This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2831176, IEEE Transactions on Automatic Control

$\mathcal{H}_\infty$ Control for 2D Markov Jump Systems in Roesser Model

Zheng-Guang Wu, Ying Shen, Peng Shi, Zhan Shu, Hongye Su

***Abstract*—This paper considers the problem of asynchronous $\mathcal{H}_\infty$ control for two-dimensional (2D) Markov jump systems. The underlying system is described based upon Roesser model. Specially, the hidden Markov model is employed when dealing with the asynchronization between controlled system and controller, and the relation between them is constructed through a conditional probability matrix. Based on Lyapunov function technique, the asymptotic mean square stability and $\mathcal{H}_\infty$ noise attenuation performance are investigated for the closed-loop 2D system. Moreover, the controller gain can be obtained by solving a convex optimization problem. An example is presented to show the effectiveness and potential of the proposed new design technique.**

***Index Terms*—$\mathcal{H}_\infty$ control, 2D system, Markov jump systems, hidden Markov model**

I. Introduction

Since the 2D model is developed for image processing by Roesser in [1], 2D system as an emerging field has received considerable concern during the past several decades. As a matter of fact, many physical dynamic processes have inherent 2D characteristic, for example, the thermal process, gas absorption, etc. The universal existence of 2D system in reality has greatly stimulate its development (see [2]–[10]). The work [2] has studied discretization problem for 2D systems utilizing higher order approaches. Output feedback control and robust stabilization have been discussed for 2D system in [4]. [6] has studied the issue of dissipative filtering and control for 2D system in Roesser model. In [7], the mixed continuous-discrete-time 2D system has been considered, and a nonconservative LMI-based condition is derived such that the robust stability is ensured.

Markov jump system (MJS) is also a prosperous research area, which excels at modelling the occurrences of abrupt structure or parameter variations. In MJSs, there are a family of subsystems and a Markov process dominates the variations among them following certain transition probability matrix. Scholars have carried out systematic research in this area. The literatures [11]–[14] have investigated filtering/estimation problems for MJSs. The topic of control/stability has been discussed in [15]–[22]. Recently, researchers have integrated the MJS with 2D system, in other words, the 2D system with Markov jump parameters has been considered. For example, the works [23] and [24] have considered $\mathcal{H}_\infty$ filtering and $\mathcal{H}_\infty$ control for 2D MJS, respectively. [25] has investigated the fault detection problem for 2D MJS. However, we find that the field of 2D MJS has not been explored thoroughly yet and is worthy of attention.

As shown in the above, control and filtering are always research hotspots in the field of MJSs. However, most of the attention has been devoted to the studies on synchronous or mode-independent controllers/filters, e.g. [11]–[14], [16]–[24]. However, the synchronous controller/filter is appropriate only in the ideal occasion that the modes of the system are completely accessible. In reality, it is always difficult to get full access to the system modes. Taking networked control system and large-scale engineering system as examples, it is particularly common that their components are scattered in different places and communicate with each other through communication channels which may be unreliable and will lead to errors between the received and original signals, thus asynchronization phenomenon will occur if they are multi-mode systems. Namely, the controller or filter's mode is unable to completely follow the transitions of the system's mode. Therefore, an asynchronous controller/filter is more realistic and desirable, and it becomes a matter of increasing concern (see [26]–[30]). Note that, though all the works [26]–[30] have involved asynchronization, different descriptions have been adopted in them when dealing with asynchronization. In [26], the asynchronization is caused by time-delay, i.e., the controlled system and controller share the same Markov chain, whereas the mode of the controller lags behind that of the controlled system. [27] has applied the nonhomogeneous Markov chain when investigating the problem of asynchronous filtering. The hidden Markov model has been proposed in [28], which is a unified framework covering asynchronous, synchronous and mode-independent cases. The work in [29] has presented the continuous hidden Markov model. Hidden Markov model has also been studied in [30]. We notice that there is little literature on asynchronous control for 2D MJS which has a strong appeal to us.

In the present paper, we devote ourselves to the issue of asynchronous $\mathcal{H}_\infty$ control for 2D MJS in Roesser model. The main contributions of this paper include the following three aspects: Firstly, it is the first time that the asynchronous $\mathcal{H}_\infty$

This work was partially supported by the Science Fund for Creative Research Groups of National Natural Science Foundation of China (61621002), the National Natural Science Foundation of China (61673339, 61773131, U1509217, 61573136, 61603133), the Australian Research Council (DP170102644), the Zhejiang Provincial Natural Science Foundation of China (LR16F030001). (Corresponding author: Zheng-Guang Wu)

Z.-G. Wu, Y. Shen and H. Su are with the National Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, Yuquan Campus, Hangzhou Zhejiang, 310027, China (e-mail: yingshen@zju.edu.cn; nashwzhg@zju.edu.cn; hysu@iipc.zju.edu.cn).

P. Shi is with the School of Electrical and Electronic Engineering, University of Adelaide, Adelaide, SA 5005, Australia (e-mail: peng.shi@adelaide.edu.au).

Z. Shu is with the School of Engineering Sciences, University of Southampton, Southampton SO17 1BJ, U.K. (e-mail: z.shu@soton.ac.uk).

control is studied for 2D MJSs, which fills the gap in the area of 2D MJSs. Secondly, the proposed asynchronous controller is more practical than synchronous or mode-independent ones, which has a wide range of applicatgions, such as networked control systems and etc. Thirdly, a unified asynchronization framework, i.e., hidden Markov model, is applied, consequently, the derived results in this paper can be easily extended to synchronous or mode-independent cases.

