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Two-dimensional Sliding Mode Control of Discrete-Time Fornasini-Marchesini Systems

Rongni Yang, *Member, IEEE* and Wei Xing Zheng, *Fellow, IEEE*

***Abstract*—This paper is concerned with the problem of discrete-time two-dimensional (2-D) sliding mode control (SMC) for Fornasini-Marchesini (FM) systems subject to exogenous nonlinear disturbances. With the introduction of the time instant and global state in the 2-D case, the discrete-time sliding surface function for 2-D systems is constructed. Then based on the corresponding sliding mode dynamics, the solvability condition for the desired sliding surface function is derived. Subsequently, a sliding mode controller is designed to guarantee that the state trajectories can enter into a neighborhood of the specified sliding surface. Meanwhile, the stability of the 2-D sliding mode dynamics is also ensured. Finally, a simulation example is provided to illustrate the feasibility and the effectiveness of the presented new 2-D SMC design method. Different from the existing results, the proposed work focuses on the features of multi-dimension and the dynamics of 2-D systems, which is expected to develop a completely different SMC design approach for discrete-time 2-D systems.**

***Index Terms*—Sliding mode control (SMC), 2-D systems, Fornasini-Marchesini local state-space (FMLSS), nonlinear disturbance.**

I. Introduction

Two-dimensional (2-D) systems can be used to describe a wide variety of phenomena such as heat, sound, image, electrodynamics, electrostatics, elasticity, and fluid flow [1], [2]. Thus, 2-D systems have remarkably contributed to various fields of information communication, electrical and electronic engineering, etc. Its extensive applications in process control, power transmission lines, vehicle platoons, grid sensor network, and signal processing have been witnessed [3]–[6]. A lot of research results have been reported on 2-D systems (see, e.g., [7]–[12] and the references therein). Different from traditional 1-D systems, 2-D systems propagate information in two independent directions and the 2-D variables have brought much difficulty and complexity in the analysis and synthesis of 2-D systems. Even stability conditions in terms of LMI are similar to classical 1-D system conditions, but the Lyapunov matrix must be chosen as the diagonal one and the resulting condition is only sufficient one. In addition, in many engineering applications, 2-D models subject to exogenous nonlinear disturbances are frequently used to describe physical systems. In particular, when external disturbances are also involved, the multi-dimensional features make it more difficult to analyze, simulate or design such 2-D systems. System performance may be degraded by the affection of exogenous disturbances. Moreover, the results and techniques in 1-D systems become inapplicable for 2-D systems. Therefore, it is necessary and meaningful to deal with the presence of external disturbances in 2-D systems based on the features of 2-D systems.

Sliding mode control (SMC), as a powerful robust control strategy for handling exogenous disturbances and uncertainties, has been widely applied to numerous complex systems and engineering [13]–[15]. The main idea of SMC is to utilize a discontinuous control to force the system state trajectories toward the sliding surface and remain on the sliding surface after reaching it. Then the dynamics restricted to the sliding surface has the desired properties such as stability, disturbance rejection capability and tracking [16]. Nevertheless, it is worth noting that, in most of the aforementioned works on SMC, the system dynamic behaviors were considered for 1-D systems, such as singular systems, uncertain systems, stochastic systems, Markovian jump systems, and switched systems [17]–[22]. Unfortunately, the SMC problem for 2-D systems subject to external disturbances has not been fully investigated up to now despite its theoretical and practical importance. Indeed, there are some preliminary results on sliding mode control of 2-D systems [23]–[26]. To be specific, the SMC problem of discrete-time 2-D systems in Roesser model was investigated in [23], [25]; Argha et al. [26] applied the discrete SMC to the 2-D systems represented by the first Fornasini-Marchesini (FM) model with the aid of the 1-D vectorial form of 2-D systems. Nevertheless, the methods and techniques proposed in [23], [25], [26] suffer from some drawbacks. First, attentions are focused on the particular form of 2-D systems in Roesser model [23], [25] and the first FM model [26]. It is well known that 2-D systems can be represented by different models, such as Roesser model, Attasi model, the first FM model, the second FM model and the general model. Note that Roesser model, Attasi model, the first FM model and the general model can all be treated as special cases of the second FM model, which is also termed as the Fornasini-Marchesini local state-space (FMLSS) model in the literature. Second, the approaches adopted are only a direct extension of 1-D SMC design to the 2-D system. The dynamics and the special structure of the 2-D system are ignored. This motivates us to develop a new SMC method for 2-D FM systems by exploring the multi-dimensional features and the special dynamics of 2-D systems.

Generally speaking, the design of the SMC scheme involves

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R. Yang is with the School of Computing, Engineering and Mathematics, Western Sydney University, Sydney, NSW 2751, Australia, and also with School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, China (e-mail: rongni.yang@westernsydney.edu.au).

