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Introduction

The eigenvalues of some specific random matrices - e.g. Gaussian Unitary Ensemble matrices (GUE) - have distinctive properties, especially when we consider them as a collection of points. We can note a special interaction between the eigenvalues, characterised by a *repulsion*, and show that the set of eigenvalues form a *determinantal point process* (DPP). Moreover, random matrix can be propagate over time regarding a matrix-valued random process as a solution of a matrix-valued *Stochastic Differential Equation* (SDE). Going further, we can link these two ideas and consider the eigenvalues processes of a matrix-valued random process. Eigenvalues processes are solutions of scalars SDEs and define some trajectories over time. In practise, the eigenvalues trajectories also display a significant repulsion.

The Dyson Brownian motion [Dys62] describes the evolution of the eigenvalues of a matrix-valued random process, the Hermitian Brownian motion. It defines a system of scalar SDEs verified by the eigenvalues, generating eigenvalues trajectories that exactly highlight an interesting repulsion. In addition, the value of the Hermitian Brownian motion at a $t = 1$ is a GUE matrix, hence its eigenvalues collection form a DPP. Here, we see the point of our work, we want to understand the link between the static repulsion in a set of eigenvalues and the dynamic repulsion in eigenvalues trajectories. We can study this connection from two points of view : the random matrix SDE and the system of SDEs satisfyed by the associated eigenvalues, which give different simulation methods with their pros and cons.

We will try to give a justification of the dynamic repulsion and an explaination of the nature of this repulsion, keeping in mind the simulation issue. A complete overview of the most classical example - the Dyson Brownian motion - is provided in Chapter 1 (mostly referring to [AGZ10]). Furthermore, we present two other examples, to spotlight the variety of these processes. Then in Chapter 2, we will focus on the link between a SDE satisfyed by a matrix-valued random process and the system of SDEs verifyed by the associated eigenvalues processes (based on [Tao12; Gra14]). Finally, we will examine two very recent papers [DE02; HP17] in Chapter 3. They propose a new approach to this problem considering tridiagonal models, which can be clever way to deal with simulations.

Chapter 1

The Dyson Brownian motion and other classical processes

In this study, we are interested in the link between the matrix-valued random process and the eigenvalues processes, and on its ability to reflect a repulsion. Before focusing on this link, we will deal with the most classical example of this type of processes, the Dyson Brownian motion [Dys62] . The objects manipulated in this study are complex and it is hard to link them in a general way.

However, it seemed to be essential to expose at least one complete example presenting those three different aspects. This first part will introduce the Dyson Brownian motion, which is the simplest illustration of the relations we want to expose here.
We will start by introducing some results for the static case, meaning that we only consider a random matrix and not a matrix-valued random process yet. Then, we will expose the general example of the Dyson Brownian motion and finally some other well-known examples.

1.1. Static random matrices and eigenvalues

We start by defining some useful matrices for this study (Section 1.1.1), and classical results about their eigenvalues (Section 1.1.2). Finally, Section 1.1.3 gives a brief introduction to determinantal point processes (sometimes written as DPP) and to the link between GUE eigenvalues and this type of point processes.

1.1.1 Random matrices

Let $\mathcal{N}_{\mathbb{R}}(0,1)$ denotes the real standard normal distribution and $\mathcal{N}_{\mathbb{C}}(0,1)$ the complex standard normal distribution. Note that $Z \sim \mathcal{N}_{\mathbb{C}}(0,1)$ if $Z = (R+\mathrm{i}S)/\sqrt{2}$ with R and S independent and $R, S \sim \mathcal{N}_{\mathbb{R}}(0,1)$.

We recall the definition of Ginibre matrices and Gaussian Unitary Ensemble (GUE) matrices.

Definition 1.1.1 (Ginibre matrices)
Considering a $N \times N$ matrix M with independent entries $(M_{ij})_{1 \leq i,j \leq N}$, M is a Ginibre matrix if $M_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, 1/N)$ for $1 \leq i, j \leq N$.

Using Ginibre matrices, we can define the Gaussian Unitary Ensemble (GUE).

Definition 1.1.2 (Gaussian Unitary Ensemble (GUE) matrices)
Let M be a $N \times N$ Ginibre matrix, we define G as follows :

$$G = \frac{M + M^*}{\sqrt{2}}$$

with $M^* = \overline{M}^T$ the conjugate transpose of M. Then, G is a GUE(N) *matrix.*

Remark 1.1.3
GUE matrices can also be described as Hermitian matrices (i.e. $G^* = G$) where the diagonal entries are i.i.d. random variables following the distribution $\mathcal{N}_{\mathbb{R}}(0, 1/N)$ and the upper-triangular entries are i.i.d. with distribution $\mathcal{N}_{\mathbb{C}}(0, 1/N)$.
Indeed, for $j = i$

$$Var(G_{ii}) = \mathbb{E}[G_{ii}^2] = \mathbb{E}\left[\left(\frac{M_{ii} + \overline{M_{ii}}}{\sqrt{2}}\right)^2\right] = \mathbb{E}[R_{ii}^2] = Var(R_{ii}) = 1/N$$

and for $j > i$

$$\begin{aligned} Var(G_{ij}) = \mathbb{E}[|G_{ij}|^2] = \mathbb{E}\left[\left|\frac{M_{ij} + \overline{M_{ji}}}{\sqrt{2}}\right|^2\right] &= \mathbb{E}\left[\left|\frac{R_{ij} + R_{ji} + \mathrm{i}(S_{ij} - S_{ji})}{2}\right|^2\right] \\ &= \mathbb{E}\left[\frac{(R_{ij} + R_{ji})^2 + (S_{ij} - S_{ji})^2}{4}\right] \\ &= \frac{1}{4}(Var(R_{ij} + R_{ji}) + Var(S_{ij} - S_{ji})) \\ &= 1/N \end{aligned}$$

GUE matrices have the following density with respect the Lebesgue measure on the

Hermitian matrices space

$$
\begin{aligned}
d\mathbb{P}(G) &= \prod_{i=1}^{N} \sqrt{\frac{N}{2\pi}} \exp\left(\frac{-Ng_{ii}^2}{2}\right) \prod_{1\leq i<j\leq N} \frac{N}{\pi} \exp\left(-N|g_{ij}|^2\right) \\
&= \left(\frac{N}{2\pi}\right)^{N/2} \left(\frac{N}{\pi}\right)^{N(N-1)/2} \exp\left(-\frac{N}{2}\left(\sum_{i=1}^{N} g_{ii}^2 + 2\sum_{1\leq i<j\leq N} |g_{ij}|^2\right)\right) \\
&= \frac{1}{2^{N/2}} \left(\frac{N}{\pi}\right)^{N^2/2} \exp\left(-\frac{N}{2} Tr(G^2)\right) \qquad (1.1)
\end{aligned}
$$

After this introduction of the fundamental matrix ensemble used further in this study - the GUE - we give some results about their eigenvalues. In fact, some precise results exist on their distribution in a static framework and one idea of our analysis is to see if these results are still available in a dynamic framework.

