H_∞ Control for 2-D Discrete State Delayed Systems in the Second FM Model

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Abstract This paper is concerned with the problem of H_∞ control for two-dimensional (2-D) discrete state delay systems described by the second Fornasini and Marchesini (FM) state-space model. A sufficient condition to have an H_∞ noise attenuation for this 2-D system is given in terms of a certain linear matrix inequality (LMI). The optimal H_∞ controller is obtained by solving a convex optimization problem. Finally, a simulation example is given to illustrate the effectiveness of the proposed result.

Key words 2-D discrete systems, state delay, H_∞ control, LMI

Two-dimensional (2-D) systems have received considerable attention due to their extensive applications of both theoretical and practical interest in the past several decades[1−3]. The key feature of a 2-D system is that the information is propagated along two independent directions. Many physical processes, such as thermal processes, image processing, signal filtering, etc, have a clear 2-D structure. The 2-D system theory is frequently used as an analysis tool to solve some problems, e.g., iterative learning control[4−5] and repetitive process control[6−7]. The H_∞ norm of the transfer function from the external inputs including noise and disturbance to the output is one of popular performance measures in system theory[8]. H_∞ control problem is to find controllers such that the H_∞ norm of the resulting closed-loop transfer function is (strictly) less than a given number for the worst exogenous signal, and many results on one-dimensional (1-D) systems have appeared in the published work. To effectively solve the noise and/or disturbance attenuation problem for 2-D systems, Sebek[9] first addressed the H_∞ control problem for 2-D systems and Du and Xie[10] established several versions of 2-D bounded real lemma.

Time-delay phenomenon often appears in various engineering systems[11]. For 2-D time-delay systems, Paszke[12] presented a sufficient stability condition and a stabilization method for discrete linear state delayed 2-D systems, and this method was extended to H_∞ control for a class of uncertain nonlinear 2-D systems[13]. On the basis of Roesser Model, Xu[14] gave an approach to the design of the optimal H_∞ controller for 2-D state delayed systems.

In this paper, we propose a method to investigate the H_∞ control problem for 2-D discrete state delayed discrete systems in the second Fornasini and Marchesini (FM) model. A sufficient condition for such a 2-D system to have a specified H_∞ noise attenuation is first presented via the LMI approach. Furthermore, a convex optimization problem with LMI constraints is formulated to design the state feedback controller such that H_∞ noise attenuation γ of the resulting closed-loop system is minimized. The simulation results demonstrate the effectiveness of the proposed method.

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1 H_∞ performance analysis

Consider a 2-D discrete linear system with state delay described by the following second FM state space model

$$\begin{aligned}\boldsymbol{x}(i+1,j+1) = {} & A_1\boldsymbol{x}(i,j+1) + A_2\boldsymbol{x}(i+1,j)+\\ & A_{1d}\boldsymbol{x}(i-d_1,j+1) + A_{2d}\boldsymbol{x}(i+1,j-d_2)+\\ & B_1\boldsymbol{w}(i,j+1) + B_2\boldsymbol{w}(i+1,j)+\\ & C_1\boldsymbol{u}(i,j+1) + C_2\boldsymbol{u}(i+1,j)\\ \boldsymbol{z}(i,j) = {} & H\boldsymbol{x}(i,j) + L\boldsymbol{w}(i,j)\end{aligned} \tag{1}$$

where $0 \le i,j \in \mathbf{Z}$ are horizontal and vertical coordinates, $\boldsymbol{x}(i,j) \in \mathbf{R}^n$ is the state vector, $\boldsymbol{u}(i,j) \in \mathbf{R}^m$ is the input vector, $\boldsymbol{z}(i,j) \in \mathbf{R}^p$ is the controlled output, $\boldsymbol{w}(i,j) \in \mathbf{R}^q$ is the noise input which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$, d_1 and d_2 are unknown positive integers representing delays along horizontal direction and vertical direction, respectively; A_1, A_2, A_{1d}, A_{2d}, B_1, B_2, C_1, C_2, H, and L are constant matrices with appropriate dimensions. The initial condition is defined as follows.

$$\begin{aligned}\boldsymbol{X}(0) = [\ & \boldsymbol{x}(-d_1,0),\ \ \boldsymbol{x}(-d_1,1),\ \ \boldsymbol{x}(-d_1,2),\ \ \cdots\\ & \boldsymbol{x}(1-d_1,0),\ \ \boldsymbol{x}(1-d_1,1),\ \ \boldsymbol{x}(1-d_1,2),\ \ \cdots\\ & \boldsymbol{x}(0,0),\ \ \boldsymbol{x}(0,1),\ \ \boldsymbol{x}(0,2),\ \ \cdots\\ & \boldsymbol{x}(0,-d_2),\ \ \boldsymbol{x}(1,-d_2),\ \ \boldsymbol{x}(2,-d_2),\ \ \cdots\\ & \boldsymbol{x}(0,1-d_2),\ \ \boldsymbol{x}(1,1-d_2),\ \ \boldsymbol{x}(2,1-d_2),\ \ \cdots\\ & \boldsymbol{x}(1,0),\ \ \boldsymbol{x}(2,0),\ \ \boldsymbol{x}(3,0),\ \ \cdots\ ]\end{aligned} \tag{2}$$

