Asychronous Control for Markov Jump Lure's Systems With Control Saturation

Abstract—**The abstract here.**

I. Introduction

II. Preliminaries

Consider a class of discrete-time MJLS on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$:

$$\begin{cases} x_{k+1} = A_{r(k)}x_k + F_{r(k)}\varphi(y_k) + B_{r(k)}u_k \\ \qquad + E^x_{r(k)}w_k, \quad y_k = C_{r(k)}x_k, \\ z_k = C^z_{r(k)}x_k + G^z_{r(k)}\varphi(y_k) + D^z_{r(k)}u_k \\ \qquad + E^z_{r(k)}w_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}, u_k \in \mathbb{R}^{n_u}, y_k \in \mathbb{R}^{n_y}, z_k \in \mathbb{R}^{n_z}$, and $w_k \in \mathbb{R}^{n_w}$ represent the state, the control input, the output related to the nonlinearity, the controlled output and the exogenous disturbance vector respectively. $A_{r(k)}, F_{r(k)}, B_{r(k)}, E^x_{r(k)}, C^z_{r(k)}, G^z_{r(k)}, D^z_{r(k)}$ and $E^z_{r(k)}$ represent the time-varying system matrices, all of which are pre-known and real. $\{r(k), k \geq 0\}$ is a Markov chain taking values in a positive integer set $\mathcal{N} = \{1, 2, \dots, N\}$ with mode TPs:

$$\Pr\{r(k+1) = j | r(k) = i\} = \pi_{ij} \tag{2}$$

Clearly, for all $i, j \in \mathcal{N}$, $\pi_{ij} \in [0, 1]$, and $\sum_{j=0}^{N} \pi_{ij} = 1$ for each mode i. The system matrices in (1) at instant k can be expressed as $A_i, F_i, B_i, E^x_i, C^z_i, G^z_i, D^z_i$ and E^z_i, which are real known constant matrices with appropriate dimensions, and the related transition probability matrix is described as $\Pi = \{\pi_{ij}\}$.

Assumption 1: The function $\varphi(\cdot) : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ satisfies:
(1) $\varphi(0) = 0$ and
(2) there exists a diagonal positive definite matrix $\Omega \in \mathbb{R}^{p\times p}$ such that:

$$\varphi_{(\ell)}(y)[\varphi(y) - \Omega y]_{(\ell)} \leq 0. \tag{3}$$

where for all $y \in \mathbb{R}^p$, $\ell \in \{1, \dots, p\}$. And in this case we say that the nonlinearity $\varphi(\cdot)$ satisfies it's own cone bounded sector conditions and to be decentralized. According to (3) we have:

$$SC(y, \Lambda) := \varphi^{'}(y)\Lambda[\varphi(y) - \Omega y] \leq 0, \tag{4}$$

where Λ is any diagonal positive semidefinite matrix, and Ω is pre-given. From the Assumption 1, we get:

$$[\Omega y]_{(\ell)}[\varphi(y) - \Omega y]_{(\ell)} \leq 0, \tag{5}$$

which implies:

$$0 \leq \varphi^{'}(y)\Lambda\varphi(y) \leq \varphi^{'}(y)\Lambda\Omega y \leq y^{'}\Omega\Lambda\Omega y \tag{6}$$

Definition 1: System (1) is said to be locally stochastically stable if for $w_(k) = 0$ and any initial condition $x_0 \in \mathcal{D}_0$, $r_0 \in \mathcal{N}$, the following formulation holds:

$$\|x\|_2^2 = \sum_{k=0}^{\infty} \mathbb{E}[\|x_k\|^2] < \infty. \tag{7}$$

Where in this case the set $\mathcal{D}_0 \subset \mathbb{R}^p$ is said to be the domain of stochastic stability of the origin.

Define $\mathcal{F}_k$ as the σ-field generated by the random variables x_k and $r(k)$. We define next the class of exogenous disturbances with bounded energy and the finite ℓ_2-induced norm.
Definition 2: For $\gamma > 0$, set $\mathcal{W}_\gamma$ is defined as follows:

$$\begin{aligned} \mathcal{W}_\gamma &:= \{w = \{w_k\}_{k\in\mathbb{N}}; w_k \in \mathbb{R}^m, k \in \mathbb{N}, w_k \text{ is} \\ &\mathcal{F}_k\text{-measureable, and } \|w\|_2^2 = \sum_{k=0}^{\infty} \mathbb{E}(\|w_k\|)^2 < \frac{1}{\gamma}\}. \end{aligned} \tag{8}$$

