qquad \lim_{s^h(i,j)\to 0^-}\Delta s^h(i,j) > 0\end{aligned} \tag{21}$$

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and

$$
\begin{aligned}
&s^v(i,j+1) = (1-q_2T)s^v(i,j) - T(s^v(i,j))^2\mathrm{sign}(s^v(i,j))\\
&\Downarrow\\
&\quad\Delta s^v(i,j) = s^v(i,j+1) - s^v(i,j)\\
&\qquad\qquad = -q_2Ts^v(i,j) - T(s^v(i,j))^2\mathrm{sign}(s^v(i,j))\\
&\Downarrow\\
&\lim_{s^v(i,j)\to 0^+}\Delta s^v(i,j) < 0, \qquad \lim_{s^v(i,j)\to 0^-}\Delta s^v(i,j) > 0
\end{aligned} \tag{22}
$$

We can see from (21) and (22) that the state trajectories of (7) can be driven onto the sliding surfaces by the controllers $u^h(i, j)$ and $u^v(i, j)$ given in (19) and (20) in finite time and be maintained there all the time. The proof is completed.

5 SMC design using choi's method

5.1 Analysis of sliding mode dynamics

Reconsider the system ($\mathcal{P}$) in (6). In this approach, we design the linear switching functions as

$$
\begin{aligned}
s^h(i,j) &= \boldsymbol{\Gamma}_1\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}x^h(i,j),\\
s^v(i,j) &= \boldsymbol{\Gamma}_2\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}x^v(i,j)
\end{aligned} \tag{23}
$$

where $\boldsymbol{Q}_1 \in \mathbb{R}^{n_1\times n_1}$, $\boldsymbol{Q}_2 \in \mathbb{R}^{n_2\times n_2}$ are positive definite matrices to be specified. $\boldsymbol{\Gamma}_1 \in \mathbb{R}^{m\times m}$, $\boldsymbol{\Gamma}_2 \in \mathbb{R}^{m\times m}$ are some nonsingular matrices. For simplicity, $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are both chosen as identity matrices in this work. Define two transformation matrices and the associated vectors $\boldsymbol{y}^h(i, j)$, $\boldsymbol{y}^v(i, j)$ as

$$
\begin{aligned}
\boldsymbol{M} &\triangleq \begin{bmatrix} \tilde{\boldsymbol{\Psi}}_1^{-1}\tilde{\boldsymbol{B}}_1^{\mathrm{T}} \\ \boldsymbol{\Psi}_1^{-1}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_1 \\ \boldsymbol{M}_2 \end{bmatrix},\\
\boldsymbol{y}^h(i,j) &\triangleq \begin{bmatrix} \boldsymbol{y}_1^h(i,j) \\ \boldsymbol{y}_2^h(i,j) \end{bmatrix} = \boldsymbol{M}\boldsymbol{x}^h(i,j)
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\boldsymbol{N} &\triangleq \begin{bmatrix} \tilde{\boldsymbol{\Psi}}_2^{-1}\tilde{\boldsymbol{B}}_2^{\mathrm{T}} \\ \boldsymbol{\Psi}_2^{-1}\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \end{bmatrix},\\
\boldsymbol{y}^v(i,j) &\triangleq \begin{bmatrix} \boldsymbol{y}_1^v(i,j) \\ \boldsymbol{y}_2^v(i,j) \end{bmatrix} = \boldsymbol{N}\boldsymbol{x}^v(i,j)
\end{aligned} \tag{25}
$$

where $\boldsymbol{\Psi}_1 \triangleq \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}\boldsymbol{B}_1$, $\boldsymbol{\Psi}_2 \triangleq \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}\boldsymbol{B}_2$, $\tilde{\boldsymbol{\Psi}}_1 \triangleq \tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1$, $\tilde{\boldsymbol{\Psi}}_2 \triangleq \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2$; $\tilde{\boldsymbol{B}}_1 \in \mathbb{R}^{n_1\times(n_1-m)}$, $\tilde{\boldsymbol{B}}_2 \in \mathbb{R}^{n_2\times(n_2-m)}$ are any basis of the null space of $\boldsymbol{B}_1^{\mathrm{T}}$ and $\boldsymbol{B}_2^{\mathrm{T}}$, respectively, that is, $\tilde{\boldsymbol{B}}_1$ (or $\tilde{\boldsymbol{B}}_2$) is an orthogonal complement of $\boldsymbol{B}_1$ (or $\boldsymbol{B}_2$) (note $\tilde{\boldsymbol{B}}_1$ (or $\tilde{\boldsymbol{B}}_2$) is non-unique, but any choice satisfying the condition is acceptable). It is easily shown that

$$
\boldsymbol{M}^{-1} = [\,\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 \quad \boldsymbol{B}_1\,], \quad \boldsymbol{N}^{-1} = [\,\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2 \quad \boldsymbol{B}_2\,]
$$

