Asychronous Control for Markov Jump Lure's Systems With Control Saturation

***Abstract*—The abstract here.**

I. Introduction

II. Preliminaries

Consider a class of discrete-time MJLS on a probability space $(\mathit{\Omega}, \mathcal{F}, \mathcal{P})$:

$$\begin{cases} x_{k+1} = A_{r(k)}x_k + F_{r(k)}\varphi(y_k) + B_{r(k)}u_k \\ \qquad + E^x_{r(k)}w_k, \quad y_k = C_{r(k)}x_k, \\ z_k = C^z_{r(k)}x_k + G^z_{r(k)}\varphi(y_k) + D^z_{r(k)}u_k \\ \qquad + E^z_{r(k)}w_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}, u_k \in \mathbb{R}^{n_u}, y_k \in \mathbb{R}^{n_y}, z_k \in \mathbb{R}^{n_z}$, and $w_k \in \mathbb{R}^{n_w}$ represent the state, the control input, the output related to the nonlinearity, the controlled output and the exogenous disturbance vector respectively. $A_{r(k)}, F_{r(k)}, B_{r(k)}, E^x_{r(k)}, C^z_{r(k)}, G^z_{r(k)}, D^z_{r(k)}$ and $E^z_{r(k)}$ represent the time-varying system matrices, all of which are pre-known and real. $\{r(k), k \geq 0\}$ is a Markov chain taking values in a positive integer set $\mathcal{N} = \{1, 2, \dots, N\}$ with mode TPs:

$$\Pr\{r(k+1) = j | r(k) = i\} = \pi_{ij} \tag{2}$$

Clearly, for all $i, j \in \mathcal{N}$, $\pi_{ij} \in [0, 1]$, and $\sum_{j=0}^{N} \pi_{ij} = 1$ for each mode i. The system matrices in (1) at instant k can be expressed as $A_i, F_i, B_i, E^x_i, C^z_i, G^z_i, D^z_i$ and E^z_i, which are real known constant matrices with appropriate dimensions, and the related transition probability matrix is described as $\mathit{\Pi} = \{\pi_{ij}\}$.

Assumption 1: The function $\varphi(\cdot) : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ satisfies:
(1) $\varphi(0) = 0$ and
(2) there exists a diagonal positive definite matrix $\mathit{\Omega} \in \mathbb{R}^{p \times p}$ such that:

$$\varphi_{(\ell)}(y)[\varphi(y) - \mathit{\Omega} y]_{(\ell)} \leq 0. \tag{3}$$

where for all $y \in \mathbb{R}^p$, $\ell \in \{1, \dots, p\}$. And in this case we say that the nonlinearity $\varphi(\cdot)$ satisfies it's own cone bounded sector conditions and to be decentralized. According to (3) we have:

$$SC(y, \mathit{\Lambda}) := \varphi^{'}(y)\mathit{\Lambda}[\varphi(y) - \mathit{\Omega} y] \leq 0, \tag{4}$$

where $\mathit{\Lambda}$ is any diagonal positive semidefinite matrix, and $\mathit{\Omega}$ is pre-given. From the Assumption 1, we get:

$$[\mathit{\Omega} y]_{(\ell)}[\varphi(y) - \mathit{\Omega} y]_{(\ell)} \leq 0, \tag{5}$$

which implies:

$$0 \leq \varphi^{'}(y)\mathit{\Lambda}\varphi(y) \leq \varphi^{'}(y)\mathit{\Lambda}\mathit{\Omega} y \leq y^{'}\mathit{\Omega}\mathit{\Lambda}\mathit{\Omega} y \tag{6}$$

Definition 1: System (1) is said to be locally stochastically stable if for $w_(k) = 0$ and any initial condition $x_0 \in \mathcal{D}_0$, $r_0 \in \mathcal{N}$, the following formulation holds:

$$\|x\|^2_2 = \sum_{k=0}^{\infty} \mathbb{E}[\|x_k\|^2] < \infty. \tag{7}$$

Where in this case the set $\mathcal{D}_0 \subset \mathbb{R}^p$ is said to be the domain of stochastic stability of the origin.