The finite ℓ_2-induced gain associated with the closed-loop system (1) with $x_0 = 0$ between the disturbance $w = \{w_k\}_{k\in\mathbb{N}}$ and controlled output $z = \{z_k\}_{k\in\mathbb{N}}$ is equal or less than $\sqrt{\varrho}$ if for every $w \in \mathcal{W}_\gamma$ then we have

$$\frac{1}{\varrho}\|z\|_2^2 = \frac{1}{\varrho}\sum_{k=0}^{\infty} \mathbb{E}\left[\|z_k\|^2\right] \leq \|w\|_2^2 = \sum_{k=0}^{\infty} \mathbb{E}\left[\|w_k\|^2\right]. \tag{9}$$

In this paper, we consider the following controller:

$$u_k = K_{\sigma(k)}x_k + \Gamma_{\sigma(k)}\varphi(y_k) \tag{10}$$

where $K_{\sigma(k)} \in \mathbb{N}^{n_u\times n_x}$ is a time-varying controller gain matrix, and $\Gamma_{\sigma(k)} \in \mathbb{N}^{n_u\times n_y}$ is a time-varying nonlinear output feedback gain matrix. The parameter $\{\sigma(k), k \geq 0\}$ takes values in another pre-given positive integer set, which is marked as $\mathcal{M} = \{1, 2, \dots, M\}$ subject to the pre-known conditional probability matrix $\Phi = \{\mu_{im}\}$, the probabilities of which are defined by

$$\Pr\{\sigma(k) = \phi | r(k) = i\} = \mu_{i\phi}. \tag{11}$$

Where for all $i \in \mathcal{N}, \phi \in \mathcal{M}, \mu_{i\phi} \in [0, 1]$, and $\sum_{\phi=1}^{M} \mu_{i\phi} = 1$ for each $i \in \mathcal{N}$.

Combing the asynchronous controller (10) and system (1) we have the following closed system:

$$\begin{cases} x_{k+1} = \bar{A}_{i\theta}x_k + \bar{F}_{i\theta}\varphi_i(C_ix_k) + E^x_iw_k \\ z_k = \bar{C}_{i\theta}x_k + \bar{G}_{i\theta}\varphi_i(C_ix_k) + E^z_iw_k \end{cases} \tag{12}$$

Where,for $i \in \mathcal{N}, \phi \in \mathcal{M}$

$$\begin{aligned} \bar{A}_{i\phi} &= A_i + B_i K_\phi, & \bar{F}_{i\phi} &= F_i + B_i \Gamma_{i\phi} \\ \bar{C}^z_{i\phi} &= C^z_i + D^z_i K_\phi, & \bar{G}^z_{i\phi} &= G^z_i + D^z_i \Gamma_\phi \end{aligned}$$

III. Main Result

$$\begin{cases} x_{k+1} = \bar{A}_{i\theta} x_k + \bar{F}_{i\theta}\varphi_i(C_i x_k) + E^x_i w_k \\ z_k = \bar{C}_{i\theta} x_k + \bar{G}_{i\theta}\varphi_i(C_i x_k) + E^z_i w_k \end{cases}$$

System (12) with $x_0 = 0$ and $w_k = 0$ is stochastically stable, if for all $i \in \mathcal{N}$ and $\phi \in \mathcal{M}$, there exist positive definite matrices $\bar{P}_i \in \mathbb{R}^{n_x \times n_x}$, $R_{i\phi} \in \mathbb{R}^{(n_x+n_y)\times(n_x+n_y)}$, matrices $K_\phi \in \mathbb{R}^{n_u \times n_x}$, $\Gamma_\phi \in \mathbb{R}^{n_u \times n_y}$ and positive semidefinite matrix $T_i \in \mathbb{R}^{n_y}$ to ensure (13) and (14) hold.

$$\begin{bmatrix} -R_{i\theta} & \mathscr{H}_{i\phi} \\ * & \mathscr{P}_i \end{bmatrix} < 0 \tag{13}$$

$$\begin{bmatrix} \mathscr{S}_{i\phi} & \mathscr{N}_{i\phi} \\ * & \mathscr{L}_{i\phi} \end{bmatrix} < 0 \tag{14}$$

where

$$\mathscr{H}_{i\phi} = \begin{bmatrix} \sqrt{\pi_{i1}}\hat{A}_{i\phi} \\ \sqrt{\pi_{i2}}\hat{A}_{i\phi} \\ \vdots \\ \sqrt{\pi_{i_N}}\hat{A}_{i\phi} \end{bmatrix}', \mathscr{P}_{i\phi} = \mathrm{diag}\{-\bar{P}_1, -\bar{P}_2, \ldots, -\bar{P}_N\}$$