II. Preliminaries

This paper is concerned with the 2D MJS established upon Roesser model, as follows:

$$\begin{cases} x^1(i,j) = A(\gamma_{i,j})x(i,j) + B(\gamma_{i,j})u(i,j) + E(\gamma_{i,j})w(i,j) \\ y(i,j) = C(\gamma_{i,j})x(i,j) + D(\gamma_{i,j})u(i,j) + F(\gamma_{i,j})w(i,j) \end{cases} \tag{1}$$

where

$$x^1(i,j) = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix},\ x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix},$$

$x^h(i,j) \in \mathbb{R}^{n_h}$ and $x^v(i,j) \in \mathbb{R}^{n_v}$ denote horizontal and vertical states respectively, $u(i,j) \in \mathbb{R}^{n_u}$ and $w(i,j) \in \mathbb{R}^{n_w}$ denote control and perturbance inputs, $y(i,j) \in \mathbb{R}^{n_y}$ denotes controlled output. The real valued matrices $A(\gamma_{i,j})$, $B(\gamma_{i,j})$, $C(\gamma_{i,j})$, $D(\gamma_{i,j})$, $E(\gamma_{i,j})$, $F(\gamma_{i,j})$ are appropriately dimensioned and pre-known, and they are functions of $\gamma_{i,j}$ which represents a Markov chain. The value of $\gamma_{i,j}$ varies in the limited set $\mathscr{K}_1 = \{1, 2, \cdots, k_1\}$ with transition probability matrix $\Lambda = \{\lambda_{pq}\}$, and λ_{pq} is given by

$$\Pr\{\gamma_{i+1,j} = q|\gamma_{i,j} = p\} = \Pr\{\gamma_{i,j+1} = q|\gamma_{i,j} = p\} = \lambda_{pq}, \tag{2}$$

According to probability theory, λ_{pq} is subject to

$$\begin{cases} \lambda_{pq} \geq 0 \\ \sum_{q=1}^{k_1} \lambda_{pq} = 1 \end{cases} \tag{3}$$

for $\forall p, q \in \mathscr{K}_1$. The boundary condition (X_0, Γ_0) of system (1) is defined as follows:

$$\begin{cases} X_0 = \{x^h(0,j), x^v(i,0)|i,j = 0,1,2,\cdots\} \\ \Gamma_0 = \{\gamma_{0,j}, \gamma_{i,0}|i,j = 0,1,2,\cdots\} \end{cases} \tag{4}$$

And we define zero boundary condition as $x^h(0,j) = 0$, $x^v(i,0) = 0$, $i,j = 0,1,2,\cdots$. We further impose following assumption on X_0.

Assumption 1. *Assume that X_0 satisfies:*

$$\lim_{L\to\infty} \mathbb{E}\Big\{\sum_{l=0}^{L}(|x^h(0,l)|^2 + |x^v(l,0)|^2)\Big\} < \infty, \tag{5}$$

where $\mathbb{E}\{\cdot\}$ denotes mathematical expectation, $|\cdot|$ denotes Euclidean vector norm.

In this paper, a commonly used assumption in state feedback control is adopted, i.e.,all $x(i,j)$ are completely accessible (see [31]). Besides, we assume that the accurate information of $\gamma_{i,j}$ is unavailable. We plan to design an asynchronous state feedback controller as follows:

$$u(i,j) = K(\eta_{i,j})x(i,j), \tag{6}$$

where $K(\eta_{i,j})$ represents the controller gain, and it depends on the parameter $\eta_{i,j} \in \mathscr{K}_2$ ($\mathscr{K}_2 = \{1,2,\cdots,k_2\}$). $\eta_{i,j}$ controls the variations of the controller mode, meanwhile, it follows some conditional probability π_{ps} given $\gamma_{i,j}$, i.e.,

$$\Pr\{\eta_{i,j} = s|\gamma_{i,j} = p\} = \pi_{ps}, \tag{7}$$

and $\Pi = \{\pi_{ps}\}$ is referred to as conditional probability matrix, which can be obtained by Monte Carlo method. There is similar restrictions for π_{ps} as (3), i.e., $\pi_{ps} \in [0,1]$, $\sum_{s=1}^{k_2} \pi_{ps} = 1$, for $\forall p \in \mathscr{K}_1$, $s \in \mathscr{K}_2$.

Remark 1. *As mentioned above, the controller is asynchronous with the controlled system. The so-called hidden Markov model $(\gamma_{i,j}, \eta_{i,j}, \Lambda, \Pi)$ is employed to depict the asynchronization phenomenon, the advantage of which lies in that it makes full use of the information about $\gamma_{i,j}$ and $\eta_{i,j}$ by taking conditional probability matrix Π into consideration. Note that the recent work [30] is also concerned with hidden Markov model denoted by $(\theta(k), \hat{\theta}(k))$, where $\theta(k)$ and $\hat{\theta}(k)$ are respectively the modes of the original system and filter, and $\hat{\theta}(k)$ seen as a detected value of $\theta(k)$ depends on $\theta(k)$ through certain conditional probability. As can be seen, the description of hidden Markov model in [30] is essentially the same as that in our paper.*

Remark 2. *Hidden Markov model is a unified framework for mode-independent, synchronous and asynchronous controller/filter. The controller (6) is instantly transformed into a synchronous one if let $\mathscr{K}_1 = \mathscr{K}_2$, $\pi_{ps} = 1$, for $p = s$, or a mode-independent one if $\mathscr{K}_2 = \{1\}$. As a consequence, the derived results in this paper can be extended to synchronous or mode-independent cases.*

In the sequel, we will make some notational simplifications: these parameters $\gamma_{i,j}$, $\gamma_{i+1,j}$ (or $\gamma_{i,j+1}$) and $\eta_{i,j}$ will be replaced by subscripts p, q, s respectively. For example, A_p is short for $A(\gamma_{i,j})$.