Then, we have

$$
\begin{aligned}
\boldsymbol{y}_2^h(i,j) &= \boldsymbol{\Psi}_1^{-1}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}x^h(i,j) = \boldsymbol{\Psi}_1^{-1}s^h(i,j)\\
\boldsymbol{y}_2^v(i,j) &= \boldsymbol{\Psi}_2^{-1}\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}x^v(i,j) = \boldsymbol{\Psi}_2^{-1}s^v(i,j)
\end{aligned}
$$

Therefore when the system state trajectories reach onto the switching surfaces, that is, $s^h(i,j) = 0$ and $s^v(i, j) = 0$, we have $\boldsymbol{y}_2^h(i, j) = 0$ and $\boldsymbol{y}_2^v(i, j) = 0$. By using these two transformations, we obtain two reduced-order subsystems, which form the sliding mode dynamics and can be written together as

$$
\mathcal{S}-2: \quad \begin{bmatrix} \boldsymbol{y}_1^h(i+1,j) \\ \boldsymbol{y}_1^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\Psi}}_1^{-1}\tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{\Psi}}_1^{-1}\tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_2\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2 \\ \tilde{\boldsymbol{\Psi}}_2^{-1}\tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_3\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{\Psi}}_2^{-1}\tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_4\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{y}_1^h(i,j) \\ \boldsymbol{y}_1^v(i,j) \end{bmatrix} \tag{26}
$$

where $\boldsymbol{y}_1^h(i,j) \in \mathbb{R}^{(n_1-m)}$, $\boldsymbol{y}_1^v(i,j) \in \mathbb{R}^{(n_2-m)}$ are the state vectors of the sliding mode dynamics along the horizontal and vertical directions, respectively.

Now, we will analyse the stability of the sliding mode dynamics. The following theorem gives a sufficient condition for the asymptotic stability of the sliding mode dynamics in (26), and by solving this condition we can obtain the parameters of the designed switching functions in (23).

Before proceeding further, we give the following Lemma, which plays a key role in deriving our main results.

Lemma 1: *Suppose matrices* $\boldsymbol{G} = \boldsymbol{G}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{U} \in \mathbb{R}^{n\times m}$ *are given and assume* $\boldsymbol{U}$ *has full rank and* $m < n$. Then, $\varepsilon\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U} - \boldsymbol{G} > 0$ *for some scalar* $\varepsilon > 0$ *if and only if* $\tilde{\boldsymbol{U}}^{\mathrm{T}}\boldsymbol{G}\tilde{\boldsymbol{U}} < 0$, *where* $\tilde{\boldsymbol{U}}$ *is any matrix whose columns form the basis of the null space of* $\boldsymbol{U}^{\mathrm{T}}$.

Theorem 4: The sliding mode dynamics in (26) is asymptotically stable if there exist matrix $\boldsymbol{Q} = \mathrm{diag}(\boldsymbol{Q}_1,\boldsymbol{Q}_2) > 0$ and a scalar $\varepsilon > 0$ such that the following LMI holds

$$
\begin{bmatrix}
-\boldsymbol{Q}_1 - \varepsilon\boldsymbol{B}_1\boldsymbol{B}_1^{\mathrm{T}} & 0 & \boldsymbol{A}_1\boldsymbol{Q}_1 & \boldsymbol{A}_2\boldsymbol{Q}_2 \\
* & -\boldsymbol{Q}_2 - \varepsilon\boldsymbol{B}_2\boldsymbol{B}_2^{\mathrm{T}} & \boldsymbol{A}_3\boldsymbol{Q}_1 & \boldsymbol{A}_4\boldsymbol{Q}_2 \\
* & * & -\boldsymbol{Q}_1 - \varepsilon\boldsymbol{B}_1\boldsymbol{B}_1^{\mathrm{T}} & 0 \\
* & * & * & -\boldsymbol{Q}_2 - \varepsilon\boldsymbol{B}_2\boldsymbol{B}_2^{\mathrm{T}}
\end{bmatrix} < 0 \tag{27}
$$

Proof: Consider the following index

$$\begin{aligned}\mathcal{J} \triangleq\ & \boldsymbol{y}_1^{h\mathrm{T}}(i+1,j)\boldsymbol{R}^h\boldsymbol{y}_1^h(i+1,j)\\ &+\boldsymbol{y}_1^{v\mathrm{T}}(i,j+1)\boldsymbol{R}^v\boldsymbol{y}_1^v(i,j+1)\\ &-\boldsymbol{y}_1^{\mathrm{T}}(i,j)\boldsymbol{R}\boldsymbol{y}_1(i,j)\end{aligned} \tag{28}$$

