

# Asynchronous Control for Markov Jump Lure's Systems With Control Saturation

**Abstract**—The abstract here.

## I. INTRODUCTION

## II. PRELIMINARIES

Consider a class of discrete-time MJLS on a probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ :

$$\begin{cases} x_{k+1} = A_{r(k)}x_k + F_{r(k)}\varphi(y_k) + B_{r(k)}u_k \\ \quad + E_{r(k)}^x w_k, \quad y_k = C_{r(k)}x_k, \\ z_k = C_{r(k)}^z x_k + G_{r(k)}^z \varphi(y_k) + D_{r(k)}^z u_k \\ \quad + E_{r(k)}^z w_k \end{cases} \quad (1)$$

where  $x_k \in \mathbb{R}^{n_x}, u_k \in \mathbb{R}^{n_u}, y_k \in \mathbb{R}^{n_y}, z_k \in \mathbb{R}^{n_z}$ , and  $w_k \in \mathbb{R}^{n_w}$  represent the state, the control input, the output related to the nonlinearity, the controlled output and the exogenous disturbance vector respectively.  $A_{r(k)}, F_{r(k)}, B_{r(k)}, E_{r(k)}^x, C_{r(k)}^z, G_{r(k)}^z, D_{r(k)}^z$  and  $E_{r(k)}^z$  represent the time-varying system matrices, all of which are pre-known and real.  $\{r(k), k \geq 0\}$  is a Markov chain taking values in a positive integer set  $\mathcal{N} = \{1, 2, \dots, N\}$  with mode TPs:

$$\Pr\{r(k+1) = j | r(k) = i\} = \pi_{ij} \quad (2)$$

Clearly, for all  $i, j \in \mathcal{N}$ ,  $\pi_{ij} \in [0, 1]$ , and  $\sum_{j=0}^N \pi_{ij} = 1$  for each mode  $i$ . The system matrices in (1) at instant  $k$  can be expressed as  $A_i, F_i, B_i, E_i^x, C_i^z, G_i^z, D_i^z$  and  $E_i^z$ , which are real known constant matrices with appropriate dimensions, and the related transition probability matrix is described as  $\Pi = \{\pi_{ij}\}$ .

**Assumption 1:** The function  $\varphi(\cdot) : \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_y}$  satisfies:

- (1)  $\varphi(0) = 0$  and
- (2) there exists a diagonal positive definite matrix  $\Omega \in \mathbb{R}^{p \times p}$  such that:

$$\varphi_{(\ell)}(y)[\varphi(y) - \Omega y]_{(\ell)} \leq 0. \quad (3)$$

where for all  $y \in \mathbb{R}^p$ ,  $\ell \in \{1, \dots, p\}$ . And in this case we say that the nonlinearity  $\varphi(\cdot)$  satisfies it's own cone bounded sector conditions and to be decentralized. According to (3) we have:

$$SC(y, \Lambda) := \varphi'(y)\Lambda[\varphi(y) - \Omega y] \leq 0, \quad (4)$$

where  $\Lambda$  is any diagonal positive semidefinite matrix, and  $\Omega$  is pre-given. From the Assumption 1, we get:

$$[\Omega y]_{(\ell)}[\varphi(y) - \Omega y]_{(\ell)} \leq 0, \quad (5)$$

which implies:

$$0 \leq \varphi'(y)\Lambda\varphi(y) \leq \varphi'(y)\Lambda\Omega y \leq y' \Omega \Lambda \Omega y \quad (6)$$

**Definition 1:** System (1) is said to be locally stochastically stable if for  $w(k) = 0$  and any initial condition  $x_0 \in \mathcal{D}_0$ ,  $r_0 \in \mathcal{N}$ , the following formulation holds:

$$\|x\|_2^2 = \sum_{k=0}^{\infty} \mathbb{E}[\|x_k\|^2] < \infty. \quad (7)$$

Where in this case the set  $\mathcal{D}_0 \subset \mathbb{R}^p$  is said to be the domain of stochastic stability of the origin.

Define  $\mathcal{F}_k$  as the  $\sigma$ -field generated by the random variables  $x_k$  and  $r(k)$ . We define next the class of exogenous disturbances with bounded energy and the finite  $\ell_2$ -induced norm.