For the 2-D system (1), assume a finite set of initial conditions, i.e., there exist positive integers L_1 and L_2, such that

$$\begin{cases}\boldsymbol{x}(i,j) = 0,\ \forall j \ge L_2,\ i = -d_1, -d_1+1, \cdots, 0\\ \boldsymbol{x}(i,j) = 0,\ \forall i \ge L_1,\ j = -d_2, -d_2+1, \cdots, 0\\ \boldsymbol{X}(0) \in \ell^2, \quad \text{i.e., } \|\boldsymbol{X}(0)\|_2 < \infty\end{cases} \tag{3}$$

Denote $X_r = \sup\{\|\boldsymbol{x}(i,j)\| : i+j = r,\ i,j \in \mathbf{Z}\}$, and we first give the definition of asymptotic stability for system (1).

Definition 1. The 2-D discrete state delayed system (1) is asymptotically stable if $\lim_{r\to\infty} X_r = 0$ with $\boldsymbol{u}(i,j) = \boldsymbol{0}$, $\boldsymbol{w}(i,j) = \boldsymbol{0}$ and the initial condition (3).

Definition 2. Consider the 2-D discrete state delayed system (1) with $\boldsymbol{u}(i,j) = 0$ and the initial condition (3). Given a scalar $\gamma > 0$, and symmetric positive definite weighting matrices Q_h, Q_v, W_h, $W_v \in \mathbf{R}^{n\times n}$, the 2-D state delayed system (1) is said to have an H_∞ noise attenuation γ if it is asymptotically stable and satisfies

$$\mathcal{J} = \sup_{0\ne(\boldsymbol{w},\boldsymbol{X}(0))\in\ell_2} \frac{\|\bar{\boldsymbol{z}}\|_2^2}{\|\bar{\boldsymbol{w}}\|_2^2 + D_h(d_1,j) + D_v(i,d_2)} < \gamma^2 \tag{4}$$

where

$$\|\bar{\boldsymbol{w}}\|_2^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left\| \begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix} \right\|^2$$
$$\|\bar{\boldsymbol{z}}\|_2^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left\| \begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix} \right\|^2$$

$$D_h(d_1,j) = \sum_{j=0}^{\infty}\Big[\boldsymbol{x}^{\mathrm{T}}(0,j+1)Q_h\boldsymbol{x}(0,j+1) + \sum_{l=-d_1}^{-1}\boldsymbol{x}^{\mathrm{T}}(l,j+1)W_h\boldsymbol{x}(l,j+1)\Big]$$
$$D_v(i,d_2) = \sum_{i=0}^{\infty}\Big[\boldsymbol{x}^{\mathrm{T}}(i+1,0)Q_v\boldsymbol{x}(i+1,0) + \sum_{l=-d_2}^{-1}\boldsymbol{x}^{\mathrm{T}}(i+1,l)W_v\boldsymbol{x}(i+1,l)\Big]$$

The following theorem presents a sufficient condition for the 2-D system (1) with any state delays d_1 and d_2 to have a specified H_∞ noise attenuation.

Theorem 1. Given a positive scalar γ, the 2-D system (1) with $\boldsymbol{u}(i,j)=\mathbf{0}$ and the initial condition (3) has an H_∞ noise attenuation γ if there exist symmetric positive define matrices $P, P_h, R_h, R_v \in \mathbf{R}^{n\times n}$ satisfying $P_h < \gamma^2 Q_h$, $0 < (P-P_h) < \gamma^2 Q_v$, $R_h < \gamma^2 W_h$, and $R_v < \gamma^2 W_v$, such that

$$\begin{bmatrix} A_1^{\mathrm T}\\ A_2^{\mathrm T}\\ A_{1d}^{\mathrm T}\\ A_{2d}^{\mathrm T}\\ B_1^{\mathrm T}\\ B_2^{\mathrm T}\end{bmatrix} P \begin{bmatrix} A_1^{\mathrm T}\\ A_2^{\mathrm T}\\ A_{1d}^{\mathrm T}\\ A_{2d}^{\mathrm T}\\ B_1^{\mathrm T}\\ B_2^{\mathrm T}\end{bmatrix}^{\mathrm T} + \begin{bmatrix} -P_h+R_h+H^{\mathrm T}H & 0 & 0 & 0 & H^{\mathrm T}L & 0\\ * & -P+P_h+R_v+H^{\mathrm T}H & 0 & 0 & 0 & H^{\mathrm T}L\\ * & * & -R_h & 0 & 0 & 0\\ * & * & * & -R_v & 0 & 0\\ * & * & * & * & L^{\mathrm T}L-\gamma^2 I & 0\\ * & * & * & * & * & L^{\mathrm T}L-\gamma^2 I \end{bmatrix} < 0 \tag{5}$$