where $\boldsymbol{R} \triangleq \mathrm{diag}(\boldsymbol{R}^h,\boldsymbol{R}^v)$, $\boldsymbol{R}^h \triangleq \tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1$ and $\boldsymbol{R}^v \triangleq \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2$. Then, along the solution of the sliding mode dynamics $(\mathcal{S}-2)$ in (26), we have

$$\mathcal{J} \triangleq \boldsymbol{y}_1^{\mathrm{T}}(i,j)\boldsymbol{\Omega}\boldsymbol{y}_1(i,j) \tag{29}$$

where

$$\begin{aligned}\boldsymbol{\Omega} \triangleq & \begin{bmatrix}\tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_2\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2\\ \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_3\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_4\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{\boldsymbol{\Psi}}_1^{-1} & 0\\ 0 & \tilde{\boldsymbol{\Psi}}_2^{-1}\end{bmatrix}\\ &\times\begin{bmatrix}\tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_2\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2\\ \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_3\boldsymbol{Q}_1\tilde{\mathrm{B}}_1 & \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_4\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2\end{bmatrix}-\begin{bmatrix}\tilde{\boldsymbol{\Psi}}_1 & 0\\ 0 & \tilde{\boldsymbol{\Psi}}_2\end{bmatrix}\end{aligned}$$

Now, we prove the equivalence between (27) and $\Omega < 0$. First, we prove the sufficiency. If $\Omega < 0$, then by Schur complement we have

$$\begin{bmatrix}-\tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & 0 & \tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{A}_2\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2\\ * & -\tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2 & \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_3\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{A}_4\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2\\ * & * & -\tilde{\boldsymbol{B}}_1^{\mathrm{T}}\boldsymbol{Q}_1\tilde{\boldsymbol{B}}_1 & 0\\ * & * & * & -\tilde{\boldsymbol{B}}_2^{\mathrm{T}}\boldsymbol{Q}_2\tilde{\boldsymbol{B}}_2\end{bmatrix} < 0 \tag{30}$$

which is equivalent to

$$\begin{aligned}&\begin{bmatrix}\tilde{\boldsymbol{B}}_1^{\mathrm{T}} & 0 & 0 & 0\\ 0 & \tilde{\boldsymbol{B}}_2^{\mathrm{T}} & 0 & 0\\ 0 & 0 & \tilde{\boldsymbol{B}}_1^{\mathrm{T}} & 0\\ 0 & 0 & 0 & \tilde{\boldsymbol{B}}_2^{\mathrm{T}}\end{bmatrix}\\ &\times\begin{bmatrix}-\boldsymbol{Q}_1 & 0 & \boldsymbol{A}_1\boldsymbol{Q}_1 & \boldsymbol{A}_2\boldsymbol{Q}_2\\ * & -\boldsymbol{Q}_2 & \boldsymbol{A}_3\boldsymbol{Q}_1 & \boldsymbol{A}_4\boldsymbol{Q}_2\\ * & * & -\boldsymbol{Q}_1 & 0\\ * & * & * & -\boldsymbol{Q}_2\end{bmatrix}\\ &\begin{bmatrix}\tilde{\boldsymbol{B}}_1 & 0 & 0 & 0\\ 0 & \tilde{\boldsymbol{B}}_2 & 0 & 0\\ 0 & 0 & \tilde{\boldsymbol{B}}_1 & 0\\ 0 & 0 & 0 & \tilde{\boldsymbol{B}}_2\end{bmatrix} < 0\end{aligned} \tag{31}$$

By Lemma 1, (31) is equivalent to (27). The necessity can be shown by inverting the above lines.

To complete the proof, the next step is to carry out some iterative operations, which can be done along the same lines as in the proof of Theorem 2. Then, the sliding mode dynamics $(\mathcal{S}-2)$ in (26) is asymptotically stable and the proof is completed.

5.2 Synthesis of sliding mode controller

The next step is to design a controller such that the system state trajectories are globally attracted to the switching surfaces. Recall (23), that is, with $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are both chosen as identity matrices, and the switching functions are $s^h(i,j)=\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}x^h(i,j)$ and $s^v(i,j)=\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}x^v(i,j)$ where $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$ are found by solving the LMI condition of (27).

If the system state trajectories reach the switching surfaces and are maintained there, it follows that $s^h(i,j)=\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}x^h(i,j)=0$, $s^h(i+1,j)=\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}x^h(i+1,j)=0$ and $s^v(i,j)=\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}x^v(i,j)=0$, $s^v(i,j+1)=\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}x^v(i,j+1)=0$. Normally, we can obtain the following two equivalent controllers by using these two attributes, respectively

$$\begin{aligned}s^h(i+1,j)&=\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}x^h(i+1,j)\\ &=\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}[\boldsymbol{A}_1x^h(i,j)+\boldsymbol{A}_2x^v(i,j)\\ &\quad+\boldsymbol{B}_1u(i,j)]=0\\ &\Downarrow\\ u_{eq}^h(i,j)&=-\boldsymbol{\Psi}_1^{-1}[\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}\boldsymbol{A}_1x^h(i,j)\\ &\quad+\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}\boldsymbol{A}_2x^v(i,j)]\end{aligned} \tag{32}$$

