Asynchronous Sliding Mode Control of Two-Dimensional Markov Jump Systems

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***Abstract*—This paper is concerned with the issue of asynchronous sliding mode control (SMC) for two-dimensional (2D) discrete-time Markov jump systems, whiAch is formulated by Roesser model. Concerning that modes of origin system are not always accessible to the controller, the hidden Markov model is utilized to describe the asynchronization between them. Based on the properties of 2D systems and Roesser model, a novel 2D sliding surface is constructed and the corresponding asynchronous SMC is designed under the framework of hidden Markov model. The Lyapunov function and linear matrix inequality technique are utilized to establish a sufficient condition that can guarantee the concerned 2D system is asymptotically mean square stable with an H_∞ disturbance attenuation performance. Moreover, this condition can simultaneously ensure the reachability of the constructed 2D sliding surface. Then, an algorithm is proposed to derive the asynchronous 2D-SMC law. Finally, the validity and effectiveness of the SMC design algorithm is verified through a numerical example.**

***Index Terms*—Markov jump systems, 2D systems, sliding mode control, hidden Markov model**

I. Introduction

Markov jump systems (MJSs), an especial class of stochastic switching systems, have received considerable attentions for its powerful ability in modeling systems with sudden changes in parameters or system structures, for instance, environmental disturbances, actuator failures, and interconnection variations in subsystems, etc. Over the past decades, a large number of results on stability analysis and the design of controller/filter have been reported in [1]–[5].

However, the aforementioned works are generally based on the implicit assumption that the information of system modes can always be fully available for the controller/filter, so that the controller/filter modes can run synchronously with system modes. Unfortunately, in practical applications, it is rather difficult to satisfy this ideal assumption because of some unexpected factors, such as, time delays, data dropouts and quantization in networked control systems. To overcome the strict limitation, two research approaches were proposed, namely, mode-independent and asynchronous methods. In mode-independent methods, see [6]–[8], the controller/filter modes are independent of system modes, which means the information of system modes is not fully utilized and may result in some conservativeness. In [9], Wu proposed the hidden Markov model, a united asynchronous framework that covers synchronous and mode-independent cases. A similar description of this model can also be found in earlier works [1] and [10]. The hidden Markov model supposes that the controller/filter modes can be detected via a hidden Markov chain, such that controllers/filters can utilize more information of the origin system modes. Base on this model, many problems of asynchronous control/filtering for MJSs have been studied in recent years [11]–[13].

Moreover, SMC, an effective control technique for its strong robustness against parameter variations, exogenous disturbances and model uncertainties, has been successfully applied to a large variety of practical systems [14]–[16]. The prime idea of SMC is to design a dis-continuous control law to drive the system state trajectories toward a predefined sliding surface and stay within a neighbourhood of the sliding surface after reaching it [17]. In the last few decades, the SMC design problems have been extensively studied, and a large number of research results for MJSs have been published [18]–[20]. Recently, some researchers have investigated the asynchronous SMC design methods for MJSs based on hidden Markov model [21]–[23].

On the other hand, two-dimensional systems, a special class of multi-dimensional systems dating back to 1970s, have been used to represent a large variety of practical applications, such as digital image processing, partial differential equations modeling and signal filtering [24]–[26], etc. Different from one-dimensional systems, the prime feature of 2D systems lie in the system information propagates along two different directions and the information in each direction is independent with each other, which brings much complexity and difficulty in system analysis and synthesis. To describe the special properties of 2D systems, several mathematical models presented in state-space framework have been proposed, for example Roesser model [24] and Fornasini-Marchesini model [27]. In Roesser model, the system state is composed of two independent states, horizontal state and vertical state, which are denoted by sub-vectors, the next horizontal state and vertical state in different directions can be deduced from current state, respectively. Unlike the Roesser model, the system state in Fornasini-Marchesini model is represented by a vector which is a function with two independent variables, and the current system state is derived from the last two adjacent states in different directions. In most circumstances, the Roesser model can be regarded as particular case of Fornasini-Marchesini model, but is much simpler and more intelligible than the latter. Based on 2D bounded real lemma, the H_∞ control theory for 2D systems was established by Du and Xie in [28] and [29]. This

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result was extended to control and filtering problems for 2D systems with time-varying delays [30], multiplicative noise [31] and uncertainty parameters [32]. Anh has studied the dissipative control and filtering problems for 2D systems in [33] by utilizing linear matrix inequalities technique. A large number of preliminary results on SMC of discrete-time 2D systems have been established [34]–[36]. In recent years, 2D systems with Markov jump parameters have attracted more interests of researchers. For example, the H_∞ control, H_∞ filtering, H_∞ mode reduction and fault detection problems for 2D-MJSs have been investigated in [37]–[41], respectively. However, we have to notice that, the issue of SMC for 2D-MJSs has not been fully studied yet, which motivates us for current work.

Based on the aforementioned considerations, we will investigate the asynchronous SMC design problem for 2D-MJSs in this paper. The primary contributions of this paper are shown as following aspects:

1) According to the properties of 2D systems, a novel 2D sliding surface is constructed, and the corresponding asynchronous 2D-SMC is designed under the framework of hidden Markov model, which means it is more generalized than the synchronous and mode-independent ones.
2) A sufficient condition that can guarantee the concerned 2D system is asymptotically mean square stable (AMSS) with an H_∞ disturbance attenuation performance is established. Moreover, this condition can simultaneously ensure the reachability of the constructed 2D sliding surface. Finally, the design procedures of SMC are summarized into an algorithm.
3) It is the first time that the asynchronous SMC is investigated for 2D-MJSs, which shows the feasibility and effectiveness of SMC for 2D-MJSs. What's more, the proposed asynchronous SMC design method can be applied not only to the Roesser model, but also to the Fornasini-Marchesini model merely through some simple modifications.

II. Preliminaries

Considering the following discrete-time 2D-MJS in Roesser model:

$$\begin{cases} \boldsymbol{x(i,j)} = A_{r(i,j)}x(i,j) + E_{r(i,j)}w(i,j) \\ \qquad\qquad + B_{r(i,j)}[(u(i,j) + f(x(i,j),r(i,j))] \\ y(i,j) = C_{r(i,j)}x(i,j) + D_{r(i,j)}w(i,j) \end{cases} \tag{1}$$

where

$$\boldsymbol{x(i,j)} = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}, \ x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$$

$x^h(i,j) \in \mathbb{R}^{n_h}$ and $x^v(i,j) \in \mathbb{R}^{n_v}$ represent horizontal and vertical sub-states respectively, $u(i,j) \in \mathbb{R}^{n_u}$ and $y(i,j) \in \mathbb{R}^{n_y}$ represent the controlled input and output respectively, and $w(i,j) \in \mathbb{R}^{n_w}$ represents the exogenous disturbance which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$. $A_{r(i,j)}$,$B_{r(i,j)}$,$C_{r(i,j)}$,$D_{r(i,j)}$ and $E_{r(i,j)}$ represent the time-varying system matrices, all of which are real known constant matrices with appropriate dimensions. Besides, we assume that the matrix $B_{r(i,j)}$ is full column rank for each $r(i,j) \in \mathcal{N}_1$, that is, rank($B_{r(i,j)}$)$= n_u$. The nonlinear function $f(x(i,j),r(i,j))$ satisfies the following property:

$$\|f(x(i,j),r(i,j))\| \le \delta_{r(i,j)}\|x(i,j)\| \tag{2}$$

where $\delta_{r(i,j)}$ is a known scalar, $\|\cdot\|$ denotes the Euclidean norm of a vector. The parameter $r(i,j)$ takes values in a finite set $\mathcal{N}_1 = \{1,2...,N_1\}$ with transition probability matrix $\varLambda = \{\lambda_{kp}\}$, and the related transition probability from mode k to mode p is given by

$$\begin{aligned} &\Pr\{r(i+1,j) = p | r(i,j) = k\} \\ =&\Pr\{r(i,j+1) = p | r(i,j) = k\} = \lambda_{kp}, \ \forall k,p \in \mathcal{N}_1 \end{aligned} \tag{3}$$

where $\lambda_{kp} \in [0,1]$, for all $k,p \in \mathcal{N}_1$, and $\sum_{p=1}^{N_1}\lambda_{kp} = 1$ for every mode k.