Define $\mathcal{F}_k$ as the σ-field generated by the random variables x_k and $r(k)$. We define next the class of exogenous disturbances with bounded energy and the finite ℓ_2-induced norm.
Definition 2: For $\gamma > 0$, set $\mathcal{W}_\gamma$ is defined as follows:

$$\begin{aligned} \mathcal{W}_\gamma := \{ & w = \{w_k\}_{k\in\mathbb{N}}; w_k \in \mathbb{R}^m, k \in \mathbb{N}, w_k \text{ is} \\ & \mathcal{F}_k\text{-measureable, and } \|w\|^2_2 = \sum_{k=0}^{\infty} \mathbb{E}(\|w_k\|)^2 < \frac{1}{\gamma}\}. \end{aligned} \tag{8}$$

The finite ℓ_2-induced gain associated with the closed-loop system (1) with $x_0 = 0$ between the disturbance $w = \{w_k\}_{k\in\mathbb{N}}$ and controlled output $z = \{z_k\}_{k\in\mathbb{N}}$ is equal or less than $\sqrt{\varrho}$ if for every $w \in \mathcal{W}_\gamma$ then we have

$$\frac{1}{\varrho}\|z\|^2_2 = \frac{1}{\varrho}\sum_{k=0}^{\infty} \mathbb{E}\left[\|z_k\|^2\right] \leq \|w\|^2_2 = \sum_{k=0}^{\infty} \mathbb{E}\left[\|w_k\|^2\right]. \tag{9}$$

In this paper, we consider the following controller:

$$u_k = K_{\sigma(k)}x_k + \mathit{\Gamma}_{\sigma(k)}\varphi(y_k) \tag{10}$$

where $K_{\sigma(k)} \in \mathbb{N}^{n_u \times n_x}$ is a time-varying controller gain matrix, and $\mathit{\Gamma}_{\sigma(k)} \in \mathbb{N}^{n_u \times n_y}$ is a time-varying nonlinear output feedback gain matrix. The parameter $\{\sigma(k), k \geq 0\}$ takes values in another pre-given positive integer set, which is marked as $\mathcal{M} = \{1, 2, \dots, M\}$ subject to the pre-known conditional probability matrix $\mathit{\Phi} = \{\mu_{im}\}$, the probabilities of which are defined by

$$\Pr\{\sigma(k) = \phi | r(k) = i\} = \mu_{i\phi}. \tag{11}$$

Where for all $i \in \mathcal{N}, \phi \in \mathcal{M}, \mu_{i\phi} \in [0, 1]$, and $\sum_{\phi=1}^{M} \mu_{i\phi} = 1$ for each $i \in \mathcal{N}$.

Combing the asynchronous controller (10) and system (1) we have the following closed system:

$$\begin{cases} x_{k+1} = \bar{A}_{i\theta}x_k + \bar{F}_{i\theta}\varphi_i(C_i x_k) + E^x_i w_k \\ z_k = \bar{C}_{i\theta}x_k + \bar{G}_{i\theta}\varphi_i(C_i x_k) + E^z_i w_k \end{cases} \tag{12}$$

Where,for $i \in \mathcal{N}, \phi \in \mathcal{M}$

$$\begin{aligned} \bar{A}_{i\phi} &= A_i + B_i K_\phi, & \bar{F}_{i\phi} &= F_i + B_i \varGamma_{i\phi} \\ \bar{C}^z_{i\phi} &= C^z_i + D^z_i K_\phi, & \bar{G}^z_{i\phi} &= G^z_i + D^z_i \varGamma_\phi \end{aligned}$$

III. Main Result

$$\left\{ \begin{array}{l} x_{k+1} = \bar{A}_{i\theta} x_k + \bar{F}_{i\theta} \varphi_i(C_i x_k) + E^x_i w_k \\ z_k = \bar{C}_{i\theta} x_k + \bar{G}_{i\theta} \varphi_i(C_i x_k) + E^z_i w_k \end{array} \right.$$

System (12) is stochastically stable, if for all $i \in \mathcal{N}$ and $\theta \in \mathcal{M}$, there exist positive definite matrices $\bar{P}_i \in \mathbb{R}^{n_x \times n_x}$, $R_{i\theta} \in \mathbb{R}^{(n_x+n_y)\times(n_x+n_y)}$, matrices $K_\theta \in \mathbb{R}^{n_u \times n_x}$, $\varGamma_\theta \in \mathbb{R}^{n_u \times n_y}$ and positive semidefinite matrix $T_i \in \mathbb{R}^{n_y}$ to ensure (13) and (14) hold.

$$\begin{bmatrix} -R_{i\theta} & \mathscr{H}_{i\phi} \\ * & \mathscr{P}_i \end{bmatrix} < 0 \tag{13}$$

$$\begin{bmatrix} \mathscr{S}_{i\phi} & \mathscr{N}_{i\phi} \\ * & \mathscr{L}_{i\phi} \end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} -R_{i\theta} & \sqrt{\pi_{i1}} \hat{A}'_{i\theta} & \cdots & \sqrt{\pi_{iN}} \hat{A}'_{i\theta} \\ \sqrt{\pi_{i1}} A_{i\theta} & -\bar{P}_1 & & \\ \vdots & & \ddots & \\ \sqrt{\pi_{iN}} A_{i\theta} & & & -\bar{P}_N \end{bmatrix} < 0$$