**Definition 2:** For  $\gamma > 0$ , set  $\mathcal{W}_\gamma$  is defined as follows:

$$\mathcal{W}_\gamma := \{w = \{w_k\}_{k \in \mathbb{N}}; w_k \in \mathbb{R}^{n_w}, k \in \mathbb{N}, w_k \text{ is}$$

$$\mathcal{F}_k\text{-measurable, and } \|w\|_2^2 = \sum_{k=0}^{\infty} \mathbb{E}(\|w_k\|)^2 < \frac{1}{\gamma}\}. \quad (8)$$

The finite  $\ell_2$ -induced gain associated with the closed-loop system (1) with  $x_0 = 0$  between the disturbance  $w = \{w_k\}_{k \in \mathbb{N}}$  and controlled output  $z = \{z_k\}_{k \in \mathbb{N}}$  is equal or less than  $\sqrt{\varrho}$  if for every  $w \in \mathcal{W}_\gamma$  then we have

$$\frac{1}{\varrho} \|z\|_2^2 = \frac{1}{\varrho} \sum_{k=0}^{\infty} \mathbb{E}[\|z_k\|^2] \leq \|w\|_2^2 = \sum_{k=0}^{\infty} \mathbb{E}[\|w_k\|^2]. \quad (9)$$

In this paper, we consider the following controller:

$$u_k = K_{\sigma(k)}x_k + \Gamma_{\sigma(k)}\varphi(y_k) \quad (10)$$

where  $K_{\sigma(k)} \in \mathbb{R}^{n_u \times n_x}$  is a time-varying controller gain matrix, and  $\Gamma_{\sigma(k)} \in \mathbb{R}^{n_u \times n_y}$  is a time-varying nonlinear output feedback gain matrix. The parameter  $\{\sigma(k), k \geq 0\}$  takes values in another pre-given positive integer set, which is marked as  $\mathcal{M} = \{1, 2, \dots, M\}$  subject to the pre-known conditional probability matrix  $\Phi = \{\mu_{im}\}$ , the probabilities of which are defined by

$$\Pr\{\sigma(k) = \phi | r(k) = i\} = \mu_{i\phi}. \quad (11)$$

Where for all  $i \in \mathcal{N}, \phi \in \mathcal{M}, \mu_{i\phi} \in [0, 1]$ , and  $\sum_{\phi=1}^M \mu_{i\phi} = 1$  for each  $i \in \mathcal{N}$ .

Combing the asynchronous controller (10) and system (1) we have the following closed system:

$$\begin{cases} x_{k+1} = \bar{A}_{i\theta}x_k + \bar{F}_{i\theta}\varphi_i(C_i x_k) + E_i^x w_k \\ z_k = \bar{C}_{i\theta}x_k + \bar{G}_{i\theta}\varphi_i(C_i x_k) + E_i^z w_k \end{cases} \quad (12)$$

Where, for  $i \in \mathcal{N}, \phi \in \mathcal{M}$

$$\begin{aligned}\bar{A}_{i\phi} &= A_i + B_i K_\phi, & \bar{F}_{i\phi} &= F_i + B_i \Gamma_{i\phi} \\ \bar{C}_{i\phi}^z &= C_i^z + D_i^z K_\phi, & \bar{G}_{i\phi}^z &= G_i^z + D_i^z \Gamma_\phi\end{aligned}$$

### III. MAIN RESULT

$$\begin{cases} x_{k+1} = \bar{A}_{i\theta} x_k + \bar{F}_{i\theta} \varphi_i(C_i x_k) + E_i^x w_k \\ z_k = \bar{C}_{i\theta} x_k + \bar{G}_{i\theta} \varphi_i(C_i x_k) + E_i^z w_k \end{cases}$$

System (12) is stochastically stable, if for all  $i \in \mathcal{N}$  and  $\theta \in \mathcal{M}$ , there exist positive definite matrices  $\bar{P}_i \in \mathbb{R}^{n_x \times n_x}$ ,  $R_{i\theta} \in \mathbb{R}^{(n_x+n_y) \times (n_x+n_y)}$ , matrices  $K_\theta \in \mathbb{R}^{n_u \times n_x}$ ,  $\Gamma_\theta \in \mathbb{R}^{n_u \times n_y}$  and positive semidefinite matrix  $T_i \in \mathbb{R}^{n_y}$  to ensure (13) and (14) hold.

$$\begin{bmatrix} -R_{i\theta} & \mathcal{H}_{i\phi} \\ * & \mathcal{P}_i \end{bmatrix} < 0 \quad (13)$$

$$\begin{bmatrix} \mathcal{S}_{i\phi} & \mathcal{N}_{i\phi} \\ * & \mathcal{L}_{i\phi} \end{bmatrix} < 0 \quad (14)$$

$$\begin{bmatrix} -R_{i\theta} & \sqrt{\pi_{i1}} \hat{A}_{i\theta}' & \cdots & \sqrt{\pi_{iN}} \hat{A}_{i\theta}' \\ \sqrt{\pi_{i1}} A_{i\theta} & -\bar{P}_1 & & \\ \vdots & & \ddots & \\ \sqrt{\pi_{iN}} A_{i\theta} & & & -\bar{P}_N \end{bmatrix} < 0$$