W. X. Zheng is with the School of Computing, Engineering and Mathematics, Western Sydney University, Sydney, NSW 2751, Australia (e-mail: w.zheng@westernsydney.edu.au).

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two steps [16]. However, the description of discrete-time SMC for 2-D FM systems should focus on the dynamics of 2-D systems and multi-dimensional features. That is, firstly, how to construct an effective 2-D function of the discrete-time sliding mode surface to ensure that the corresponding discrete-time 2-D FM system is asymptotically stable? secondly, how to design an appropriate discrete-time 2-D controller to drive the trajectories to a band of the specified sliding surface and remain within it? In fact, these pose challenging and difficult issues, since one is faced with 2-D dynamic function and special structure of two independent variables. Necessarily none of two independent variables can be regarded as time in 1-D systems. In the traditional SMC systems, the sliding surface function is usually assumed to be a linear combination of the system state $x(k)$. However, since there exist two-dimensional horizontal and vertical coefficients, a direct extension of the available SMC method in 1-D systems to 2-D FM systems is infeasible as this would lead to the coupling of the system matrices A_1, A_2, B_1, B_2 and the difficulty in developing control synthesis methods.

In this work, the new SMC method is proposed for discrete-time 2-D systems described by the FMLSS model. In detail, the global states in the 2-D case are firstly introduced based on the definition of time instant κ in 2-D systems. Then depending on the key features of 2-D state recursion and SMC, a discrete-time 2-D sliding function is constructed for FM systems and a 2-D SMC design algorithm is derived in the presence of external nonlinear disturbances. Finally, an example is provided to illustrate the effectiveness of the proposed SMC method. The contributions of this study are summarized as follows: 1) The definitions of time instant κ and the global states in the 2-D case are introduced, which lays down the basis for the subsequent analysis; 2) A new solution is proposed for the discrete-time SMC problem of 2-D FM systems subject to external nonlinear disturbances; 3) Different from the existing results [23], [25], [26], the proposed SMC design method focuses on the dynamics of 2-D system and multi-dimensional features, thus it is expected to develop a completely new SMC design scheme for 2-D systems.

Notations. The notations used throughout the paper are fairly standard. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$; the notation $P > 0$ means that matrix P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrix, respectively; $\|\cdot\|$ refers to the Euclidean vector norm. In symmetric block matrices or long matrix expressions, an asterisk '$*$' is used to denote a term that is induced by symmetry; diag$\{\ldots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. System Description and Preliminaries

Consider the following commonly used discrete-time 2-D system described by the Fornasini-Marchesini local state space (FMLSS) model [1]:

$$\begin{aligned} x(i+1,j+1) &= A_1x(i,j+1) + A_2x(i+1,j) \\ &\quad +B_1[u(i,j+1) + f(x(i,j+1))] \\ &\quad +B_2[u(i+1,j) + f(x(i+1,j))], \end{aligned} \tag{1}$$

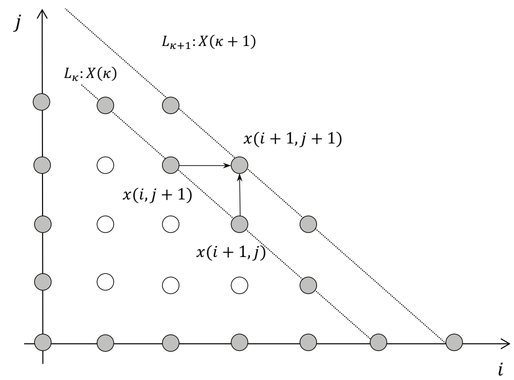

Fig. 1. Discrete-time 2-D FMLSS model and global state X_κ.

where $x(i,j) \in \mathbb{R}^n$ is the local state vector; $u(i,j) \in \mathbb{R}^m$ is the control input with $i,j \in \mathbb{N}$; $f(x(i,j)) \in \mathbb{R}^m$ is an external disturbances satisfying $\|f(x(i,j))\| \le \gamma\|x(i,j)\|$, with $\gamma > 0$ being a known constant; A_1, A_2, B_1 and B_2 are known system matrices.

Remark 1: The 2-D systems depend on two independent variables and the presence of 2-D systems in many practical problems has prompted numerous studies. In this work, we concentrate on the widely used 2-D local state space FM model in (1) and many successful applications can be found in practice. To mention a few, examples include thermal engineering, metal rolling process, vehicle platoons, grid sensor network [3]–[6] and so on.

Remark 2: According to the 2-D FMLSS model in (1), the major difference between 1-D and 2-D systems is that 2-D systems propagate information in two independent directions as depicted in Fig. 1. Based on this important two-dimensional feature, we introduce a global state denoted here by X_κ, which is defined as collection of all local states along $L_\kappa = (i,j) : i+j = \kappa$, that is

$$X(\kappa) = \{x(i,j) : i+j = \kappa\}, \tag{2}$$

and $x(i,j)$ is a local state in the 2-D case. As shown in Fig. 1, the global state X_κ is of infinite dimension in the diagonal line L_κ and preserves all past information, while the local state $x(i,j)$ tells the size of recursion. On the other hand, the evolution of the discrete-time 2-D system in (1) exhibits that the local state $x(i+1,j+1)$ results from a global state assignment $x(i,j+1)$ and $x(i+1,j)$. Thus, the 2-D system causality imposes an increment depending on κ.