1.1.2 Eigenvalues of random matrices

In this part, we aim to give fundamental results about GUE eigenvalues. The first result concerns the eigenvalues *empirical distribution*.
If we naively represent the histogram of eigenvalues of a GUE matrix, we can see on Figure 1.1 that a particular type of distribution appears.

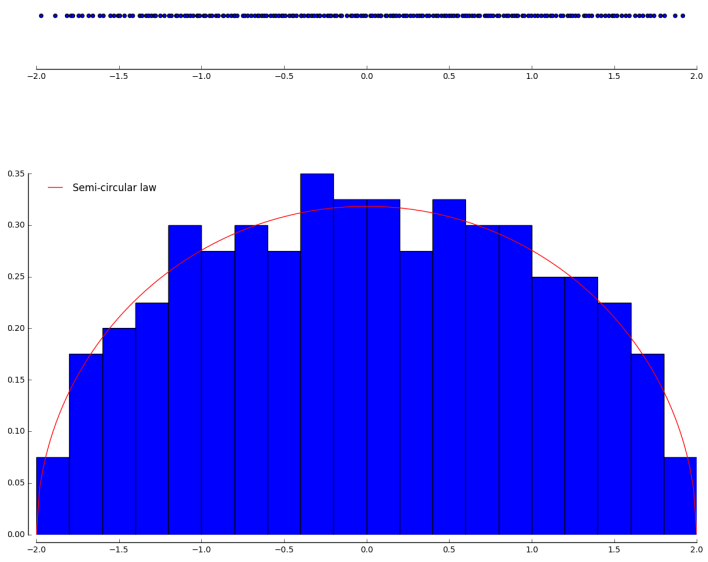

Figure 1.1 – **Eigenvalues histogram of a GUE matrix**, generated by GUE(200) matrix diagonalisation.

Now, let us explain mathematically this phenomenon. We denote by $(\lambda_i)_{1\leq i\leq N}$ the

(real) eigenvalues of a GUE matrix, ordered such that $\lambda_1 < \lambda_2 < ... < \lambda_N$. Note that because $(\lambda_i)_{1\leq i\leq N}$ are eigenvalues of a GUE matrix, they are almost surely distincts. We define the empirical distribution of the eigenvalues as the probability measure on $\mathbb{R}$

$$\widehat{\mu}_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \tag{1.2}$$

We also define the *semicircle distribution* as the probability distribution $\sigma(x)dx$ on $\mathbb{R}$ with density

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{|x|\leq 2}, \tag{1.3}$$

Then we can present the Wigner theorem in the particular case of GUE matrices. Note that Theorem 1.1.4 has been proved for more general matrices than GUE matrices in [Wig55; AGZ10]. For this theorem, let us define a sequence of GUE matrices $(G_N)_{N>0}$ with growing dimensions: $G_1 \sim$ GUE(1), $G_2 \sim$ GUE(2) until $G_N \sim$ GUE(N). On this sequence of matrices, we define the sequence of eigenvalues empirical distributions $(\widehat{\mu}_N)_{N>0}$ of these matrices.

Theorem 1.1.4 (Wigner Theorem)
For a sequence $(G_N)_{N>0}$ of GUE *matrices (as previously defined), the sequence of empirical measures $(\widehat{\mu}_N)_{N>0}$ converges weakly, in probability, to the semicircle distribution.*

Another well-known aspect of the GUE eigenvalues is their joint distribution. The following result was given in [AGZ10].

Theorem 1.1.5 (Joint distribution of GUE eigenvalues)
Let G be a GUE(N) *matrix, the joint distribution of the eigenvalues $\lambda_1 < \lambda_2 < ... < \lambda_N$ has density with respect to the Lebesgue measure which equals*

$$u_N(x_1, ..., x_N) = C_N\Delta_N(x)^2\prod_{i=1}^{N}e^{-Nx_i^2/2}\mathbb{1}_{\{x_1<x_2<\cdots<x_N\}} \tag{1.4}$$

where

$$\begin{aligned} C_N &= N!\left(\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\Delta_N(x)^2\prod_{i=1}^{N}e^{-Nx_i^2/2}dx_i\right)^{-1} \\ &= \left(\frac{N^N}{2\pi}\right)^{N/2}\frac{1}{\prod_{j=1}^{N}(j-1)!} \end{aligned} \tag{1.5}$$

and $\Delta_N(x) = \prod_{1\leq i<j\leq N}(x_i - x_j)$ *is the Vandermonde determinant.*

Having the joint distribution of the eigenvalues, we can go further and see the set of GUE eigenvalues on a different point of view. Indeed, the collection of eigenvalues

of a GUE matrix can be considered as a configuration of points on $\mathbb{R}$, namely a *point process*. Let us introduce the very basics of this kind of processes in the next section.

1.1.3 Determinantal point processes

In this study, we introduce very briefly real determinantal point processes. For a more general study, the reader could refer to [AGZ10], Section 4.2. Let us start by defining configuration of points before exposing the definition of a point process.

Definition 1.1.6 (Configuration (of points))
A configuration of points γ is a subset of $\mathbb{R}$, dicrete and locally finite :
- *Discrete : $\forall x \in \gamma, \exists r > 0, \quad B(x,r) \cap \gamma = \{x\}$*
- *Locally finite : $\forall K \subset \mathbb{R}$, K compact, $\quad \#(\gamma \cap K) < \infty$*

Definition 1.1.7 (Point process (simple))
A point process is a random configuration of points γ. By random, we mean that it exists a probability measure on the set of configurations, equipped with a standard σ-algebra, which make $\gamma \to \#(\gamma \cap K)$ mesurable for all compact K.

From now, let us note μ the reference measure on $\mathbb{R}$. Determinantal point processes have their name because of the particular form of their *joint intensities*.

Definition 1.1.8 (Joint intensities)
Considering a point process defined with respect to μ, for all $k \geq 1$, its k-correlation function is defined by

$$\mathbb{E}\left[\sum_{\substack{(X_1,\dots,X_k)\\ X_1 \neq \dots \neq X_k \in \gamma}} f(X_1,\dots,X_k)\right] = \int_{\mathbb{X}^k} f(x_1,\dots,x_k)\rho_k(x_1,\dots,x_k)\mu(dx_1)\cdots\mu(dx_k)$$

for all $f : \mathbb{R}^k \to \mathbb{C}$ mesurable with compact support.

Then, we can introduce a particular type of point processes, the *determinantal point processes*.

Definition 1.1.9
A point process on $\mathbb{R}$ with respect to μ is determinantal if kernel $K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ exists so that

$$\rho_k(x_1, ..., x_k) = \det[K(x_i, x_j)]_{1 \leq i,j \leq k}$$

for all $k \geq 1$ and $x_1, ..., x_k \in \mathbb{R}$.

As the last definition shows, the singularity of determinantal point processes is to have their joint intensities which can be written as a determinant of a matrix, namely the *kernel*.