Proof. Suppose now that there exist $P_h > 0$, $(P-P_h) > 0$, $R_h > 0$ and $R_v > 0$ such that LMI (5) holds. We define a Lyapunov-Krasovski functional

$$V(\boldsymbol{x}(i,j)) = V_h(\boldsymbol{x}(i,j)) + V_v(\boldsymbol{x}(i,j)) \tag{6}$$

where

$$\begin{aligned} V_h(\boldsymbol{x}(i,j)) &= \boldsymbol{x}^{\mathrm T}(i,j)P_h\boldsymbol{x}(i,j) + \sum_{l=-d_1}^{-1}\boldsymbol{x}^{\mathrm T}(i+l,j)R_h\boldsymbol{x}(i+l,j)\\ V_v(\boldsymbol{x}(i,j)) &= \boldsymbol{x}^{\mathrm T}(i,j)(P-P_h)\boldsymbol{x}(i,j) + \sum_{l=-d_2}^{-1}\boldsymbol{x}^{\mathrm T}(i,j+l)R_v\boldsymbol{x}(i,j+l) \end{aligned}$$

It is clear that $V(\boldsymbol{x}(i,j))$ is positive.

The increment $\triangle V(i+1,j+1)$ along any trajectory of system (1) with $\boldsymbol{u}(i,j)=\mathbf{0}$ and $\boldsymbol{w}(i,j)=\mathbf{0}$ satisfies

$$\begin{aligned} &\triangle V(i+1,j+1) = \\ &V_h(\boldsymbol{x}(i+1,j+1)) + V_v(\boldsymbol{x}(i+1,j+1)) - \\ &V_h(\boldsymbol{x}(i,j+1)) - V_v(\boldsymbol{x}(i+1,j)) = \\ &\begin{bmatrix}\boldsymbol{x}(i,j+1)\\ \boldsymbol{x}(i+1,j)\\ \boldsymbol{x}(i-d_1,j+1)\\ \boldsymbol{x}(i+1,j-d_2)\end{bmatrix}^{\mathrm T} \left(\begin{bmatrix} A_1^{\mathrm T}\\ A_2^{\mathrm T}\\ A_{1d}^{\mathrm T}\\ A_{2d}^{\mathrm T}\end{bmatrix} P \begin{bmatrix} A_1^{\mathrm T}\\ A_2^{\mathrm T}\\ A_{1d}^{\mathrm T}\\ A_{2d}^{\mathrm T}\end{bmatrix}^{\mathrm T} + \begin{bmatrix} -P_h+R_h & 0 & 0 & 0\\ 0 & -P+P_h+R_v & 0 & 0\\ 0 & 0 & -R_h & 0\\ 0 & 0 & 0 & -R_v \end{bmatrix}\right) \begin{bmatrix}\boldsymbol{x}(i,j+1)\\ \boldsymbol{x}(i+1,j)\\ \boldsymbol{x}(i-d_1,j+1)\\ \boldsymbol{x}(i+1,j-d_2)\end{bmatrix} \end{aligned}$$

It follows from the LMI (5) that $\triangle V(i+1,j+1) \le 0$, i.e.,

$$\begin{aligned} &V_h(\boldsymbol{x}(i+1,j+1)) + V_v(\boldsymbol{x}(i+1,j+1)) \le \\ &V_h(\boldsymbol{x}(i,j+1)) + V_v(\boldsymbol{x}(i+1,j)) \end{aligned} \tag{7}$$

where the equality sign holds only when $\boldsymbol{x}(i,j+1)=0$, $\boldsymbol{x}(i+1,j)=0$, $\boldsymbol{x}(i-d_1,j+1)=0$, and $\boldsymbol{x}(i+1,j-d_2)=0$.

Let $D(r)$ denote the set defined by

$$D(r) = \{(i,j): i+j=r, i\ge 0, j\ge 0\} \tag{8}$$

For any integer $r \ge \max\{L_1, L_2\}$, it follows from (7) and the initial condition (3) that