Throughout the paper, the following assumption is made on the boundary condition.

Assumption 1: The boundary condition is assumed to satisfy

$$\lim_{\Gamma\to\infty}\sum_{i=0}^{\Gamma}\left(\|x(0,i)\|^2 + \|x(i,0)\|^2\right) < \infty.$$

Definition 1: The discrete-time 2-D FM systems (1) under the zero input with Assumption 1 is said to be asymptotically stable if

$$\lim_{i+j\to\infty}\|x(i,j)\|^2 = 0.$$

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The aim of this work is to design an SMC law $u(i,j)$ for the discrete-time 2-D FMLSS system in (1) such that

(a) The systems state trajectories can be driven to a neighborhood of the designed sliding surface $s(i,j)=0$ in a finite time and maintained there for all subsequent time;
(b) The dynamics in sliding surface is asymptotically stable.

Remark 3: It should be noted that lots of results on the control problem for the 2-D systems have sprung up in recent years (see, e.g., [8]–[12]), since the 2-D system theory can be used to solve many practical problems in a more efficient manner. Nevertheless, there are still important problems in this field that remain to be very challenging, such as sliding mode control for 2-D systems which are subjected to uncertainties and external disturbances. Although there are some preliminary results for the SMC of 2-D systems, those works are focused on the specific form of 2-D systems in Roesser model [23], [25] and the first FM model [26]. Both models are special cases of FMLSS model. Furthermore, the available results mentioned above are obtained by the model transformation approach [23], [25] or using the 1-D vectorial form [26], which is only a direct extension of the existing method for 1-D systems to 2-D systems. All this motivates our present study.

III. Main Results

When a discrete sliding mode control is applied to the 2-D systems, there are two phases for the state trajectories of 2-D systems: the reaching phase and sliding motion phase. In this section, the sliding surface design method for discrete-time 2-D systems in FMLSS model will be given and the sufficient condition ensuring the stability of the sliding motion will be derived. Then, a 2-D SMC law will be developed and the reachability of the specified sliding surface will be analyzed.

For illustration convenience, we denote

$$\bar{x}(i,j)=\begin{bmatrix}x(i,j+1)\\x(i+1,j)\end{bmatrix},\ \bar{u}(i,j)=\begin{bmatrix}u(i,j+1)\\u(i+1,j)\end{bmatrix},$$
$$\bar{f}(i,j)=\begin{bmatrix}f(x(i,j+1))\\f(x(i+1,j))\end{bmatrix},$$
$$\bar{A}=\begin{bmatrix}A_1 & A_2\end{bmatrix},\ \bar{B}=\begin{bmatrix}B_1 & B_2\end{bmatrix}.$$

Then the FMLSS model (1) can be recast as

$$x(i+1,j+1)=\bar{A}\bar{x}(i,j)+\bar{B}[\bar{u}(i,j)+\bar{f}(i,j)]. \tag{3}$$

A. Sliding Surface Design

For the SMC problem of 1-D systems, the sliding surface function is usually assumed to be a linear combination of the system state $x(k)$. Similarly, we construct the following sliding surface function for the discrete-time 2D FM systems in (1):

$$s(i,j)=Gx(i,j), \tag{4}$$

where the matrix G will be designed later to ensure the nonsingularity of $G\bar{B}$. However, a direct extension of the available SMC method in 1-D systems to 2-D FM systems based on the sliding surface function $s(i,j)$ will lead to the coupling of the system matrices A_1, A_2, B_1, B_2 and the difficulty in calculation of the sliding surface parameter G and design of SMC law.

On the other hand, it should be noted that the evolution of the discrete-time 2-D FM system in (1) exhibits that the local state $x(i+1,j+1)$ results from a global state assignment $x(i,j+1)$ and $x(i+1,j)$, which means that the 2-D system causality imposes an increment depending on κ. Now based on the global instant κ and global state $X(\kappa)$ in (2), the corresponding sliding variable s in (4) along L_κ becomes

$$s(\kappa)=GX(\kappa),\ \kappa=i+j, \tag{5}$$

where $X(\kappa)$ is given in (2). For the next global 2-D instant $\kappa+1$, the sliding variable $s(\kappa+1)$ can be represented as

$$s(\kappa+1)=GX(\kappa+1),\ \kappa=i+j. \tag{6}$$

Remark 4: The discrete-time 2-D FM system dynamics in (1) also reveals the distinct relationship between $X(\kappa+1)$ and $X(\kappa)$ since the 2-D system causality imposes an increment depending on κ. Then the subsequent analysis can be conducted based on the global instant κ and the state $X(\kappa)$.

