Asynchronous Siding Mode Control of Two-dimensional Markov Jump Systems in Roesser Model

***Abstract*—abstract**

***Index Terms*—Markov jump systems, 2D systems, Siding mode control, Hidden Markov model**

I. Introduction

This part is introduciton.

II. Preliminaries

In this paper, we consider the following two-dimensional Markov jump systems in Roesser model:

$$\begin{cases} \textbf{\textit{x(i, j)}} = A_{r(i,j)}x(i,j) + E_{r(i,j)}w(i,j) \\ \qquad + B_{r(i,j)}[(u(i,j) + f(x(i,j), r(i,j))] \\ y(i,j) = C_{r(i,j)}x(i,j) + D_{r(i,j)}w(i,j) \end{cases} \tag{1}$$

where

$$\textbf{\textit{x(i, j)}} = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix},\ x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$$

$x^h(i,h) \in \mathbb{R}^{n_h}$ and $x^v(i,h) \in \mathbb{R}^{n_v}$ represent horizontal and vertical states respectively, $u(i,j) \in \mathbb{R}^{n_u}$ and $y(i,j) \in \mathbb{R}^{n_y}$ represent the controlled input and output respectively, and $w(i,j) \in \mathbb{R}^{n_w}$ represents the exogenous disturbance which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$. $A_{r(i,j)}$,$B_{r(i,j)}$,$C_{r(i,j)}$,$D_{r(i,j)}$ and $E_{r(i,j)}$ represent the time-varying system matrices, all of which are real known constant matrices with appropriate dimensions. Besides, we assume that the matrix $B_{r(i,j)}$ is full column rank for each $r(i,j) \in \mathcal{N}_1$, that is, rank($B_{r(i,j)}$)= n_u. The nonlinear function $f(x(i,j), r(i,j))$ satisfying the following property:

$$\|f(x(i,j), r(i,j)\| \leq \delta_{r(i,j)}\|x(i,j)\| \tag{2}$$

where $\delta_{r(i,j)}$ is a known scalar, $\|\cdot\|$ denotes the Euclidean norm of a vector. The parameter $r(i,j)$ takes values in a finite set $\mathcal{N}_1 = \{1,2...,N_1\}$ with transition probability matrix $\Lambda = \{\lambda_{k\tau}\}$, and the related transition probability from mode k to mode τ is given by

$$\begin{aligned} &\Pr\{r(i+1,j) = \tau | r(i,j) = k\} \\ =&\Pr\{r(i,j+1) = \tau | r(i,j) = k\} = \lambda_{k\tau},\ \forall k,\tau \in \mathcal{N}_1 \end{aligned} \tag{3}$$

where $\lambda_{k\tau} \in [0,1]$, for all $k,\tau \in \mathcal{N}_1$, and $\sum_{\tau=1}^{N_1} \lambda_{k\tau} = 1$ for every mode k.

We define the boundary condition (X_0, Γ_0) of system (1), as follows:

$$\begin{cases} X_0 = \{x^h(0,j), x^v(i,0) |\ i,j = 0,1,2...\} \\ \mathit{\Gamma}_0 = \{r(0,j), r(i,0) |\ i,j = 0,1,2...\} \end{cases} \tag{4}$$

And the corresponding zero boundary condition is assumed as $x^h(0,j) = 0, x^v(i,0) = 0, i,j = 0,1,2....$ Besides, we further impose following assumption on X_0.

Assumption 1. *The boundary condition X_0 satisfies:*

$$\lim_{L\to\infty} \mathbb{E}\Big\{\sum_{\ell=1}^{L}(\|x^h(0,\ell)\|^2 + \|x^v(\ell,0)\|^2)\Big\} < \infty \tag{5}$$

where $\mathbb{E}\{\cdot\}$ stands for mathematical expectation.

In practical applications, the complete information of $r(i,j)$ can not always be available to the controller. Hence, in this paper, the hidden Markov model $(r(i,j), \sigma(i,j), \mathit{\Lambda}, \mathit{\Psi})$ as in [refto] is introduced to characterize the asynchronous phenomenon between the controller and the system. The parameter $\sigma(i,j)$, refers to controller mode, takes values in another finite set $\mathcal{N}_2 = \{1,2...N_2\}$, and satisfies the conditional probability matrix $\mathit{\Psi} = \{\mu_{ks}\}$ with conditional mode transition probabilities

$$\Pr\{\sigma(i,j) = s | r(i,j) = k\} = \mu_{ks} \tag{6}$$

where $\mu_{ks} \in [0,1]$ for all $k \in \mathcal{N}_1, s \in \mathcal{N}_2$, and $\sum_{s=1}^{N2} \mu_{ks} = 1$ for any mode k.

