Asynchronous Sliding Mode Control of Two-Dimensional Markov Jump Systems

***Abstract*—This paper is concerned with the problem of asynchronous sliding mode control (SMC) for two-dimensional (2D) discrete-time Markov jump systems, which is described by Roesser model. A hidden Markov model is introduced to describe the asynchronization appears between the designed sliding model controller and the original system. Based on the constructed 2D sliding surface, an asynchronous 2D-SMC law is designed. The Lyapunov function and Linear matrix inequality technique are utilized to establish sufficient conditions of the asymptotic mean square stability with a prescribed H_∞ disturbance attenuation performance for the closed-loop 2D system. Moreover, the reachability of the constructed 2D sliding surface is ensured simultaneously. Then, the asynchronous 2D-SMC law can be obtained by solving a convex optimization problem. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed asynchronous 2D-SMC design method.**

***Index Terms*—Markov jump systems, 2D systems, Sliding mode control, Hidden Markov model**

I. Introduction

Markov jump systems (MJSs), an important class of stochastic switching systems, have received considerable attentions for its powerful ability in modeling systems with abrupt change in parameters or system structures, for instance, environmental disturbances, actuator failures, and interconnection variations in subsystems, etc. Over the past decades, a large number of results on stability analysis and the design of controller/filter have been reported in [**?**].

However, the aforementioned works are generally based on the implicit assumption that the information of system modes can always be fully available for the controller/filter, so that the controller/filter modes can run synchronously with system modes. Unfortunately, in practical applications, it is rather difficult to satisfy this ideal assumption because of some unexpected factors, such as, time delays, data dropouts and quantization in networked control systems. In order to overcome the strict limitation, two research approaches were proposed, namely, mode-independent and asynchronous methods. In mode-independent methods, see [1] and [2], the controller/filter modes is independent of system modes, which means the information of system modes is not fully utilized and may result in some conservativeness. In [3], Wu proposed the hidden Markov model, a united asynchronous framework that covering synchronous and mode-independent cases. The hidden Markov model supposes that the controller/filter modes can be detected via a hidden Markov chain, such that controllers/filters can utilize more information of system modes. Therefore, many problems of asynchronous control/filtering for MJSs have been studies in recent years [4]-[6].

Moreover, sliding mode control (SMC), an effective control method for its strong robustness against parameter variations, exogenous disturbances and model uncertainties, has been successfully applied to a variety of practical systems [7]-[9]. The main idea of SMC is to design a discontinuous control law to force the system state trajectories toward the sliding surface and remain within a neighbourhood of the sliding surface after reaching it [10]. In the last few decades, the SMC design problems have been extensively studied, and a large number of research results for MJSs have been published [11]-[13]. Recently, some researchers have investigated the asynchronous SMC design methods for MJSs based on hidden Markov model [14]-[16].

On the other hand, two-dimensional (2D) systems, a special class of multi-dimensional systems dating back to 1970s, have be used to describe a wide variety of practical applications, such as digital image processing, partial differential equations modeling and signal filtering [17]-[19], etc. Different from one dimensional (1D) systems, the key feature of 2D systems is that the system information are independent and propagated along two different directions, which will brought much complexity and difficulty in system analysis and synthesis. To describe the special properties of 2D systems, server different mathematical models presented in state-space framework, such as Roesser model [17] and Fornasini-Marchesini model [20], have been proposed. In Roesser model, the system state is composed of two independent states, horizontal state and vertical state, which are denoted by sub-vectors, the next horizontal state and vertical state in different directions can be deduced from current state, respectively. Different from the Roesser model, the system state in Fornasini-Marchesini model is represented by a vector which is a function with two independent variables, and the current system state is derived from the adjacent last two states. In most circumstances , the Roesser model can be considered as a special case of Fornasini-Marchesini model, but is much simpler and more intelligible than the latter. Based on 2D bounded real lemma, the H_∞ control theory for 2D systems was established by Du and Xie in [21] and [22]. This result was extended to control and filtering problems for 2D systems with time-varying delays [23], multiplicative noise [24] and uncertainty parameters [25]. Anh has studied the dissipative control and filtering problems for 2D systems in [26] by utilizing Linear matrix inequalities technique. A number of preliminary results on sliding mode control of discrete-time 2D systems have been studied in [27]-[29]. In recent years, 2D systems with Markov jump parameters have attract more interests of researchers. For example, the H_∞ control, H_∞ filtering, H_∞ mode reduction and fault detection problems for 2D-MJSs have been investigated in [30]-[34], respectively. However, we have to notice that, the issue of SMC for 2D-MJSs has not been fully studied yet, which motivates us for current work.

Based on the aforementioned considerations, we will investigate the asynchronous SMC design problems for 2D-MJSs in this paper. The main contributions of this paper include the following three aspects:

1) For the concerned 2D-MJSs, the asymptotically mean square stability with H_∞ disturbance attenuation performance is analyzed and the reachability of the 2D sliding mode dynamics are ensured simultaneously. It is the first time that the asynchronous SMC is studies for 2D-MJSs, which shows the feasibility and effectiveness of 2D-SMC for 2D control systems.
2) Based on the constructed 2D sliding surface, an asynchronous 2D-SMC law is designed under the framework of hidden Markov model, which means it is more generalized than the synchronous and mode-independent ones.
3) The proposed asynchronous 2D-SMC law design method can be applied not only to the Roesser model, but also to the Fornasini-Marchesini model through some simple modifications.

II. Preliminaries

In this paper, we consider the following two-dimensional Markov jump system (2D-MJS) in Roesser model:

$$\begin{cases} \boldsymbol{x(i,\, j)} = A_{r(i,j)}x(i,j) + E_{r(i,j)}w(i,j) \\ \qquad\quad + B_{r(i,j)}[(u(i,j) + f(x(i,j), r(i,j))] \\ y(i,j) = C_{r(i,j)}x(i,j) + D_{r(i,j)}w(i,j) \end{cases} \tag{1}$$

where

$$\boldsymbol{x(i,\, j)} = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix},\ x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$$

$x^h(i,h) \in \mathbb{R}^{n_h}$ and $x^v(i,h) \in \mathbb{R}^{n_v}$ represent horizontal and vertical states respectively, $u(i,j) \in \mathbb{R}^{n_u}$ and $y(i,j) \in \mathbb{R}^{n_y}$ represent the controlled input and output respectively, and $w(i,j) \in \mathbb{R}^{n_w}$ represents the exogenous disturbance which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$. $A_{r(i,j)}$,$B_{r(i,j)}$,$C_{r(i,j)}$,$D_{r(i,j)}$ and $E_{r(i,j)}$ represent the time-varying system matrices, all of which are real known constant matrices with appropriate dimensions. Besides, we assume that the matrix $B_{r(i,j)}$ is full column rank for each $r(i,j) \in \mathcal{N}_1$, that is, rank($B_{r(i,j)}$)= n_u. The nonlinear function $f(x(i,j), r(i,j))$ satisfying the following property:

$$\|f(x(i,j), r(i,j)\| \leq \delta_{r(i,j)}\|x(i,j)\| \tag{2}$$

where $\delta_{r(i,j)}$ is a known scalar, $\|\cdot\|$ denotes the Euclidean norm of a vector. The parameter $r(i,j)$ takes values in a finite set $\mathcal{N}_1 = \{1, 2..., N_1\}$ with transition probability matrix $\Lambda = \{\lambda_{kp}\}$, and the related transition probability from mode k to mode p is given by

$$\begin{aligned} &\Pr\{r(i+1,j) = p|r(i,j) = k\} \\ =&\Pr\{r(i,j+1) = p|r(i,j) = p\} = \lambda_{kp},\ \forall k,p \in \mathcal{N}_1 \end{aligned} \tag{3}$$

where $\lambda_{kp} \in [0,1]$, for all $k,p \in \mathcal{N}_1$, and $\sum_{p=1}^{N_1} \lambda_{kp} = 1$ for every mode k.