Similar to the discrete-time 1-D system case [16], the ideal quasi-sliding mode in 2-D systems should also satisfy the following formula

$$s(\kappa+1)=s(\kappa)=0. \tag{7}$$

When the state trajectories enter the ideal quasi-sliding mode, an equivalent control law is obtained from (3) and (7) as

$$\bar{u}_{eq}(i,j)=-(G\bar{B})^{-1}G\bar{A}\bar{x}(i,j)-\bar{f}(i,j). \tag{8}$$

By substituting (8) into (3), we get the following discrete-time 2-D sliding mode dynamics

$$x(i+1,j+1)=[\bar{A}-\bar{B}(G\bar{B})^{-1}G\bar{A}]\bar{x}(i,j). \tag{9}$$

Then the next theorem holds.

Theorem 1: Consider the discrete-time 2-D FM system in (1) and the 2-D sliding surface specified by (5). If there exist positive-definite matrices $P_1\in\mathbb{R}^{n\times n}$ and $P_2\in\mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix}-\bar{P} & \sqrt{2}\bar{A}^TP & \sqrt{2}\bar{A}^TG^T\\ * & -P & 0\\ * & * & -G\bar{B}\end{bmatrix}<0, \tag{10}$$

where $P=P_1+P_2$, $\bar{P}=\operatorname{diag}\{P_1,\ P_2\}$ and $G=\bar{B}^TP$, then the sliding mode dynamics (9) is asymptotically stable.

Proof: Choose the following Lyapunov functional candidate for system (9) at global instant κ:

$$V(\kappa)=\bar{x}^T(i,j)\bar{P}\bar{x}(i,j),\ \kappa=i+j.$$

For the next global instant $\kappa+1$, according to the discrete-time 2-D state space model (1), we have

$$V(\kappa+1)=x^T(i+1,j+1)Px(i+1,j+1).$$

It follows from (9) and $G=\bar{B}^TP$ that

$$\begin{aligned}\Delta V(\kappa)&=V(\kappa+1)-V(\kappa)\\&=x^T(i+1,j+1)Px(i+1,j+1)\\&\quad-\bar{x}^T(i,j)\bar{P}\bar{x}(i,j)\end{aligned}$$

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$$
\begin{aligned}
&= \bar{x}^T(i,j)[\bar{A} - \bar{B}(G\bar{B})^{-1}G\bar{A}]^T P \\
&\quad \cdot [\bar{A} - \bar{B}(G\bar{B})^{-1}G\bar{A}]\bar{x}(i,j) - \bar{x}^T(i,j)\bar{P}\bar{x}(i,j) \\
&= \bar{x}^T(i,j)\big[\bar{A}^T P\bar{A} - 2\bar{A}^T P\bar{B}(G\bar{B})^{-1}G\bar{A} \\
&\quad + \bar{A}^T G^T(G\bar{B})^{-T}\bar{B}^T P\bar{B}(G\bar{B})^{-1}G\bar{A}\big]\bar{x}(i,j) \\
&\quad - \bar{x}^T(i,j)\bar{P}\bar{x}(i,j) \\
&\leq \bar{x}^T(i,j)\big[2\bar{A}^T P\bar{A} + 2\bar{A}^T G^T(G\bar{B})^{-T}\bar{B}^T P\bar{B} \\
&\quad \cdot (G\bar{B})^{-1}G\bar{A} - \bar{P}\big]\bar{x}(i,j) \\
&= \bar{x}^T(i,j)\big[2\bar{A}^T P\bar{A} + 2\bar{A}^T G^T(G\bar{B})^{-1}G\bar{A} - \bar{P}\big] \\
&\quad \cdot \bar{x}(i,j).
\end{aligned}
$$

By Schur's complement, we obtain from (10) that

$$\Delta V(\kappa) < 0,$$

which means $\lim_{\kappa\to\infty}\|\bar{x}(i,j)\|^2 = 0$, that is, $\|x(i,j)\|^2 \to 0$ as $i+j \to \infty$. Hence, the discrete-time 2-D sliding mode dynamics (9) is asymptotically stable and the proof is complete. ■

Remark 5: In 2-D systems, there exist horizontal and vertical coefficients. Thus, the direct extension of the traditional SMC method in 1-D systems is infeasible because this leads to the coupling of the system matrices A_1, A_2, B_1, B_2. In this work, we introduce the time κ and global state $X(\kappa)$ in the 2-D case and then the discrete-time sliding function for 2-D FM systems in (1) is constructed. The sufficient conditions in Theorem 1 are proposed to guarantee the stability of the sliding motion on the basis of global state $X(\kappa)$. It is worthy to mention that the introduction of κ and $X(\kappa)$ help conduct the analysis of sliding mode dynamics to make G solvable since the 2-D system causality imposes an increment depending on κ according to (1).