We define the boundary condition $(X_0, \varGamma_0)$ of the 2D-MJS (1), as follows:

$$\begin{cases} X_0 = \{x^h(0,j), x^v(i,0) |\ i,j = 0,1,2...\} \\ \varGamma_0 = \{r(0,j), r(i,0) |\ i,j = 0,1,2...\} \end{cases} \tag{4}$$

And the corresponding zero boundary condition is assumed as $x^h(0,j) = 0, x^v(i,0) = 0$, for every nonnegative integer i,j. Besides, the following assumption is imposed on X_0:

Assumption 1. *The boundary condition X_0 satisfies:*

$$\lim_{Z\to\infty} \mathbb{E}\Big\{\sum_{z=1}^{Z}(\|x^h(0,z)\|^2 + \|x^v(z,0)\|^2)\Big\} < \infty \tag{5}$$

where $\mathbb{E}\{\cdot\}$ represents the mathematical expectation.

In practical applications, the complete information of $r(i,j)$ can not always be available to the controller. Hence, in this paper, the hidden Markov model $(r(i,j), \sigma(i,j), \varLambda, \varPsi)$ as in [9] is introduced to characterize the asynchronous phenomenon between the controller and the original system. The parameter $\sigma(i,j)$, refers to controller mode, takes values in another finite set $\mathcal{N}_2 = \{1,2...N_2\}$, and satisfies the conditional probability matrix $\varPsi = \{\mu_{k\tau}\}$ with conditional mode transition probabilities

$$\Pr\{\sigma(i,j) = \tau | r(i,j) = k\} = \mu_{k\tau} \tag{6}$$

where $\mu_{k\tau} \in [0,1]$ for all $k \in \mathcal{N}_1, \tau \in \mathcal{N}_2$, and $\sum_{\tau=1}^{N_2}\mu_{k\tau} = 1$ for any mode k.

Remark 1. *The hidden Markov model is a united framework that covers synchronous and mode-independent cases. Which case the controller belongs to depends on the conditional probability matrix $\varPsi$. The controller belongs to synchronous case if and only if $\varPsi$ satisfies:*

$$\mu_{k\tau} = \begin{cases} 1, & \tau = \tau_k \\ 0, & \tau \neq \tau_k \end{cases} \qquad \tau_k \in \mathcal{N}_2$$

for every mode $k \in \mathcal{N}_1$. The controller becomes a mode-independent one when $\mathcal{N}_2 = \{1\}$. Thus, the established results can be readily extended to synchronous and mode-independent cases by adjusting the conditional probability matrix $\varPsi$.

In the following, the definitions of AMSS and H_∞ disturbance attenuation performance for 2D systems will be given in Definition 1 and Definition 2, respectively.

Definition 1. *For $w(i,j) \equiv 0$, the 2D-MJS (1) under Assumption 1 is said to be AMSS, if the following formulation holds:*

$$\lim_{i+j\to\infty} \mathbb{E}\{\|x(i,j)\|^2\} = 0 \tag{7}$$

for any boundary condition X_0.

Definition 2. *Given a scalar $\gamma > 0$, the 2D-MJS (1) under zero boundary condition is said to be AMSS with an H_∞ disturbance attenuation performance γ, if condition (7) and the following formulation hold:*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|y(i,j)\|^2\} < \gamma^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\{\|w(i,j)\|^2\} \tag{8}$$

for all $w(i,j) \in \ell_2\{[0,\infty),[0,\infty)\}$.

Now, we will make some notational simplification for convenience. The parameter $r(i,j)$ will be represented by k, $r(i+1,j)$ and $r(i,j+1)$ will be represented by p, and $\sigma(i,j)$ will be represented by s.

One objective of this work is to design an asynchronous SMC, such that the 2D-MJS (1) is AMSS with an H_∞ disturbance attenuation performance γ.

III. Main Result

A. Sliding Surface and Asynchronous SMC Design

In this work, we construct a novel 2D sliding surface as follows:

$$s(i,j) = \begin{bmatrix} s^h(i,j) \\ s^v(i,j) \end{bmatrix} = Gx(i,j) \tag{9}$$

where $s^h(i,j)$ and $s^v(i,j)$ represent horizontal and vertical sub-sliding surface respectively, the matrix $G = \sum_{k=1}^{N_1} \beta_k B_k^T$, and scalars β_k should be chosen properly such that GB_k is nonsingular for any $k \in \mathcal{N}_1$. Based on the assumption that B_k is full column rank for any $k \in \mathcal{N}_1$, we can find that the above condition can be guaranteed easily with properly selected parameter β_k.

Considering that modes of original system can not be directly accessed to the controller, thus, we design an asynchronous 2D-SMC law as follows:

$$u(i,j) = K_\tau x(i,j) - \rho(i,j)\frac{s(i,j)}{\|s(i,j)\|} \tag{10}$$

where the matrix $K_\tau \in \mathbb{R}^{n_u \times n_x}$ with $n_x = n_h + n_v$ will be determined later, and the parameter $\rho(i,j)$ is set to

$$\rho(i,j) = \varrho_1\|x(i,j)\| + \varrho_2\|w(i,j)\| \tag{11}$$

with $\varrho_1 = \max_{k\in\mathcal{N}_1}\{\delta_k\}$, $\varrho_2 = \max_{k\in\mathcal{N}_1}\{\|(GB_k)^{-1}GE_k\|\}$, and the parameter δ_k is given in (2).

Considering system (1) with asynchronous 2D-SMC law (9), we can derive the following closed-loop 2D system

$$\boldsymbol{x(i,\, j)} = \bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_k w(i,j) \tag{12}$$

where $\bar{A}_{k\tau} = A_k + B_k K_\tau$, and $\bar{\rho}_k(i,j)$ as follows:

$$\bar{\rho}_k(i,j) = f_k(x(i,j)) - (\varrho_1\|x(i,j)\| + \varrho_2\|w(i,j)\|)\cdot\frac{s(i,j)}{\|s(i,j)\|}.$$

Then, based on the properties of norm, following condition can be deduced easily

$$\|\bar{\rho}_k(i,j)\| \le (\varrho_1 + \delta_k)\|x(i,j)\| + \varrho_2\|w(i,j)\|. \tag{13}$$

Remark 2. *As mentioned above, the SMC designed in this paper is asynchronous with the original system, and this phenomenon that lies between them is described by the hidden Markov model $(r(i,j), \sigma(i,j), \Lambda, \Psi)$. The model consists of two Markov chain, the first Markov chain $\{r(i,j), i,j = 1,2\ \ldots\}$ governs the mode jumps of original system and the second $\{\sigma(i,j), i,j = 1,2\ldots\}$ governs the mode jumps of controller. Noting that, the second Markov chain is related to a conditional probability matrix Ψ which establishes a connection between the modes of controller and original system, which means the asynchronous SMC can maximumly utilize the information of original system.*

Remark 3. *Different from SMC for 1D systems, the SMC designed in this paper is an asynchronous 2D-SMC. First, it is asynchronous and described by the hidden Markov model. Secondly, following the idea of Roesser model, $s^h(i,j)$ and $s^v(i,j)$ that denote horizontal and vertical sub-sliding surface respectively, are employed to construct a new 2D sliding surface $s(i,j)$. The property will be inherited by the 2D-SMC law (10) since the $s(i,j)$ is included, which is the reason we name it as 2D-SMC.*

B. Analysis of Stability and H_∞ Attenuation Performance

In this subsection, we focus on the stability and H_∞ disturbance attenuation performance analysis for the concerned closed-loop 2D system (12). A sufficient condition will be established to guarantee the concerned 2D system is AMSS with an H_∞ attenuation performance γ.