$$\hat{A}_{i\theta} = \begin{bmatrix} \bar{A}_{i\theta} & \bar{F}_{i\theta} \end{bmatrix}$$

$$\mathscr{S}_{i\phi} = \begin{bmatrix} -P_i & 0 \\ * & H_{i\phi} \end{bmatrix}$$

$$\mathscr{N}_{i\phi} = \begin{bmatrix} \sqrt{u_{i1}} \begin{bmatrix} \bar{P}_i & \\ & R_{i1} \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} \bar{P}_i & \\ & R_{iM} \end{bmatrix} \end{bmatrix}$$

$$\mathscr{L}_{i\phi} = \mathrm{diag}\left\{ \begin{bmatrix} -I & \\ & -R_{i1} \end{bmatrix}, \ldots, \begin{bmatrix} -I & \\ & -R_{iM} \end{bmatrix} \right\}$$

$$H_{i\theta} = \begin{bmatrix} -I & C_i' \Omega_i T_i \\ * & -2T_i \end{bmatrix}$$

Proof: Construct a Lyapunov function in the form of (15).

$$V(k, x_k, r(k)) = x_k' P_{r(k)} x_k \tag{15}$$

Where $P_{r(k)} = \bar{P}^{-1}_{r(k)}$. Let $\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{V(k+1, x_{k+1}, r(k+1) = j | x_k, r(k) = i\} - V(k, x_k, i)$ and it is easy to find that

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{x_{k+1}' X_i x_{k+1}\} - V(k, x_k, i) \tag{16}$$

Where $X_i = \sum_{j=1}^N \pi_{ij} P_j$. Based on system (12) we can further obtain

$$\begin{aligned} \mathbb{E}\{x_{k+1}' X_i x_{k+1}\} &= \sum_{\phi=1}^M u_{i\phi} \hat{x}_k' \hat{A}_{i\phi}' X_i \hat{A}_{i\phi} \hat{x}_k \\ &= \hat{x}_k' \Big(\sum_{\phi=1}^M u_{i\phi} \hat{A}_{i\phi}' X_i \hat{A}_{i\phi} \Big) \hat{x}_k \end{aligned} \tag{17}$$

Where $\hat{x}_k = (x_k', \varphi_i'(C_i x_k))'$. Which implies

$$\mathbb{E}\{\Delta V(k)\} = \hat{x}_k' \Big(\sum_{\phi=1}^M u_{i\phi} \hat{A}_{i\phi}' X_i \hat{A}_{i\phi} \Big) \hat{x}_k - V(k, x_k, i) \tag{18}$$

Denoting $h_{i\phi} = \mathrm{diag}\{P_i, I_{n_x+n_y}, \{\mathrm{diag}(I_{n_x}, R^{-1}_{i\phi})^M_{\phi=1}\}\}$ and using $h_{i\phi}$ to pre- and post-multiplying the inequality given in (14), respectively, then we can get

$$\begin{bmatrix} \begin{bmatrix} -P_i & 0 \\ * & H_{i\phi} \end{bmatrix} & \sqrt{u_{i1}} \begin{bmatrix} I & \\ & I \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} I & \\ & I \end{bmatrix} \\ \sqrt{u_{i1}} \begin{bmatrix} I & \\ & I \end{bmatrix} & \begin{bmatrix} -I & \\ & -R^{-1}_{i1} \end{bmatrix} & & \\ \vdots & & \ddots & \\ \sqrt{u_{iM}} \begin{bmatrix} I & \\ & I \end{bmatrix} & & & \begin{bmatrix} -I & \\ & -R^{-1}_{iM} \end{bmatrix} \end{bmatrix} < 0$$

Applying Schur complement, we can get

$$\sum_{\phi=1}^M u_{i\phi} \begin{bmatrix} I & 0 \\ * & R_{i\phi} \end{bmatrix} + \begin{bmatrix} -P_i & 0 \\ * & H_{i\phi} \end{bmatrix} < 0 \tag{19}$$

(19) multiply $(x_k', \hat{x}_k')'$ on left and it's transpose on the right. We get

$$\hat{x}_k' \sum_{\phi=1}^M u_{i\phi} R_{i\phi} \hat{x}_k - x_k' P_i x_k - 2SC(i, x_k, T_i) < 0 \tag{20}$$

Applying Schur complement to inequality (13). We can obtain

$$\hat{A}_{i\phi} X_i \hat{A}_{i\phi} < R_{i\phi} \tag{21}$$

Combining inequalities (20) and (21), it's easy to find that

$$\begin{aligned} \hat{x}_k' \sum_{\phi=1}^M u_{i\phi} (\hat{A}_{i\phi} X_i \hat{A}_{i\phi}) \hat{x}_k - x_k' P_i x_k \\ - 2SC(i, x_k, T_i) < 0 \end{aligned} \tag{22}$$

Combining inequalities (18) and (22), it is easy to obtain that

$$\mathbb{E}\{\Delta V(k)\} - 2SC(i, x_k, T_i) < 0 \tag{23}$$

Furthermore, noticing the condition given in (4), we further have $\mathbb{E}\{\Delta V(k)\} < 0$, which completes the proof.