B. Reachability Analysis

In this subsection, attention is directed at designing a suitable discrete-time 2-D SMC law to ensure the reachability of the specified sliding surface in a finite time and maintain it there for subsequent time.

Theorem 2: For the discrete-time 2-D FM system in (1), if G is chosen by (10) such that $G\bar{B}$ is nonsingular, then the state trajectories of the system can be driven to a band of the specified sliding surface (5) by the following discrete-time 2-D SMC law:

$$
\begin{aligned}
\bar{u}(i,j) = &-(G\bar{B})^{-1}G\bar{A}\bar{x}(i,j) \\
&-(\rho + \gamma\|\bar{x}(i,j)\|)\cdot \mathrm{sgn}(s(i,j)),
\end{aligned}
\tag{11}
$$

where ρ is a positive constant.

Proof: For the analysis of reachability, we select the following Lyapunov function

$$V(\kappa) = \frac{1}{2}s^T(\kappa)(G\bar{B})^{-1}s(\kappa). \tag{12}$$

Then the incremental $\Delta V(\kappa)$ is given by

$$
\begin{aligned}
\Delta V(\kappa) &= s^T(\kappa)(G\bar{B})^{-1}\Delta s(\kappa) \\
&\quad + \frac{1}{2}\Delta s^T(\kappa)(G\bar{B})^{-1}\Delta s(\kappa) \\
&= s^T(\kappa)(G\bar{B})^{-1}\big[s(\kappa+1) - s(\kappa)\big] \\
&\quad + \frac{1}{2}\Delta s^T(\kappa)(G\bar{B})^{-1}\Delta s(\kappa).
\end{aligned}
$$

By using (3), (6), (11) and noting $\|s(X(\kappa))\| \leq |s(X(\kappa))|$, it follows that

$$
\begin{aligned}
\Delta V(\kappa) &= s^T(\kappa)[-(\rho + \gamma\|\bar{x}(i,j)\|)\mathrm{sgn}(s(i,j) + \bar{f}(i,j)] \\
&\quad - s^T(\kappa)(G\bar{B})^{-1}s(\kappa) \\
&\quad + \frac{1}{2}\Delta s^T(\kappa)(G\bar{B})^{-1}\Delta s(\kappa) \\
&\leq \gamma\|s(\kappa)\|\|\bar{x}(i,j)\| - \big[\rho + \gamma\|\bar{x}(i,j)\|\big]\|s(\kappa)\| \\
&\quad - s^T(\kappa)(G\bar{B})^{-1}s(\kappa) \\
&\quad + \frac{1}{2}\Delta s^T(\kappa)(G\bar{B})^{-1}\Delta s(\kappa) \\
&\leq -\rho\|s(\kappa)\| + \frac{1}{2}\Delta s^T(\kappa)(G\bar{B})^{-1}\Delta s(\kappa).
\end{aligned}
\tag{13}
$$

Since ρ is a positive constant to be selected, an appropriate ρ can be chosen large enough to guarantee $\Delta V(\kappa) < 0$ when $s(\kappa)$ is out of a certain bounded region. Then $\Delta s(\kappa)$ is reasonably bounded although it is not asymptotically convergent to zero. Hence, under controller (11), the system state can be driven into a neighborhood of the specified sliding surface and maintained there for subsequent time. This completes the proof. ■

Remark 6: It is proven in Theorem 2 that the reachability of the specified sliding surface can be ensured by the designed SMC, which also presents a general framework to solve SMC design problem for discrete-time 2-D systems. To the authors' knowledge, it is possibly the first attempt to solve the SMC problem for the discrete-time 2-D FM systems. Furthermore, more complex situations and problems will be discussed in our future work, such as the effect of the faults and switching phenomenon, network communications, hybrid (continuous-discrete) case, etc [6], [27]–[29].

IV. Illustrative Example

It is known that the Darboux equation can describe some dynamical processes such as gas absorption, water stream heating and air drying [3]. Consider the following Darboux equation

$$
\begin{aligned}
\frac{\partial^2 s(x,t)}{\partial x \partial t} &= a_1\frac{\partial s(x,t)}{\partial t} + a_2\frac{\partial s(x,t)}{\partial x} \\
&\quad + a_0 s(x,t) + bg(x,t),
\end{aligned}
\tag{14}
$$

where $s(x,t)$ is an unknown function at $x(\cdot) \in [0, x_f]$ and $t \in [0,\infty]$; $g(x,t)$ is the input function; a_0, a_1, a_2 and b are real coefficients. Note that (14) is a partial differential equation (PDE). Similar to the technique used in [2], define

$$
\begin{aligned}
h(x,t) &\triangleq \frac{\partial s(x,t)}{\partial t} - a_2 s(x,t), \\
x_{i,j} &\triangleq \begin{bmatrix} h(i,j) \\ s(i,j) \end{bmatrix},
\end{aligned}
$$

where $x_{i,j} = x_{i\Delta x, j\Delta t}$. Then the PDE model (14) can be easily converted into a discrete-time 2-D system in terms of FMLSS type of the form (1). Now, subject to the selection of