$$\begin{aligned}s^v(i,j+1)&=\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}x^v(i,j+1)\\ &=\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}[\boldsymbol{A}_3x^h(i,j)+\boldsymbol{A}_4x^v(i,j)\\ &\quad+\boldsymbol{B}_2u(i,j)]=0\\ &\Downarrow\\ u_{eq}^v(i,j)&=-\boldsymbol{\Psi}_2^{-1}[\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}\boldsymbol{A}_3x^h(i,j)\\ &\quad+\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}\boldsymbol{A}_4x^v(i,j)]\end{aligned} \tag{33}$$

Now, we design the following two switched feedback controller, by which the system state trajectories can be driven onto the predefined switching surfaces along the horizontal and vertical directions, respectively

$$u^h(i,j)=\begin{cases}-\boldsymbol{\Psi}_1^{-1}\{[\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}\boldsymbol{A}_1x^h(i,j)+\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}\boldsymbol{A}_2x^v(i,j)]\\ \quad+\rho_1(s^h(i,j))^2\mathrm{sign}(s^h(i,j))\},\qquad(\rho_1>0)\\ \qquad s^h(i,j)\neq 0\\ -\boldsymbol{\Psi}_1^{-1}[\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}\boldsymbol{A}_1x^h(i,j)+\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}\boldsymbol{A}_2x^v(i,j)]\\ \qquad s^h(i,j)=0\end{cases} \tag{34}$$

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$$u^v(i,j) = \begin{cases} -\boldsymbol{\Psi}_2^{-1}\{[\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}\boldsymbol{A}_3 x^h(i,j) + \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}\boldsymbol{A}_4 x^v(i,j)] \\ \quad +\rho_2(s^v(i,j))^2\operatorname{sign}(s^v(i,j))\}, \quad (\rho_2 > 0) \\ \qquad s^v(i,j) \neq 0 \\ -\boldsymbol{\Psi}_2^{-1}[\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}\boldsymbol{A}_3 x^h(i,j) + \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}\boldsymbol{A}_4 x^v(i,j)] \\ \qquad s^v(i,j) = 0 \end{cases} \tag{35}$$

Theorem: Suppose that the LMI (27) has a set of feasible solutions ($\boldsymbol{Q}_1$,$\boldsymbol{Q}_2$,ε) and the linear switching functions are given by (23). Then, by the controller given in (34) and (35) the state trajectories of the closed-loop system ($\mathcal{P}$) in (6) can be driven onto the switching surfaces $s^h(i,j) = 0$ and $s^v(i,j) = 0$, respectively, in finite time and be maintained there all the time.

Proof: First, consider (34) and (35) we have

$$\begin{aligned} \Delta s^h(i,j) &= s^h(i+1,j) - s^h(i,j) \\ &= \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{Q}_1^{-1}[\boldsymbol{A}_1 x^h(i,j) + \boldsymbol{A}_2 x^v(i,j) \\ &\quad + \boldsymbol{B}_1 u(i,j)] - s^h(i,j) \\ &= -s^h(i,j) - \rho_1(s^h(i,j))^2\operatorname{sign}(s^h(i,j)) \end{aligned} \tag{36}$$

and

$$\begin{aligned} \Delta s^v(i,j) &= s^v(i+1,j) - s^v(i,j) \\ &= \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{Q}_2^{-1}[\boldsymbol{A}_3 x^h(i,j) + \boldsymbol{A}_4 x^v(i,j) \\ &\quad + \boldsymbol{B}_2 u(i,j)] - s^v(i,j) \\ &= -s^v(i,j) - \rho_2(s^v(i,j))^2\operatorname{sign}(s^v(i,j)) \end{aligned} \tag{37}$$

Then, obviously, we have

$$\lim_{s^h(i,j)\to 0^+} \Delta s^h(i,j) < 0, \qquad \lim_{s^h(i,j)\to 0^-} \Delta s^h(i,j) > 0$$
$$\lim_{s^v(i,j)\to 0^+} \Delta s^v(i,j) < 0, \qquad \lim_{s^v(i,j)\to 0^-} \Delta s^v(i,j) > 0$$

Therefore according to [25] the controllers (34) and (35) can drive the system state trajectories onto the predefined switching surfaces $s^h(i,j) = 0$ and $s^v(i,j) = 0$ along the horizontal and vertical directions, respectively. This completes the proof.

Remark 3: Sections 4 and 5 provide two different methods of SMC designs for 2D discrete-time systems, respectively. Both of them introduce a model transformation to obtain the reduced-order sliding mode dynamics. The first method applies the extended reaching law approach to design SMC, while the second method uses the existence conditions of sliding mode dynamics directly to synthesise SMC.