Let us momentarily set $N = 2$ and consider a collection of 2 random variables, in order to understand the influence of the kernel. We compute some classical probability densities.
The joint density of (X_1, X_2) is

$$f_2(x, y) = \frac{1}{2}\left(K(x, x)K(y, y) - |K(x, y)|^2\right)$$

to compute the marginal density, we integrate with respect to one of the variable and get

$$f_1(x) = \frac{1}{2}K(x, x)$$

The conditional density of $X_2|X_1$ is then given by

$$f_{2|1}(y) = \frac{f_2(X_1, y)}{f_1(X_1)} = K(y, y) - \frac{|K(X_1, y)|^2}{K(X_1, X_1)}$$

In this last formula, we notice the *repulsion* between the two random variables. Indeed, the more y is close to X_1 (i.e. y and X_1 are highly correlated), the more the conditional density is small.

Getting back to GUE matrices, the collection of their eigenvalues can actually be viewed as a determinantal point process with a particular kernel. The following theorem expose this result.

Theorem 1.1.10 (Kernel of the DPP generated by the eigenvalues of a GUE matrix) *Considering a* GUE(N) *matrix, the collection of its eigenvalues* $\{\lambda_1, ..., \lambda_N\}$ *is a determinantal point process with a kernel* K_N *defined by*

$$K_N(x, y) = \sum_{i=0}^{N-1} \frac{h_i(x)h_i(x)}{||h_i||^2} e^{-N(x^2+y^2)/4} \tag{1.6}$$

with $(h_i)_{0\leq i\leq N-1}$ *the orthogonal polynomial with respect to the measure* $e^{-Nx^2/2}dx$ *and with leading coefficient 1 (a.k.a the Hermite polynomials).*

This result is not straightforward and we usually employ the following theorem, presented in [Joh06], to prove it.

Theorem 1.1.11
Let us consider $(\phi_i)_{1\leq i\leq N}$ *and* $(\psi_j)_{1\leq j\leq N}$ *be measurable functions such as* $\phi_i\psi_j$ *is integrable for any* i, j*. Supposing that the random vector* $(X_1, ..., X_N) \in \mathbb{R}^N$ *has a density with respect to the reference measure :*

$$u_N(x_1, ..., x_N) = \frac{1}{Z_N N!} \det[\phi_i(x_j)]_{i,j=1}^N \det[\psi_i(x_j)]_{i,j=1}^N \tag{1.7}$$

where

$$Z_N = \frac{1}{N!}\int_{\mathbb{X}^N} \det[\phi_i(x_j)]_{i,j=1}^N \det[\psi_i(x_j)]_{i,j=1}^N \mu^{\otimes N}(dx) \tag{1.8}$$

Then, the associated point process $\{X_1, ..., X_N\}$ *is determinantal and its kernel is given by :*

$$K_N(x,y) = \sum_{i,j=1}^N \phi_i(x)[A^{-1}]_{ij}\psi_j(x) \tag{1.9}$$

where A *is a* $N \times N$ *matrix with entries*

$$A_{ij} = \int_{\mathbb{X}} \psi_i(x)\phi_j(x)\mu(dx) \tag{1.10}$$

Proof of 1.1.11. We do not detail the proof here but the reader could refer to [Joh06]. □

Proof of 1.1.10. Now, we aim to apply Theorem 1.1.11 to Equation (1.4). Forgetting about the normalisation factor C_N, we have

$$\begin{aligned}\Delta_N(x)^2 \prod_{i=1}^N e^{-Nx_i^2/2} = \left(\det[x_i^{j-1}]_{i,j=1}^N\right)^2 \prod_{i=1}^N e^{-Nx_i^2/2} &= \left(\det[x_i^{j-1}e^{-Nx_i^2/4}]_{i,j=1}^N\right)^2 \\ &= \left(\det[x_j^{i-1}e^{-Nx_j^2/4}]_{i,j=1}^N\right)^2 \end{aligned} \tag{1.11}$$

We can now set

$$\phi_i(x) = \psi_i(x) = x^{i-1}e^{-Nx^2/4} \tag{1.12}$$

So that

$$A_{ij} = \int_{\mathbb{R}} \phi_i(x)\psi_i(x)dx = \int_{\mathbb{R}} x^{i+j-2}e^{-Nx^2/2}dx$$

With these choices of ϕ_i and ψ_i, the matrix A can be tough to inverse. Recall that these choices are motivated by the value of the determinant in Equation (1.11), the classical property of the determinant allow us to do row operations - like linear combinations - without changing its value. Hence, we have

$$\det[x_j^{i-1}e^{-Nx_j^2/4}]_{i,j=1}^N = \det[h_{i-1}(x)e^{-Nx_j^2/4}]_{i,j=1}^N \tag{1.13}$$

with $(h_i)_{0\le i\le N-1}$ an arbitrary sequence of polynomials of degree $i-1$ and leading coefficient 1. Thus, by chosing for $(h_i)_{0\le i\le N-1}$ the orthogonal polynomial with respect to the measure $e^{-Nx^2/2}dx$ and with leading coefficient 1, we set

$$\phi_i(x) = \psi_i(x) = h_{i-1}(x)e^{-Nx^2/4}$$

and the entries of matrix A become

$$A_{ij} = \int_{\mathbb{R}} \phi_i(x)\psi_i(x)dx = \int_{\mathbb{R}} h_{i-1}(x)h_{j-1}(x)e^{-Nx^2/2}dx = ||h_{i-1}||^2\delta_{ij}$$

where $||h_i||^2 = \int_{\mathbb{R}} h_i(x)^2 e^{-Nx^2/2} dx$. A becomes a diagonal matrix and is now easy to inverse.
According to the general expression of the kernel (1.9), we do not need to care about the normalisation factor C_N. Indeed, if we decide to separate it in two and put it in the expression of ϕ_i and ψ_i, it will be balanced by the expression of $[A^{-1}]_{ij}$.

We can now apply Theorem 1.1.11 and write the kernel K_N of the point process $\{\lambda_1, ..., \lambda_N\}$

$$K_N(x,y) = \sum_{i=0}^{N-1} \frac{h_i(x)h_i(y)}{||h_i||^2} e^{-N(x^2+y^2)/4}$$

□

Regarding that GUE eigenvalues are determinantal processes, they also have the particular repulsion property. Our idea is to understand how this property can be spread dynamically. For this purpose, we will now present the Dyson Brownian motion. From now, we will exposed some processes varying over time. They will also be called processes but they are very different from point processes in nature (Definition 1.1.7). Scalar-valued processes and matrix-valued processes refer to processes varying with time while determinantal point processes indicate point process at a fixed time.

1.2. The Dyson Brownian motion

Before exhibiting the Dyson Brownian motion, we need to introduce some notations in Section 1.2.1 and to motivate our analysis in Section 1.2.2 and 1.2.3. Section 1.2.3 also links the dynamic framework to the static one, concerning the eigenvalues joint distribution and we finally present the Dyson Brownian motion in Section 1.2.4.