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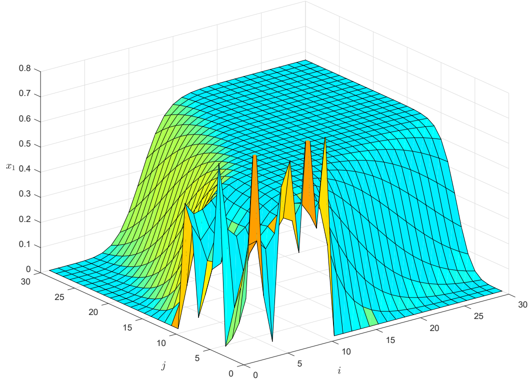

Fig. 2. Open-loop state response of x_1.

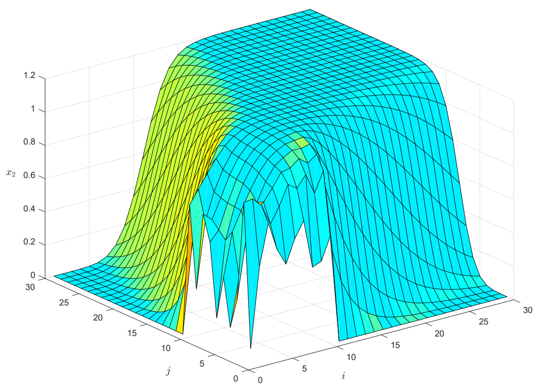

Fig. 3. Open-loop state response of x_2.

the parameters a_0, a_1, a_2 and b [2], the system matrices in (1) are given as follows:

$$A_1 = \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}.$$

Let the external disturbance $f(x(i,j))$ be

$$f(x(i,j)) = 0.7 \sin \sqrt{x_1^2(i,j) + x_2^2(i,j)}.$$

Thus, γ can be chosen as 0.7. Figs. 2–3 describe the state responses of the open-loop system in (1) under $u(i,j) = 0$. Obviously, the considered system is unstable under zero control input.

Now the task is to design a discrete-time 2-D SMC law such that the underlying system is asymptotically stable. Solving the linear matrix inequality (LMI) in (10) yields

$$P_1 = \begin{bmatrix} 11.8845 & -0.9512 \\ -0.9512 & 2.1144 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 6.3569 & -0.5005 \\ -0.5005 & 2.6669 \end{bmatrix},$$
$$G = \begin{bmatrix} 4.7466 & 1.9551 \\ 3.0676 & 1.6222 \end{bmatrix}.$$

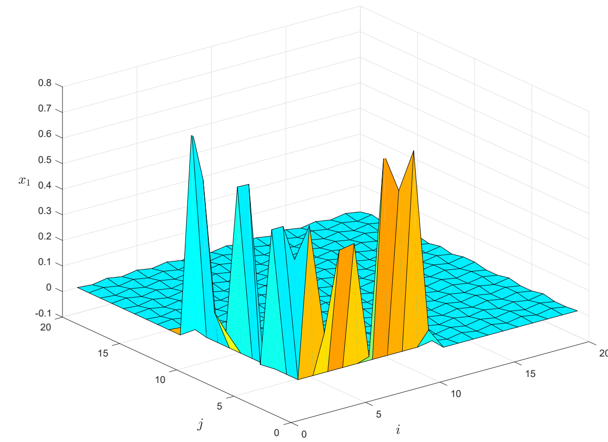

Fig. 4. Closed-loop state response of x_1.

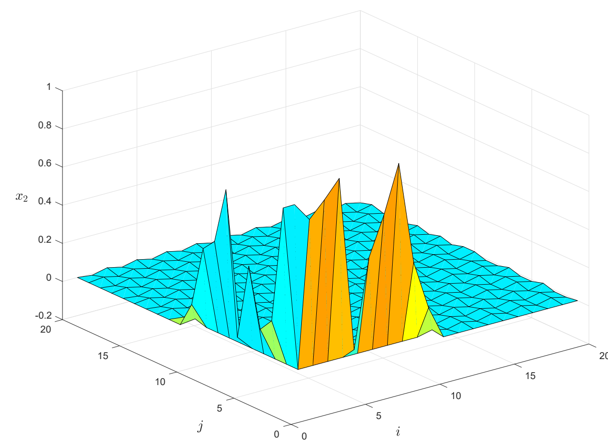

Fig. 5. Closed-loop state response of x_2.