Theorem 1. *Consider the 2D-MJS (1) under Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma > 0$, if there exist matrices $K_\tau \in \mathbb{R}^{n_u\times n_x}$, $R_k = \mathrm{diag}\{R_k^h, R_k^v\} > 0$, $Q_{k\tau} > 0$, $T_{k\tau} > 0$ and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, \tau \in \mathcal{N}_2$, such that the following inequalities hold:*

$$B_k^T \mathcal{R}_k B_k - \epsilon_k I \le 0 \tag{14}$$

$$\mathcal{A} + 2\Big(\sum_{\tau=0}^{N_2} \mu_{k\tau}\mathrm{diag}\{Q_{k\tau}, T_{k\tau}\}\Big) < 0 \tag{15}$$

$$\hat{A}_{k\tau}^T \mathcal{R}_k \hat{A}_{k\tau} - \mathrm{diag}\{Q_{k\tau}, T_{k\tau}\} < 0 \tag{16}$$

where

$$\mathcal{A} = \begin{bmatrix} \varPi_1 & \varPi_3 \\ * & \varPi_2 \end{bmatrix}$$

with

$$\begin{cases} \varPi_1 = -R_k + 4(\delta_k + \varrho_1)^2\epsilon_k I + C_k^T C_k \\ \varPi_2 = -\gamma^2 I + D_k^T D_k + 4\varrho_2^2\epsilon_k I \\ \varPi_3 = C_k^T D_k \end{cases}$$

and $\mathcal{R}_k = \sum_{p=1}^{N1} \lambda_{kp} R_p$, $\hat{A}_{k\tau} = \begin{bmatrix} \bar{A}_{k\tau} & E_k \end{bmatrix}$, *then the closed-loop system* (12) *is AMSS with an* H_∞ *disturbance attenuation performance* γ.

Proof. Let's start the proof with the stability of system. Selecting the Lyapunov candidate as $V_1(i,j) = x^T(i,j) R_k x(i,j)$, then, define

$$\varDelta V_1(i,j) = \boldsymbol{x(i, j)}^T R_p \boldsymbol{x(i, j)} - x^T(i,j) R_k x(i,j) \tag{17}$$

Based on the closed-loop system equation (12) with $w(i,j) = 0$, we have

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j)\} \\
&= \sum_{\tau=0}^{N_2} \mu_{k\tau} \Big\{ \big[\bar{A}_{k\tau} x(i,j) + B_k \bar{\rho}_k(i,j)\big]^T \mathcal{R}_k \\
&\times \big[\bar{A}_{k\tau} x(i,j) + B_k \bar{\rho}_k(i,j)\big] \Big\} \\
&- x^T(i,j) R_k x(i,j) \\
&\le x^T(i,j) \Big\{ 2\big(\sum_{\tau=1}^{N_2} \mu_{k\tau} \bar{A}_{k\tau}^T(i,j) \mathcal{R}_k \bar{A}_{k\tau} \big) \Big\} x(i,j) \\
&+ 2\bar{\rho}_k^T(i,j) B_k^T \mathcal{R}_k B_k \bar{\rho}_k(i,j) \\
&- x^T(i,j) R_k x(i,j)
\end{aligned} \tag{18}$$

Recalling the conditions given in (13) and (14), the following inequality can be further obtained

$$\mathbb{E}\{\varDelta V_1(i,j)\} \le x^T(i,j) \mathcal{G}_{k\tau} x(i,j) \tag{19}$$

where $\mathcal{G}_{k\tau} = 2\big(\sum_{\tau=0}^{N_2} \mu_{k\tau} \bar{A}_{k\tau}^T \mathcal{R}_k \bar{A}_{k\tau}\big) + 2\epsilon_k(\delta_k + \varrho_1)^2 I - R_k$. The following inequality can be deduced from (15) based on the properties of matrix quadratic

$$2\big(\sum_{\tau=1}^{N_2} \mu_{k\tau} Q_{k\tau}\big) + 4\epsilon_k(\delta_k + \varrho_1)^2 I + C_k^T C_k - R_k < 0 \tag{20}$$

which will further deduce

$$2\big(\sum_{\tau=1}^{N_2} \mu_{k\tau} Q_{k\tau}\big) + 2\epsilon_k(\delta_k + \varrho_1)^2 I - R_k < 0 \tag{21}$$

The following inequality can be inferred directly from condition (16)

$$\bar{A}_{k\tau}^T \mathcal{R}_k \bar{A}_{k\tau} - Q_{k\tau} < 0 \tag{22}$$

Combining (21) and (22), we can infer that $\mathcal{G}_{k\tau} < 0$, which is equivalent to

$$\mathcal{G}_{k\tau} \le -\alpha I \tag{23}$$

with scalar $\alpha > 0$. Recalling (19), we can further infer that

$$\mathbb{E}\{\varDelta V_1(i,j)\} \le -\alpha \mathbb{E}\{\|x(i,j)\|^2\} \tag{24}$$

Summing up on the both sides of (24), we have

$$\mathbb{E}\Big\{ \sum_{i=0}^{\hat{\kappa}_1} \sum_{j=0}^{\hat{\kappa}_2} \|x(i,j)\|^2 \Big\} \le -\frac{1}{\alpha} \mathbb{E}\Big\{ \sum_{i=0}^{\hat{\kappa}_1} \sum_{j=0}^{\hat{\kappa}_2} \varDelta V_1(i,j) \Big\} \tag{25}$$

where parameters $\hat{\kappa}_1$, $\hat{\kappa}_2$ are any positive integers. By substituting $\varDelta V_1$ with (17) and let $R_k = \mathrm{diag}\{R_k^{\mathrm{h}}, R_k^{\mathrm{v}}\}$, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\hat{\kappa}_1} \sum_{j=0}^{\hat{\kappa}_2} \varDelta V_1(i,j) \\
&= \sum_{i=0}^{\hat{\kappa}_1} \big\{ V_1^v(i, \hat{\kappa}_2 + 1) - V_1^v(i, 0) \big\} \\
&- \sum_{j=0}^{\hat{\kappa}_2} \big\{ V_1^h(\hat{\kappa}_1 + 1, j) - V_1^h(0, j) \big\} \\
&\le -\big(\sum_{i=0}^{\hat{\kappa}_1} V_1^v(i,0) + \sum_{j=0}^{\hat{\kappa}_2} V_1^h(0,j) \big)
\end{aligned} \tag{26}$$

where $V_1^h(i,j)$ and $V_1^v(i,j)$ are defined as

$$\begin{cases} V_1^h(i,j) = x^{hT}(i,j) R_{r(i,j)}^h x^h(i,j) \\ V_1^v(i,j) = x^{vT}(i,j) R_{r(i,j)}^v x^v(i,j) \end{cases}$$

Recalling the boundary condition in Assumption 1, and let $\hat{\kappa}_1$, $\hat{\kappa}_2$ tend to infinity, it follows from (25) and (26) that

$$\begin{aligned}
&\mathbb{E}\Big\{ \sum_{i=0}^{\hat{\kappa}_1} \sum_{j=0}^{\hat{\kappa}_2} \|x(i,j)\|^2 \Big\} \\
&\le -\frac{\beta}{\alpha} \sum_{\ell=0}^{\infty} \big(\|x^v(\ell, 0)\|^2 + \|x^h(0,\ell)\|^2 \big) \\
&< \infty
\end{aligned} \tag{27}$$

where β is the maximum eigenvalue of $R^h(0,\ell)$ and $R^v(\ell, 0)$, for any $\ell = 0, 1, 2 \ldots$, which indicates that (7) holds. Therefore, the asymptotic mean square stability of the concerned closed-loop 2D system is proved.