We define the boundary condition (X_0, Γ_0) of system (1), as follows:

$$\begin{cases} X_0 = \{x^h(0,j), x^v(i,0)|\ i,j = 0,1,2...\} \\ \Gamma_0 = \{r(0,j), r(i,0)|\ i,j = 0,1,2...\} \end{cases} \tag{4}$$

And the corresponding zero boundary condition is assumed as $x^h(0,j) = 0, x^v(i,0) = 0, i,j = 0,1,2....$ Besides, we further impose following assumption on X_0.

Assumption 1. *The boundary condition X_0 satisfies:*

$$\lim_{L\to\infty} \mathbb{E}\Big\{\sum_{\ell=1}^{L}(\|x^h(0,\ell)\|^2 + \|x^v(\ell,0)\|^2)\Big\} < \infty \tag{5}$$

where $\mathbb{E}\{\cdot\}$ stands for mathematical expectation.

In practical applications, the complete information of $r(i,j)$ can not always be available to the controller. Hence, in this paper, the hidden Markov model $(r(i,j), \sigma(i,j), \Lambda, \Psi)$ as in [3] is introduced to characterize the asynchronous phenomenon between the controller and the original system. The parameter $\sigma(i,j)$, refers to controller mode, takes values in another finite set $\mathcal{N}_2 = \{1, 2...N_2\}$, and satisfies the conditional probability matrix $\Psi = \{\mu_{k\tau}\}$ with conditional mode transition probabilities

$$\Pr\{\sigma(i,j) = s|r(i,j) = k\} = \mu_{k\tau} \tag{6}$$

where $\mu_{k\tau} \in [0,1]$ for all $k \in \mathcal{N}_1, \tau \in \mathcal{N}_2$, and $\sum_{\tau=1}^{N2} \mu_{k\tau} = 1$ for any mode k.

Remark 1. *The hidden Markov model is a united framework that covering synchronous and mode-independent cases. The controller belongs to which case depends on the conditional probability matrix Ψ. The necessary and sufficient condition for synchronous case is as follows:*

$$\mu_{k\tau} = \begin{cases} 1, & s = s_k \\ 0, & s \neq s_k \end{cases} \qquad s_k \in \mathcal{N}_2$$

for every mode $k \in \mathcal{N}_1$. The controller becomes a mode-independent one when $N_2 = \{1\}$. Thus, the result derived in this work can be easily applied to synchronous or mode-independent cases.

Next, the definitions of asymptotically mean square stability and H_∞ performance for 2D systems will be given in Definition 1 and Definition 2, respectively.

Definition 1. *The 2D-MJS (1) with $w(i,j) \equiv 0$ is said to be asymptotically mean square stable if the following holds:*

$$\lim_{i+j\to\infty} \mathbb{E}\{\|x(i,j)\|^2\} = 0 \tag{7}$$

for any boundary condition X_0 with Assumption 1.

Definition 2. *Given a scalar $\gamma > 0$, the 2D-MJS (1) is said to be asymptotically mean square stable with an H_∞ disturbance attenuation performance γ if the system is under zero boundary condition, satisfies (7) and the following inequality:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|y(i,j)\|^2\} < \gamma^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|w(i,j)\|^2\} \tag{8}$$

for all $w(i,j) \in \ell_2\{[0,\infty),[0,\infty)\}$.

Now, we will make some notational simplification for convenience. The parameter $r(i,j)$ will be represented by k, $r(i+1,j)$ and $r(i,j+1)$ will be represented by p, and $\sigma(i,j)$ will be represented by s.

The objective of this work is to devise an asynchronous SMC law, such that the 2D-MJS (1) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

III. Main Result

A. Sliding surface and sliding mode controller

In this paper, a novel 2D sliding surface function is constructed as follows:

$$s(i,j) = \begin{bmatrix} s^h(i,j) \\ s^v(i,j) \end{bmatrix} = Gx(i,j) \tag{9}$$

where $s^h(i,j)$ and $s^v(i,j)$ represent horizontal and vertical sub-sliding surface respectively, the matrix $G = \sum_{k=1}^{N_1}\beta_k B_k^T$, and scalars β_k should be chosen properly such that GB_k is nonsingular for any $k \in \mathcal{N}_1$. Based on the assumption that B_k is full column rank for any $k \in \mathcal{N}_1$, we can find that the above condition can be guaranteed easily with properly selected parameter β_k.

Then, an asynchronous 2D-SMC law is designed as follows:

$$u(i,j) = K_\tau x(i,j) - \rho(i,j)\frac{s(i,j)}{\|s(i,j)\|} \tag{10}$$

where the matrix $K_\tau \in \mathbb{R}^{n_u \times n_x}$ with $n_x = n_h + n_v$ will be determined later, and the parameter $\rho(i,j)$ is given as

$$\rho(i,j) = \varrho_1\|x(i,j)\| + \varrho_2\|w(i,j)\| \tag{11}$$

with $\varrho_1 = \max_{k\in\mathcal{N}_1}\{\delta_k\}$, $\varrho_2 = \max_{k\in\mathcal{N}_1}\{\|(GB_k)^{-1}GE_k\|\}$, and the parameter δ_k is given in (2).

Combining system (1) and the asynchronous 2D-SMC law (9), the closed-loop 2D-MJS can be obtained easily as follows:

$$\boldsymbol{x(i,\,j)} = \bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_k w(i,j) \tag{12}$$

where $\bar{A}_{k\tau} = A_k + B_k K_\tau$, and $\bar{\rho}_k(i,j)$ as follows

$$\bar{\rho}_k(i,j) = f_k(x(i,j)) - (\varrho_1\|x(i,j)\| + \varrho_2\|w(i,j)\|)\cdot\frac{s(i,j)}{\|s(i,j)\|}.$$

Then, based on the properties of norm, the following condition can be deduced easily

$$\|\bar{\rho}_k(i,j)\| \le (\varrho_1 + \delta_k)\|x(i,j)\| + \varrho_2\|w(i,j)\|. \tag{13}$$

Remark 2. *As mentioned above, the SMC designed in this paper is asynchronous with the original system, and the phenomenon that lies between them is described by the hidden Markov model $(r(i,j),\sigma(i,j),\Lambda,\Psi)$. The model consists of two Markov chain, the first Markov chain $\{r(i,j), i,j = 1,2\ \ldots\}$ governs the mode jumps of original system and the second $\{\sigma(i,j), i,j = 1,2\ldots\}$ governs the mode jumps of controller. Noting that, the second Markov chain is related to a conditional probability matrix Ψ which establishes a connection between the modes of controller and original system, and this mean the asynchronous SMC can fully utilize the information of original system.*

Remark 3. *Different from SMC for 1D systems, the SMC designed in this paper is an asynchronous 2D-SMC. First, it is asynchronous and described by the hidden Markov model. Secondly, following the idea of Roesser model, $s^h(i,j)$ and $s^v(i,j)$ that denote horizontal and vertical sliding surface respectively, are employed to construct a new 2D sliding surface function $s(i,j)$. The property will be inherited by the 2D-SMC law (10) since the $s(i,j)$ is included, this is the reason we name it as 2D-SMC.*

B. Analysis of Stability and H_∞ attenuation performance

In this subsection, we focus on the stability and H_∞ attenuation performance analysis for the closed-loop 2D-MJS system (12). A sufficient condition will be derived to guarantee the considered system is asymptotically mean square stable with an H_∞ attenuation performance γ.