Consequently, the sliding surface function in (4) is expressed as

$$s(i,j) = \begin{bmatrix} 4.7466 & 1.9551 \\ 3.0676 & 1.6222 \end{bmatrix} x(i,j).$$

Choose $\rho = 0.01$ and then we obtain the desired SMC law in (11). The simulation results are presented in Figs. 4–7. Among them, Figs. 4–5 plot the state responses of the resultant closed-loop system, which show that the underlying system is asymptotically stable under the proposed SMC law. The proposed SMC law generates desirable control signals to quickly drive the state trajectories toward the sliding surface $s(i,j)$=0. Further, Figs. 6–7 display the trajectories of sliding variable $s(i,j)$.

V. Conclusion

In this work, we have investigated the 2-D SMC problem for the discrete-time FMLSS model subject to external nonlinear disturbances. A new 2-D sliding surface function involving the introduced global state has been constructed to guarantee the sliding mode dynamics to be asymptotically stable. Subsequently, using the Lyapunov function method, a SMC law has been developed to ensure that the trajectories of system can

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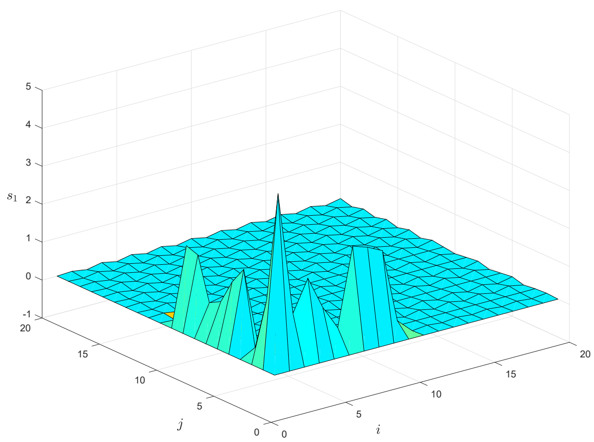

Fig. 6. Sliding surface function s_1.

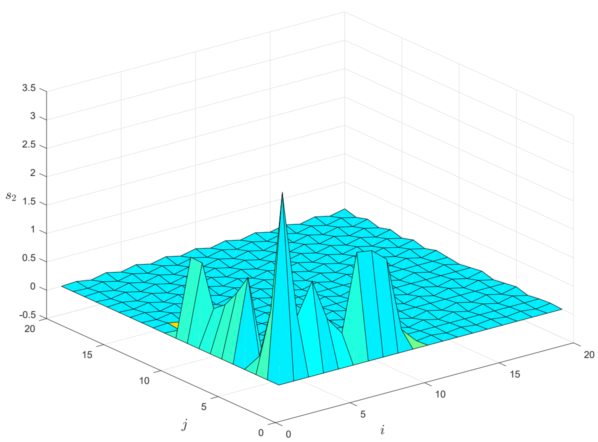

Fig. 7. Sliding surface function s_2.

be driven to a neighborhood of the specified sliding surface in a finite time. Finally, a simulation example has been given to demonstrate the feasibility of the proposed approach. In fact, this work can be extended to several more comprehensive 2-D models, which will be pursued in our future research.

References

[1] T. Kaczorek, *Two-Dimensional Linear Systems*. Berlin: Springer, 1985.

[2] C. Du and L. Xie, *$\mathcal{H}_\infty$ Control and Filtering of Two-Dimensional Systems*. Berlin: Springer, 2002.

[3] W. Marszalek, "Two-dimensional state-space discrete models for hyperbolic partial differential equations," *Applied Mathematical Modeling*, vol. 8, no. 1, pp.11–14, 1984.

[4] M. G. Buddika Sumanasena and P. H. Bauer, "Realization using the Fornasini-Marchesini model for implementations in distributed grid sensor networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 58, no. 11, pp. 2708–2717, 2011.

[5] S. Knorn and R. H. Middleton, "Two-Dimensional Analysis of String Stability of Nonlinear Vehicle Strings," *The 52nd IEEE Conference on Decision and Control*, pp. vol. 58, no. 5, pp. 5864-5869, 2013.

[6] E. Rogers, K. Galkowski, W. Paszke, K. L. Moore, P. H. Bauer, L. Hladowski, and P. Dabkowski, "Multidimensional control systems: case studies in design and evaluation," *Multidimensional Systems and Signal Processing*, vol. 26, no. 4, pp. 895–939, 2015.

[7] N. Yeganefar, N. Yeganefar, M. Ghamgui, and E. Moulay, "Lyapunov theory for 2D nonlinear Roesser models: application to asymptotic and exponential stability," *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1299–1304, 2013.

[8] C. K. Ahn, P. Shi, and M. Basin, "Two-dimensional dissipative control and filtering for Roesser model," *IEEE Transactions on Automatic Control*, vol. 60, no. 7, pp. 1745–1759, 2015.