Next, let's focus on the H_∞ disturbance attenuation performance when the system is under zero boundary condition. Based on the closed-loop system equation (12), it is easy to find that

$$\begin{aligned}
&\mathbb{E}\{\varDelta V_1(i,j)\} \\
&= \sum_{\tau=0}^{N_2} \mu_{k\tau} \Big\{ \big[\bar{A}_{k\tau} x(i,j) + B_k \bar{\rho}_k(i,j) + E_k w(i,j)\big]^T \\
&\times \mathcal{R}_k \big[\bar{A}_{k\tau} x(i,j) + B_k \bar{\rho}_k(i,j) + E_k w(i,j)\big] \Big\} \\
&- x^T(i,j) R_k x(i,j) \\
&\le \hat{x}^T(i,j) \Big\{ 2\big(\sum_{\tau=1}^{N_2} \mu_{k\tau} \hat{A}_{k\tau}^T(i,j) \mathcal{R}_k \hat{A}_{k\tau} \big) \Big\} \hat{x}(i,j) \\
&+ 2\bar{\rho}_k^T(i,j) B_k^T \mathcal{R}_k B_k \bar{\rho}_k(i,j) \\
&- x^T(i,j) R_k x(i,j)
\end{aligned} \tag{28}$$

where

$$\hat{x}(i,j) = \begin{bmatrix} x(i,j) \\ w(i,j) \end{bmatrix}, \ \hat{A}_{k\tau}(i,j) = \begin{bmatrix} \bar{A}_{k\tau} & E_k \end{bmatrix}$$

Notice that from (13) and (14), we have

$$\begin{aligned}
&\bar{\rho}_k^T(i,j) B_k^T \mathcal{R}_k B_k \bar{\rho}_k(i,j) \\
&\le 2\epsilon_k \big((\delta_k + \varrho_1)^2 \|x(i,j)\|^2 + \varrho_2^2 \|w(i,j)\|^2 \big)
\end{aligned} \tag{29}$$

The following condition can be deduced easily from (15) and (16)

$$\Xi_{k\tau} < 0 \tag{30}$$

where $\Xi_{k\tau} \equiv \mathcal{A} + 2\sum_{\tau=1}^{N_2} \mu_{k\tau} \hat{A}_{k\tau}^T \mathcal{R}_k \hat{A}_{k\tau}$. Recalling the system (1), and substituting (29) into (28) yields

$$\begin{aligned} &\mathbb{E}\{\Delta V_1(i,j) + \|y(i,j)\|^2 - \gamma^2 \|w(i,j)\|^2\} \\ &\le \hat{x}^T(i,j)\Xi_{k\tau}\hat{x}(i,j) < 0 \end{aligned} \tag{31}$$

Noting (26) with the zero boundary condition, we can infer that

$$\begin{aligned} &\sum_{i=0}^{\kappa_1}\sum_{j=0}^{\kappa_2} \Delta V_1(i,j) \\ &= \sum_{i=0}^{\kappa_1} V_1^v(i,\kappa_2+1) + \sum_{j=0}^{\kappa_2} V_1^h(\kappa_1+1,j) \\ &\ge 0 \qquad \forall \kappa_1,\kappa_2 = 1,2,3... \end{aligned} \tag{32}$$

Then, we can further deduce from (31) and (32) that

$$\begin{aligned} &\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\{\|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\} \\ &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\{\Delta V_1(i,j) + \|y(i,j)\|^2 - \gamma^2\|w(i,j)\|^2\} \\ &< 0 \end{aligned} \tag{33}$$

which suggests that (8) is satisfied. And this completes the proof of Theorem 1.

□

Remark 4. *As mentioned above, the hidden Markov model consists of two Markov chains which will result in some difficulties in SMC design. Following a similar method proposed in [9], the matrices $Q_{k\tau}$ and $T_{k\tau}$ are introduced to separate those two Markov chains.*

C. Analysis of Reachability

The reachability of the designed asynchronous 2D-SMC law for the concerned closed-loop 2D system (12) is going to be discussed in this subsection. By using a stochastic Lyapunov method, we provide a sufficient condition which will confirm that the designed asynchronous 2D-SMC law (10) can force the state trajectories of the concerned closed-loop 2D system (12) into a time-varying sliding region around the predefined 2D sliding surface (9).

Theorem 2. *Consider the closed-loop 2D-MJS (12) with asynchronous 2D-SMC law (10). If there exist matrices $K_\tau \in \mathbb{R}^{n_u \times n_x}$, $R_k > 0$, $F_k > 0$, and scalars $\epsilon_k > 0$, for any $k \in \mathcal{N}_1, \tau \in \mathcal{N}_2$, such that the condition (14) and the following inequality holds*

$$2\sum_{\tau=1}^{N_2} \bar{A}_{k\tau}^T\big(\mathcal{R}_k + G^T\mathcal{F}_kG\big)\bar{A}_{k\tau} - R_k < 0 \tag{34}$$

where $\mathcal{R}_k$ is defined in Theorem 1, and $\mathcal{F}_k = \sum_{p=1}^{N_1}\lambda_{kp}F_p$. Then, the state trajectories of the concerned closed-loop 2D system (12) will be forced into a sliding region $\mathcal{O}$, around the predefined sliding surface (9), where $\mathcal{O}$ is given as:

$$\mathcal{O} \equiv \Big\{\|s(i,j)\| \le \ \rho^*(i,j)\Big\} \tag{35}$$

and $\rho^*(i,j) = \max_{k\in\mathcal{N}_1}\sqrt{\hat{\rho}_k(i,j)/\lambda_{\min}(F_k)}$ with

$$\begin{aligned} \hat{\rho}_k(i,j) = 4\big(&\|E_k^T\mathcal{R}_kE_k\| + \|E_k^TG^T\mathcal{R}_kGE_k\| \\ &+ 2\varrho_2^2(\|B_k^T\mathcal{F}_kB_k\| + \|B_k^TG^T\mathcal{F}_kGB_k\|)\big)\|w(i,j)\|^2 \\ &+ 8\big(\|B_k^T\mathcal{R}_kB_k\| + \|B_k^TG^T\mathcal{R}_kGB_k\|\big) \\ &\times(\varrho_1+\delta_k)^2\|x(i,j)\|^2. \end{aligned}$$

and $\lambda_{\min}(F_k)$ here denotes the minimum eigenvalue of F_k.