Theorem 1. *Consider the 2D-MJS (1) under Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma > 0$, if there exist matrices $K_\tau \in \mathbb{R}^{n_u\times n_x}$, $R_k = \mathrm{diag}\{R_k^h, R_k^v\} > 0$, $Q_{k\tau} > 0$, $T_{k\tau} > 0$ and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, \tau \in \mathcal{N}_2$, such that the following inequalities hold:*

$$B_k^T\mathcal{R}_k B_k - \epsilon_k I \le 0 \tag{14}$$

,

$$\mathcal{A} + 2\Big(\sum_{\tau=0}^{N_2}\mu_{k\tau}\mathrm{diag}\{Q_{k\tau},T_{k\tau}\}\Big) < 0 \tag{15}$$

,

$$\hat{A}_{k\tau}^T\mathcal{R}_k\hat{A}_{k\tau} - \mathrm{diag}\{Q_{k\tau},T_{k\tau}\} < 0 \tag{16}$$

where

$$\mathcal{A} = \begin{bmatrix} \Pi_1 & \Pi_3 \\ * & \Pi_2 \end{bmatrix}$$

with

$$\begin{cases} \Pi_1 = -R_k + 4(\delta_k+\varrho_1)^2\epsilon_k I + C_k^T C_k \\ \Pi_2 = -\gamma^2 I + D_k^T D_k + 4\varrho_2^2\epsilon_k I \\ \Pi_3 = C_k^T D_k \end{cases}$$

and $\mathcal{R}_k = \sum_{p=1}^{N1}\lambda_{kp}R_p$, $\hat{A}_{k\tau} = \begin{bmatrix}\bar{A}_{k\tau} & E_k\end{bmatrix}$, then the closed-loop system (12) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

Proof. Let's start the proof with the stability of system. We select the Lyapunov candidate as $V_1(i,j) = x^T(i,j)R_k x(i,j)$, then, define

$$\varDelta V_1(i,j) = \boldsymbol{x(i,j)}^T R_p \boldsymbol{x(i,j)} - x^T(i,j)R_k x(i,j) \tag{17}$$

Based on the closed-loop system equation (12) with $w(i,j) = 0$, it is easy to find that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j)\}\\
&= \sum_{\tau=0}^{N_2} \mu_{k\tau}\Big\{\big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j)\big]^T \mathcal{R}_k\\
&\times \big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j)\big]\Big\}\\
&- x^T(i,j)R_k x(i,j)\\
&\leq x^T(i,j)\Big\{2\big(\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T(i,j)\mathcal{R}_k\bar{A}_{k\tau}\big)\Big\}x(i,j)\\
&+ 2\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&- x^T(i,j)R_k x(i,j)
\end{aligned} \tag{18}$$

Recalling the conditions given in (13) and (14), the following inequality can be further obtained

$$\mathbb{E}\{\varDelta V_1(i,j)\} \leq x^T(i,j)\mathcal{G}_{k\tau}x(i,j) \tag{19}$$

where $\mathcal{G}_{k\tau} = 2\big(\sum_{\tau=0}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\mathcal{R}_k\bar{A}_{k\tau}\big) + 2\epsilon_k(\delta_k+\varrho_1)^2 I - R_k$. The following inequality can be deduced from (15) based on the properties of matrix quadratic

$$2\big(\sum_{\tau=1}^{N_2}\mu_{k\tau}Q_{k\tau}\big) + 4\epsilon_k(\delta_k+\varrho_1)^2 I + C_k^T C_k - R_k < 0 \tag{20}$$

which will further deduce

$$2\big(\sum_{\tau=1}^{N_2}\mu_{k\tau}Q_{k\tau}\big) + 2\epsilon_k(\delta_k+\varrho_1)^2 I - R_k < 0 \tag{21}$$

The following inequality can be inferred directly from condition (16)

$$\bar{A}_{k\tau}^T\mathcal{R}_k\bar{A}_{k\tau} - Q_{k\tau} < 0 \tag{22}$$

Combine (21) and (22), we can infer that $\mathcal{G}_{k\tau} < 0$, which is equivalent to

$$\mathcal{G}_{k\tau} \leq -\alpha I \tag{23}$$

with scalar $\alpha > 0$. Recalling (19), we can further infer that

$$\mathbb{E}\{\varDelta V_1(i,j)\} \leq -\alpha\mathbb{E}\{\|x(i,j)\|^2\} \tag{24}$$

Summing up on the both side of (24), we have

$$\mathbb{E}\Big\{\sum_{i=0}^{\hat{\kappa}_1}\sum_{j=0}^{\hat{\kappa}_2}\|x(i,j)\|^2\Big\} \leq -\frac{1}{\alpha}\mathbb{E}\Big\{\sum_{i=0}^{\hat{\kappa}_1}\sum_{j=0}^{\hat{\kappa}_2}\varDelta V_1(i,j)\Big\} \tag{25}$$

where parameters $\hat{\kappa}_1$, $\hat{\kappa}_2$ are any positive integers. By substituting $\varDelta V_1$ with (17) and $R_k = \mathrm{diag}\{R_k^{\mathrm{h}}, R_k^{\mathrm{v}}\}$ respectively, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\hat{\kappa}_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\\
&= \sum_{i=0}^{\hat{\kappa}_1}\big\{V_1^v(i,\hat{\kappa}_2+1) - V_1^v(i,0)\big\}\\
&- \sum_{j=0}^{\hat{\kappa}_2}\big\{V_1^h(\hat{\kappa}_1+1,j) - V_1^h(0,j)\big\}\\
&\leq -\big(\sum_{i=0}^{\hat{\kappa}_1}V_1^v(i,0) + \sum_{j=0}^{\hat{\kappa}_2}V_1^h(0,j)\big)
\end{aligned} \tag{26}$$

where $V_1^h(i,j)$ and $V_1^v(i,j)$ are defined as

$$\begin{cases} V_1^h(i,j) = x^{hT}(i,j)R_{r(i,j)}^h x^h(i,j)\\ V_1^v(i,j) = x^{vT}(i,j)R_{r(i,j)}^v x^v(i,j)\end{cases}$$

Recalling the boundary condition in Assumption 1, and let $\hat{\kappa}_1$, $\hat{\kappa}_2$ tend to infinity, it follows from (25) and (26) that

$$\begin{aligned}
&\mathbb{E}\Big\{\sum_{i=0}^{\hat{\kappa}_1}\sum_{j=0}^{\hat{\kappa}_2}\|x(i,j)\|^2\Big\}\\
&\leq -\frac{\beta}{\alpha}\sum_{\ell=0}^{\infty}\big(\|x^v(\ell,0)\|^2 + \|x^h(0,\ell)\|^2\big)\\
&< \infty
\end{aligned} \tag{27}$$

where β is the maximum eigenvalue of $R^h(0,\ell)$ and $R^v(\ell,0)$, for any $\ell = 0,1,2...$, which implies that (7) holds. Thus, the asymptotically mean square stability of the considered closed-loop system is proved.