Next, the definitions of asymptotically mean square stable and H_∞ performance for 2D systems will be given in Definition 1 and Definition 2, respectively.

Definition 1. *The 2D Markov jump system* (1) *with $w(i,j) \equiv 0$ is said to be asymptotically mean square stable if the following holds:*

$$\lim_{i+j\to\infty} \mathbb{E}\{\|x(i,j)\|^2\} = 0 \tag{7}$$

for any boundary condition X_0 with Assumption 1.

Definition 2. *Given a scalar $\gamma > 0$, the 2D Markov jump system* (1) *is said to be asymptotically mean square stable with an H_∞ disturbance attenuation performance γ if the system satisfies* (7)*, and under zero boundary condition, the following holds:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|y(i,j)\|^2\} < \gamma^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|w(i,j)\|^2\} \tag{8}$$

for all $w(i,j) \in \ell_2\{[0,\infty),[0,\infty)\}$.

Now, we will make some notational simplification for convenience. The parameter $r(i,j)$ is represented by k, $r(i+1,j)$

and $r(i,j+1)$ are represented by τ, $\sigma(i,j)$ is represented by s.

The objective of this work is to devise an asynchronous SMC law $u(i,j)$, such that the 2D Markov jump system (1) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

III. Main Result

A. Sliding surface and sliding mode controller

In this paper, a novel Two-dimensional sliding surface function is constructed as follows:

$$s(i,j)=\begin{bmatrix} s^h(i,j)\\ s^v(i,j)\end{bmatrix}=Gx(i,j) \tag{9}$$

where $G=\sum_{k=1}^{N_1}\beta_k G_k^T$, and scalars β_k should be chosen such that GB_k is nonsingular for any $k\in\mathcal{N}_1$. Based on the the assumption that B_k is full column rank for any $k\in\mathcal{N}_1$, we can find that the above condition can be guaranteed easily with the properly selected parameter β_k.

An asynchronous 2D-SMC law is designed as follows:

$$u(i,j)=K_s x(i,j)-\rho(i,j)\frac{s(i,j)}{\|s(i,j)\|} \tag{10}$$

for any $s\in\mathcal{N}_2$, where the matrix $K_s\in\mathbb{R}^{n_u\times n_x}$ with $n_x=n_h+n_v$ will be determined later, and the parameter $\rho(i,j)$ is given as

$$\rho(i,j)=\varrho_1\|x(i,j)\|+\varrho_2\|w(i,j)\| \tag{11}$$

with $\varrho_1=\max_{k\in\mathcal{N}_1}\{\delta_k\}$, $\varrho_2=\max_{k\in\mathcal{N}_1}\{\|(GB_k)^{-1}GE_k\|\}$, and the parameter δ_k is given in (2).

Combining system (1) and the asynchronous 2D-SMC low (9), the closed-loop 2D markov jump system can be obtained easily as follows:

$$\boldsymbol{x(i,\,j)}=\bar{A}_{ks}x(i,j)+B_k\bar{\rho}_k(i,j)+E_k w(i,j) \tag{12}$$

where $\bar{A}_{ks}=A_k+B_kK_s$, and $\bar{\rho}_k(i,j)$ as follows

$$\bar{\rho}_k(i,j)=f_k(x(i,j))-(\varrho_1\|x(i,j)\|+\varrho_2\|w(i,j)\|)\cdot\frac{s(i,j)}{\|s(i,j)\|}.$$

Then, based on the properties of norm, the following condition can be deduced easily

$$\|\bar{\rho}_k(i,j)\|\le(\varrho_1+\delta_k)\|x(i,j)\|+\varrho_2\|w(i,j)\|. \tag{13}$$

B. Analysis of Stability and H_∞ attenuation performance

In this subsection, we focus on the stability and H_∞ attenuation performance analysis for the closed-loop 2D system (12). A sufficient condition will be derived to guarantee the considered system is asymptotically mean square stable with an H_∞ attenuation performance γ.

Theorem 1. *Consider the Markov jump system (1) under the Assumption (1) and with the asynchronous 2D-SMC low (10). For a given scalar $\gamma>0$, if there exist matrices $K_s\in\mathbb{R}^{n_u\times n_x}$, $R_k=\mathrm{diag}\{R_k^h,R_k^v\}>0$, $Q_{ks}>0$, $T_{ks}>0$ and scalars $\epsilon_k>0$, for any $k\in\mathcal{N}_1, s\in\mathcal{N}_2$, such that the following inequalities hold:*

$$B_k^T\mathcal{R}_kB_k-\epsilon_kI\le 0 \tag{14}$$

,

$$\mathcal{A}+2\Big(\sum_{s=0}^{N_2}\mu_{ks}\mathrm{diag}\{Q_{ks},T_{ks}\}\Big)<0 \tag{15}$$