Proof. Above all, let's define ***s(i, j)*** $= \begin{bmatrix} s^h(i+1,j) \\ s^v(i,j+1)\end{bmatrix}$, it is easy to find that ***s(i, j)*** $=$ G***x(i, j)***. Then, we select the Lyapunov candidate as

$$V(i,j) = V_1(i,j) + V_2(i,j) \tag{36}$$

where $V_1(i,j)$ is defined in Theorem 1, and $V_2(i,j) = s^T(i,j)F_ks(i,j)$. Similar with the proof procedure of Theorem 1, it is easy to find that

$$\begin{aligned} &\mathbb{E}\{\Delta V_1(i,j)\} \\ &= \mathbb{E}\Big\{\boldsymbol{x(i,\,j)}^TR_p\boldsymbol{x(i,\,j)} - x^T(i,j)R_kx(i,j)\Big\} \\ &= \sum_{\tau=0}^{N_2}\mu_{k\tau}\Big\{\big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T \\ &\times\mathcal{R}_k\big[\bar{A}_{k\tau}x(i,j) + B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]\Big\} \\ &- x^T(i,j)R_kx(i,j) \\ &\le 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\mathcal{R}_k\bar{A}_{k\tau}x(i,j) \\ &+ 2\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big]^T\mathcal{R}_k \\ &\times\big[B_k\bar{\rho}_k(i,j) + E_kw(i,j)\big] \\ &- x^T(i,j)R_kx(i,j) \\ &\le 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\mathcal{R}_k\bar{A}_{k\tau}x(i,j) \\ &+ \bar{\rho}_k^T(i,j)B_k^T\mathcal{R}_kB_k\bar{\rho}_k(i,j) \\ &+ w^T(i,j)E_k^T\mathcal{R}_kE_kw(i,j) \\ &- x^T(i,j)R_kx(i,j) \end{aligned} \tag{37}$$

Along with the sliding surface function (9), we can deduce that

$$\begin{aligned}
&\mathbb{E}\{\Delta V_2(i,j)\}\\
&=\mathbb{E}\Big\{\boldsymbol{s(i,\,j)}^T F_p \boldsymbol{s(i,\,j)}-s^T(i,j)F_k s(i,j)\Big\}\\
&=\sum_{\tau=0}^{N_2}\mu_{k\tau}\Big\{\big[\bar{A}_{k\tau}x(i,j)+B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]^T\\
&\times G^T\mathcal{F}_k G\big[\bar{A}_{k\tau}x(i,j)+B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]\Big\}\\
&-s^T(i,j)F_k s(i,j)\\
&\le 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T G^T\mathcal{F}_k G\bar{A}_{k\tau}x(i,j)\\
&+2\big[B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]^T G^T\mathcal{F}_k\\
&\times G\big[B_k\bar{\rho}_k(i,j)+E_k w(i,j)\big]\\
&-s^T(i,j)F_k s(i,j)\\
&\le 2x^T(i,j)\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T G^T\mathcal{F}_k G\bar{A}_{k\tau}x(i,j)\\
&+\bar{\rho}_k^T(i,j)B_k^T G^T\mathcal{F}_k G B_k\bar{\rho}_k(i,j)\\
&+w^T(i,j)E_k^T G^T\mathcal{F}_k G E_k w(i,j)\\
&-s^T(i,j)F_k s(i,j)
\end{aligned}\tag{38}$$

Combing conditions (36)-(38), we can infer that

$$\begin{aligned}
&\mathbb{E}\{\Delta V(i,j)\}\\
&=\mathbb{E}\Big\{\Delta_1(i,j)+\Delta V_2(i,j)\Big\}\\
&\le x^T(i,j)\Big\{2\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\big(\mathcal{R}_k+G^T\mathcal{F}_k G\big)\bar{A}_{k\tau}\Big\}x(i,j)\\
&+\vec{\rho}_k(i,j)-x^T(i,j)R_k x(i,j)-\lambda_{\min}(F_k)\|s(i,j)\|^2
\end{aligned}\tag{39}$$

where

$$\begin{aligned}
&\vec{\rho}_k(i,j)\\
&=4\big(\|B_k^T\mathcal{F}_k B_k\|+\|B_k^T G^T\mathcal{F}_k G B_k\|\big)\|\bar{\rho}_k(i,j)\|^2\\
&+4\big(\|E_k^T\mathcal{R}_k E_k\|+\|E_k^T G^T\mathcal{R}_k G E_k\|\big)\|w(i,j)\|^2
\end{aligned}$$

Recalling the condition (13), we can get an inequality as follows

$$\|\bar{\rho}_k(i,j)\|^2\le 2(\varrho_1+\delta_k)^2\|x(i,j)\|^2+2\varrho_2^2\|w(i,j\|^2 \tag{40}$$

It is obvious that $\vec{\rho}_k(i,j)<\hat{\rho}_k(i,j)$ for any $k\in\mathcal{N}_1$ after substituting $\vec{\rho}_k(i,j)$ with (40). Then, based on the condition (15), when the state trajectories are out of the region $\mathcal{O}$, we can infer that

$$-\lambda_{\min}(F_k)\|s(i,j)\|^2+\vec{\rho}_k(i,j)<0 \tag{41}$$

It yields from (34), (37), (39) and (41) that

$$\begin{aligned}
&\mathbb{E}\{\Delta V(i,j)\}\\
&\le x^T(i,j)\Big\{2\sum_{\tau=1}^{N_2}\mu_{k\tau}\bar{A}_{k\tau}^T\big(\mathcal{R}_k+G^T\mathcal{F}_k G\big)\bar{A}_{k\tau}\\
&-R_k\Big\}x(i,j)<0
\end{aligned}\tag{42}$$

which means that, when outside the region $\mathcal{O}$, the state trajectories of the concerned close-loop 2D system (12) will strictly decrease in the sense of mean square. Now, the proof is completed.

□

D. *Synthesis of Asynchronous 2D-SMC Law*

It is obvious that, if the theorem 1 and the theorem 2 hold simultaneously, then, the asymptotic mean square stability with an H_∞ performance γ of the closed-loop 2D system (12) and the reachability of the predefined 2D sliding function (9) can be guaranteed simultaneously. That is, the to-be-determined matrix K_τ in 2D-SMC law (10) should ensure that theorem 1, 2 are established at the same time. Now, in this subsection, we will continue our study with this idea.

Theorem 3. *Consider the 2D-MJS (1) under the Assumption (1) and with the asynchronous 2D-SMC law (10). For a given scalar $\gamma>0$, if there exist matrices $\tilde{K}_\tau\in\mathbb{R}^{n_u\times n_x}$, $L_\tau\in\mathbb{R}^{n_x\times n_x}$, $\tilde{R}_k=\mathrm{diag}\{\tilde{R}_k^h,\tilde{R}_k^v\}>0$, $\tilde{F}_k>0$, $\tilde{Q}_{k\tau}>0$, $\tilde{T}_{k\tau}>0$ and scalars $\tilde{\epsilon}_k>0$, for any $k\in\mathcal{N}_1,\tau\in\mathcal{N}_2$, such that the following inequalities hold:*

$$\begin{bmatrix}-\tilde{\epsilon}_k I & \mathscr{B}_k\\ * & \mathscr{R}_k\end{bmatrix}<0 \tag{43}$$

$$\begin{bmatrix}\mathscr{L}_{k\tau} & \mathscr{A}_{k\tau} & \mathscr{G}_{k\tau}\\ * & \mathscr{R}_k & 0\\ * & * & \mathscr{F}_k\end{bmatrix}<0 \tag{44}$$

$$\begin{bmatrix}\mathscr{H}_k & \mathscr{D}_k & \mathscr{P}_{k\tau} & \mathscr{Y}_{k\tau}\\ * & \mathscr{I}_k & 0 & 0\\ * & * & \mathscr{Q}_{k\tau} & 0\\ * & * & * & \mathscr{T}_{k\tau}\end{bmatrix}<0 \tag{45}$$