1.2.1 Notation

In the following study, we deal with several different mathematical objects, that is why it seems important to directly clarify the notation for these objects. We will use the same notation as in [HP17]. Matrix-valued processes $\boldsymbol{X} = (X_t)_{t\geq 0}$ will be noted in bold whereas X_t denote the value of the process $\boldsymbol{X}$ at a fixed time t. Then $\boldsymbol{X}_{ij}$ refers to the scalar-valued process of the entry (i,j) of $\boldsymbol{X}$ and the value of $\boldsymbol{X}_{ij}$ at the time t will be denoted $X_{ij,t}$.
Concerning the eigenvalues processes of a matrix process, we consider them ordered. In most cases, they will be strictly ordered except for the initial conditions. For example, if there are N processes, we order them so that $\lambda_1(t) < \lambda_2(t) < ... < \lambda_{N-1}(t) < \lambda_N(t)$ for $t > 0$ and $\lambda_1(0) \leq \lambda_2(0) \leq ... \leq \lambda_{N-1}(0) \leq \lambda_N(0)$. The ith eigenvalue process $(\lambda_i(t))_{t\geq 0}$ of the matrix-valued process will be noted $\lambda_i(t)$.

Moreover, if we consider a simple scalar-valued random process which is not an entry of a matrix neither an eigenvalue of a matrix, we will denote it simply $(Z(t))_{t\geq 0}$.

1.2.2 Random matrix-valued process and its eigenvalues processes

Now, we introduce the Hermitian Brownian motion (as in [AGZ10]) which is the very classical matrix-valued process and the starting point of this study.

Definition 1.2.1 (Hermitian Brownian motion)
Let $(\boldsymbol{B}^1_{kl})_{1\leq k\leq l\leq N}$ and $(\boldsymbol{B}^2_{kl})_{1\leq k<l\leq N}$ be two independent collections of i.i.d. real valued standard Brownian motions. The Hermitian Brownian motion denoted $\boldsymbol{H}$ is the random process with entries $(\boldsymbol{H}_{kl})_{1\leq k,l\leq N}$ so that

$$\boldsymbol{H}_{kl} = \begin{cases} \frac{1}{\sqrt{2N}}(\boldsymbol{B}^1_{kl} + \mathrm{i}\boldsymbol{B}^2_{kl}) & \text{if } k < l \\ \frac{1}{\sqrt{2N}}(\boldsymbol{B}^1_{lk} - \mathrm{i}\boldsymbol{B}^2_{lk}) & \text{if } k > l \\ \frac{1}{\sqrt{N}}\boldsymbol{B}^1_{kk} & \text{if } k = l \end{cases}$$

With this definition, we can deduce an exact way of simulating such kind of process. Indeed, using the independence and the law of the Brownian motion's increments, we can explain the following theorem, useful for the simulations.

Lemma 1.2.2
Let $\boldsymbol{H}$ be a $N \times N$ Hermitian Brownian motion. The increments of $\boldsymbol{H}$ are independent and follow the law $H_t - H_s \sim \sqrt{t-s}\, G$ with $G \sim$ GUE(N).

Assuming that we have a random matrix SDE satisfied by a $N \times N$ matrix process $\boldsymbol{X}$, for example : $X_t = A + H_t$ with A a $N \times N$ Hermitian matrix and $\boldsymbol{H}$ a $N \times N$ Hermitian Brownian motion as defined in 1.2.1.

Remark 1.2.3 (First definition of a matrix SDE)
At this point of the study, let us assume that a SDE is define by an equation like $X_t = A + H_t$, where A is an initial condition and $\boldsymbol{H}$ a $N \times N$ Hermitian Brownian motion. SDE will be define in a more general way, for scalars and matrices in Section 1.3.1.

Lemma 1.2.2 provides a way to **exactly simulate** eigenvalues trajectories of $\boldsymbol{X}$, using the random matrix SDE, as the following algorithm shows :

Algorithm 1 Eigenvalues trajectories simulation with random matrix SDE

input : $A,\ N,\ n_{samples},\ t_f$
initialisation :
$dt \leftarrow t_f / n_{samples}$
$\Lambda \leftarrow \{OrderedSpectrum(A)\}$
$X \leftarrow A$
for i=1 **to** n_{sample} **do**
  draw $G \sim$ GUE(N)
  $X \leftarrow X + \sqrt{dt} * G$
  $\Lambda \leftarrow \Lambda \cup \{OrderedSpectrum(X)\}$
end for
output : Λ

By transforming Λ into a $N \times n_{samples}$ matrix, each row represents the value of an eigenvalue over time. The complete algorithm on Python language is available on [Ohl17]. Figure 1.2 was generated using it.

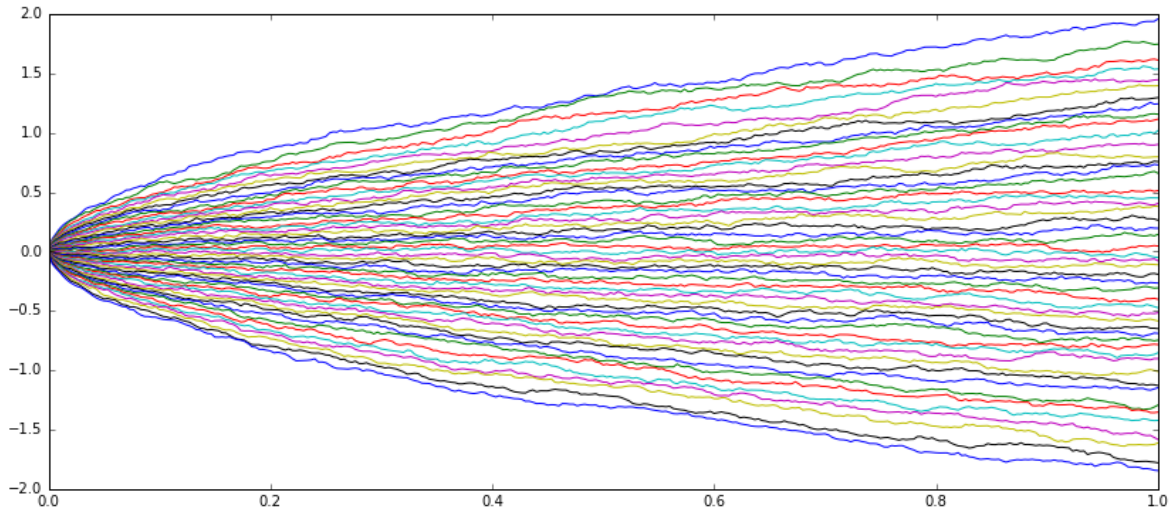

Figure 1.2 – **Eigenvalues trajectories of an Hermitian Brownian motion generated with random matrix SDE.** 50 eigenvalues, with 300 points in $[0, 1]$ and initial condition $A = 0_{\mathcal{M}_N(\mathbb{R})}$.