By substituting (6) into (1), we readily obtain the closed-loop dynamics as follows:

$$\mathscr{G}: \begin{cases} x^1(i,j) = \bar{A}_{ps}x(i,j) + E_p w(i,j) \\ y(i,j) = \bar{C}_{ps}x(i,j) + F_p w(i,j) \end{cases} \tag{8}$$

where

$$\bar{A}_{ps} = A_p + B_p K_s,\ \bar{C}_{ps} = C_p + D_p K_s.$$

Next, by extending the canonical definitions for 1D system, we have following definitions for 2D system.

Definition 1. *The closed-loop 2D system $\mathscr{G}$ with $w(i,j) \equiv 0$ is said to be asymptotically mean square stable if the following condition holds:*

$$\lim_{i+j\to\infty} \mathbb{E}\{|x(i,j)|^2\} = 0, \tag{9}$$

for any boundary condition (X_0, Γ_0).

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2831176, IEEE Transactions on Automatic Control

Definition 2. *Assume that the closed-loop 2D system $\mathscr{G}$ satisfies* (9). *Then $\mathscr{G}$ is said to have an $\mathcal{H}_\infty$ noise attenuation performance μ, if under zero boundary condition and $w(i,j) \in l_2\{[0,\infty),[0,\infty)\}$, the following condition is satisfied:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\{|y(i,j)|^2\} < \mu^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|w(i,j)|^2, \tag{10}$$

where μ is a positive scalar.

In this paper, we intend to devise an asynchronous controller in the form (6) for 2D system (1) with the hope that the resulting closed-loop 2D system $\mathscr{G}$ possesses asymptotic mean square stability and an $\mathcal{H}_\infty$ noise attenuation performance μ, i.e., (9) and (10) are simultaneously satisfied.

III. Main Results

In this section we will investigate the asymptotic mean square stability and $\mathcal{H}_\infty$ noise attenuation performance for closed-loop 2D system $\mathscr{G}$, and try to propose a controller design approach. Firstly, we present following sufficient condition:

Theorem 1. *Consider the closed-loop 2D system $\mathscr{G}$ under Assumption 1. For a given scalar $\mu > 0$, if there exist matrices $R_p = diag\{R_p^h, R_p^v\} > 0$, $Q_{ps} > 0$ and K_s, for $\forall p \in \mathscr{K}_1$, $s \in \mathscr{K}_2$, such that the following conditions hold:*

$$\sum_{s=1}^{k_2}\pi_{ps}Q_{ps} < R_p, \tag{11}$$

$$\begin{bmatrix} -\bar{R}_p^{-1} & 0 & \bar{A}_{ps} & E_p \\ * & -I & \bar{C}_{ps} & F_p \\ * & * & -Q_{ps} & 0 \\ * & * & * & -\mu^2 I \end{bmatrix} < 0, \tag{12}$$

where $\bar{R}_p = \sum_{q=1}^{k_1}\lambda_{pq}R_q$, then system $\mathscr{G}$ is asymptotically mean square stable with an $\mathcal{H}_\infty$ noise attenuation performance μ.

Proof. The proof will begin with the verification of asymptotic mean square stability. We introduce Lyapunov function $V(i,j) = x^{\mathrm{T}}(i,j)R_p x(i,j)$, and define

$$\triangle V(i,j) = x^{1\mathrm{T}}(i,j)R_q x^1(i,j) - x^{\mathrm{T}}(i,j)R_p x(i,j). \tag{13}$$

According to the state equation in (8) with $w(i,j) \equiv 0$, it is easy to verify that

$$\triangle V(i,j) = x^{\mathrm{T}}(i,j)(\bar{A}_{ps}^{\mathrm{T}}R_q\bar{A}_{ps} - R_p)x(i,j). \tag{14}$$

Taking expectation operation of $\triangle V(i,j)$, we obtain

$$\mathbb{E}\{\triangle V(i,j)\} = \mathbb{E}\Big\{x^{\mathrm{T}}(i,j)\Big(\sum_{s=1}^{k_2}\pi_{ps}\bar{A}_{ps}^{\mathrm{T}}\bar{R}_p\bar{A}_{ps} - R_p\Big)x(i,j)\Big\}, \tag{15}$$

By means of Schur Complement, we are able to infer from (12) that

$$\bar{A}_{ps}^{\mathrm{T}}\bar{R}_p\bar{A}_{ps} < Q_{ps}. \tag{16}$$

Combining (15), (16), we have

$$\begin{aligned}\mathbb{E}\{\triangle V(i,j)\} <& \mathbb{E}\Big\{x^{\mathrm{T}}(i,j)\Big(\sum_{s=1}^{k_2}\pi_{ps}Q_{ps} - R_p\Big)x(i,j)\Big\} \\ \le& -\alpha\mathbb{E}\{|x(i,j)|^2\},\end{aligned} \tag{17}$$

where α is the minimum eigenvalue of $(R_p - \sum_{s=1}^{k_2}\pi_{ps}Q_{ps})$. As a result of (11), $\alpha > 0$. Adding both sides of (17), we get

$$\mathbb{E}\Big\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}|x(i,j)|^2\Big\} \le -\frac{1}{\alpha}\mathbb{E}\Big\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\triangle V(i,j)\Big\}. \tag{18}$$