6 Numerical example

In a real world, some dynamical processes in gas absorption, water stream heating and air drying can be described by the Darboux equation [27]

$$\frac{\partial^2 s(x,t)}{\partial x \partial t} = a_0 s(x,t) + a_1 \frac{\partial s(x,t)}{\partial t} + a_2 \frac{\partial s(x,t)}{\partial x} + bf(x,t) \tag{38}$$

where $s(x, t)$ is an unknown function at x(space) $\in [0, x_f]$ and t(time) $\in [0, \infty)$; a_0, a_1, a_2 and b are real coefficients, $f(x, t)$ is the input function.

Note that (38) is a partial differential equation (PDE). Similar to the technique used in [28], we define

$$r(x,t) \triangleq \frac{\partial s(x,t)}{\partial t} - a_2 s(x,t)$$
$$x^h(i,j) \triangleq r(i,j) \triangleq r(i\Delta x, j\Delta t),\ x^v(i,j) \triangleq s(i,j) \triangleq s(i\Delta x, j\Delta t)$$

and then the PDE model (38) can be converted into a 2D Roesser model of the form (1).

As discussed in [28], the discrepancy between the PDE model and its 2D difference approximation depends on the step sizes Δx and Δt which may be treated as uncertainty in the difference model. Obviously, the smaller the step sizes Δx and Δt, the closer between the PDE model and the difference model.

Now, subject to the selection of the parameters a_0, a_1, a_2 and b, we let the system matrices be given as follows

$$\begin{aligned} &\boldsymbol{A}_1 = \begin{bmatrix} 0.65 & -0.25 & 0.32 \\ -0.20 & 0.75 & -0.15 \\ 0.26 & 0.34 & 0.80 \end{bmatrix}, \\ &\boldsymbol{A}_2 = \begin{bmatrix} 0.25 & -0.30 & 0.20 \\ -0.30 & 0.15 & 0.24 \\ 0.15 & 0.36 & -0.48 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\ &\boldsymbol{A}_3 = \begin{bmatrix} 0.45 & 0.20 & -0.15 \\ 0.25 & -0.30 & 0.20 \\ -0.20 & 0.65 & 0.25 \end{bmatrix}, \\ &\boldsymbol{A}_4 = \begin{bmatrix} 0.60 & 0.25 & 0.18 \\ -0.75 & -0.40 & 0.14 \\ 0.20 & 0.15 & -0.37 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \end{aligned} \tag{39}$$

The considered system ($\mathcal{P}$) with (39) and $u(i, j) = 0$ is not stable, our attention is to design sliding mode controllers using the aforementioned two methods such that the closed-loop system is asymptotically stable.

Note that the system ($\mathcal{P}$) with (39) has the so-called 'regular' form, thus, we don't need to take model transformation, that is, $z(i,j) = x(i,j)$. Defining $s^h(i,j) = \boldsymbol{C}_1 x_1^h(i,j) + x_2^h(i,j)$, $s^v(i,j) = \boldsymbol{C}_2 x_1^v(i,j) + x_2^v(i,j)$, thus, the sliding mode dynamics can be described by (9) with

$$\tilde{\boldsymbol{A}}_{11} = \begin{bmatrix} \bar{\boldsymbol{A}}_{111} & \bar{\boldsymbol{A}}_{211} \\ \bar{\boldsymbol{A}}_{311} & \bar{\boldsymbol{A}}_{411} \end{bmatrix} = \begin{bmatrix} 0.65 & -0.25 & 0.25 & -0.30 \\ -0.20 & 0.75 & -0.30 & 0.15 \\ 0.45 & 0.20 & 0.60 & 0.25 \\ 0.25 & -0.30 & -0.75 & -0.40 \end{bmatrix}$$

$$\tilde{\boldsymbol{A}}_{12} = \begin{bmatrix} \bar{\boldsymbol{A}}_{112} & \bar{\boldsymbol{A}}_{212} \\ \bar{\boldsymbol{A}}_{312} & \bar{\boldsymbol{A}}_{412} \end{bmatrix} = \begin{bmatrix} 0.32 & 0.20 \\ -0.15 & 0.24 \\ -0.15 & 0.18 \\ 0.20 & 0.14 \end{bmatrix}$$

Solving condition (10) by using LMI Toolbox in MATLAB yields

$$\boldsymbol{C}_1 = \begin{bmatrix} 1.3420 & -1.3852 \end{bmatrix}, \quad \boldsymbol{C}_2 = \begin{bmatrix} 0.7635 & -1.2455 \end{bmatrix}$$

thus, we have

$$\begin{aligned} s^h(i,j) &= 1.3420 x_1^h(i,j) - 1.3852 x_2^h(i,j) + x_3^h(i,j), \\ s^v(i,j) &= 0.7635 x_1^v(i,j) - 1.2455 x_2^v(i,j) + x_3^v(i,j) \end{aligned} \tag{40}$$