We also note that the eigenvalues seem to "grow" as $\sqrt{t}$. Let us explain this observation by the following reasoning. Remember that in Equation (1.2), we defined the empirical distribution. Going further, we define the second moment of $\widehat{\mu}_N$

$$\widehat{\mu}_{2,N} = \mathbb{E}_{\widehat{\mu}_N}[\lambda^2] = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^2$$

Eigenvalues are function of time, then by taking the expected value we have

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\lambda_i^2(t)\right] = \mathbb{E}\left[\frac{1}{N}Tr(X^2)\right] = \mathbb{E}\left[\frac{1}{N}\sum_{1\leq i,j\leq N}|x_{ij}|^2\right] = \frac{1}{N}\sum_{1\leq i,j\leq N}\mathbb{E}\left[|x_{ij}|^2\right]$$
$$= \frac{1}{N}\sum_{1\leq i,j\leq N}\frac{t}{N} = t$$

Hence, we represent $\mu_i(t) = \lambda_i(t)/\sqrt{t}$ for $t > 0$ on Figure 1.3. This renormalisation maintains the trajectories in the interval $[-2, 2]$ and allows us to observe the empirical distribution converging to the semicircule law at each time t and especially at final time $t_f = 1$ as the histogram shows.

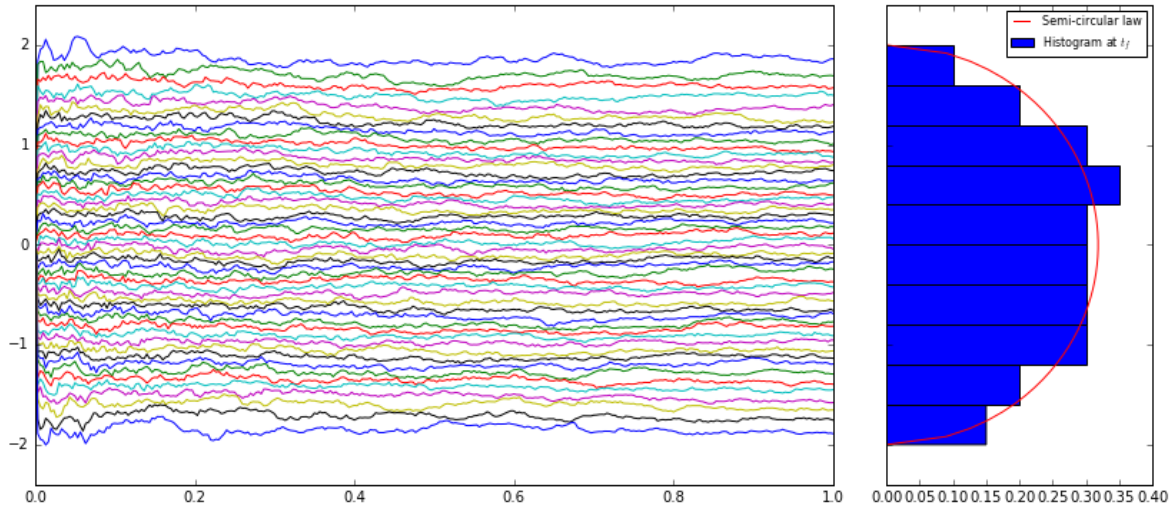

Figure 1.3 – **Rescaled eigenvalues trajectories of an Hermitian Brownian motion generated with random matrix SDE.** 50 eigenvalues, with 300 points in $[0, 1]$ and initial condition $A = 0_{\mathcal{M}_N(\mathbb{R})}$.

Now that we have a random-matrix process with eigenvalues trajectories, it can be interesting to observe the evolution of the joint distribution of these eigenvalues. Because we know that collection of eigenvalues can have particular properties, we can study the possibility for the static repulsion noticed in Section 1.1.3 to be preserved dynamically by the Hermitian Brownian motion.

1.2.3 Eigenvalues joint distribution at a fixed time

In this part, we study the evolution of the eigenvalues joint distribution over time. In fact, if this joint distribution has a form adapted to Theorem 1.1.11 (e.g. joint distribution of GUE eigenvalues Equation 1.4), we could find some interesting kernel and characterise the set of eigenvalues at a fixed time as a determinantal point process.

Considering a classical Dyson Brownian motion $\boldsymbol{X}$ with initial condition $X_0 = A$ as decribed in Theorem 1.2.6, we remark that at each time t, $\boldsymbol{X}$ has **the same law** as $Y_t = A + t^{1/2}G$ with G a GUE matrix. Considering this observation, the eigenvalues joint distribution is given by the following Theorem, presented in [Tao12; Joh01].

Theorem 1.2.4
Let A be a $N \times N$ Hermitian matrix, let $t > 0$, and let $Y_t = A + t^{1/2}G$ where G is drawn from GUE(N). *Then if A has a simple spectrum $\nu = (\nu_1, ..., \nu_N)$, the spectrum $\lambda = (\lambda_1, ..., \lambda_N)$ of Y_t has probability density function :*

$$\rho_\nu(t, \lambda) = \left(\frac{N}{2\pi t}\right)^{N/2} \frac{\Delta_N(\lambda)}{\Delta_N(\nu)} \det[e^{-\frac{N}{2t}(\lambda_i - \nu_j)^2}]_{i,j=1}^N \tag{1.14}$$

on $\mathbb{R}^N_{\geq}$, and $\Delta_N(\nu) = \prod_{1 \leq i < j \leq N}(\nu_i - \nu_j)$ is the Vandermonde determinant.

The proof of this theorem will not be detailed here but can be easily achieve following [Joh01] and using our particular distribution of GUE matrices (1.1). However, we will focus on the case when $A = 0$ and $t = 1$. This particular case is not covered by the Theorem 1.2.4 but we already know that the joint distribution is given by Equation (1.4) because $Y_1 = G$ where $G \sim$ GUE(N).

Nevertheless, we will check $\lim_{\nu \to 0} \rho_\nu(1, \lambda) = u_N(\lambda)$ where u_N is given by Equation (1.4), following Exercice 3.1.9. in [Tao12]. An intermediate result concerning the determinant behaviour in this case is presented in the following Lemma.

Lemma 1.2.5
Considering $\lambda = (\lambda_1, ..., \lambda_N)$ and $\nu = (\nu_1, ..., \nu_N)$ as previously defined, we have

$$\det[e^{-\frac{N}{2}(\lambda_i - \nu_j)^2}]_{i,j=1}^N = \prod_{i=1}^N e^{-N\lambda_i^2/2} \prod_{i=1}^N e^{-N\nu_j^2/2} \det[e^{N\lambda_i\nu_j}]_{i,j=1}^N \tag{1.15}$$

and

$$\det\left[e^{N\lambda_i\nu_j}\right]_{i,j=1}^N = \frac{N^{\frac{(N-1)N}{2}}}{\prod_{i=1}^N (j-1)!} \Delta_N(\lambda)\Delta_N(\nu) + o(\Delta_N(\nu)) \tag{1.16}$$

Proof. Equation (1.15) is forthwith

$$\begin{aligned}\det[e^{-\frac{N}{2}(\lambda_i-\nu_j)^2}]_{i,j=1}^N &= \det[e^{-N\lambda_i^2/2}e^{-N\nu_j^2/2}e^{N\lambda_i\nu_j}]_{i,j=1}^N \\ &= \prod_{i=1}^N e^{-N\lambda_i^2/2}\prod_{i=1}^N e^{-N\nu_j^2/2}\det[e^{N\lambda_i\nu_j}]_{i,j=1}^N\end{aligned}$$