$$\begin{aligned} &\sum_{(i+j)\in D(r)} V(\boldsymbol{x}(i,j)) = \\ &\sum_{(i+j)\in D(r)} \Big[V_h(\boldsymbol{x}(i,j)) + V_v(\boldsymbol{x}(i,j))\Big] = \\ &\quad V_h(\boldsymbol{x}(r,0)) + V_h(\boldsymbol{x}(r-1,1)) + V_h(\boldsymbol{x}(r-2,2)) + \cdots + \\ &\quad V_h(\boldsymbol{x}(1,r-1)) + V_h(\boldsymbol{x}(0,r)) + \\ &\quad V_v(\boldsymbol{x}(r,0)) + V_v(\boldsymbol{x}(r-1,1)) + V_v(\boldsymbol{x}(r-2,2)) + \cdots + \\ &\quad V_v(\boldsymbol{x}(1,r-1)) + V_v(\boldsymbol{x}(0,r)) \ge \\ &\quad V_h(\boldsymbol{x}(r+1,0)) + V_h(\boldsymbol{x}(r,1)) + V_h(\boldsymbol{x}(r-1,2)) + \cdots + \\ &\quad V_h(\boldsymbol{x}(1,r)) + V_h(\boldsymbol{x}(0,r+1)) + \\ &\quad V_v(\boldsymbol{x}(r+1,0)) + V_v(\boldsymbol{x}(r,1)) + V_v(\boldsymbol{x}(r-1,2)) + \cdots + \\ &\quad V_v(\boldsymbol{x}(1,r)) + V_v(\boldsymbol{x}(0,r+1)) - \\ &\quad V_h(\boldsymbol{x}(-1,r+1)) - V_v(\boldsymbol{x}(r+1,-1)) = \\ &\quad \sum_{(i+j)\in D(r+1)} V(\boldsymbol{x}(i,j)) \end{aligned} \tag{9}$$

where the equality sign holds only when

$$\sum_{(i+j)\in D(r)} V(\boldsymbol{x}(i,j)) = 0$$

This implies that the whole energies stored at the points $\{(i,j): i+j=r+1\}$ is strictly less than those at the points $\{(i,j): i+j=r\}$ unless all $\boldsymbol{x}(i,j)=\mathbf{0}$. Thus, we obtain

$$\lim_{r\to\infty} \sum_{(i+j)\in D(r)} V(\boldsymbol{x}(i,j)) = 0 \tag{10}$$

It follows that

$$\lim_{i+j\to\infty} V(\boldsymbol{x}(i,j)) = 0, \qquad \lim_{i+j\to\infty} \|\boldsymbol{x}(i,j)\| = 0$$

Consequently, we conclude from Definition 1 that the system (1) is asymptotically stable.

To establish the H_∞ performance of system (1) with the control input $\boldsymbol{u}(i,j)=\mathbf{0}$ for $\boldsymbol{w}(i,j)\in \ell_2\{[0,\infty),[0,\infty)\}$, we consider

$$\begin{aligned} &\triangle V(i+1,j+1) + \begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix}^{\mathrm T}\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix} - \\ &\quad\gamma^2\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix}^{\mathrm T}\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix} = \\ &\quad\begin{bmatrix}\boldsymbol{x}(i,j+1)\\ \boldsymbol{x}(i+1,j)\\ \boldsymbol{x}(i-d_1,j+1)\\ \boldsymbol{x}(i+1,j-d_2)\\ \boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix}^{\mathrm T} \left(\begin{bmatrix} A_1^{\mathrm T}\\ A_2^{\mathrm T}\\ A_{1d}^{\mathrm T}\\ A_{2d}^{\mathrm T}\\ B_1^{\mathrm T}\\ B_2^{\mathrm T}\end{bmatrix} P \begin{bmatrix} A_1^{\mathrm T}\\ A_2^{\mathrm T}\\ A_{1d}^{\mathrm T}\\ A_{2d}^{\mathrm T}\\ B_1^{\mathrm T}\\ B_2^{\mathrm T}\end{bmatrix}^{\mathrm T} + \right.\\ &\quad\begin{bmatrix} -P_h+R_h+H^{\mathrm T}H & 0 & 0 & 0\\ 0 & -P+P_h+R_v+H^{\mathrm T}H & 0 & 0\\ 0 & 0 & -R_h & 0\\ 0 & 0 & 0 & -R_v\\ L^{\mathrm T}H & 0 & 0 & 0\\ 0 & L^{\mathrm T}H & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{matrix} H^{\mathrm{T}}L & 0 \\ 0 & H^{\mathrm{T}}L \\ 0 & 0 \\ 0 & 0 \\ L^{\mathrm{T}}L-\gamma^2 I & 0 \\ 0 & L^{\mathrm{T}}L-\gamma^2 I \end{matrix}\Bigg]\Bigg)\begin{bmatrix} \boldsymbol{x}(i,j+1) \\ \boldsymbol{x}(i+1,j) \\ \boldsymbol{x}(i-d_1,j+1) \\ \boldsymbol{x}(i+1,j-d_2) \\ \boldsymbol{w}(i,j+1) \\ \boldsymbol{w}(i+1,j) \end{bmatrix} \tag{11}$$

It follows from the LMI (5) that

$$\triangle V(i+1,j+1)+\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix}- \gamma^2\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix}<0$$

Therefore, for any integers $T_1, T_2>0$, we have

$$\sum_{i=0}^{T_1-1}\sum_{j=0}^{T_2-1}\left(\triangle V(i+1,j+1)+\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix}^{\mathrm{T}}\times \begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix}-\gamma^2\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix}\right)<0 \tag{12}$$

where

$$\sum_{i=0}^{T_1-1}\sum_{j=0}^{T_2-1}\triangle V(i+1,j+1)= \sum_{i=0}^{T_1-1}\Big[V_v(\boldsymbol{x}(i+1,T_2))-V_v(\boldsymbol{x}(i+1,0))\Big]+ \sum_{j=0}^{T_2-1}\Big[V_h(\boldsymbol{x}(T_1,j+1))-V_h(\boldsymbol{x}(0,j+1))\Big]$$