where

$$\begin{aligned}
\mathscr{B}_k&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{\epsilon}_k B_k^T & \sqrt{\lambda_{k2}}\tilde{\epsilon}_k B_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{\epsilon}_k B_k^T\end{bmatrix}\\
\mathscr{R}_k&=\mathrm{diag}\{-\tilde{R}_1,-\tilde{R}_2,\cdots,-\tilde{R}_{N_1}\}\\
\mathscr{F}_k&=\mathrm{diag}\{-\tilde{F}_1,-\tilde{F}_2,\cdots,-\tilde{F}_{N_1}\}\\
\mathscr{I}_k&=\mathrm{diag}\{-\tilde{\epsilon}_k,-I,-I,-\tilde{\epsilon}_k\}\\
\mathscr{Q}_{k\tau}&=\mathrm{diag}\{-\tilde{Q}_{k1},-\tilde{Q}_{k2},\cdots,-\tilde{Q}_{kN_2}\}\\
\mathscr{T}_{k\tau}&=\mathrm{diag}\{-\tilde{T}_{k1},-\tilde{T}_{k2},\cdots,-\tilde{T}_{kN_2}\}\\
\mathscr{L}_{k\tau}&=\mathrm{diag}\{\tilde{Q}_{k\tau}-L_\tau^T-L_\tau,-\tilde{T}_{k\tau}\}\\
\mathscr{H}_k&=\begin{bmatrix}-\tilde{R}_k & \tilde{R}C_k^T D_k^T\\ * & -\gamma^2 I\end{bmatrix}\\
\mathscr{A}_{k\tau}&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{k\tau}^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{k\tau}^T\\ \sqrt{\lambda_{k1}}\tilde{T}_{k\tau}B_k^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{T}_{k\tau}B_k^T\end{bmatrix}\\
\mathscr{G}_{k\tau}&=\begin{bmatrix}\sqrt{\lambda_{k1}}\tilde{A}_{k\tau}^T G^T & \cdots & \sqrt{\lambda_{kN_1}}\tilde{A}_{k\tau}^T G^T\\ 0 & \cdots & 0\end{bmatrix}\\
\mathscr{D}_{k\tau}&=\begin{bmatrix}2(\varrho_1+\delta_k)\tilde{R}_k & \tilde{R}_k C_k^T & 0 & 0\\ 0 & 0 & D_k^T & 2\varrho_2\end{bmatrix}\\
\mathscr{P}_{k\tau}&=\begin{bmatrix}\sqrt{2\mu_{k1}}\tilde{R}_k & \cdots & \sqrt{2\mu_{kN_2}}\tilde{R}_k\\ 0 & \cdots & 0\end{bmatrix}\\
\mathscr{Y}_{k\tau}&=\begin{bmatrix}0 & \cdots & 0\\ \sqrt{2\mu_{k1}}I & \cdots & \sqrt{2\mu_{kN_2}}I\end{bmatrix}
\end{aligned}$$

and $\tilde{A}_{k\tau}=A_k L_\tau+B_k\tilde{K}_\tau$. Then, the concerned closed-loop 2D-MJS (12) is AMSS with an H_∞ disturbance attenuation *performance γ, and the state trajectories of the concerned closed-loop 2D-MJS will be forced into a sliding region $\mathcal{O}$, around the predefined sliding surface (9). Moreover, the to-be-determined matrix K_τ in 2D-SMC law (10) can be chosen as*

$$K_\tau = \tilde{K}_\tau L_\tau^{-1} \tag{46}$$

if the LMIs (43)-(45) have feasible solutions.

Proof. As we discussed above, the objective is to testify that, the conditions (14)-(16) in Theorem 1 and (34) in Theorem 2 can be guaranteed simultaneously by (43)-(45). Before that, let's make some notations as $\tilde{K}_\tau = K_\tau L_\tau$, $\tilde{R}_k = R_k^{-1}$, $\tilde{F}_k = F_k^{-1}$, $\tilde{Q}_{k\tau} = Q_{k\tau}^{-1}$, $\tilde{T}_{k\tau} = T_{k\tau}^{-1}$ and $\tilde{\epsilon} = \epsilon_k^{-1}$. First of all, let's prove that (43) is equivalent to (14). Pre- and post- multiplying the inequalities given in (43) by $\mathrm{diag}\{\epsilon_k I, I, I, \cdots, I\}$, respectively, and applying Schur complement after that, then, we can see (14) is satisfied. Next, we will verity that (44)-(45) are sufficient to ensure (15)-(16) and (34) hold simultaneously. Using $\mathrm{diag}\{R_k, I, I, \cdots, I\}$ to pre- and post-multiply the inequality given in (45), and applying Schur complement after that, then we will have (15) satisfied. It follows from (44) that $\tilde{Q}_{k\tau} - L_\tau^T - L_\tau < 0$, that is $L_\tau^T + L_\tau$ is positive definite, which can further deduces that the matrix L_τ is invertible. Noting that $Q_{k\tau} > 0$, then we can infer the following formulation holds

$$(\tilde{Q}_{k\tau} - L_\tau)^T \tilde{Q}_{k\tau}^{-1} (\tilde{Q}_{k\tau} - L_\tau) \geq 0 \tag{47}$$

which means

$$-L_\tau^T \tilde{Q}_{k\tau} L_\tau \leq \tilde{Q}_{k\tau} - L_\tau^T - L_\tau \tag{48}$$

Recalling the condition given in (45), then we can infer that

$$\begin{bmatrix} \tilde{\mathscr{L}}_{k\tau} & \mathscr{A}_{k\tau} & \mathscr{G}_{k\tau} \\ * & \mathscr{R}_k & 0 \\ * & * & \mathscr{F}_k \end{bmatrix} < 0 \tag{49}$$

where $\tilde{\mathscr{L}}_{k\tau} = \mathrm{diag}\{-L_\tau^T \tilde{Q}_{k\tau} L_\tau, -\tilde{T}_{k\tau}\}$.
Noting that the slack matrix L_τ is invertible, we denote $h_{k\tau} = \mathrm{diag}\{L_\tau^{-1}, T_{k\tau}, I, I, \cdots, I\}$. Using $h_{k\tau}$ to pre- and post-multiply the inequality given in (49), and applying Schur complement after that, then the following inequality will be obtained

$$\hat{A}_{k\tau}^T \mathcal{R}_k \hat{A}_{k\tau} + \check{A}_{k\tau}^T \mathcal{F}_k \check{A}_{k\tau} - \mathrm{diag}\{Q_{k\tau}, T_{k\tau}\} < 0 \tag{50}$$

where $\check{A}_{k\tau} = \begin{bmatrix} \bar{A}_{k\tau} & 0 \end{bmatrix}$. Combing (15) and (50), we will have

$$\mathcal{A} + 2\sum_{\tau=0}^{N_2} \mu_{k\tau} \left\{ \hat{A}_{k\tau}^T \mathcal{R}_k \hat{A}_{k\tau} + \check{A}_{k\tau}^T \mathcal{F}_k \check{A}_{k\tau} \right\} < 0 \tag{51}$$

which further implies (34) holds based on the property of positive definite matrix. It is clear that, the gain matrix K_τ can not be obtained directly from LMIs in Theorem 3 while $\tilde{K}_\tau$ is obtained. Thanks to the fact that matrix L_τ is invertible, K_τ can be calculated indirectly with $K_\tau = \tilde{K}_\tau L_\tau^{-1}$. Now, the proof is finished. □

It is clear that, the conditions in Theorem 3 are presented in the form of linear matrix inequality (LMI), which can be easily solved with the help of Matlab LMI toolbox. Next, we are going to propose an algorithm to obtain the asynchronous 2D-SMC law with the minimum disturbance attenuation performance γ^*. The design procedures of the 2D-SMC are summarized into an algorithm as follows:

- ▶ Step 1: Select the parameters β_k properly and compute matrix G, such that, for any $k \in \mathcal{N}_1$, the matrix GB_k is nonsingular.
- ▶ Step 2: Figure out the scalars ϱ_1 and ϱ_2 in (11).
- ▶ Step 3: Get the matrices $\tilde{K}_\tau$ and L_τ by solving the following optimization problem

$$\min\ \tilde{\sigma} \text{ subject to (43)-(45) with } \tilde{\sigma} = \gamma^2\ . \tag{52}$$

- ▶ Step 4: Finally, if Step 3 has feasible solutions, then, the sliding mode controller matrix K_τ can be obtained with $K_\tau = \tilde{K}_\tau L_\tau^{-1}$.