Let us now focus on the limit (1.16). Using Taylor expansion, we have

$$\begin{aligned}\det\left[e^{N\lambda_i\nu_j}\right]_{i,j=1}^N &= \det\left[\sum_{k=0}^{N-1}\frac{(N\lambda_i\nu_j)^k}{k!}+\sum_{k=N}^{\infty}\frac{(N\lambda_i\nu_j)^k}{k!}\right]_{i,j=1}^N \\ &= \det(\Lambda V^t + D) \\ &= \det(C+D) \qquad (1.17)\end{aligned}$$

where $\Lambda = \left[\frac{(N\lambda_i)^{j-1}}{(j-1)!}\right]_{i,j=1}^N$, $V = \left[\nu_i^{j-1}\right]_{i,j=1}^N$, $C = \Lambda V^t = [c_{ij}]_{i,j=1}^N$ and $D = [o(\nu_j^{N-1})]_{i,j=1}^N = [d_{ij}]_{i,j=1}^N$.
We denote by S_N the permutation group, and by $\mathcal{P}_N$ the power set of the set $\{1,...,N\}$. When $I \in \mathcal{P}_N$, we note $\overline{I}$ the ensemble $\{1,...,N\}\backslash I$. Using the Leibniz formula, we have

$$\begin{aligned}\det(C+D) &= \sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{i=1}^N(c_{\sigma(i),i}+d_{\sigma(i),i}) \\ &= \sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\sum_{I\in\mathcal{P}_N}\left(\prod_{i\in I}c_{\sigma(i),i}\prod_{i\in\overline{I}}d_{\sigma(i),i}\right) \\ &= \left(\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{i\in\{1,...,N\}}c_{\sigma(i),i}\right)+\left(\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\sum_{I\subsetneq\{1,...,N\}}\prod_{i\in I}c_{\sigma(i),i}\prod_{i\in\overline{I}}d_{\sigma(i),i}\right) \\ &= \det(C)+\left(\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\sum_{I\subsetneq\{1,...,N\}}\prod_{i\in I}c_{\sigma(i),i}\prod_{i\in\overline{I}}d_{\sigma(i),i}\right) \qquad (1.18)\end{aligned}$$

Yet, we have

$$\begin{aligned}\det(C) = \det(\Lambda N^t) = \det(\Lambda)\det(N^t) &= \det(\Lambda)\det(N)\\ &= \det\left[\frac{(N\lambda_i)^{j-1}}{(j-1)!}\right]_{i,j=1}^{N} \det\left[\nu_i^{j-1}\right]_{i,j=1}^{N}\\ &= \prod_{j=1}^{N}\frac{N^{j-1}}{(j-1)!}\det\left[\lambda_i^{j-1}\right]_{i,j=1}^{N}\det\left[\nu_i^{j-1}\right]_{i,j=1}^{N}\\ &= \frac{N^{\frac{(N-1)N}{2}}}{\prod_{j=1}^{N}(j-1)!}\Delta_N(\lambda)\Delta_N(\nu)\end{aligned}$$

Finally, we have to prove

$$\frac{\det(C+D)}{\Delta_N(\nu)} = \frac{1}{\prod_{i=1}^{N}(j-1)!}\Delta_N(\lambda) + o(1)$$

We found the first term, them we have to prove

$$\frac{1}{\Delta_N(\nu))}\left(\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\sum_{I\subsetneq\{1,\dots,N\}}\prod_{i\in I}c_{\sigma(i),i}\prod_{i\in\bar{I}}d_{\sigma(i),i}\right) = o(1) \tag{1.19}$$

This part was not detailed is this study, and we admit Equation (1.19). □

Applying Lemma 1.2.5 to Theorem 1.2.4, we can compute the eigenvalues joint distribution at $t = 1$ when $\nu \to 0$

$$\rho(1,\lambda) = \left(\frac{N^N}{2\pi}\right)^{N/2}\frac{1}{\prod_{j=1}^{N}(j-1)!}\Delta_N(\lambda)^2\prod_{i=1}^{N}e^{-\lambda_i^2/2} \tag{1.20}$$

Which is exactly the joint distribution of GUE matrices, see Equation (1.4). Hence, at $t = 1$, we can apply every result on GUE matrices highlighted in Section 1.1. Actually, the collection of eigenvalues at $t = 1$ of a Hermitian Brownian motion is a determinantal point process, with kernel given in Theorem 1.1.10. This is undoubtely an interesting result, however it is only valid for $t = 1$. Later, Section 1.3.1 will present a model, slightly different from the Hermitian Brownian motion, able to preserve the GUE distribution at each time when $t \to +\infty$.

This section and the previous one motivate the analysis of the eigenvalues processes generated by the Hermitian Brownian motion. Nevertheless, one can imagine a way to simulate directly the eigenvalues processes, without considering any matrix-valued random process. Next section spotlight a fundamental result related to this remark.

1.2.4 Random matrix SDE and associated eigenvalues SDEs

After this brief introduction of some useful definitions and concepts for this study, we now present the most classical process correponding to our context : the Dyson Brownian motion. The first step in the Dyson Brownian motion presentation highlights the link between random matrix SDE and system of eigenvalues SDEs, following [Tao12; AGZ10].

The classical Dyson Brownian motion is defined as follows in Theorem 1.2.6.

Theorem 1.2.6 (Dyson Brownian motion)
Let A be a $N \times N$ Hermitian matrix and $\boldsymbol{X}$ the matrix-valued process defined by $X_t = A + H_t$ with $\boldsymbol{H}$ a $N \times N$ Hermitian Brownian motion. The eigenvalues $\{\lambda_1(t), ..., \lambda_N(t)\}$ of $\boldsymbol{X}$ are almost surely distinct. Ordering $(\lambda_i(t))_{1 \leq i \leq N}$ so that $\lambda_1(t) < \lambda_2(t) < ... < \lambda_{N-1}(t) < \lambda_N(t)$ for $t > 0$, they satisfy the following system of SDEs :

$$d\lambda_i(t) = \frac{1}{\sqrt{N}} dw_i(t) + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt \tag{1.21}$$

with $(w_i(t))_{1 \leq i \leq N}$ a collection of independent real standard Brownian motions and initial conditions $(\lambda_i(0))_{1 \leq i \leq N}$ the eigenvalues of A so that $\lambda_1(0) \leq \lambda_2(0) \leq ... \leq \lambda_{N-1}(0) \leq \lambda_N(0)$.

This theorem is proved in [AGZ10; Tao12] and we will give an idea of proof following [Tao12] and the proof of a more general theorem following [Gra14] in Chapter 2.