Furthermore, according to (19) and (20) and letting $T = 1$, $q_1 = q_2 = 0.5$, we have

$$\begin{aligned} u^h(i,j) = -\frac{1}{2}\{&1.4093 x_1^h(i,j) - 1.0344 x_2^h(i,j) \\ &+ 1.4372 x_3^h(i,j) + 0.9011 x_1^v(i,j) \\ &- 0.2504 x_2^v(i,j) - 0.5440 x_3^v(i,j) \\ &- 0.5 s^h(i,j) + (s^h(i,j))^2 \operatorname{sign}(s^h(i,j))\} \end{aligned} \tag{41}$$

$$\begin{aligned} u^v(i,j) = -\frac{1}{3}\{&0.0322 x_1^h(i,j) + 0.5264 x_2^h(i,j) \\ &- 0.3636 x_3^h(i,j + 1.3922 x_1^v(i,j)) \\ &+ 0.6891 x_2^v(i,j) - 0.0369 x_3^v(i,j) \\ &- 0.5 s^v(i,j) + (s^v(i,j))^2 \operatorname{sign}(s^v(i,j))\} \end{aligned} \tag{42}$$

Let the initial and boundary conditions to be

$$x(i,1) = x(1,i) = \begin{cases} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}, & 1 \le i \le 20 \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, & i > 20 \end{cases} \tag{43}$$

Fig. 1 shows the states of the closed-loop system with controllers (41) and (42).

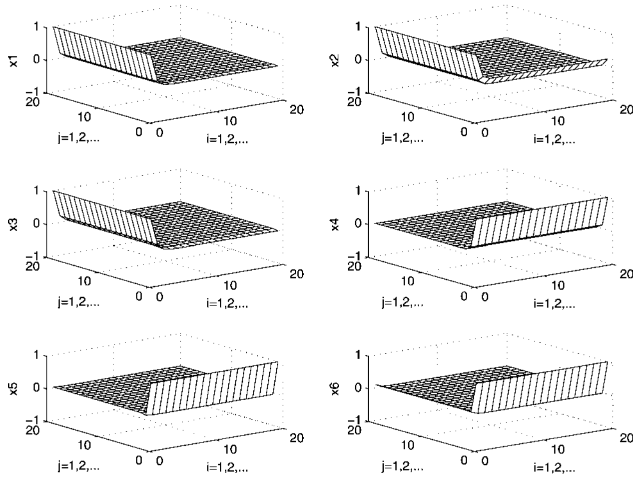

Figure 1 *States of the closed-loop system with controller (41) and (42)*

Now, consider the second method, by solving LMI (27) in Theorem 4, we obtain

$$\boldsymbol{Q}_1 = \begin{bmatrix} 0.7762 & -0.0814 & -0.7581 \\ -0.0814 & 1.0783 & 0.6448 \\ -0.7581 & 0.6448 & 1.7194 \end{bmatrix},$$

$$\boldsymbol{Q}_2 = \begin{bmatrix} 0.7661 & -0.2956 & -0.1788 \\ -0.2956 & 1.4504 & 0.0034 \\ -0.1788 & 0.0034 & 1.0029 \end{bmatrix},$$

$$\varepsilon = 0.3917$$

Then, from (23) the switching functions can be descried by

$$\begin{aligned} s^h(i,j) &= 2.7085 x_1^h(i,j) - 1.5535 x_2^h(i,j) + 2.9399 x_3^h(i,j), \\ s^v(i,j) &= 0.7901 x_1^v(i,j) + 0.1537 x_2^v(i,j) + 3.1316 x_3^v(i,j) \end{aligned} \tag{44}$$

Furthermore, according to (34) and (35) and letting $\rho_1 = \rho_2 = 1$, we have

$$u^h(i,j) = \begin{cases} \begin{aligned} -0.1701\{[&2.8356 x_1^h(i,j) \\ &-0.8427 x_2^h(i,j) \\ &+3.4517 x_3^h(i,j) \\ &+1.5842 x_1^v(i,j) \\ &+0.0128 x_2^v(i,j) \\ &-1.2423 x_3^v(i,j)] \\ +(s^h&(i,j))^2 \operatorname{sign}(s^h(i,j))\} \end{aligned} & s^h(i,j) \neq 0 \\ \begin{aligned} -0.1701\{[&2.8356 x_1^h(i,j) \\ &-0.8427 x_2^h(i,j) \\ &+3.4517 x_3^h(i,j) \\ &+1.5842 x_1^v(i,j) \\ &+0.0128 x_2^v(i,j) \\ &-1.2423 x_3^v(i,j)]\} \end{aligned} & s^h(i,j) = 0 \end{cases} \tag{45}$$