Next, let's focus on the H_∞ attenuation performance under zero boundary condition. Based on the closed-loop system equation (12), it is easy to find that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j)\}\\
&= \sum_{\tau=0}^{N_2}\mu_{k\tau}\Big\{\big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_k w(i,j)\big]^T\\
&\times \mathcal{R}_k\big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_k w(i,j)\big]\Big\}\\
&- x^T(i,j)R_k x(i,j)\\
&\leq \hat{x}^T(i,j)\Big\{2\big(\sum_{\tau=1}^{N_2}\mu_{k\tau}\hat{A}_{k\tau}^T(i,j)\mathcal{R}_k\hat{A}_{k\tau}\big)\Big\}\hat{x}(i,j)\\
&+ 2\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&- x^T(i,j)R_k x(i,j)
\end{aligned} \tag{28}$$

where

$$\hat{x}(i,j) = \begin{bmatrix} x(i,j)\\ w(i,j)\end{bmatrix}, \quad \hat{A}_{k\tau}(i,j) = \begin{bmatrix}\bar{A}_{k\tau} & E_k\end{bmatrix}$$

Notice that from (13) and (14), we have

$$\begin{aligned}
&\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&\leq 2\epsilon_k\big((\delta_k+\varrho_1)^2\|x(i,j)\|^2 + \varrho_2^2\|w(i,j)\|^2\big)
\end{aligned} \tag{29}$$

The following condition can be deduced easily from (15) and (16)

$$\varXi_{k\tau} < 0 \tag{30}$$

where $\varXi_{k\tau} \equiv \mathcal{A} + 2\sum_{\tau=1}^{N_2} \mu_{k\tau}\hat{A}_{k\tau}^T \mathcal{R}_k \hat{A}_{k\tau}$. Recalling the system (1), and substituting (29) into (28) yields

$$\begin{aligned} &\mathbb{E}\{\varDelta V_1(i,j) + \|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\} \\ &\leq \hat{x}^T(i,j)\varXi_{k\tau}\hat{x}(i,j) < 0 \end{aligned} \tag{31}$$

Noting (26) with the zero boundary condition, we can infer that

$$\begin{aligned} &\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2} \varDelta V_1(i,j) \\ &= \sum_{i=0}^{\kappa_1} V_1^v(i,\kappa_2+1) + \sum_{j=0}^{\kappa_2} V_1^h(\kappa_1+1,j) \\ &\geq 0 \qquad \forall \kappa_1,\kappa_2 = 1,2,3... \end{aligned} \tag{32}$$

Then, we can further deduce from (31) and (32) that

$$\begin{aligned} &\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\{\|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\} \\ &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\{\varDelta V_1(i,j) + \|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\} \\ &< 0 \end{aligned} \tag{33}$$

which implies (8) holds. And this completes the proof of Theorem 1.

□

Remark 4. *As we mentioned above, the hidden Markov model is consists of two Markov chain and will result in some difficulties in SMC design. Follow a similar method imposed in [3], the matrices $Q_{k\tau}$ and $T_{k\tau}$ are introduced to separate those two Markov chains.*

C. Analysis of reachability

The reachability of the designed asynchronous 2D-SMC law for the closed-loop system (12) will be discussed in this subsection. By using a stochastic Lyapunov method, we provide a sufficient condition which will confirm that the designed asynchronous 2D-SMC law (10) can force the state trajectories of the closed-loop system (12) into a time-varying sliding region around the specified 2D sliding surface (9).

Theorem 2. *Consider the closed-loop 2D-MJS (12) with asynchronous 2D-SMC law (10). If there exists matrices $K_\tau \in \mathbb{R}^{n_u\times n_x}$, $R_k > 0$, $F_k > 0$, and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, \tau \in \mathcal{N}_2$, such that the condition (14) and the following inequality hold*

$$2\sum_{\tau=1}^{N_2} \bar{A}_{k\tau}^T\big(\mathcal{R}_k + G^T\mathcal{F}_kG\big)\bar{A}_{k\tau} - R_k < 0 \tag{34}$$

where $\mathcal{R}_k$ is defined in Theorem 1, and $\mathcal{F}_k = \sum_{p=1}^{N_1}\lambda_{kp}F_p$. Then, the state trajectories of the considered closed-loop system will be driven into the following sliding region $\mathcal{O}$, around the predefined sliding surface (9):

$$\mathcal{O} \equiv \Big\{\|s(i,j)\| \leq \ \rho^*(i,j)\Big\} \tag{35}$$

where $\rho^*(i,j) = \max_{k\in\mathcal{N}_1}\sqrt{\hat{\rho}_k(i,j)/\lambda_{\min}(F_k)}$ with

$$\begin{aligned} \hat{\rho}_k(i,j) = 4\big(&\|E_k^T\mathcal{R}_kE_k\| + \|E_k^TG^T\mathcal{R}_kGE_k\| \\ &+ 2\varrho_2^2(\|B_k^T\mathcal{F}_kB_k\| + \|B_k^TG^T\mathcal{F}_kGB_k\|)\big)\|w(i,j)\|^2 \\ &+ 8\big(\|B_k^T\mathcal{R}_kB_k\| + \|B_k^TG^T\mathcal{R}_kGB_k\|\big) \\ &\times(\varrho_1+\delta_k)^2\|x(i,j)\|^2. \end{aligned}$$

and $\lambda_{\min}(F_k)$ here denotes the minimum eigenvalue of F_k.

Proof. First, let's define ***s(i, j)*** $= \begin{bmatrix} s^h(i+1,j) \\ s^v(i,j+1)\end{bmatrix}$, it is easy find that ***s(i, j)*** $= G$***x(i, j)***. Then, we select the Lyapunov candidate as

$$V(i,j) = V_1(i,j) + V_2(i,j) \tag{36}$$

where $V_1(i,j)$ is defined in Theorem 1, $V_2(i,j) = s^T(i,j)F_ks(i,j)$. Similar with the proof in Theorem 1, it is easy to find that

$$\begin{aligned} &\mathbb{E}\{\varDelta V_1(i,j)\} \\ &= \mathbb{E}\Big\{\boldsymbol{x(i,j)}^TR_p\boldsymbol{x(i,j)} - x^T(i,j)R_kx(i,j)\Big\} \\ &= \sum_{\tau=0}^{N_2}\mu_{k\tau}\Big\{\big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T \\ &\times\mathcal{R}_k\big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]\Big\} \\ &- x^T(i,j)R_kx(i,j) \\ &\leq 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\mathcal{R}_k\bar{A}_{k\tau}x(i,j) \\ &+ 2\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T\mathcal{R}_k \\ &\times\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big] \\ &- x^T(i,j)R_kx(i,j) \\ &\leq 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\mathcal{R}_k\bar{A}_{k\tau}x(i,j) \\ &+ \bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_kB_k\bar{\rho}_k(i,j) \\ &+ w^T(i,j)E_k^T\mathcal{R}_kE_kw(i,j) \\ &- x^T(i,j)R_kx(i,j) \end{aligned} \tag{37}$$