where d_1, d_2 are arbitrary positive integers. By substituting $x^1(i,j) = \begin{bmatrix} x^{h\mathrm{T}}(i+1,j) & x^{v\mathrm{T}}(i,j+1)\end{bmatrix}^{\mathrm{T}}$ and $R_p = diag\{R_p^h, R_p^v\}$ into $\triangle V(i,j)$, we have

$$\begin{aligned}&\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\triangle V(i,j) \\ &= \sum_{j=0}^{d_2}\big[x^{h\mathrm{T}}(d_1+1,j)R^h_{\gamma_{d_1+1,j}}x^h(d_1+1,j) \\ &-x^{h\mathrm{T}}(0,j)R^h_{\gamma_{0,j}}x^h(0,j)\big] \\ &+\sum_{i=0}^{d_1}\big[x^{v\mathrm{T}}(i,d_2+1)R^v_{\gamma_{i,d_2+1}}x^v(i,d_2+1) \\ &-x^{v\mathrm{T}}(i,0)R^v_{\gamma_{i,0}}x^v(i,0)\big]\end{aligned} \tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned}\mathbb{E}\Big\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}|x(i,j)|^2\Big\} \le \frac{1}{\alpha}\mathbb{E}\Big\{&\sum_{j=0}^{d_2}x^{h\mathrm{T}}(0,j)R^h_{\gamma_{0,j}}x^h(0,j) \\ &+\sum_{i=0}^{d_1}x^{v\mathrm{T}}(i,0)R^v_{\gamma_{i,0}}x^v(i,0)\Big\}.\end{aligned} \tag{20}$$

Let β be the maximum eigenvalue of $R^h_{\gamma_{0,j}}$ and $R^v_{\gamma_{i,0}}$, $i,j = 0,1,2,\cdots$, and let d_1, d_2 tend to infinity in (20), then the following inequality holds

$$\mathbb{E}\Big\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|x(i,j)|^2\Big\} \le \frac{\beta}{\alpha}\mathbb{E}\Big\{\sum_{i=0}^{\infty}|x^h(0,i)|^2 + |x^v(i,0)|^2\Big\}. \tag{21}$$

Recalling (5), we know that

$$\mathbb{E}\Big\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|x(i,j)|^2\Big\} < \infty, \tag{22}$$

which implies that (9) holds. Thus the asymptotic mean square stability is proved.

Next, we will divert our attention to $\mathcal{H}_\infty$ noise attenuation performance under zero boundary condition. Denote

$$\xi(i,j) = \begin{bmatrix} x(i,j) \\ w(i,j)\end{bmatrix}, Y_1 = \begin{bmatrix}\bar{A}_{ps} & E_p\end{bmatrix}, Y_2 = \begin{bmatrix}\bar{C}_{ps} & F_p\end{bmatrix}$$
$$\bar{I} = diag\{0,\mu^2 I\}, \bar{Q}_{ps} = diag\{Q_{ps}, 0\},$$

then (12) is equivalent to the following matrix inequality:

$$Y_1^{\mathrm{T}}\bar{R}_p Y_1 + Y_2^{\mathrm{T}}Y_2 - \bar{I} < \bar{Q}_{ps} \tag{23}$$

by Schur Complement. In line with the dynamics of system $\mathscr{G}$, we figure out that

$$\triangle V(i,j) = \xi^{\mathrm{T}}(i,j)Y_1^{\mathrm{T}} R_q Y_1 \xi(i,j) - x^{\mathrm{T}}(i,j)R_p x(i,j), \quad (24)$$

$$y^{\mathrm{T}}(i,j)y(i,j) - \mu^2 w^{\mathrm{T}}(i,j)w(i,j) = \xi^{\mathrm{T}}(i,j)(Y_2^{\mathrm{T}} Y_2 - \bar{I})\xi(i,j). \quad (25)$$

Recalling (19) with zero boundary condition, we know that $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \triangle V(i,j) \geq 0$, based on which, we introduce following performance index and have

$$\begin{aligned} J =& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\{y^{\mathrm{T}}(i,j)y(i,j) - \mu^2 w^{\mathrm{T}}(i,j)w(i,j)\} \\ \leq& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\{y^{\mathrm{T}}(i,j)y(i,j) - \mu^2 w^{\mathrm{T}}(i,j)w(i,j) + \triangle V(i,j)\} \end{aligned} \quad (26)$$

Then (24) and (25) yield that

$$\begin{aligned} &\mathbb{E}\{y^{\mathrm{T}}(i,j)y(i,j) - \mu^2 w^{\mathrm{T}}(i,j)w(i,j) + \triangle V(i,j)\} \\ =&\mathbb{E}\Big\{\xi^{\mathrm{T}}(i,j)\sum_{s=1}^{k_2} \pi_{ps}\big[Y_1^{\mathrm{T}} \bar{R}_p Y_1 + Y_2^{\mathrm{T}} Y_2 - \bar{I}\big]\xi(i,j) \\ &- x^{\mathrm{T}}(i,j)R_p x(i,j)\Big\} \\ <&\mathbb{E}\Big\{\xi^{\mathrm{T}}(i,j)\sum_{s=1}^{k_2} \pi_{ps}\bar{Q}_{ps}\xi(i,j) - x^{\mathrm{T}}(i,j)R_p x(i,j)\Big\} \\ =&\mathbb{E}\Big\{x^{\mathrm{T}}(i,j)\Big[\sum_{s=1}^{k_2} \pi_{ps} Q_{ps} - R_p\Big]x(i,j)\Big\} \end{aligned} \quad (27)$$

where "$<$" holds owing to (23). Finally, we can arrive at $J < 0$ from (11), (26) and (27), thus (10) is satisfied. With this, the proof is finished. □

Remark 3. *The matrix Q_{ps} is introduced in the sufficient condition proposed by Theorem 1, it enables the matrices $\bar{A}_{ps}$ and $\bar{C}_{ps}$ which contain the controller gain K_s to be separated from the conditional probability π_{ps} successfully. This reduces the dimension of the matrix inequality and hence simplifies the controller design.*

Though a sufficient condition in concise form is derived in Theorem 1, it is difficult to be used for controller design directly because of the nonlinearity. Next, we will further study the controller design method.