[9] W. Paszke, J. Lam, K. Galkowski, S. Xu, and Z. Lin, "Robust stability and stabilisation of 2D discrete state-delayed systems," *Systems & Control Letters*, vol. 51, no. 3, pp. 277–291, 2004.

[10] O. Bachelier, W. Paszke, N. Yeganefar, D. Mehdi, and A. Cherifi, "LMI Stability Conditions for 2D Roesser Models," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 766–770, 2016.

[11] S. Huang, Z. Xiang, and H. R. Karimi, "Robust l_2-gain control for 2D nonlinear stochastic systems with time-varying delays and actuator saturation," *Journal of the Franklin Institute*, vol. 350, no. 7, pp. 1865–1885, 2013.

[12] Z. Duan, Z. Xiang, and H. R. Karimi, "Delay-dependent exponential stabilization of positive 2D switched state-delayed systems in the Roesser model," *Information Sciences*, vol. 272, pp. 173–184, 2014.

[13] J. Liu, S. Laghrouche, M. Harmouche, and M. Wack, "Adaptive gain second-order sliding mode observer design for switching power converters," *Control Engineering Practice*, vol. 30, pp. 124–131, 2014.

[14] J. Liu, S. Vazquez, L. Wu, A. Marquez, H. Gao, and L. G. Franquelo, "Extended state observer-based sliding-mode control for three-phase power converters," *IEEE Transactions on Industrial Electronics*, vol. 64, no. 1, pp. 22–31, 2017.

[15] S. Li, X. Yu, L. Fridman, Z. Man, and X. Wang (Eds.), *Advances in Variable Structure Systems and Sliding Mode Control–Theory and Applications*. Berlin: Springer, 2017.

[16] W. Gao, Y. Wang, and A. Homaifa, "Discrete-time variable structure control systems," *IEEE Transactions on Industrial Electronics*, vol. 42, no. 2, pp.117–122, 1995.

[17] M. Basin, J. Rodriguez-Gonzalez, L. Fridman, and P. Acosta, "Integral sliding mode design for robust filtering and control of linear stochastic time-delay systems," *International Journal of Robust and Nonlinear Control*, vol. 15, no. 9, pp. 407–421, 2005.

[18] M. Basin and P. C. Rodriguez-Ramirez, "Sliding mode controller design for stochastic polynomial systems with unmeasured states," *IEEE Transactions on Industrial Electronics*, vol. 61, no. 1, pp. 387–396, 2014.

[19] Y. Niu and D. W. C. Ho, "Design of sliding mode control subject to packet losses," *IEEE Transactions on Automatic Control*, vol. 55, no. 11, pp. 2623–2628, 2010.

[20] J. Song, Y. Niu, and Y. Zou, "Finite-time stabilization via sliding mode control," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1478–1483, 2017.

[21] X. Su, X. Liu, P. Shi and R. Yang, "Sliding mode control of discrete-time switched systems with repeated scalar nonlinearities," *IEEE Transactions on Automatic Control*, vol. 62, no. 9, pp. 4604–4610, 2017.

[22] L. Wu, Y. Gao, J. Liu, and H. Li, "Event-triggered sliding mode control of stochastic systems via output feedback," *Automatica*, vol. 82, pp. 79–92, 2017.

[23] L. Wu and H. Gao, "Sliding mode control of two-dimensional systems in Roesser model," *IET Control Theory and Applications*, vol. 2, no. 4, pp. 352–364, 2008.

[24] L. Wu, H. Gao, and C. Wang, "Quasi-sliding mode control of differential linear repetitive processes with unknown input disturbance," *IEEE Transactions on Industrial Electronics*, vol. 58, no. 7, pp. 3059–3068, 2011.

[25] H. Adloo, P. Karimaghaee, and A. S. Sarvestani, "An extension of sliding mode control design for the 2-D systems in Roesser model," in *Proceeding of the 48th IEEE Conference on Decision and Control*, Shanghai, China, December 2009, pp. 7753–7758.

[26] A. Argha, L. Li, and S. W. Su, "A new approach to applying discrete sliding mode control to 2D systems," in *Proceeding of the 52nd IEEE Conference on Decision and Control*, Florence, Italy, December 2013, pp. 3584–3589.

[27] R. Sakthive, P. Selvara, Y. Lim, and H. R. Karimi, "Adaptive reliable output tracking of networked control systems against actuator faults," *Journal of the Franklin Institute*, vol. 354, no. 9, pp. 3813–3837, 2017.

[28] C. K. Ahn, P. Shi, and M. Basin, "Two-dimensional peak-to-peak filtering for stochastic Fornasini-Marchesini systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 5, pp. 1472-1479, 2018.

[29] S. Ding, W. X. Zheng, J. Sun, and J. Wang, "Second-order sliding mode controller design and its implementation for buck converters," *IEEE Transactions on Industrial Informatics*, vol. 14, no. 5, pp. 1990–2000, 2018.

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