Along with the sliding function in (9), we have

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_2(i,j)\}\\
&=\mathbb{E}\Big\{\boldsymbol{s(i,j)}^T F_p \boldsymbol{s(i,j)} - s^T(i,j)F_k s(i,j)\Big\}\\
&=\sum_{\tau=0}^{N_2}\mu_{k\tau}\Big\{\big[\bar{A}_{k\tau}x(i,j)+B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]^T\\
&\times G^T\mathcal{F}_k G\big[\bar{A}_{k\tau}x(i,j)+B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]\Big\}\\
&-s^T(i,j)F_k s(i,j)\\
&\leq 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T G^T\mathcal{F}_k G\bar{A}_{k\tau}x(i,j)\\
&+2\big[B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]^T G^T\mathcal{F}_k\\
&\times G\big[B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]\\
&-s^T(i,j)F_k s(i,j)\\
&\leq 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T G^T\mathcal{F}_k G\bar{A}_{k\tau}x(i,j)\\
&+\bar{\rho}_k^T(i,j)B_k^T G^T\mathcal{F}_k G B_k\bar{\rho}_k(i,j)\\
&+w^T(i,j)E_k^T G^T\mathcal{F}_k G E_k w(i,j)\\
&-s^T(i,j)F_k s(i,j)
\end{aligned}\tag{38}$$

Combing (37) and (38), we can infer that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V(i,j)\}\\
&=\mathbb{E}\Big\{\varDelta_1(i,j)+\varDelta V_2(i,j)\Big\}\\
&\leq x^T(i,j)\Big\{2\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\big(\mathcal{R}_k+G^T\mathcal{F}_k G\big)\bar{A}_{k\tau}\Big\}x(i,j)\\
&+\vec{\rho}_k(i,j)-x^T(i,j)R_k x(i,j)-\lambda_{\min}(F_k)\|s(i,j)\|^2
\end{aligned}\tag{39}$$

where

$$\begin{aligned}
&\vec{\rho}_k(i,j)\\
&=4\big(\|B_k^T\mathcal{F}_k B_k\|+\|B_k^T G^T\mathcal{F}_k G B_k\|\big)\|\bar{\rho}_k(i,j)\|^2\\
&+4\big(\|E_k^T\mathcal{R}_k E_k\|+\|E_k^T G^T\mathcal{R}_k G E_k\|\big)\|w(i,j)\|^2
\end{aligned}$$

Recalling the condition (13), we can get an inequality as follows

$$\|\bar{\rho}_k(i,j)\|^2\leq 2(\varrho_1+\delta_k)^2\|x(i,j)\|^2+2\varrho_2^2\|w(i,j)\|^2 \tag{40}$$

It is obvious that $\vec{\rho}_k(i,j)<\hat{\rho}_k(i,j)$ for any $k\in\mathcal{N}_1$ after substitute (40) into $\vec{\rho}_k(i,j)$. Then, based on the condition (15), when the state trajectories is out of the region $\mathcal{O}$, we can infer that

$$-\lambda_{\min}(F_k)\|s(i,j)\|^2+\vec{\rho}_k(i,j)<0 \tag{41}$$

It yields from (34), (37), (39) and (41) that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V(i,j)\}\\
&\leq x^T(i,j)\Big\{2\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\big(\mathcal{R}_k+G^T\mathcal{F}_k G\big)\bar{A}_{k\tau}\\
&-R_k\Big\}x(i,j)<0
\end{aligned}\tag{42}$$

which means the state trajectories of the close-loop (12) are strictly decreasing (with mean square) outside the region $\mathcal{O}$ defined in (35). Now, the proof is complete.

□

D. Synthesis of Asynchronous 2D-SMC Law

It is obvious that, if the theorem 1 and the theorem 2 hold simultaneously, then, the asymptotically mean square stability with an H_∞ disturbance attenuation performance γ of the closed-loop 2D system (12) and the reachability of the predefined sliding function (9) can be guaranteed simultaneously. That is, the to be determined matrix K_τ in 2D-SMC law (10) should ensure that theorem 1, 2 are established at the same time. Now, in this subsection, we will continue our study with this idea.

Theorem 3. *Consider the 2D-MJS (1) under the Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma>0$, if there exist matrices $\tilde{K}_\tau\in\mathbb{R}^{n_u\times n_x}$, $L_\tau\in\mathbb{R}^{n_x\times n_x}$, $\tilde{R}_k=\mathrm{diag}\{\tilde{R}_k^h,\tilde{R}_k^v\}>0$, $\tilde{F}_k>0$, $\tilde{Q}_{k\tau}>0$, $\tilde{T}_{k\tau}>0$ and scalars $\tilde{\epsilon}_k>0$, for any $k\in\mathcal{N}_1,\tau\in\mathcal{N}_2$, such that the following inequalities hold:*

$$\begin{bmatrix}-\tilde{\epsilon}_k I & \mathscr{B}_k\\ * & \mathscr{R}_k\end{bmatrix}<0 \tag{43}$$

$$\begin{bmatrix}\mathscr{L}_{k\tau} & \mathscr{A}_{k\tau} & \mathscr{G}_{k\tau}\\ * & \mathscr{R}_k & 0\\ * & * & \mathscr{F}_k\end{bmatrix}<0 \tag{44}$$

$$\begin{bmatrix}\mathscr{H}_k & \mathscr{D}_k & \mathscr{P}_{k\tau} & \mathscr{Y}_{k\tau}\\ * & \mathscr{I}_k & 0 & 0\\ * & * & \mathscr{Q}_{k\tau} & 0\\ * & * & * & \mathscr{T}_{k\tau}\end{bmatrix}<0 \tag{45}$$

where

$$\begin{aligned}
\mathscr{B}_k &= \begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{\epsilon}_k B_k^T & \sqrt{\lambda_{k2}}\tilde{\epsilon}_k B_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{\epsilon}_k B_k^T\end{bmatrix}\\
\mathscr{R}_k &= \mathrm{diag}\{-\tilde{R}_1,-\tilde{R}_2,\cdots,-\tilde{R}_{N_1}\}\\
\mathscr{F}_k &= \mathrm{diag}\{-\tilde{F}_1,-\tilde{F}_2,\cdots,-\tilde{F}_{N_1}\}\\
\mathscr{I}_k &= \mathrm{diag}\{-\tilde{\epsilon}_k,-I,-I,-\tilde{\epsilon}_k\}\\
\mathscr{Q}_{k\tau} &= \mathrm{diag}\{-\tilde{Q}_{k1},-\tilde{Q}_{k2},\cdots,-\tilde{Q}_{kN_2}\}\\
\mathscr{T}_{k\tau} &= \mathrm{diag}\{-\tilde{T}_{k1},-\tilde{T}_{k2},\cdots,-\tilde{T}_{kN_2}\}\\
\mathscr{L}_{k\tau} &= \mathrm{diag}\{\tilde{Q}_{k\tau}-L_\tau^T-L_\tau,-\tilde{T}_{k\tau}\}\\
\mathscr{H}_k &= \begin{bmatrix}-\tilde{R}_k & \tilde{R}C_k^T D_k^T\\ * & -\gamma^2 I\end{bmatrix}\\
\mathscr{A}_{k\tau} &= \begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{k\tau}^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{k\tau}^T\\ \sqrt{\lambda_{k1}}\tilde{T}_{k\tau}B_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{T}_{k\tau}B_k^T\end{bmatrix}\\
\mathscr{G}_{k\tau} &= \begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{k\tau}^T G^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{k\tau}^T G^T\\ 0 & \cdots & 0\end{bmatrix}\\
\mathscr{D}_{k\tau} &= \begin{bmatrix}2(\varrho_1+\delta_k)\tilde{R}_k & \tilde{R}_k C_k^T & 0 & 0\\ 0 & 0 & D_k^T & 2\varrho_2\end{bmatrix}\\
\mathscr{P}_{k\tau} &= \begin{bmatrix}\sqrt{2\mu_{k1}}\tilde{R}_k & \cdots & \sqrt{2\mu_{kN_2}}\tilde{R}_k\\ 0 & \cdots & 0\end{bmatrix}\\
\mathscr{Y}_{k\tau} &= \begin{bmatrix}0 & \cdots & 0\\ \sqrt{2\mu_{k1}}I & \cdots & \sqrt{2\mu_{kN_2}}I\end{bmatrix}
\end{aligned}$$