Remark 1.2.7
Unlike in Remark 1.2.3, Equation (1.21) is now a scalar SDE. Let assume that a scalar SDE is just an initial condition, a Brownian increment and a *drift*. In Section 1.3.1, we will give a more general definition of scalar SDEs.
The solution of a SDE as described in Equation (1.21) are called *diffusions*. Let us try to understand the role of the two terms appearing in Equation (1.21). The first one represents the term of diffusion, and it is remarkable that the *coefficient of diffusion* is a constant $1/\sqrt{N}$. The second - corresponding to the drift - introduces an noticeable idea of *repulsion* between the eigenvalue, which recall the repulsion presented in Section 1.1.3.

This last result allows us to consider the second way of generating the eigenvalues trajectories : the simulation using the eigenvalues system of SDEs. Instead of having a random matrix SDE, here we have got a system of N scalar SDEs satisfied by the eigenvalues of a random matrix-valued process (a.k.a. the Hermitian Brownian motion).

However, we cannot have the same guarantee provided by Lemma 1.2.2 for Algorithm 1. In this second case, we have to use an Euler discretisation (see [KPS12], Chapitre 9) to simulate these trajectories which make these simulations **approximate** and no

longer exact. Algorithm 2 explains how these simulations have been performed and is also available on [Ohl17].

Algorithm 2 Eigenvalues trajectories simulation with system of eigenvalues SDEs

input : $A,\ N,\ n_{samples},\ t_f$
initialisation :
$dt \leftarrow t_f / n_{samples}$
$\Lambda \leftarrow \{OrderedSpectrum(A)\}$
for t=1 **to** n_{sample} **do**
  for i=1 **to** N **do**
    draw $B \sim \mathcal{N}(0,1)$
    $D \leftarrow \frac{1}{N} \sum_{j \neq i} \frac{1}{\Lambda(t,i) - \Lambda(t,j)}$
    $\Lambda(t+1,i) \leftarrow \Lambda(t,i) + \sqrt{dt/N} * B + D * dt$
  end for
end for
output : Λ

Equally to Algorithm 1, by transforming Λ into a $N \times n_{samples}$ matrix, each row represent the value of an eigenvalue over time.

Figure 1.4 shows 50 eigenvalues processes, simulated using their SDE. Unfortunately,

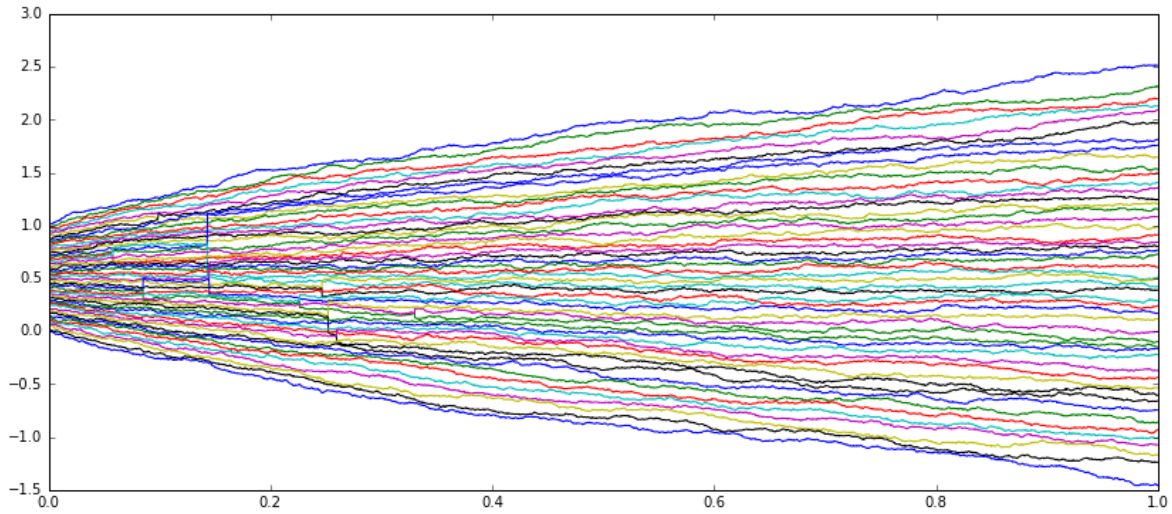

Figure 1.4 – **Eigenvalues trajectories generated with the Dyson Brownian motion eigenvalues system of SDEs.** 50 eigenvalues, with 3000 points in $[0, 1]$.

because of the approximate nature of these simulations, we can observe some "jumps" of trajectories. These jumps are due to the Euler discretisation created to generate them.

Indeed, the generation of these trajectories needs the computation of $d\lambda_i(t)$ at each step and for $i \in \{i, ...n\}$, then add these increments to the current values $\lambda_i(t)$. However, these computations are obtained separatedly and cannot guarantee each trajectory to not intersect with others. Consequently, after an intersection we observe a "jump". This phenomenom can be limited by using a very thin discretisation (i.e. fixing dt to a very small value) and by initialising eigenvalues not too close to each other - as in Figure 1.4 - but it cannot be fully avoided.

We can still quickly comment simulations using the random matrix SDE and using the system of eigenvalues SDEs. Indeed, these two methods provide a way to compute the eigenvalues trajectories by defining discretisation policy - choosing $n_{samples}$ and t_f. Let us remark that if we have the system of eigenvalues SDEs corresponding to a random matrix SDE, this second method is far less computationaly costly than the first one, as Table 1.1 shows.

	Matrix SDE	Eigenvalues SDEs
Type of simulation	Exact	Approximate
# simulations $\sim \mathcal{N}$	$2N^2$	N
Diagonalisation	$O(N^3)$	None
Robustness	High	Low ("jumps")

Table 1.1 – Comparison of the two simulation methods

First because we only need N generations of random variables following a normal distribution and secondly because we do not need to diagonalise at each step. However, this simulation method is less robust than the simulation with the random matrix SDE. To conclude, it seems that there is not a perfect method but the choice between the one or the other can depend on the application. Still, the link between both is complex and will motivate the following questioning.

After this complete presentation of the Dyson Brownian motion we will now focus on other examples of these type of processes. In fact, the Dyson Brownian motion is a starting point and several other processes derived from it. In this study, we choose to present two other particular processes : the Ornstein-Ulhenbeck process and the Wishart process.

1.3. Other classical examples

The two other examples exposed here have different interesting properties. The first one, the Ornstein Ulhenbeck process is remarkable because it is preserve the GUE distribution at each time t. Otherwise, the Wishart is notable because it generates only non-negative trajectories which can be attractive for some simulations.
Note that every simulations in this part had been performed using algorithms available on [Ohl17].

1.3.1 The Ornstein-Ulhenbeck process

The previous part aims to consider a matricial form of the Brownian motion and leads us to analyze the Dyson Brownian motion. In this part we try to reproduce this analysis with another simple example : the Ornstein-Ulhenbeck process. However, we need to go further in the definition of scalar and matrix Stochastic Differential Equations because this example is slightly more complicated than the Dyson Brownian motion.
Thus, introducing the Ornstein-Ulhenbeck process gives us the opportunity to remind some few points concerning SDEs. There is a wide development of SDEs in the literaturea and the reader can refer to [JS05; Øks03] for instance.