IV. Numerical Example

In this section, we are going to provide an simulation example to verify the effectiveness of the proposed asynchronous 2D-SMC design method. As is well-known that several dynamical processes, for instance, gas absorption, water stream heating and air drying can be represented by the Darboux equation [25]. It is generally presented in the form of partial differential equations, and can be easily transformed into a 2D system in Roesser model with a similar approach applied in [28]. In this example, the concerned 2D-MJS and the asynchronous 2D-SMC both consist of two operation modes, that is, $\mathcal{N}_1 = \{1, 2\}$, $\mathcal{N}_2 = \{1, 2\}$, and the relevant system parameters are given as follows:
Mode 1:

$$A_1 = \begin{bmatrix} -1.0 & 0.4 \\ 0.2 & -1.0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & -0.3 \end{bmatrix}, \ D_1 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.1 \end{bmatrix}, \ E_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

Mode 2:

$$A_2 = \begin{bmatrix} -0.8 & 0.6 \\ 0.2 & -1.2 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.2 & -0.2 \\ -0.1 & -0.2 \end{bmatrix}, \ D_2 = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -0.2 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \ E_2 = E_1$$

The nonlinear function $f(x(i,j), r(i,j))$ is set as

$$f((x(i,j), r(i,j)) = \begin{cases} 0.2 \sin \|x(i,j)\|, & r(i,j) = 1 \\ 0.3 \sin \|x(i,j)\|, & r(i,j) = 2 \end{cases}$$

Thus, the parameter δ_k can be selected as first mode with $\delta_1 = 0.2$ and second mode with $\delta_2 = 0.3$.
The mode jumps of the considered 2D-MJS and 2D-SMC are governed by the transition probability matrix Λ and Ψ, respectively, which are set as follows:

$$\Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \ \Psi = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

A possible time sequences with two different directions of the system modes and the asynchronous 2D-SMC modes are depicted in Fig.1-2. By comparison, it is clear that the designed 2D-SMC run asynchronously with the original 2D system.

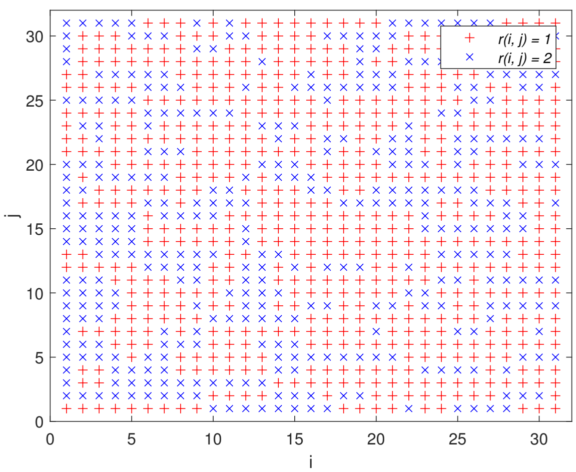

Fig. 1. Modes of the considered 2D system

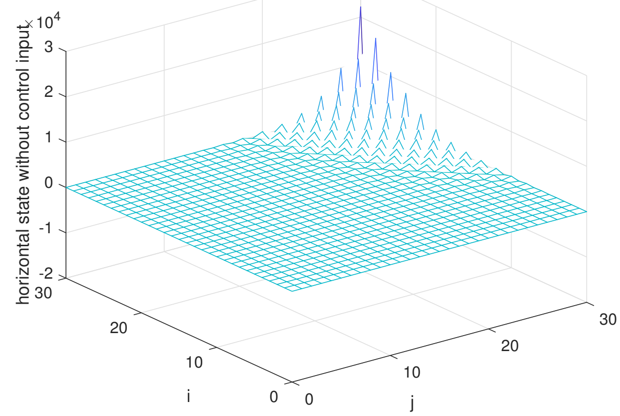

Fig. 3. Horizontal state $x^h(i,j)$ with $u(i,j)=0$

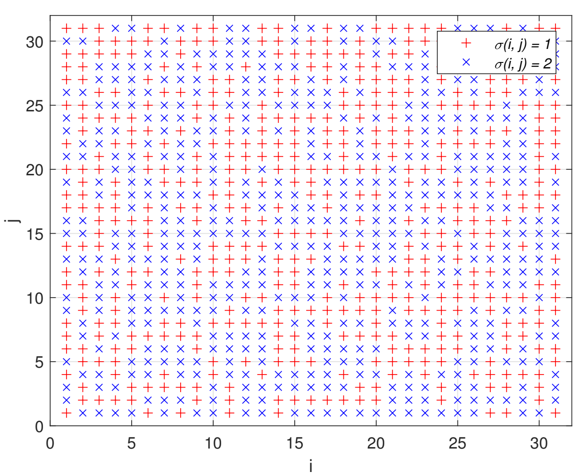

Fig. 2. Modes of the 2D-SMC

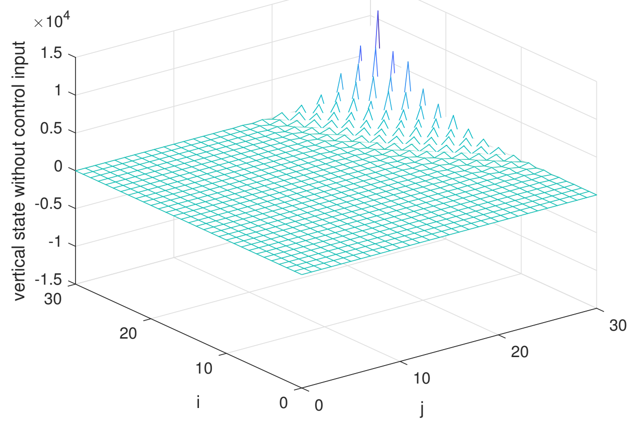

Fig. 4. Vertical state $x^v(i,j)$ with $u(i,j)=0$

Based on the corresponding assumptions, the boundary condition X_0 and the exogenous disturbance $w(i,j)$ are assumed to be

$$x^h(0,j) = \begin{cases} 0.1, & 0 \le j \le 10 \\ 0, & elsewhere \end{cases}$$
$$x^v(i,0) = \begin{cases} 0.1, & 0 \le i \le 10 \\ 0, & elsewhere \end{cases}$$
$$w(i,j) \ = \begin{cases} 0.2, & 0 \le i,j \le 10 \\ 0, & elsewhere \end{cases}$$

The horizontal and vertical state responses of the open-loop system (1) under $u(i,j)=0$ are depicted in Fig.3 and Fig.4, respectively. The obtained result implies the considered 2D system is unstable under zero control input. Next, let's follow the procedures of asynchronous 2D-SMC law design algorithm, and obtain the sliding model controller gain. Letting $\beta_1 = 0.6$ and $\beta_2 = 0.4$, then the matrix G can be calculated as follows:

$$G = \begin{bmatrix} 0.2 & -0.16 \\ -0.02 & -0.26 \end{bmatrix}$$

It's easy to verify that the non-singularity of the matrix GB_k is satisfied for any $k \in \mathcal{N}_1$. The parameters ϱ_1 and ϱ_2 can be calculated as $\varrho_1 = 0.3$ and $\varrho_2 = 0.6872$, respectively. By solving optimization problem (52), we can obtain the following sliding mode controller gains
Mode1

$$K_1 = \begin{bmatrix} 4.4147 & -4.3512 \\ -0.6827 & -2.1886 \end{bmatrix}$$

Mode2

$$K_2 = \begin{bmatrix} 4.3706 & -3.8453 \\ -0.6826 & -2.3252 \end{bmatrix}$$

with the minimum H_∞ disturbance attenuation performance $\gamma^* = 0.3164$.
The horizontal and vertical state response of the closed-loop system (12) with asynchronous 2D-SMC law are depicted in Fig.5-6. It is clear that both horizontal and vertical system state