Theorem 2. *Consider the closed-loop 2D system $\mathscr{G}$ under Assumption 1. If there exist a scalar $\tilde{\mu} > 0$, matrices $\tilde{R}_p = diag\{\tilde{R}_p^h, \tilde{R}_p^v\} > 0$, $\tilde{Q}_{ps} > 0$, $\tilde{K}_s$ and M_s, for $\forall p \in \mathscr{K}_1$, $s \in \mathscr{K}_2$, such that the following conditions hold:*

$$\begin{bmatrix} -\tilde{R}_p & T_p \\ * & -\mathscr{Q}_p \end{bmatrix} < 0, \quad (28)$$

$$\begin{bmatrix} \mathscr{A}_{ps} & \mathscr{B}_{ps} & \mathscr{C}_{ps} \\ * & -I & 0 \\ * & * & -\mathscr{R} \end{bmatrix} < 0, \quad (29)$$

where

$$\begin{aligned} T_p &= \big[\sqrt{\pi_{p1}}\tilde{R}_p \quad \sqrt{\pi_{p2}}\tilde{R}_p \quad \cdots \quad \sqrt{\pi_{pk_2}}\tilde{R}_p\big] \\ \mathscr{Q}_p &= diag\{\tilde{Q}_{p1}, \tilde{Q}_{p2}, \cdots, \tilde{Q}_{pk_2}\} \\ \mathscr{A}_{ps} &= diag\{\tilde{Q}_{ps} - M_s^{\mathrm{T}} - M_s, -\tilde{\mu} I\} \\ \mathscr{B}_{ps} &= \big[C_p M_s + D_p \tilde{K}_s \quad F_p\big]^{\mathrm{T}} \\ \mathscr{C}_{ps} &= \big[\sqrt{\lambda_{p1}}U_{ps} \quad \sqrt{\lambda_{p2}}U_{ps} \quad \cdots \sqrt{\lambda_{pk_1}}U_{ps}\big] \\ U_{ps} &= \big[A_p M_s + B_p \tilde{K}_s \quad E_p\big]^{\mathrm{T}} \\ \mathscr{R} &= diag\{\tilde{R}_1, \tilde{R}_2, \cdots, \tilde{R}_{k_1}\} \end{aligned}$$

then system $\mathscr{G}$ is asymptotically mean square stable with $\mathcal{H}_\infty$ noise attenuation performance μ, $\mu = \sqrt{\tilde{\mu}}$. Moreover, if the LMIs (28) *and* (29) *have feasible solutions, then the controller gain can be obtained by*

$$K_s = \tilde{K}_s M_s^{-1}. \quad (30)$$

Proof. In Theorem 2, we introduce a slack matrix M_s. It follows from (29) that $\tilde{Q}_{ps} - M_s^{\mathrm{T}} - M_s < 0$, that is, $M_s^{\mathrm{T}} + M_s$ is positive definite, which guarantees that M_s is invertible. Here we use following notations:

$$\tilde{R}_p = R_p^{-1}, \ \tilde{Q}_{ps} = Q_{ps}^{-1}, \ \tilde{K}_s = K_s M_s, \ \tilde{\mu} = \mu^2. \quad (31)$$

Firstly, we will prove that (11) and (28) are equivalent. Note that, by applying Schur Complement, we can establish the equivalence between (11) and the following inequality:

$$\begin{bmatrix} -R_p & \bar{T}_p \\ * & -\mathscr{Q}_p \end{bmatrix} < 0, \quad (32)$$

where $\bar{T}_p = \big[\sqrt{\pi_{p1}}I \quad \sqrt{\pi_{p2}}I \quad \cdots \quad \sqrt{\pi_{pk_2}}I\big]$. Then (28) will be obtained by pre- and post-multiplying (32) with $diag\{\tilde{R}_p, I\}$.

Next, we will verify that (29) is sufficient to ensure (12) holds. Based on the fact that

$$(\tilde{Q}_{ps} - M_s)^{\mathrm{T}} \tilde{Q}_{ps}^{-1} (\tilde{Q}_{ps} - M_s) \geq 0, \quad (33)$$

namely,

$$-M_s^{\mathrm{T}} \tilde{Q}_{ps}^{-1} M_s \leq \tilde{Q}_{ps} - M_s^{\mathrm{T}} - M_s, \quad (34)$$

we can come to the conclusion that (29) guarantees the inequality below is tenable:

$$\begin{bmatrix} \mathscr{D}_{ps} & \mathscr{B}_{ps} & \mathscr{C}_{ps} \\ * & -I & 0 \\ * & * & -\mathscr{R} \end{bmatrix} < 0, \quad (35)$$

where $\mathscr{D}_{ps} = diag\{-M_s^{\mathrm{T}} \tilde{Q}_{ps}^{-1} M_s, -\mu^2 I\}$. Pre- and post-multiplying (35) by $diag\{(M_s^{\mathrm{T}})^{-1}, I, I, I, \cdots, I\}$ and its transposed matrix, (35) is equivalent to