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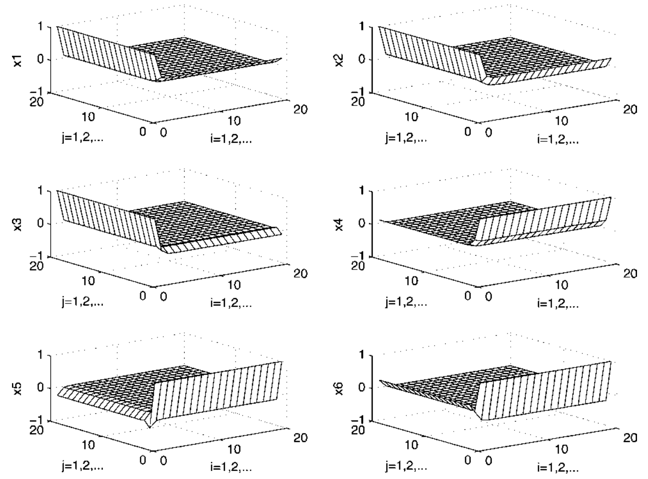

Figure 2 *States of the closed-loop system with controller (45) and (46)*

$$u^v(i,j) = \begin{cases} -0.1064\{[-0.2323x_1^h(i,j) \\ \quad +2.1474x_2^h(i,j) \\ \quad +0.6951x_3^h(i,j) \\ \quad +0.9851x_1^v(i,j) & s^v(i,j) \neq 0 \\ \quad +0.6058x_2^v(i,j) \\ \quad -0.9949x_3^v(i,j)] \\ +(s^v(i,j))^2 \mathrm{sign}(s^v(i,j))\} \\ -0.1064\{[-0.2323x_1^h(i,j) \\ \quad +2.1474x_2^h(i,j) \\ \quad +0.6951x_3^h(i,j) & s^v(i,j) = 0 \\ \quad +0.9851x_1^v(i,j) \\ \quad +0.6058x_2^v(i,j) \\ \quad -0.9949x_3^v(i,j)]\} \end{cases} \tag{46}$$

Fig. 2 shows the states of the closed-loop system with controller (45) and (46). The switching functions $s^h(i, j)$ and $s^v(i, j)$ in (44) are shown in Figs. 3*a*–3*b* and 3*c*–3*d*, respectively.

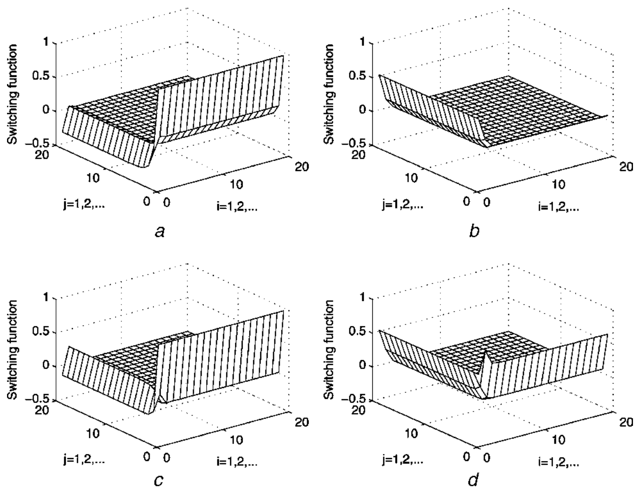

Figure 3 *Switching functions $s^h(i,j)$ and $s^v(i,j)$ in using two methods*

7 Conclusion

In this paper, the problem of SMC of 2D discrete-time systems has been investigated. This problem is solved by using two different methods: model transformation method and Choi's 1997 method. First, some extension and improvement have been made on the reaching law approach, which plays a key role in designing sliding mode controllers subsequently. Some sufficient conditions have been proposed for the existence of ideal quasi-sliding mode dynamics in terms of LMI, which can be solved effeciently via standard numerical software. A numerical example has been provided to show the effectiveness of the proposed SMC design methods.

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9 References

[1] CHOI H.H.: 'Variable structure control of dynamical systems with mismatched norm-bounded uncertainties: an LMI approach', *Int. J. Control*, 2001, **74**, (13), pp. 1324–1334

[2] WU Y., YU X.: 'Variable structure control design for uncertain dynamic systems with disturbances in input and output channels', *Automatica.*, 1999, **35**, pp. 311–319

[3] XIA Y.Q., JIA Y.M.: 'Robust sliding-mode control for uncertain time-delay systems: an LMI approach', *IEEE Trans. Autom. Control*, 2003, **48**, (6), pp. 1086–1092

[4] GOUAISBAUT F., DAMBRINE M., RICHARD J.P.: 'Robust control of delay systems: a sliding mode control design via LMI', *Syst. Control Lett.*, 2002, **46**, pp. 219–230