For $T_1\geq T_2\geq \max\{L_1+d_1, L_2+d_2\}$, it follows from (7) and the initial condition (3) that

$$\begin{aligned}&\sum_{i=0}^{T_1-1}V_v(\boldsymbol{x}(i+1,T_2))\leq \\ &\sum_{i=0}^{T_1-1}\Big[V_h(\boldsymbol{x}(i,T_2))+V_v(\boldsymbol{x}(i+1,T_2-1))- V_h(\boldsymbol{x}(i+1,T_2))\Big]= \\ &V_h(\boldsymbol{x}(0,T_2))-V_h(\boldsymbol{x}(T_1,T_2))+V_v(\boldsymbol{x}(1,T_2-1))+ \sum_{i=1}^{T_1-1}V_v(\boldsymbol{x}(i+1,T_2-1))\leq \\ &V_h(\boldsymbol{x}(0,T_2))-V_h(\boldsymbol{x}(T_1,T_2))+V_v(\boldsymbol{x}(1,T_2-1))+ \\ &\sum_{i=1}^{T_1-1}\Big[V_h(\boldsymbol{x}(i,T_2-1))+V_v(\boldsymbol{x}(i+1,T_2-2))- V_h(\boldsymbol{x}(i+1,T_2-1))\Big]= \\ &V_h(\boldsymbol{x}(0,T_2))+V_v(\boldsymbol{x}(1,T_2-1))+V_h(\boldsymbol{x}(1,T_2-1))- V_h(\boldsymbol{x}(T_1,T_2))-V_h(\boldsymbol{x}(T_1,T_2-1))+ \\ &V_v(\boldsymbol{x}(2,T_2-2))+\sum_{i=2}^{T_1-1}V_v(\boldsymbol{x}(i+1,T_2-2))\leq\cdots\leq \\ &\sum_{j=0}^{T_2}\Big[V_h(\boldsymbol{x}(j,T_2-j))+V_v(\boldsymbol{x}(j,T_2-j))\Big]- V_h(\boldsymbol{x}(T_2,0))-V_v(\boldsymbol{x}(0,T_2))- \\ &\sum_{j=0}^{T_2-1}V_h(\boldsymbol{x}(T_1,j+1))+\sum_{i=T_2}^{T_1-1}V_v(\boldsymbol{x}(i+1,0))= \\ &\sum_{(i+j)\in D(T_2)}V(\boldsymbol{x}(i,j))-\sum_{j=0}^{T_2-1}V_h(\boldsymbol{x}(T_1,j+1))\end{aligned} \tag{13}$$

where

$$V_h(\boldsymbol{x}(T_2,0))=0,\quad V_v(\boldsymbol{x}(0,T_2))=0$$
$$\sum_{i=T_2}^{T_1-1}V_v(\boldsymbol{x}(i+1,0))=0$$

This implies that

$$\sum_{i=0}^{T_1-1}V_v(\boldsymbol{x}(i+1,T_2))+\sum_{j=0}^{T_2-1}V_h(\boldsymbol{x}(T_1,j+1))\leq \sum_{(i+j)\in D(T_2)}V(\boldsymbol{x}(i,j)) \tag{14}$$

Thus, when $T_1, T_2\rightarrow\infty$, it follows from (10) $\sim$ (12) and (14) that

$$\begin{aligned}&\|\bar{\boldsymbol{z}}\|_2^2-\gamma^2\|\bar{\boldsymbol{w}}\|_2^2< \\ &\sum_{i=0}^{\infty}V_v(\boldsymbol{x}(i+1,0))+\sum_{j=0}^{\infty}V_h(\boldsymbol{x}(0,j+1))= \\ &\sum_{j=0}^{\infty}\Big[\boldsymbol{x}^{\mathrm{T}}(0,j+1)P_h\boldsymbol{x}(0,j+1)+ \sum_{l=-d_1}^{-1}\boldsymbol{x}^{\mathrm{T}}(l,j+1)R_h\boldsymbol{x}(l,j+1)\Big]+ \\ &\sum_{i=0}^{\infty}\Big[\boldsymbol{x}^{\mathrm{T}}(i+1,0)(P-P_h)\boldsymbol{x}(i+1,0)+ \sum_{l=-d_2}^{-1}\boldsymbol{x}^{\mathrm{T}}(i+1,l)R_v\boldsymbol{x}(i+1,l)\Big]\end{aligned} \tag{15}$$