Theorem 2: For each mode $i \in \mathcal{N}$ and $\phi \in \mathcal{M}$, if there exist positive-definite matrices $\bar{P}_i \in \mathbb{R}^{n_x \times n_x}$ and $R_{i\phi} \in \mathbb{R}^{(n_x+n_y+n_w)\times(n_x+n_y+n_w)}$, matrices $K_\phi \in \mathbb{N}^{n_u \times n_x}, \Gamma_\phi \in \mathbb{R}^{n_u \times n_y}$, positive semi-definite matrix $T_i \in \mathbb{R}^{n_y \times n_y}$ and a positive scalar γ, such that the following LMIs are verified

$$\begin{bmatrix} -R_{i\theta} & \mathscr{H}_{i\phi} \\ * & \mathscr{P}_i \end{bmatrix} < 0 \tag{24}$$

$$\begin{bmatrix} \mathscr{S}_{i\phi} & \mathscr{N}_{i\phi} \\ * & \mathscr{L}_{i\phi} \end{bmatrix} < 0 \tag{25}$$

Where

$$\mathscr{H}_{i\phi} = \begin{bmatrix} \sqrt{\pi_{i1}}\hat{A}_{i\phi} \\ \sqrt{\pi_{i2}}\hat{A}_{i\phi} \\ \vdots \\ \sqrt{\pi_{iN}}\hat{A}_{i\phi} \\ \hat{A}^z_{i\phi} \end{bmatrix}^{'}, \quad \mathscr{S}_{i\phi} = \begin{bmatrix} -P_i & 0 \\ * & H_{i\phi} \end{bmatrix}$$

$$\mathscr{P}_{i\phi} = \mathrm{diag}\{-\bar{P}_1, -\bar{P}_2, \ldots, -\bar{P}_N, -I^{n_w \times n_w}\}$$

$$\hat{A}_{i\theta} = \begin{bmatrix} \bar{A}_{i\theta} & \bar{F}_{i\theta} & E^x_i \end{bmatrix}, \quad \hat{A}^z_{i\theta} = \begin{bmatrix} \bar{C}^z_{i\theta} & \bar{G}^z_{i\theta} & E^z_i \end{bmatrix}$$

$$\mathscr{N}_{i\phi} = \begin{bmatrix} \sqrt{u_{i1}}\begin{bmatrix} \bar{P}_i & \\ & R_{i1} \end{bmatrix} & \cdots & \sqrt{u_{iM}}\begin{bmatrix} \bar{P}_i & \\ & R_{iM} \end{bmatrix} & \end{bmatrix}$$

$$\mathscr{L}_{i\phi} = \mathrm{diag}\left\{ \begin{bmatrix} -I & \\ & -R_{i1} \end{bmatrix}, \ldots, \begin{bmatrix} -I & \\ & -R_{iM} \end{bmatrix} \right\}$$

$$H_{i\theta} = \begin{bmatrix} -I_{n_x} & C_i^{'}\Omega_i T_i & 0 \\ * & -2T_i & 0 \\ * & * & -\gamma^2 I_{n_w} \end{bmatrix}$$

then system (12) with $x_0 = 0$ and $w \in \mathcal{W}_\gamma$ is stochastically stable and the l_2-gain of the closed system is strictly less or equal to γ.

Proof: We first select (16) as the Lyapunov function and $P_{r(k)} = \bar{P}^{-1}_{r(k)}$. Similar to the proof of theorem 1, we can obtain that

$$\mathbb{E}\{\Delta V(k)\} = \hat{x}_k^{'}\Big(\sum_{\phi=1}^{M} u_{i\phi}\hat{A}^{'}_{i\phi}X_i\hat{A}_{i\phi}\Big)\hat{x}_k - V(k, x_k, i) \tag{26}$$

Where $\hat{x}_k = (x_k^{'}, \varphi_i^{'}(C_i x_k), w_k^{'})^{'}$, $X_i = \sum_{j=1}^{N}\pi_{ij}P_j$