$$\begin{bmatrix} \begin{bmatrix} -\bar{P}_i & 0 \\ * & H_{i\theta} \end{bmatrix} & \sqrt{u_{i1}} \begin{bmatrix} \bar{P}_i & R_{i1} \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} \bar{P}_i & R_{iM} \end{bmatrix} \\ \sqrt{u_{i1}} \begin{bmatrix} \bar{P}_i & R_{i1} \end{bmatrix} & \begin{bmatrix} -I & \\ & -R_{i1} \end{bmatrix} & & \\ \vdots & & \ddots & \\ \sqrt{u_{iM}} \begin{bmatrix} \bar{P}_i & R_{iM} \end{bmatrix} & & & \begin{bmatrix} -I & \\ & -R_{iM} \end{bmatrix} \end{bmatrix} < 0 \quad (17)$$

where

$$\mathcal{H}_{i\phi} = \begin{bmatrix} \sqrt{\pi_{i1}} \hat{A}_{i\phi} \\ \sqrt{\pi_{i2}} \hat{A}_{i\phi} \\ \vdots \\ \sqrt{\pi_{iN}} \hat{A}_{i\phi} \end{bmatrix}', \quad \mathcal{P}_i = \text{diag}\{-\bar{P}_1, -\bar{P}_2, \dots, -\bar{P}_N\}$$

$$\begin{aligned}\hat{A}_{i\theta} &= [\bar{A}_{i\theta} \quad \bar{F}_{i\theta}] \\ \mathcal{S}_{i\phi} &= \begin{bmatrix} -P_i & 0 & 0 \\ * & -I & C_i' \Delta_i T_i \\ * & * & -2T_i \end{bmatrix} \\ \mathcal{N}_{i\phi} &= \begin{bmatrix} \sqrt{u_{i1}} \begin{bmatrix} \bar{P}_i & R_{i1} \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} \bar{P}_i & R_{iM} \end{bmatrix} \end{bmatrix} \\ \mathcal{L}_{i\phi} &= \text{diag}\left\{ \begin{bmatrix} -I & \\ & -R_{i1} \end{bmatrix}, \dots, \begin{bmatrix} -I & \\ & -R_{iM} \end{bmatrix} \right\} \\ H_{i\theta} &= \begin{bmatrix} -I & C_i' \Delta_i T_i \\ * & -2T_i \end{bmatrix}\end{aligned}$$

Proof: Construct a Lyapunov function in the form of (15).

$$V(k, x_k, r(k)) = x_k' P_{r(k)} x_k \quad (15)$$

Where  $P_{r(k)} = \bar{P}_{r(k)}^{-1}$ . Denoting  $h_{i\phi} = \text{diag}\{P_i, I_{n_x+n_y}, \{\text{diag}(I_{n_x}, R_{i\phi}^{-1})_{\phi=1}^M\}\}$  (14) multiply  $\text{diag}(P_i, I_{n_x+n_y}, \{\text{diag}(I_{n_x}, R_{i\theta}^{-1})_{\theta=1}^M\})$  in the left and right with it's transpose, then we get:

$$\begin{bmatrix} \begin{bmatrix} -P_i & 0 \\ * & H_{i\theta} \end{bmatrix} & \sqrt{u_{i1}} \begin{bmatrix} I & \\ & I \end{bmatrix} & \cdots & \sqrt{u_{iM}} \begin{bmatrix} I & \\ & I \end{bmatrix} \\ \sqrt{u_{i1}} \begin{bmatrix} I & \\ & I \end{bmatrix} & \begin{bmatrix} -I & \\ & -R_{i1}^{-1} \end{bmatrix} & & \\ \vdots & & \ddots & \\ \sqrt{u_{iM}} \begin{bmatrix} I & \\ & I \end{bmatrix} & & & \begin{bmatrix} -I & \\ & -R_{iM}^{-1} \end{bmatrix} \end{bmatrix} < 0$$

According to Schur we get:

$$\sum_{\theta=1}^M u_{i\theta} \begin{bmatrix} I & \\ & R_{i\theta} \end{bmatrix} + \begin{bmatrix} -P_i & \\ & H_{i\theta} \end{bmatrix} < 0 \quad (16)$$

(16) multiply  $(x_k, x_k, \varphi_i(C_i x_k))'$  on left and it's transpose on the right. We get

$$\hat{x}_k' \sum_{\theta=1}^M u_{i\theta} R_{i\theta} \hat{x}_k - x_k' P_i x_k - 2SC(i, x_k, T_i) < 0$$

According (13), By Schur complements. We get:

$$\hat{A}_{i\theta}' \sum_{j=1}^N \pi_{ij} P_j \hat{A}_{i\theta} < R_{i\theta}$$

< 0

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### REFERENCES

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