Because $P_h<\gamma^2 Q_h$, $(P-P_h)<\gamma^2 Q_v$, $R_h<\gamma^2 W_h$, $R_v<\gamma^2 W_v$, the inequality (15) leads to

$$\begin{aligned}&\|\bar{\boldsymbol{z}}\|_2^2<\gamma^2\Bigg\{\|\bar{\boldsymbol{w}}\|_2^2+\sum_{j=0}^{\infty}\Big[\boldsymbol{x}^{\mathrm{T}}(0,j+1)Q_h\boldsymbol{x}(0,j+1)+ \\ &\sum_{l=-d_1}^{-1}\boldsymbol{x}^{\mathrm{T}}(l,j+1)W_h\boldsymbol{x}(l,j+1)\Big]+ \\ &\sum_{i=0}^{\infty}\Big[\boldsymbol{x}^{\mathrm{T}}(i+1,0)Q_v\boldsymbol{x}(i+1,0)+ \\ &\sum_{l=-d_2}^{-1}\boldsymbol{x}^{\mathrm{T}}(i+1,l)W_v\boldsymbol{x}(i+1,l)\Big]\Bigg\}\end{aligned} \tag{16}$$

Therefore, it follows from Definition 2 that the result of this theorem is true. $\square$

In the case where the initial condition is known to be zero, i.e., $\boldsymbol{X}(0)=\boldsymbol{0}$, the conditions for $P_h<\gamma^2 Q_h$, $(P-P_h)<\gamma^2 Q_v$, $R_h<\gamma^2 W_h$ and $R_v<\gamma^2 W_v$ in Theorem 1 are no longer needed. It follows from (16) that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\end{bmatrix}\right\|^2<\gamma^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\end{bmatrix}\right\|^2 \tag{17}$$

i.e.,

$$\begin{aligned}&2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|\boldsymbol{z}(i,j)\|^2-\sum_{i=0}^{\infty}\|\boldsymbol{z}(i,0)\|^2-\sum_{j=0}^{\infty}\|\boldsymbol{z}(0,j)\|^2< \\ &2\gamma^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|\boldsymbol{w}(i,j)\|^2-\gamma^2\sum_{i=0}^{\infty}\|\boldsymbol{w}(i,0)\|^2-\gamma^2\sum_{j=0}^{\infty}\|\boldsymbol{w}(0,j)\|^2\end{aligned} \tag{18}$$

By considering the zero initial conditions, $\boldsymbol{x}(i,0)=\boldsymbol{x}(0,j)=\boldsymbol{0}$, $i,j=0,1,\cdots$. Then, from system (1) we have that $\boldsymbol{z}(i,0)=L\boldsymbol{w}(i,0)$ and $\boldsymbol{z}(0,j)=L\boldsymbol{w}(0,j)$. Thus, it follows from (18) that

$$2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|\boldsymbol{z}(i,j)\|^2 < 2\gamma^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|\boldsymbol{w}(i,j)\|^2 - \sum_{i=0}^{\infty}\boldsymbol{w}^{\mathrm{T}}(i,0)(\gamma^2 I - L^{\mathrm{T}}L)\boldsymbol{w}(i,0) - \sum_{j=0}^{\infty}\boldsymbol{w}^{\mathrm{T}}(0,j)(\gamma^2 I - L^{\mathrm{T}}L)\boldsymbol{w}(0,j)$$

It can be known from (5) that $\gamma^2 I - L^{\mathrm{T}}L > 0$. Thus, for all nonzero $w(i,j)$, we have

$$\|\boldsymbol{z}\|_2 < \gamma\|\boldsymbol{w}\|_2 \tag{19}$$

where

$$\|\boldsymbol{w}\|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|\boldsymbol{w}(i,j)\|^2}, \quad \|\boldsymbol{z}\|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|\boldsymbol{z}(i,j)\|^2}.$$

It follows from that the 2-D Parseval′s theorem[3] that (19) is equivalent to

$$\|G(z_1,z_2)\|_{\infty} = \sup_{\omega_1,\omega_2\in[0,2\pi]}\sigma_{\max}\left[G(\mathrm{e}^{\mathrm{j}\omega_1},\mathrm{e}^{\mathrm{j}\omega_2})\right] < \gamma \tag{20}$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value of the corresponding matrix, and

$$G(z_1,z_2) = H(z_1z_2I - z_2A_1 - z_1A_2 - z_1^{-d_1}z_2A_{1d} - z_1z_2^{-d_2}A_{2d})^{-1}(z_2B_1 + z_1B_2) + L \tag{21}$$

is the transfer function from the noise input $\boldsymbol{w}(i,j)$ to the controlled output $\boldsymbol{z}(i,j)$ for the 2-D system (1).