,

$$\hat{A}_{ks}^T\mathcal{R}_k\hat{A}_{ks}-\mathrm{diag}\{Q_{ks},T_{ks}\}<0 \tag{16}$$

where

$$\mathcal{A}=\begin{bmatrix}\mathit{\Pi}_1 & \mathit{\Pi}_3\\ * & \mathit{\Pi}_2\end{bmatrix}$$

with

$$\begin{cases}\mathit{\Pi}_1=-R_k+4(\delta_k+\varrho_1)^2\epsilon_kI+C_k^TC_k\\ \mathit{\Pi}_2=-\gamma^2I+D_k^TD_k+4\varrho_2^2\epsilon_kI\\ \mathit{\Pi}_3=C_k^TD_k\end{cases}$$

and $\mathcal{R}_k=\sum_{\tau=1}^{N1}\lambda_{k\tau}R_\tau$, $\hat{A}_{ks}=\begin{bmatrix}\bar{A}_{ks} & E_k\end{bmatrix}$, then, the closed-loop system (12) is asymptotically mean square stable with an H_∞ disturbance attenuation performance γ.

Proof. Let's start the proof with the stability of system. We select the Lyapunov candidate as $V_1(i,j)=x^(i,j)R_kx(i,j)$, then, define

$$\mathit{\Delta}V_1(i,j)=\boldsymbol{x(i,\,j)}^TR_\tau\boldsymbol{x(i,\,j)}-x^T(i,j)R_kx(i,j) \tag{17}$$

Based on the closed-loop system equation (12) with $w(i,j)=0$, it is easy to find that

$$\begin{aligned}&\mathbb{E}\{\mathit{\Delta}V_1(i,j)\}\\ &=\sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j)+B_k\bar{\rho}_k(i,j)\big]^T\mathcal{R}_k\\ &\times\big[\bar{A}_{ks}x(i,j)+B_k\bar{\rho}_k(i,j)\big]\Big\}\\ &-x^T(i,j)R_kx(i,j)\\ &\le x^T(i,j)\Big\{2\Big(\sum_{s=1}^{N_2}\mu_{ks}\bar{A}_{ks}^T(i,j)\mathcal{R}_k\bar{A}_{ks}\Big)\Big\}x(i,j)\\ &+2\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_kB_k\bar{\rho}_k(i,j)\\ &-x^T(i,j)R_kx(i,j)\end{aligned} \tag{18}$$

Recalling the conditions given in (13) and (14), the following inequality can be further obtained

$$\mathbb{E}\{\mathit{\Delta}V_1(i,j)\}\le x^T(i,j)\mathcal{G}_{ks}x(i,j) \tag{19}$$

where $\mathcal{G}_{ks}=2\big(\sum_{s=0}^{N_2}\mu_{ks}\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks}\big)+2\epsilon_k(\delta_k+\varrho_1)^2I-R_k$. The following inequality can be deduced from (15) based on the properties of matrix quadratic

$$2\Big(\sum_{s=1}^{N_2}\mu_{ks}Q_{ks}\Big)+4\epsilon_k(\delta_k+\varrho_1)^2I+C_k^TC_k-R_k<0 \tag{20}$$

which will further deduce

$$2\Big(\sum_{s=1}^{N_2}\mu_{ks}Q_{ks}\Big)+2\epsilon_k(\delta_k+\varrho_1)^2I-R_k<0 \tag{21}$$

The following inequality can be inferred directly from condition (16)

$$\bar{A}_{ks}^T\mathcal{R}_k\bar{A}_{ks}-Q_{ks}<0 \tag{22}$$

Combine (21) and (22), we can infer that $\mathcal{G}_{ks} < 0$, which is equivalent to

$$\mathcal{G}_{ks} \leq -\alpha I \tag{23}$$

with scalar $\alpha > 0$. Recalling (19), we can further infer that

$$\mathbb{E}\{\varDelta V_1(i,j)\} \leq -\alpha \mathbb{E}\{\|x(i,j)\|^2\} \tag{24}$$

Summing up on the both side of (24), we have

$$\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\|x(i,j)\|^2\Big\} \leq -\frac{1}{\alpha}\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\Big\} \tag{25}$$

where parameters κ_1, κ_2 are any positive integers. By substituting $\varDelta V_1$ and R_k with (17) and $R_k = \mathrm{diag}\{R_k^{\mathrm{h}}, R_k^{\mathrm{v}}\}$ respectively, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\\
&= \sum_{i=0}^{\kappa_1}\big\{V_1^v(i,\kappa_2+1) - V_1^v(i,0)\big\}\\
&- \sum_{j=0}^{\kappa_2}\big\{V_1^h(\kappa_1+1,j) - V_1^h(0,j)\big\}\\
&\leq -\Big(\sum_{i=0}^{\kappa_1}V_1^v(i,0) + \sum_{j=0}^{\kappa_2}V_1^h(0,j)\Big)
\end{aligned} \tag{26}$$

where $V_1^h(i,j)$ and $V_1^v(i,j)$ are defined as

$$\begin{cases} V_1^h(i,j) = x^{hT}(i,j)R^h_{r(i,j)}x^h(i,j)\\ V_1^v(i,j) = x^{vT}(i,j)R^v_{r(i,j)}x^v(i,j)\end{cases}$$