$$\begin{bmatrix} -\bar{Q}_{ps} - \bar{I} & Y_2^{\mathrm{T}} & \mathscr{W}_{ps} \\ * & -I & 0 \\ * & * & -\mathscr{R} \end{bmatrix} < 0, \quad (36)$$

where $\mathscr{W}_{ps} = \big[\sqrt{\lambda_{p1}}Y_1^{\mathrm{T}} \quad \sqrt{\lambda_{p2}}Y_1^{\mathrm{T}} \quad \cdots \sqrt{\lambda_{pk_1}}Y_1^{\mathrm{T}}\big]$, and the definitions of $\bar{Q}_{ps}$, $\bar{I}$, Y_1 and Y_2 are the same as in (23). Then using Schur Complement operation again, (36) is equivalent to (23) and (12). In conclusion, (28) and (29) can guarantee that system $\mathscr{G}$ is asymptotically mean square stable with $\mathcal{H}_\infty$ noise attenuation performance μ. Besides, since M_s is invertible, the controller gain K_s can be obtained with ease through an inversion operation according to (31). This ends the proof. □

Remark 4. *Theorem 2 deals with the nonlinearity involved in the matrix inequalities* (11) *and* (12) *through slack matrix technique and variable replacement, and successfully converts the control design issue to a LMI based problem which can be settled effortlessly thanks to Matlab. Meanwhile,* $\tilde{\mu}$ *can be minimized subject to* (28) *and* (29)*, then the optimized* $\mathcal{H}_\infty$ *noise attenuation performance can be obtained by* $\mu^* = \sqrt{\tilde{\mu}_{min}}$.

IV. Numerical Example

In this section, we are going to show the validness of the designed controller by conducting a simulation concerning Darboux equation [32] which is often applied to describe a number of practical dynamical processes, such as air drying and gas absorption. Darboux equation is described in the form of partial differential equation, and it can be converted to a 2D MJS (1) using the similar technique in [4]. Here this 2D MJS model is composed of two operation modes, the corresponding system matrices are listed in the following:
Mode 1:

$$A_1 = \begin{bmatrix} -0.9 & 0.5 \\ -0.2 & -1.5 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}, \ E_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0.6 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, \ F_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

Mode 2:

$$A_2 = \begin{bmatrix} -1.1 & 0.6 \\ -0.3 & -2.2 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$C_2 = C_1, \ D_2 = D_1, \ F_2 = F_1$$

The controller to be designed has two modes. The jumps of the controlled system and controller accord with the following transition probability matrix Λ and conditional probability matrix Π:

$$\Lambda = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}, \ \Pi = \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix},$$

Then we utilize the controller design method proposed in Theorem 2 and obtain following controller gains:
Mode 1:

$$K_1 = \begin{bmatrix} 0.5703 & 3.3186 \end{bmatrix}$$

Mode 2:

$$K_2 = \begin{bmatrix} 0.5627 & 3.9023 \end{bmatrix}$$

with optimized $\mathcal{H}_\infty$ noise attenuation performance $\mu^* = 5.0519$. Since there exists a feasible solution, we are able to conclude according to Theorem 2 that this closed-loop 2D system is asymptotically mean square stable with $\mathcal{H}_\infty$ noise attenuation performance.

Next, we will further show the effectiveness by comparing the evolutions of system states with and without control input. Accordingly, it is necessary to present the boundary condition and disturbance input:

$$x^h(0,j) = \begin{cases} 0.2, & 0 \le j \le 9 \\ 0, & elsewhere \end{cases}$$

$$x^v(i,0) = \begin{cases} 0.3, & 0 \le i \le 9 \\ 0, & elsewhere \end{cases}$$

$$w(i,j) \ = \begin{cases} 0.1, & 2 \le i,j \le 9 \\ 0, & elsewhere \end{cases}$$

With these parameters, we have obtained the dynamics of system states in the open-loop system, which are displayed in Fig.1 and Fig 2. It can be observed from the two figures that both the horizontal and vertical state tend to infinity. Fig.3 and Fig.4 have displayed the dynamics of system states in the closed-loop system, which show that the system states converge to zero after undergoing a period of fluctuation. These observations indicate that the open-loop system has been effectively stabilized by the designed controller. In addition, the control effect is exhibited in Fig.5.

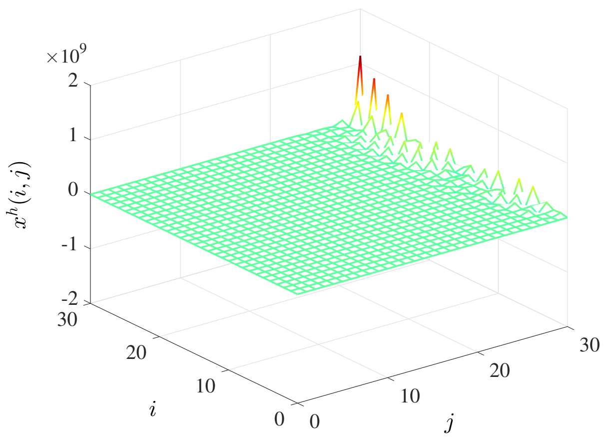

Fig. 1. Horizontal state without control input

V. Conclusion

This paper has investigated the $\mathcal{H}_\infty$ control problem of 2D MJS constructed on Roesser model. Considering the information of system modes are unavailable, we focus our attention on asynchronous control, and hidden Markov model has been constructed for the asynchronization between the controlled 2D system and the controller. By extending the canonical definitions for 1D system to 2D system, a sufficient condition has been obtained, which can ensure the asymptotic mean square stability and $\mathcal{H}_\infty$ noise attenuation performance. Furthermore, the controller has been devised via optimization technique. Finally, the feasibility and applicability of the obtained theoretic results have been verified by an example associated with practical systems.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2831176, IEEE Transactions on Automatic Control