2 H_∞ controller design

Consider the 2-D state delayed system (1) and the following controller

$$\boldsymbol{u}(i,j) = K\boldsymbol{x}(i,j) \tag{22}$$

The corresponding closed-loop system is given by

$$\begin{aligned}\boldsymbol{x}(i+1,j+1) = {} & (A_1+C_1K)\boldsymbol{x}(i,j+1) + \\ & (A_2+C_2K)\boldsymbol{x}(i+1,j) + \\ & A_{1d}\boldsymbol{x}(i-d_1,j+1) + A_{2d}\boldsymbol{x}(i+1,j-d_2) + \\ & B_1\boldsymbol{w}(i,j+1) + B_2\boldsymbol{w}(i+1,j) \\ \boldsymbol{z}(i,j) = {} & H\boldsymbol{x}(i,j) + L\boldsymbol{w}(i,j)\end{aligned} \tag{23}$$

If there exists the controller (22) such that the closed-loop system (23) is asymptotically stable, and the H_∞ norm of the transfer function (21) from the noise input $\boldsymbol{w}(i,j)$ to the controlled output $\boldsymbol{z}(i,j)$ for the closed-loop system (23) is smaller than γ, then the closed-loop system (23) has a specified H_∞ noise attenuation γ, and the controller (22) is said to be a γ-suboptimal H_∞ state feedback controller for the 2-D state delayed system (1).

Theorem 2. Consider the 2-D state delay system (1) with the zero initial condition. Given a positive scalar γ, if there exist a matrix $N \in \mathbf{R}^{m\times n}$ and symmetric define matrices $\bar{P}$, $\bar{P}_h$, $\bar{R}_h$, $\bar{R}_v \in \mathbf{R}^{n\times n}$ such that

$$\begin{bmatrix} -\bar{P}_h+\bar{R}_h & 0 & 0 & 0 & 0 & 0 & \bar{P}A_1^{\mathrm{T}}+N^{\mathrm{T}}C_1^{\mathrm{T}} & \bar{P}H^{\mathrm{T}} & 0 \\ * & -\bar{P}+\bar{P}_h+\bar{R}_v & 0 & 0 & 0 & 0 & \bar{P}A_2^{\mathrm{T}}+N^{\mathrm{T}}C_2^{\mathrm{T}} & 0 & \bar{P}H^{\mathrm{T}} \\ * & * & -\bar{R}_h & 0 & 0 & 0 & \bar{P}A_{1d}^{\mathrm{T}} & 0 & 0 \\ * & * & * & -\bar{R}_v & 0 & 0 & \bar{P}A_{2d}^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & B_1^{\mathrm{T}} & L^{\mathrm{T}} & 0 \\ * & * & * & * & * & -\gamma^2 I & B_2^{\mathrm{T}} & 0 & L^{\mathrm{T}} \\ * & * & * & * & * & * & -\bar{P} & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{24}$$

then the close-loop system (23) has a specified H_∞ noise attenuation γ and

$$\boldsymbol{u}(i,j) = N\bar{P}^{-1}\boldsymbol{x}(i,j) \tag{25}$$

is a γ-suboptimal state feedback H_∞ controller for the 2-D state delayed system (1).

Proof. By applying Theorem 1 and Schur complement, a sufficient condition for the closed-loop system (23) to have a specified H_∞ noise attenuation γ is that there exist $P_h > 0$, $(P - P_h) > 0$, $R_h > 0$, and $R_v > 0$ such that

$$\begin{bmatrix} -P_h+R_h & 0 & 0 & 0 & 0 & 0 & A_1^{\mathrm{T}}+K^{\mathrm{T}}C_1^{\mathrm{T}} & H^{\mathrm{T}} & 0 \\ 0 & -P+P_h+R_v & 0 & 0 & 0 & 0 & A_2^{\mathrm{T}}+K^{\mathrm{T}}C_2^{\mathrm{T}} & 0 & H^{\mathrm{T}} \\ 0 & 0 & -R_h & 0 & 0 & 0 & A_{1d}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & -R_v & 0 & 0 & A_{2d}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma^2 I & 0 & B_1^{\mathrm{T}} & L^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & B_2^{\mathrm{T}} & 0 & L^{\mathrm{T}} \\ A_1+C_1K & A_2+C_2K & A_{1d} & A_{2d} & B_1 & B_2 & -P^{-1} & 0 & 0 \\ H & 0 & 0 & 0 & L & 0 & 0 & -I & 0 \\ 0 & H & 0 & 0 & 0 & L & 0 & 0 & -I \end{bmatrix} < 0 \tag{26}$$

Pre- and post-multiplying both sides of the inequality (26) by $\mathrm{diag}\{P^{-1},P^{-1},P^{-1},P^{-1},I,I,I,I,I\}$ and denoting $\bar{P} = P^{-1}$, $\bar{P}_h = \bar{P}P_h\bar{P}$, $\bar{R}_h = \bar{P}R_h\bar{P}$, $\bar{R}_v = \bar{P}R_v\bar{P}$, and $N = K\bar{P}$, it follows that the inequality (26) is equal to the LMI (24). □

In addition, by solving the following optimization problem:

$$\begin{aligned} \min_{\bar{P},\bar{P}_h,\bar{R}_h,\bar{R}_v,N} \ & \gamma^2 \\ \text{s.t.} \ & (24) \end{aligned} \tag{27}$$

we can obtain a state feedback controller such that the H_∞ noise attenuation γ of the resulting closed-loop system is minimized. This controller (25) is said to be the optimal H_∞ controller for the 2-D discrete state delayed system (1).