and $\tilde{A}_{k\tau} = A_k L_\tau + B_k \tilde{K}_\tau$. Then, the closed-loop system (12) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ, and the state trajectories of the considered closed-loop system will be driven into a sliding region $\mathcal{O}$, around the predefined sliding surface (9). Moreover, the to be determined matrix K_τ in 2D-SMC law (10) can be chosen as

$$K_\tau = \tilde{K}_\tau L_\tau^{-1} \tag{46}$$

if the LMIs (43)-(45) have feasible solutions.

Proof. As we discussed above, the objective is to testify that, the conditions (14)-(16) in Theorem 1 and (34) in Theorem 2 can be guaranteed simultaneously by (43)-(45). Before that, let's make some notations as $\tilde{K}_\tau = K_\tau L_\tau$, $\tilde{R}_k = R_k^{-1}$, $\tilde{F}_k = F_k^{-1}$, $\tilde{Q}_{k\tau} = Q_{k\tau}^{-1}$, $\tilde{T}_{k\tau} = T_{k\tau}^{-1}$ and $\tilde{\epsilon} = \epsilon_k^{-1}$. Firstly, we will prove that (14) and (43) are equivalent. Pre- and post- multiplying the inequalities given in (43) by $\mathrm{diag}\{\epsilon_k I, I, I, \cdots, I\}$, respectively, and applying Schur complement after that, then, we can see (14) satisfied. Next, we will verity that (44)-(45) are sufficient to ensure (15)-(16) and (34) hold simultaneously. Using $\mathrm{diag}\{R_k, I, I, \cdots, I\}$ to pre- and post-multiply the inequality given in (45), and applying Schur complement after that, then we will have (15) satisfied. It follows from (44) that $\tilde{Q}_{k\tau} - L_\tau^T - L_\tau < 0$, that is $L_\tau^T + L_\tau$ is positive definite, which guarantees that the matrix L_τ is invertible. We can infer the following formulation based on $Q_{k\tau} > 0$

$$(\tilde{Q}_{k\tau} - L_\tau)^T \tilde{Q}_{k\tau}^{-1} (\tilde{Q}_{k\tau} - L_\tau) \geq 0 \tag{47}$$

which means

$$-L_\tau^T \tilde{Q}_{k\tau} L_\tau \leq \tilde{Q}_{k\tau} - L_\tau^T - L_\tau \tag{48}$$

Recalling the condition give in (45), then we can infer that

$$\begin{bmatrix} \tilde{\mathscr{L}}_{k\tau} & \mathscr{A}_{k\tau} & \mathscr{G}_{k\tau} \\ * & \mathscr{R}_k & 0 \\ * & * & \mathscr{F}_k \end{bmatrix} < 0 \tag{49}$$

where $\tilde{\mathscr{L}}_{k\tau} = \mathrm{diag}\{-L_\tau^T \tilde{Q}_{k\tau} L_\tau, -\tilde{T}_{k\tau}\}$.
Noting that the slack matrix L_τ is invertible, we denote $h_{k\tau} = \mathrm{diag}\{L_\tau^{-1}, T_{k\tau}, I, I, \cdots, I\}$. Using $h_{k\tau}$ to pre- and post-multiply the inequality given in (49), and applying Schur complement after that, then the following inequality will be obtained

$$\hat{A}_{k\tau}^T \mathcal{R}_k \hat{A}_{k\tau} + \check{A}_{k\tau}^T \mathcal{F}_k \check{A}_{k\tau} - \mathrm{diag}\{Q_{k\tau}, T_{k\tau}\} < 0 \tag{50}$$

where $\check{A}_{k\tau} = \begin{bmatrix} \bar{A}_{k\tau} & 0 \end{bmatrix}$. Combing (15) and (50), we will have

$$\mathcal{A} + 2 \sum_{\tau=0}^{N_2} \mu_{k\tau} \left\{ \hat{A}_{k\tau}^T \mathcal{R}_k \hat{A}_{k\tau} + \check{A}_{k\tau}^T \mathcal{F}_k \check{A}_{k\tau} \right\} < 0 \tag{51}$$

which further implies (34) holds based on the property of positive definite matrix. It is clear that, the gain matrix K_τ can not obtained directly from LMIs in Theorem 3 while $\tilde{K}_\tau$ is obtained. Thanks to the matrix L_τ is invertible, K_τ can be calculated indirectly with $K_\tau = \tilde{K}_\tau L_\tau^{-1}$. Now, the proof is finished. □

It is clear that, the conditions in Theorem 3 are presented in the form of Linear matrix inequality, which can be easily solved with the help of Matlab LMI toolbox. Next, we are going to propose an algorithm to obtain the asynchronous 2D-SMC law with the minimum disturbance attenuation performance γ^*. The design procedures of the 2D-SMC are summarized into an algorithm as follows.

1. Select the parameters β_k properly and compute matrix G, such that the matrix GB_k is nonsingular for any $k \in \mathcal{N}_1$.
2. Compute the scalars ϱ_1 and ϱ_2 in (11).
3. Get the matrices $\tilde{K}_\tau$ and L_τ by solving the following optimization problem

$$\min \tilde{\sigma} \text{ subject to (43)-(45) with } \tilde{\sigma} = \gamma^2 \,. \tag{52}$$

4. Then, the sliding mode controller matrix K_τ can be obtained with $K_\tau = \tilde{K}_\tau L_\tau^{-1}$.

IV. Numerical Example

In this section, we provide an example to verify the effectiveness of the proposed asynchronous 2D-SMC law design method. It is known that some dynamical processes, for instance, gas absorption, air drying and water stream heating can be described by the Darboux equation [18]. It is generally described in the form of partial differential equation, and can be easily converted into a 2D-MJS in Roesser model with a similar approach applied in [21]. In this example, the considered 2D-MJS and the asynchronous 2D-SMC law both consist of two operation modes, that is, $\mathcal{N}_1 = \{1, 2\}$, $\mathcal{N}_2 = \{1, 2\}$, and the corresponding system matrices are listed as follows:
Mode 1:

$$A_1 = \begin{bmatrix} -1.0 & 0.4 \\ 0.2 & -1.0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & -0.3 \end{bmatrix}, \; D_1 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.1 \end{bmatrix}, \; E_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