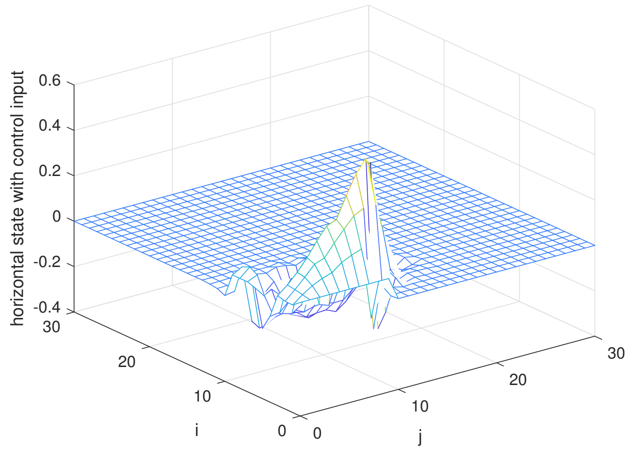

Fig. 5. Horizontal state $x^h(i,j)$ with asynchronous 2D-SMC

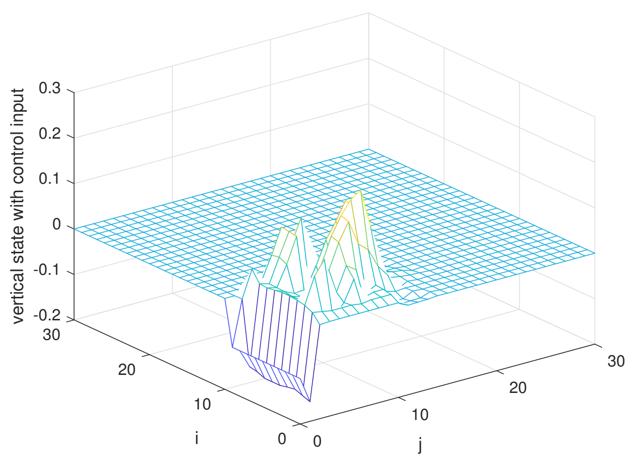

Fig. 6. Vertical state $x^v(i,j)$ with asynchronous 2D-SMC

converge to zero within finite steps. The sliding surface $s(i,j)$ and the asynchronous 2D-SMC law can be obtained based on the conditions given in (9) and (10), respectively. Then, the horizontal 2D-SMC input, vertical 2D-SMC input, horizontal sliding surface and vertical sliding surface are depicted in Fig.7-10. Clearly, the observed simulation results imply that the proposed asynchronous 2D-SMC law is effective for the considered 2D-MJS.

V. Conclusions

This work has investigated the problem of asynchronous SMC for 2D-MJSs in Roesser model. In consideration that the system modes are not always accessible to the controller, an asynchronous 2D-SMC design method is explored with hidden Markov model. By utilizing Lyapunov function, the asymptotic mean square stability for the concerned 2D-MJS and the reachability of the sliding mode dynamics are analyzed, respectively. A sufficient condition is derived that can guarantee the concerned closed-loop 2D system is AMSS with H_∞ disturbance attenuation performance , and the reachability

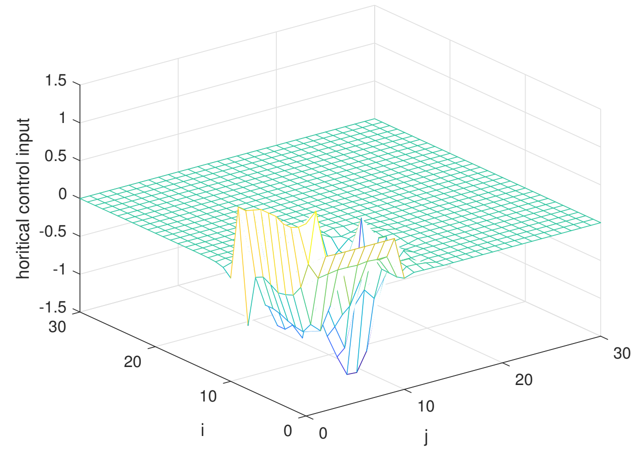

Fig. 7. Horizontal asynchronous 2D-SMC input $u^h(i,j)$

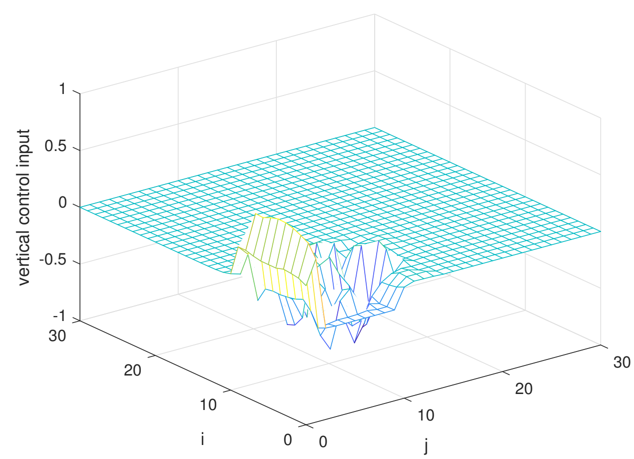

Fig. 8. Vertical asynchronous 2D-SMC input $u^v(i,j)$

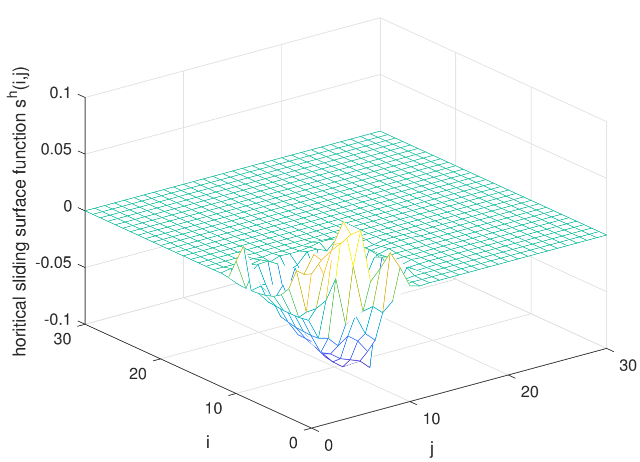

Fig. 9. Horizontal sliding surface $s^h(i,j)$

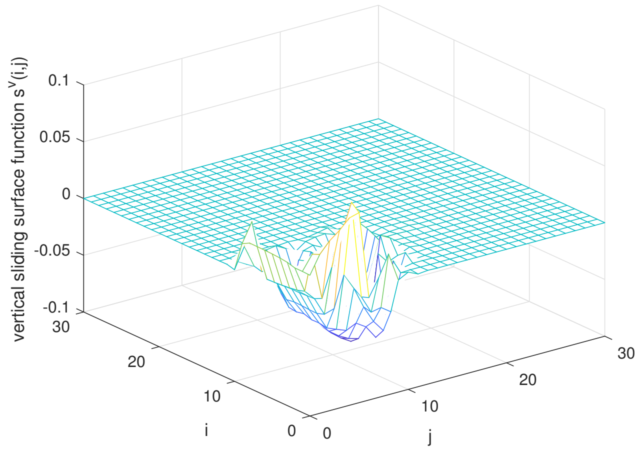

Fig. 10. Vertical sliding surface $s^v(i,j)$

of the 2D sliding mode dynamics is ensured simultaneously. Finally, the asynchronous SMC design procedures are summarized as an algorithm whose effectiveness is verified by a numerical example.

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