Recalling the boundary condition in Assumption 1, and let κ_1, κ_2 tend to infinity, it follows from (25) and (26) that

$$\begin{aligned}
&\mathbb{E}\Big\{\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\|x(i,j)\|^2\Big\}\\
&\leq -\frac{\beta}{\alpha}\sum_{\ell=0}^{\infty}\big(\|x^v(\ell,0)\|^2 + \|x^h(0,\ell)\|^2\big)\\
&< \infty
\end{aligned} \tag{27}$$

where β is the maximum value of $R^h(0,\ell)$ and $R^v(\ell,0)$, for any $\ell = 0,1,2...$, which implies that (7) holds. Thus, the asymptotically mean square stable of the considered system is proved.

Next, let's focus on the H_∞ attenuation performance under zero boundary condition. Based on the closed-loop system equation (12), it is easy to find that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j)\}\\
&= \sum_{s=0}^{N_2}\mu_{ks}\Big\{\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_p w(i,j)\big]^T\\
&\times \mathcal{R}_k\big[\bar{A}_{ks}x(i,j) + B_k\bar{\rho}_k(i,j) + E_p w(i,j)\big]\Big\}\\
&- x^T(i,j)R_k x(i,j)\\
&\leq \hat{x}^T(i,j)\Big\{2\Big(\sum_{s=1}^{N_2}\mu_{ks}\hat{A}_{ks}^T(i,j)\mathcal{R}_k\hat{A}_{ks}\Big)\Big\}\hat{x}(i,j)\\
&+ 2\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&- x^T(i,j)R_k x(i,j)
\end{aligned} \tag{28}$$

where

$$\hat{x}(i,j) = \begin{bmatrix} x(i,j)\\ w(i,j)\end{bmatrix},\ \hat{A}_{ks}(i,j) = \begin{bmatrix}\bar{A}_{ks} & E_k\end{bmatrix}$$

Notice that from (13) and (14), we have

$$\begin{aligned}
&\bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_k B_k\bar{\rho}_k(i,j)\\
&\leq 2\epsilon_k\big((\delta_k + \varrho_1)^2\|x(i,j)\|^2 + \varrho_2^2\|w(i,j)\|^2\big)
\end{aligned} \tag{29}$$

The following condition can be deduced easily from (15) and (16)

$$\varXi_{ks} < 0 \tag{30}$$

where $\varXi_{ks} \equiv \mathcal{A} + 2\sum_{s=1}^{N_2}\mu_{ks}\hat{A}_{ks}^T\mathcal{R}_k\hat{A}_{ks}$. Recalling the system (1), and substituting (29) into (28) yields

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j) + \|z(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&\leq \hat{x}^T(i,j)\varXi_{ks}\hat{x}(i,j) < 0
\end{aligned} \tag{31}$$

Noting (26) with the zero boundary condition, we can infer that

$$\begin{aligned}
&\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2}\varDelta V_1(i,j)\\
&= \sum_{i=0}^{\kappa_1}V_1^v(i,\kappa_2+1) + \sum_{j=0}^{\kappa_2}V_1^h(\kappa_1+1,j)\\
&\geq 0 \qquad \forall \kappa_1,\kappa_2 = 1,2,3...
\end{aligned} \tag{32}$$

Then, we can further deduce from (31) and (32) that

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\{\|z(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\{\varDelta V_1(i,j) + \|z(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\}\\
&<0
\end{aligned} \tag{33}$$

which implies (8) holds. And this completes the proof of Theorem 1.

□

Remark 1. *Remark.*

C. Analysis of reachability

The reachability of the designed asynchronous 2D-SMC low for the closed-loop system (12) will be discussed in this subsection. By using a stochastic Lyapunov method, we provide a sufficient condition which will confirm that the designed asynchronous 2D-SMC law (10) can force the state trajectories of the closed-loop system (12) into a time-varying sliding region around the specified 2D sliding surface (9).

Theorem 2. *Theroem.*

Proof. Proof of theorem. □

Remark 2. *Remark.*

IV. Numerical Example

In this section, we provide an example to verify the validity of the proposed method.

V. Conclusions

References

[1] Zhang, Guangming, et al. "Finite-time H static output control of Markov jump systems with an auxiliary approach." Applied Mathematics & Computation 273.C(2016):553-561.