Mode 2:

$$A_2 = \begin{bmatrix} -0.8 & 0.6 \\ 0.2 & -1.2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.2 & -0.2 \\ -0.1 & -0.2 \end{bmatrix}, \; D_2 = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -0.2 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \; E_2 = E_1$$

The nonlinear function $f(x(i,j), r(i,j))$ is set as

$$f((x(i,j), r(i,j)) = \begin{cases} 0.2 \sin \|x(i,j)\|, & r(i,j) = 1 \\ 0.3 \sin \|x(i,j)\|, & r(i,j) = 2 \end{cases}$$

Thus, the parameter δ_k can be selected as first mode with $\delta_1 = 0.2$ and second mode with $\delta_2 = 0.3$.
The mode jumps of the considered 2D-MJS and 2D-SMC are governed by the transition probability matrix Λ and Ψ, respectively, which are set as follows:

$$\Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \; \Psi = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

A possible time sequences with two different directions of the system modes and the asynchronous 2D-SMC modes are

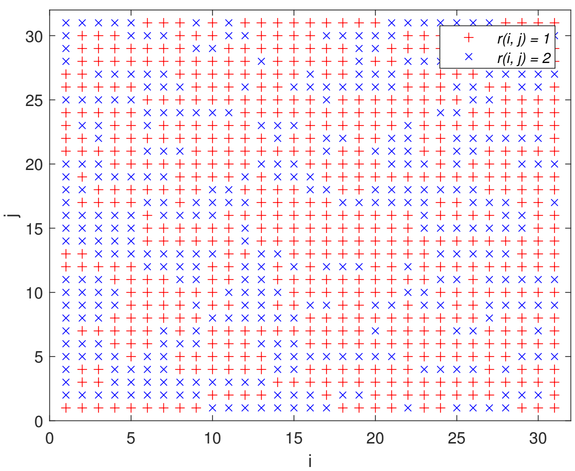

Fig. 1. Modes of the considered 2D system

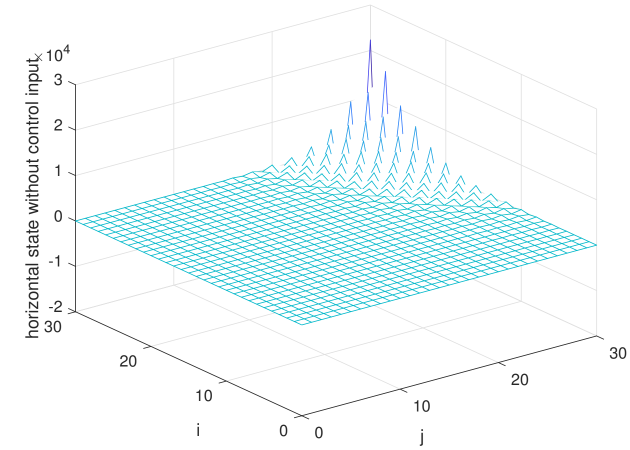

Fig. 3. Horizontal state $x^h(i,j)$ with control input $u(i,j)=0$

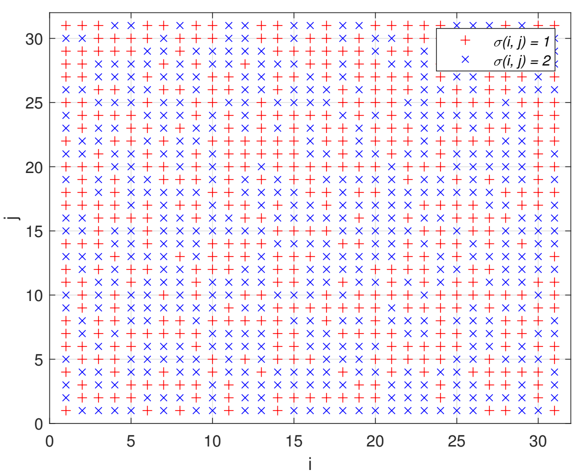

Fig. 2. Modes of the 2D-SMC law

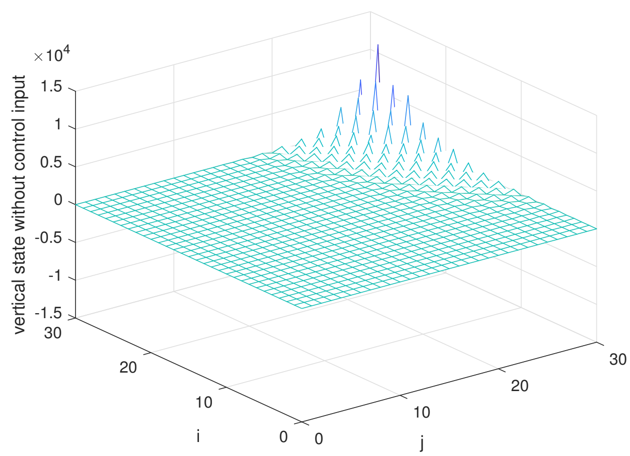

Fig. 4. Vertical state $x^v(i,j)$ without control input $u(i,j)=0$

depicted in Fig.1-2. By comparison, it is clear that the designed 2D-SMC run asynchronously with the original 2D system. Based on the corresponding assumptions, the boundary condition X_0 and the exogenous disturbance $w(i,j)$ are assumed to be

$$x^h(0,j)=\begin{cases}0.1, & 0\le j\le 10\\ 0, & elsewhere\end{cases}$$

$$x^v(i,0)=\begin{cases}0.1, & 0\le i\le 10\\ 0, & elsewhere\end{cases}$$

$$w(i,j)\ =\begin{cases}0.2, & 0\le i,j\le 10\\ 0, & elsewhere\end{cases}$$

The horizontal and vertical state responses of the open-loop system (1) under $u(i,j)=0$ are depicted in Fig.3 and Fig.4, respectively. The obtained result implies the considered 2D system is unstable under zero control input. Next, let's follow the procedures of asynchronous 2D-SMC law design algorithm, and obtain the sliding model controller gain. Letting $\beta_1=0.6$ and $\beta_2=0.4$, then the matrix G can be calculated as follows:

$$G=\begin{bmatrix}0.2 & -0.16\\ -0.02 & -0.26\end{bmatrix}$$

It's easy to verify that the non-singularity of the matrix GB_k is satisfied for any $k\in\mathcal{N}_1$. The parameters ϱ_1 and ϱ_2 can be calculated as $\varrho_1=0.3$ and $\varrho_2=0.6872$, respectively. By solving optimization problem (52), we can obtain the following sliding mode controller gains

Mode1

$$K_1=\begin{bmatrix}4.4147 & -4.3512\\ -0.6827 & -2.1886\end{bmatrix}$$

Mode2

$$K_2=\begin{bmatrix}4.3706 & -3.8453\\ -0.6826 & -2.3252\end{bmatrix}$$

with the minimum H_∞ disturbance attenuation performance $\gamma^*=0.3164$.