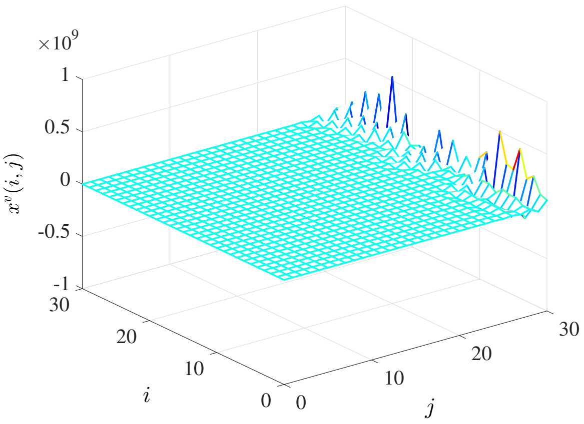

Fig. 2. Vertical state without control input

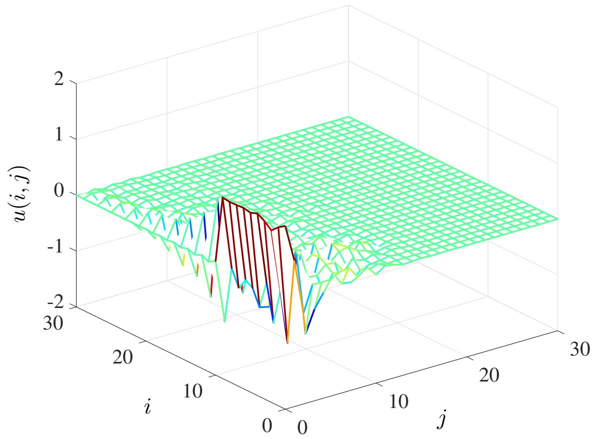

Fig. 5. Control input

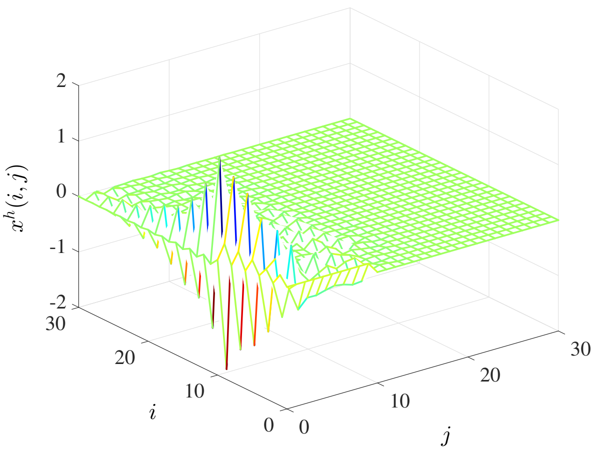

Fig. 3. Horizontal state with control input

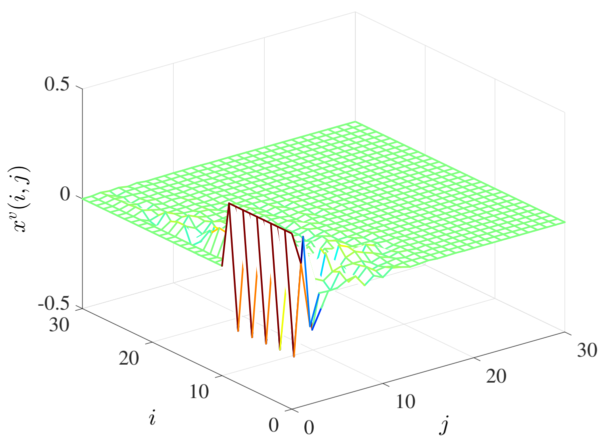

Fig. 4. Vertical state with control input

REFERENCES

[1] R. Roesser, "A discrete state-space model for linear image processing," *IEEE Transactions on Automatic Control*, vol. 20, no. 1, pp. 1–10, 1975.

[2] K. Galkowski, "Higher order discretization of 2-D systems," *IEEE Transactions on Circuits & Systems I: Fundamental Theory & Applications*, vol. 47, no. 5, pp. 713–722, 2000.

[3] C. Du, L. Xie and Y. Soh, "H_∞ reduced-order approximation of 2-D digital filters," *IEEE Transactions on Circuits & Systems I: Fundamental Theory & Applications*, vol. 48, no. 6, pp. 688–698, 2001.

[4] C. Du, L. Xie and C. Zhang, "H_∞ control and robust stabilization of two-dimensional systems in Roesser models," *Automatica*, vol. 37, no. 2, pp. 205–211, 2001.

[5] R. Yang, L. Xie and C. Zhang, "H_2 and mixed H_2/H_∞ control of two-dimensional systems in Roesser model," *Automatica*, vol. 42, no. 9, pp. 1507–1514, 2006.

[6] C. Ahn, P. Shi and M. Basin, "Two-dimensional dissipative control and filtering for Roesser model," *IEEE Transactions on Automatic Control*, vol. 60, no. 7, pp. 1745–1759, 2015.