3 An illustrative example

This section applies the main results on H_∞ control to the thermal processes[10] in chemical reactors, heat exchangers, and pipe furnaces, which can be expressed in the partial differential equation with time delays:

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - a_0T(x,t) - a_1T(x,t-\tau) + bu(x,t) \tag{28}$$

where $T(x,t)$ is usually the temperature at x (space) $\in [0,x_f]$ and t (time) $\in [0,\infty]$, $u(x,t)$ is a given force function, τ is the time delay, and a_0, a_1, and b are real coefficients.

Denote $x^{\mathrm{T}}(i,j) = [T^{\mathrm{T}}(i-1,j) \ \ T^{\mathrm{T}}(i,j)]$, where $T(i,j) = T(i\triangle x, j\triangle t)$. It is easy to verify that equation (28) can be converted into a 2-D FM state space model (1) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ \dfrac{\triangle t}{\triangle x} & 1 - \dfrac{\triangle t}{\triangle x} - a_0 \triangle t \end{bmatrix}$$

$$A_{1d} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{2d} = \begin{bmatrix} 0 & 0 \\ 0 & -a_1 \triangle t \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ b\triangle t \end{bmatrix}, d_2 = \mathrm{int}(\tau / \triangle t + 1)$$

where $\mathrm{int}(\cdot)$ is the integer function.

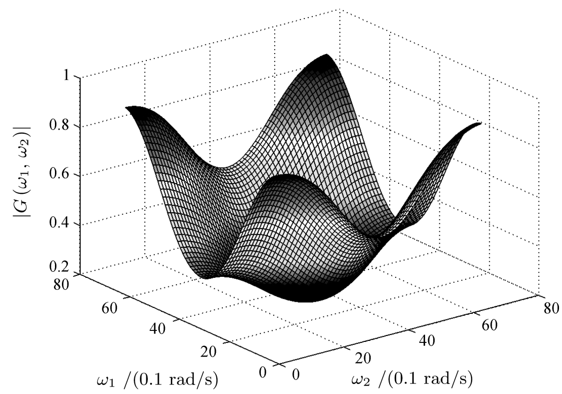
Let $\triangle t = 0.1, \triangle x = 0.4, a_0 = 1, a_1 = 0.4$, and $b = 1$. By considering the problem of H_∞ disturbance attenuation, the thermal process is modeled in the form (1) with

$$B_1 = \begin{bmatrix} 0.2 \\ 0.04 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.04 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad L = 0.5$$

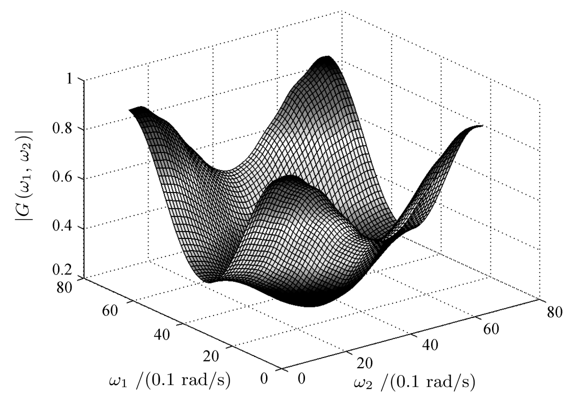
Solving the optimization problem (27), we obtain delay-independent H_∞ noise attenuation $\gamma = 0.9751$ and the optimal H_∞ controller

$$u(i,j) = \begin{bmatrix} -2.5000 & -6.500 \end{bmatrix} \boldsymbol{x}(i,j) \tag{29}$$

For time delay $d_2 = 1$ and $d_2 = 6$, (a) and (b) of Fig. 1, respectively, show the frequency responses from the disturbance input $\boldsymbol{w}(i,j)$ to the controlled output $\boldsymbol{z}(i,j)$ for the corresponding closed-loop system over all frequencies, i.e., $|G(\mathrm{e}^{\mathrm{j}\omega_1}, \mathrm{e}^{\mathrm{j}\omega_2})|$, $0 \le \omega_1 \le 2\pi$, $0 \le \omega_2 \le 2\pi$, the corresponding maximum values of $|G(\mathrm{e}^{\mathrm{j}\omega_1}, \mathrm{e}^{\mathrm{j}\omega_2})|$ are, respectively, 0.9538 and 0.9570, all below the specified level of attenuation $\gamma = 0.9751$.



(a) For $d_2 = 1$



(b) For $d_2 = 6$

Fig. 1 The frequency responses of the disturbance transfer function

4 Conclusions

This paper has presented a solution to the problem of delay-independent H_∞ control for 2-D state delay systems described by the second FM model. A sufficient condition for this 2-D system to have a specified H_∞ noise attenuation is proposed in terms of a certain LMI. The optimal H_∞ controller is obtained by solving a convex optimization problem. The results can be extended to the robust H_∞ control problem of 2-D uncertain systems with state delay.

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