[5] NIU Y., LAM J., WANG X.: 'Sliding-mode control for uncertain neutral delay systems', *IEE Proc. D, Control Theory Appl.*, 2004, **151**, (1), pp. 38–44

[6] NIU Y., HO D.W.C.: 'Robust observer design for ito stochastic time-delay systems via sliding mode control', *Syst. Control Lett.*, 2006, **55**, pp. 781–793

[7] NIU Y., HO D.W.C., LAM J.: 'Robust integral sliding mode control for uncertain stochastic systems with time-varying delay', *Automatica.*, 2005, **41**, pp. 873–880

[8] SHI P., XIA Y., LIU G.P., REES D.: 'On designing of sliding-mode control for stochastic jump systems', *IEEE Trans. Automat. Control*, 2006, **51**, (1), pp. 97–103

[9] GOLO G., MILOSAVLJEVIC C.: 'Robust discrete-time chattering free sliding mode control', *Syst. Control Lett.*, 2000, **41**, (1), pp. 19–28

[10] HUI S., ZAK S.H.: 'On discrete-time variable structure sliding mode control', *Syst. Control Lett.*, 1999, **38**, pp. 283–288

[11] HA Q.P., TRINH H., NGUYEN H.T., TUAN H.D.: 'Dynamic output feedback sliding-mode control using pole placement and linear functional observers', *IEEE Trans. Ind. Electron.*, 2003, **50**, (5), pp. 1030–1037

[12] ZAK S.H., HUI S.: 'Output feedback in variable structure controllers and state estimators for uncertain/nonlinear dynamical systems', *IEE Proc. D, Control Theory Appl.*, 1993, **140**, (1), pp. 41–50

[13] KACZOREK T.: 'Two-dimensional linear systems' (Springer-Verlag, Berlin, Germany, 1985)

[14] LU W.S., ANTONIOU A.: 'Two-dimensional digital filters' (Marcel Dekker, New York, 1992)

[15] GAO H., LAM J., XU S., WANG C.: 'Stability and stabilization of uncertain 2-D discrete systems with stochastic perturbation', *Multidimens. Syst. Signal Process.*, 2005, **16**, pp. 85–106

[16] LU W.S.: 'On a Lyapunov approach to stability analysis of 2-D digital filters', *IEEE Trans. Circuits and Syst. I*, 1994, **41**, (10), pp. 665–669

[17] DU C., XIE L.: '$\mathcal{H}_\infty$ control and filtering of two-dimensional Systems' in 'Lecture Notes in Control and Information Sciences' (Springer-Verlag, Berlin, Germany, 2002), Vol. 278

[18] XIE L., DU C., ZHANG C., SOH Y.C.: '$\mathcal{H}_\infty$ deconvolution filtering of 2-D digital systems', *IEEE Trans. Signal Process.*, 2002, **50**, (9), pp. 2319–2332

[19] GAO H., LAM J., WANG C., XU S.: '$\mathcal{H}_\infty$ model reduction for uncertain two-dimensional discrete systems', *Optim. Control Appl. Meth.*, 2005, **26**, pp. 199–227

[20] WU L., SHI P., GAO H., WANG C.: '$\mathcal{H}_\infty$ mode reduction for two-dimensional discrete state-delayed systems', *IEE Proc. J, Vis. Image Signal Process.*, 2006, **153**, (6), pp. 769–784

[21] CHOI H.H.: 'A new method for variable structure control system design: a linear matrix inequality approach', *Automatica.*, 1997, **33**, pp. 2089–2092

[22] EDWARDS C.: 'A practical method for the design of sliding mode controllers using linear matrix inequalities', *Automatica.*, 1994, **40**, pp. 1761–1769

[23] EDWARD C., SPURGEON S.K.: 'Linear matrix inequality methods for designing sliding mode output feedback controllers', *IEE Proc. D, Control Theory Appl.*, 2003, **150**, (5), pp. 539–545

[24] GAO W., HUNG J.C.: 'Variable structure control of non-linear systems: a new approach', *IEEE Trans. Ind. Electron.*, 1993, **40**, (1), pp. 45–55

[25] GAO W., WANG Y.F., HOMAIFA A.: 'Discrete-time variable structure control system', *IEEE Trans. Ind. Electron.*, 1995, **42**, (2), pp. 117–122

[26] BOYD S., EL GHAOUI L., FERON E., BALAKRISHNAN V.: 'Linear matrix inequalities in systems and control theory' (SIAM, Philadelphia, PA, 1994)

[27] MARSZALEK W.: 'Two-dimensional state-space discrete models for hyperbolic partial differential equations', *Appl. Math. Model.*, 1984, **8**, pp. 11–14

[28] DU C., XIE L.: '$\mathcal{H}_\infty$ control and robust stabilization of two-dimensional systems in Roesser models', *Automatica.*, 2001, **37**, pp. 205–211

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