[7] G. Chesi and R. Middleton, "Robust stability and performance analysis of 2D mixed continuous-discrete-time systems with uncertainty," *Automatica*, vol. 67, pp. 233–243, 2016.

[8] A. Argha, L. Li, S. Su and H. Nguyen, "Controllability analysis of two-dimensional systems using 1D approaches," *IEEE Transactions on Automatic Control*, vol. 60, no. 11, pp. 2977–2982, 2015.

[9] C. Ahn, L. Wu and P. Shi, "Stochastic stability analysis for 2-D Roesser systems with multiplicative noise," *Automatica*, vol. 69, pp. 356–363, 2016.

[10] L. Wu, Z. Wang, H. Gao and C. Wang, "Robust H_∞ and L_2-H_∞ filtering for two-dimensional linear parameter-varying systems," *International Journal of Robust & Nonlinear Control*, vol. 17, no. 12, pp. 1129–1154, 2007.

[11] S. Aberkane and V. Dragan, "H_∞ filtering of periodic Markovian jump systems: Application to filtering with communication constraints," *Automatica*, vol. 48, no. 12, pp. 3151–3156, 2012.

[12] L. Zhang, "H_∞ estimation for discrete-time piecewise homogeneous Markov jump linear systems," *Automatica*, vol. 45, no. 11, pp. 2570–2576, 2009.

[13] A. Goncalves, A. Fioravanti and J. Geromel, "H_∞ filtering of discrete-time Markov jump linear systems through linear matrix inequalities," *IEEE Transactions on Automatic Control*, vol. 54, no. 6, pp. 1347–1351, 2009.

[14] B. Zhang, W. Zheng and S. Xu, "Filtering of Markovian jump delay systems based on a new performance index," *IEEE Transactions on Circuits & Systems I Regular Papers*, vol. 60, no. 5, pp. 1250–1263, 2013.

[15] J. Xiong, J. Lam, Z. Shu and X. Mao, "Stability analysis of continuous-time switched systems with a random switching signal," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 180–186, 2013.

[16] J. Tao, R. Lu, P. Shi, H. Su and Z. Wu, "Dissipativity-based reliable control for fuzzy Markov jump systems with actuator faults," *IEEE Transactions on Cybernetics*, 2016.

[17] H. Ma, W. Zhang and T. Hou, "Infinite horizon H_2/H_∞ control for discrete-time time-varying Markov jump systems with multiplicative noise," *Automatica*, vol. 48, no. 7, pp. 1447–1454, 2012.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2018.2831176, IEEE Transactions on Automatic Control

[18] X. Luan, S. Zhao and F. Liu, "H_∞ control for discrete-time Markov jump systems with uncertain transition probabilities," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1566–1572, 2013.

[19] R. Oliveira, A. Vargas, J. Do Val and P. Peres, "Mode-independent H_2 control of a DC motor modeled as a Markov jump linear system," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 5, pp. 1915–1919, 2014.

[20] X. Zhong, H. He, H. Zhang and Z. Wang, "Optimal control for unknown discrete-time nonlinear Markov jump systems using adaptive dynamic programming," *IEEE Transactions on Neural Networks & Learning Systems*, vol. 25, no. 12, pp. 2141–2155, 2014.

[21] T. Hou and H. Ma, "Exponential stability for discrete-time infinite Markov jump systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 4241–4246, 2016.

[22] L. Wu, X. Su and P. Shi, "Sliding mode control with bounded L_2 gain performance of Markovian jump singular time-delay systems," *Automatica*, vol. 48, no. 8, pp. 1929–1933., 2012.

[23] L. Wu, P. Shi, H. Gao and C. Wang, "H_∞ filtering for 2D Markovian jump systems," *Automatica*, vol. 44, no. 7, pp. 1849–1858, 2008.

[24] H. Gao, J. Lam, S. Xu and C. Wang, "Stabilization and H_∞ control of two-dimensional Markovian jump systems," *IMA Journal of Mathematical Control & Information*, vol. 21, no. 4, pp. 377–392, 2004.

[25] L. Wu, X. Yao and W. Zheng, "Generalized H_2 fault detection for two-dimensional Markovian jump systems," *Automatica*, vol. 48, no. 8, pp. 1741–1750, 2012.

[26] L. Zhang and H. Gao, "Asynchronously switched control of switched linear systems with average dwell time," *Automatica*, vol. 46, no. 5, pp. 953–958, 2010.

[27] Z. Wu, P. Shi, H. Su and J. Chu, "Asynchronous $l_2 - l_\infty$ filtering for discrete-time stochastic Markov jump systems with randomly occurred sensor nonlinearities," *Automatica*, vol. 50, no. 1, pp. 180–186, 2014.

[28] Z. Wu, P. Shi, Z. Shu, H. Su and R. Lu, "Passivity-based asynchronous control for Markov jump systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 2020–2025, 2017.

[29] F. Stadtmann and O. Costa, "H_2 control of continuous-time hidden Markov jump linear systems," *IEEE Transactions on Automatic Control*, 2016.

[30] A. de Oliveira and O. Costa, "H_2-Filtering for discrete-time hidden Markov jump systems", *International Journal of Control*. vol. 90, no. 3, pp. 599–615, 2017.

[31] D. Yue, Q. Han and C. Peng, "State feedback controller design of networked control systems," *IEEE Transactions on Circuits & Systems II Express Briefs*, vol. 51, no. 11, pp. 640–644, 2004.

[32] W. Marszalek, "Two-dimensional state-space discrete models for hyperbolic partial differential equations," *Applied Mathematical Modelling*, vol. 8, no. 1, pp. 11–14, 1984.