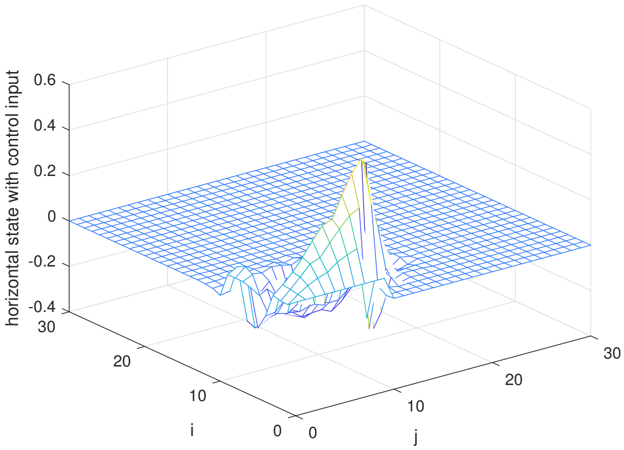

Fig. 5. Horizontal state $x^h(i,j)$ with asynchronous 2D-SMC law

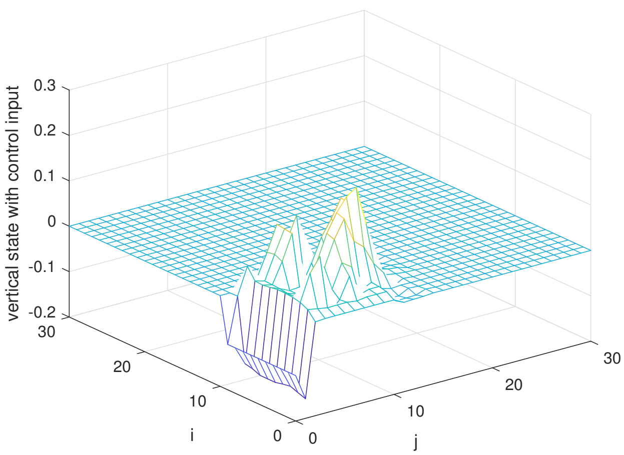

Fig. 6. Vertical state $x^v(i,j)$ with asynchronous 2D-SMC law

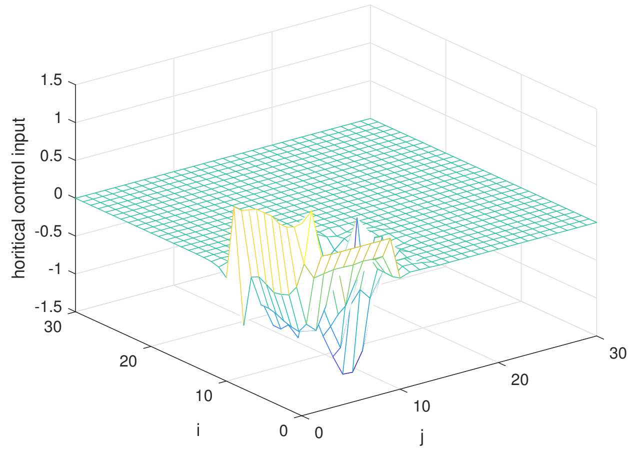

Fig. 7. Horizontal asynchronous 2D-SMC input $u^h(i,j)$

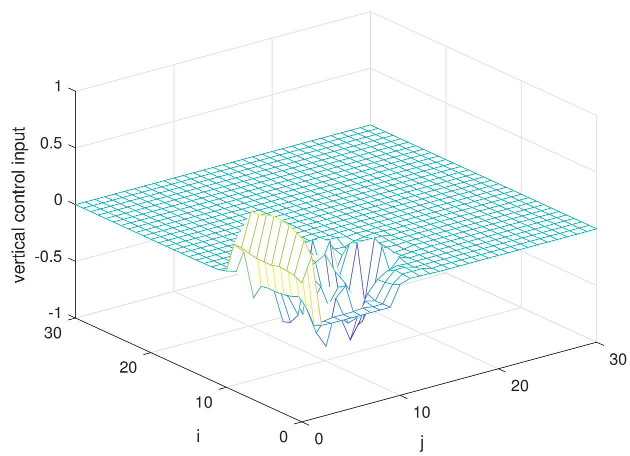

Fig. 8. Vertical asynchronous 2D-SMC input $u^v(i,j)$

The horizontal and vertical state response of the closed-loop system (12) with asynchronous 2D-SMC law are depicted in Fig.5-6. It is clear that both horizontal and vertical system state converge to zero within finite steps. The sliding surface $s(i,j)$ and the asynchronous 2D-SMC law can be obtained based on the conditions given in (9) and (10), respectively. Then, the horizontal 2D-SMC input, vertical 2D-SMC input, horizontal sliding surface and vertical sliding surface are depicted in Fig.7-10. Clearly, the observed simulation results imply that the proposed asynchronous 2D-SMC law is effective for the considered 2D-MJS.

V. Conclusions

This work has investigate the problem of asynchronous SMC for 2D-MJSs on Roesser model. In consideration of the system modes is is not always accessible to the controller, an asynchronous 2D-SMC design method is explored with hidden Markov model. By utilizing Lyapunov function, the asymptotic mean square stability for the concerned 2D-MJS and the reachability of the sliding mode dynamics are analyzed, respectively. A sufficient condition is derived that can guarantee the asymptotic mean square stability for the closed-loop system with H_∞ disturbance attenuation performance and the reachability of the 2D sliding mode dynamics, simultaneously. Finaly, the asynchronous 2D-SMC design procedures are summarized as an algorithm whose effectiveness is verified by a numerical example.

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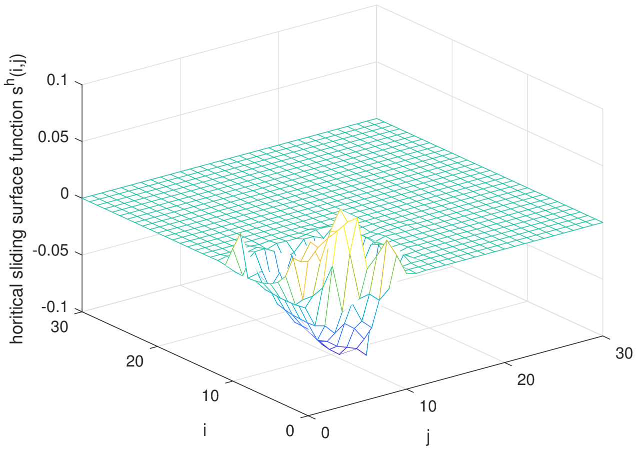

Fig. 9. Horizontal sliding surface $s^h(i, j)$

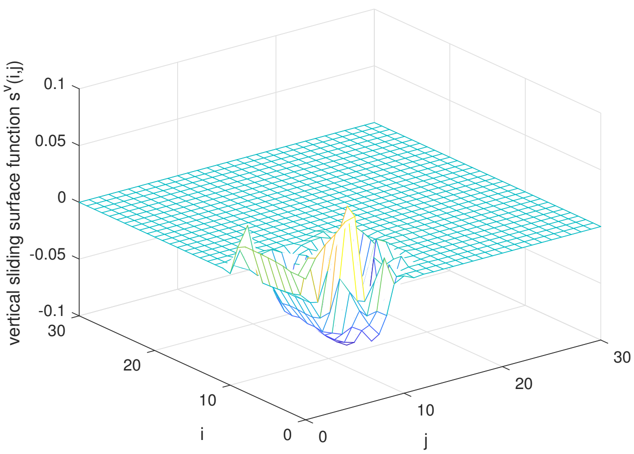

Fig. 10. Vertical sliding surface $s^v(i, j)$

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