Denoting $h_{i\phi} = \mathrm{diag}\{P_i, I_{n_x+n_y+n_w}, \{\mathrm{diag}(I_{n_x}, R^{-1}_{i\phi})^{M}_{\phi=1}\}\}$ and using $h_{i\phi}$ to pre- and post-multiplying the inequality given in (25), respectively, then we can obtain

$$\begin{bmatrix} \begin{bmatrix} -P_i & 0 \\ * & H_{i\phi} \end{bmatrix} & \sqrt{u_{i1}}\begin{bmatrix} I & \\ & I \end{bmatrix} & \cdots & \sqrt{u_{iM}}\begin{bmatrix} I & \\ & I \end{bmatrix} \\ \sqrt{u_{i1}}\begin{bmatrix} I & \\ & I \end{bmatrix} & \begin{bmatrix} -I & \\ & -R^{-1}_{i1} \end{bmatrix} & & \\ \vdots & & \ddots & \\ \sqrt{u_{iM}}\begin{bmatrix} I & \\ & I \end{bmatrix} & & & \begin{bmatrix} -I & \\ & -R^{-1}_{iM} \end{bmatrix} \end{bmatrix} < 0$$

Applying Schur complement, we can get

$$\sum_{\phi=1}^{M} u_{i\phi}\begin{bmatrix} I & 0 \\ * & R_{i\phi} \end{bmatrix} + \begin{bmatrix} -P_i & 0 \\ * & H_{i\phi} \end{bmatrix} < 0 \tag{27}$$

id	name	sex	id	name	sex	id	name	sex
1			1			1		
2		male	2		male	2		male

(27) multiply $(x_k^{'}, \hat{x}_k^{'})^{'}$ on left and it's transpose on the right. We get

$$\hat{x}_k^{'}(\sum_{\phi=1}^{M} u_{i\phi}R_{i\phi})\hat{x}_k - x_k^{'}P_i x_k - 2SC(i, x_k, T_i) - \gamma^2 w_k^{'}w_k < 0 \tag{28}$$

Applying Schur complement to inequality (24). We can obtain

$$\hat{A}_{i\phi}X_i\hat{A}_{i\phi} + (\hat{A}^z_{i\phi})^{'}\hat{A}^z_{i\phi} < R_{i\phi} \tag{29}$$

Combining inequalities (28) and (29), it's easy to find that

$$\begin{aligned} \hat{x}_k^{'}\Big(\sum_{\phi=1}^{M} u_{i\phi}(\hat{A}_{i\phi}X_i\hat{A}_{i\phi} + (\hat{A}^z_{i\phi})^{'}\hat{A}^z_{i\phi})\Big)\hat{x}_k - x_k^{'}P_i x_k \\ - 2SC(i, x_k, T_i) - \gamma^2 w_k^{'}w_k < 0 \end{aligned} \tag{30}$$

Which implies

$$\mathbb{E}\{\Delta V(k)\} + \mathbb{E}\{z_k^{'}z_k\} - \gamma^2 w_k^{'}w_k - 2SC(i, x_k, T_i) < 0 \tag{31}$$

Furthermore, noticing the condition given in (4), we further have $\mathbb{E}\{\Delta V(k)\} + \mathbb{E}\{z_k^{'}z_k\} - \gamma^2 w_k^{'}w_k < 0$, which means

$$\mathbb{E}\{\Delta V(k) + z_k^{'}z_k - \gamma^2 w_k^{'}w_k | x_k, r(k) = i\} < 0 \tag{32}$$

Noting the zero initial condition , we can conclude that $\frac{\|z\|_2}{\|w\|_2} < \gamma$ is satisfied, which completes the proof.

IV. Numerical Example

In this section, we provide an example to verify the validity of proposed methods. Consider the Markov jump Lur'e system (1) with following data: $\mathcal{N} = \{1, 2\}$, $\mathcal{M} = \{1, 2\}$

$$A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.1 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.9 \\ 0.5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1 \\ 0.7 \end{bmatrix}, \quad C^z_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \quad C^z_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

$$G^z_1 = 0.3, \quad G^z_2 = 0.4, \quad D^z_1 = 0, \quad D^z_2 = 0.3;$$

$$E^z_1 = 1.3, \quad E^z_2 = -0.8, \quad E^x_1 = E^x_2 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}^{'}$$

$$\Omega_1 = 1.3, \quad \varphi_1(y) = 0.5\Omega_1 y(1 + cos(25y))$$

$$\Omega_2 = 1.5, \quad \varphi_2(y) = 0.5\Omega_2 y(1 - sin(20y))$$

$$\Pi = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

Furthermore, three cases of conditional probability matrix Φ are given in TABLEtable1

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References

[1] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.