$$\begin{bmatrix} \begin{bmatrix} -\bar{P}_i & 0 \\ * & H_{i\theta} \end{bmatrix} & \sqrt{u_{i1}} \begin{bmatrix} \bar{P}_i & \\ & R_{i1} \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} \bar{P}_i & \\ & R_{iM} \end{bmatrix} \\ \sqrt{u_{i1}} \begin{bmatrix} \bar{P}_i & \\ & R_{i1} \end{bmatrix} & \begin{bmatrix} -I & \\ & -R_{i1} \end{bmatrix} & & \\ \vdots & & \ddots & \\ \sqrt{u_{iM}} \begin{bmatrix} \bar{P}_i & \\ & R_{iM} \end{bmatrix} & & & \begin{bmatrix} -I & \\ & -R_{iM} \end{bmatrix} \end{bmatrix} < 0$$

where

$$\mathscr{H}_{i\phi} = \begin{bmatrix} \sqrt{\pi_{i1}} \hat{A}_{i\phi} \\ \sqrt{\pi_{i2}} \hat{A}_{i\phi} \\ \vdots \\ \sqrt{\pi_{i_N}} \hat{A}_{i\phi} \end{bmatrix}', \mathscr{P}_{i\phi} = \operatorname{diag}\{-\bar{P}_1, -\bar{P}_2, \ldots, -\bar{P}_N\}$$

$$\hat{A}_{i\theta} = \begin{bmatrix} \bar{A}_{i\theta} & \bar{F}_{i\theta} \end{bmatrix}$$

$$\mathscr{S}_{i\phi} = \begin{bmatrix} -P_i & 0 & 0 \\ * & -I & C'_i \varDelta_i T_i \\ * & * & -2T_i \end{bmatrix}$$

$$\mathscr{N}_{i\phi} = \begin{bmatrix} \sqrt{u_{i1}} \begin{bmatrix} \bar{P}_i & \\ & R_{i1} \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} \bar{P}_i & \\ & R_{iM} \end{bmatrix} & \end{bmatrix}$$

$$\mathscr{L}_{i\phi} = \operatorname{diag}\{ \begin{bmatrix} -I & \\ & -R_{i1} \end{bmatrix}, \ldots, \begin{bmatrix} -I & \\ & -R_{iM} \end{bmatrix} \}$$

$$H_{i\theta} = \begin{bmatrix} -I & C'_i \varDelta_i T_i \\ * & -2T_i \end{bmatrix}$$

Proof: Construct a Lyapunov function in the form of (15).

$$V(k, x_k, r(k)) = x'_k P_{r(k)} x_k \tag{15}$$

Where $P_{r(k)} = \bar{P}^{-1}_{r(k)}$. Denoting $h_{i\phi} = \operatorname{diag}\{P_i, I_{n_x+n_y}, \{\operatorname{diag}(I_{n_x}, R^{-1}_{i\phi})^M_{\phi=1}\}\}$

(14) mulitply $diag(P_i, I_{nx+ny}, \{diag(I_{nx}, R^{-1}_{i\theta}\}^M_{\theta=1}))$ in the left and right with it's transpose, then we get:

$$\begin{bmatrix} \begin{bmatrix} -P_i & 0 \\ * & H_{i\theta} \end{bmatrix} & \sqrt{u_{i1}} \begin{bmatrix} I & \\ & I \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} I & \\ & I \end{bmatrix} \\ \sqrt{u_{i1}} \begin{bmatrix} I & \\ & I \end{bmatrix} & \begin{bmatrix} -I & \\ & -R^{-1}_{i1} \end{bmatrix} & & \\ \vdots & & \ddots & \\ \sqrt{u_{iM}} \begin{bmatrix} I & \\ & I \end{bmatrix} & & & \begin{bmatrix} -I & \\ & -R^{-1}_{iM} \end{bmatrix} \end{bmatrix} < 0$$

According to Schur we get:

$$\sum_{\theta=1}^{M} u_{i\theta} \begin{bmatrix} I & \\ & R_{i\theta} \end{bmatrix} + \begin{bmatrix} -Pi & \\ & H_{i\theta} \end{bmatrix} < 0 \tag{16}$$

(16) multiply $(x_k, x_k, \varphi_i(C_i x_k))'$ on left and it's transpose on the right. We get

$\hat{x}_k \sum_{\theta=1}^M u_{i\theta} R_{i\theta} \hat{x}_k - x'_k P_i x_k - 2SC(i, x_k, T_i) < 0$

According (13), By Schur complements. We get:

$\hat{A}_{i\theta} \sum_{j=1}^N \pi_{ij} P_j \hat{A}_{i\theta} < R_{i\theta}$

$$\ldots \tag{17}$$

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References

[1] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.