Stochastic Differential Equations

In order to understand the motivation of SDEs, let us start by considering an Ordinary Differential Equation namely a system

$$\begin{cases} \frac{dz}{dt} = b(t, z(t)) \\ z(0) = u \end{cases}$$

where $z : \mathbb{R}^+ \to \mathbb{R}$ is the unknown function and $b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a given function. Now, imagine that we want a "noisy" version of this previous equation. The simplest way to do that is by adding a Brownian motion. However, because the Brownian motion is almost surely non derivable, we define the Stochastic Differential Equation as follows

$$Z(t) = u + \int_0^t b(s, Z(s))\ ds + \sigma(B(t)) \tag{1.22}$$

with $\sigma \in \mathbb{R}$ corresponding to the diffusion coefficient and $(B(t))_{t\geq 0}$ a real Brownian motion. Recall that $(Z(t))_{t\geq 0}$ is now a scalar-valued random process. This is the integral form of SDEs, and the only form which is mathematically correct. Nevertheless, the following notation is generally used to point this SDE

$$\begin{cases} dZ(t) = b(t, Z(t))\ dt + \sigma dB(t) \\ Z(0) = u \end{cases}$$

Remark 1.3.1
Stochastic Differential Equations are usually defined in a more general context, where σ is a function of the time and the process. Here we will only develop the case related to the Ornstein-Ulhenbeck process so that the reader can intuitively understand SDEs and because we did not introduce Itô integral so far. Itô calculus will only be introduced in section 2.2.1, for the need of the proof on Theorem 2.2.9.

Focusing on the particular process studied in this part, the Ornstein-Ulhenbeck process is the solution of this equation, namely the *Langevin equation*

$$Z(t) = U - a\int_0^t Z(s)\ ds + \sigma B(t) \tag{1.23}$$

where $a, \sigma \in \mathbb{R}$, $Z(0) = U$ a random variable and $(B(t))_{t\geq 0}$ a real Brownian motion independent from U. We can write this equation as follows

$$\begin{cases} dZ(t) = -aZ(t)\ dt + \sigma dB(t) \\ Z(0) = U \end{cases}$$

In the particular case where $U \sim \mathcal{N}(m, \sigma_0^2)$ independent of $(B(t))_{t\geq 0}$, we have the following result

Proposition 1.3.2
The Langevin equation (1.23) *has a unique solution given by*

$$Z(t) = e^{-at}\left(U + \sigma\int_0^t e^{as} dB(s)\right) \tag{1.24}$$

Assuming that $U \sim \mathcal{N}(m, \sigma_0^2)$ *independent of* $(B(t))_{t\geq 0}$, $(Z(t))_{t\geq 0}$ *is a Gaussian process of expected value*

$$\mathbb{E}[Z(t)] = me^{-at}$$

and covariance function

$$\mathrm{Cov}(Z(s), Z(t)) = e^{-a(t+s)}\left(\sigma_0^2 + \frac{\sigma^2}{2a}(e^{2a(s\wedge t)} - 1)\right)$$

with $t, s \geq 0$ *and* $s \wedge t = min(s, t)$.

By showing that $(Z(t))_{t\geq 0}$ is a Gaussian process, we implicitly explain that the solution of Langevin equation preserves the normal random distribution when $t \to \infty$. This result will have its matrix equivalent later (Proposition 1.3.3). Let us give a proof of the expression of the expected value and the covariance function.

Proof. The solution of the Langevin equation (1.24) is admitted.
We know that $t \to e^{-at}U$ is a Gaussian process. Then, by the properties of the Itô integral,

$$t \to \sigma\int_0^t e^{-a(t-s)} dB(s)$$

is also a Gaussian process with zero expected value. Hence, $(Z(t))_{t\geq 0}$ which is the sum of the two previous processes is also a Gaussian process of expected value

$$\mathbb{E}[Z(t)] = \mathbb{E}[e^{-at}U] = e^{-at}\mathbb{E}[U] = me^{-at}$$

Regarding the covariance function and using the isometry property of the Itô integral we have

$$\begin{aligned}
\mathrm{Cov}(Z(s), Z(t)) &= \mathrm{Cov}(Ue^{-as}, Ue^{-at}) + \mathrm{Cov}\left(\sigma \int_0^s e^{a(v-s)} dB(u), \sigma \int_0^t e^{a(v-t)} dB(v)\right) \\
&= e^{-a(t+s)}\left(\mathrm{Var}(U) + \sigma^2 \int_0^{s\wedge t} e^{2av} dv\right) \\
&= e^{-a(t+s)}\left(\sigma_0^2 + \frac{\sigma^2}{2a}(e^{2a(s\wedge t)} - 1)\right)
\end{aligned}$$

□

The reader must notice that the classical Langevin equation is define for real scalar-valued stochastic processes. Hence, it can be extended to matrix-valued processes. Then, we define the Langevin equation for the matrix-valued random process $\boldsymbol{X}$ as

$$X_t = U - a\int_0^t X_s \, ds + H_t \tag{1.25}$$

or equally

$$\begin{cases} X_0 = U \\ dX_t = -aX_t \, dt + dH_t \end{cases}$$

where $b \in \mathbb{R}$, $X_0 = U$ a random matrix and $\boldsymbol{H}$ a Hermitian Brownian motion as defined in Definition 1.2.1 and independent from U. Here, the solution of the SDE (1.25) will no longer be a real-valued process but a $N \times N$ matrix-valued process. As mentioned above, Proposition 1.3.2 has its parallel for matrix-valued process, as Proposition 1.3.3 shows.

Proposition 1.3.3
The Langevin equation (1.25) *has a unique solution* $\boldsymbol{X}$ *and if* $U \sim$ GUE(N) *and* $a = 1/2$, *then* $X_t \sim$ GUE(N) *when* $t \geq 0$

Proof. Let us now assume that $U \sim$ GUE(N) (see Definition 1.1.2) and observe the behaviour of the entries $\boldsymbol{X}_{kl}$ with $k, l \in \{1, ..., N\}$.
Equation (1.25) means that every diagonal entries of $\boldsymbol{X}_{kk}$ is a Gaussian process, verifying (1.23) with $\sigma = 1/\sqrt{N}$. Using Proposition 1.3.2 we have

$$\begin{aligned}
\mathbb{E}[X_{kk,t}] &= 0 \\
\mathrm{Var}(X_{kk,t}) = \mathrm{Cov}(X_{kk,t}, X_{kk,t}) &= \frac{1}{N}e^{-2at} + \frac{1}{2aN}(1 - e^{-2at})
\end{aligned}$$

Because we set $a = 1/2$, we have $X_{kk,t} \sim \mathcal{N}_{\mathbb{R}}(0, 1/N)$.
Moreover, every entries above the diagonal - when $k < l$ - has its real part and imaginary part verifying (1.23) with $\sigma = 1/\sqrt{2N}$ (recall that the two Brownians in the definition of the Hermitian Brownian motion are independent, see 1.2.1). It means that

$$\mathbb{E}\left[\mathrm{Re}(X_{kl,t})\right] = 0$$
$$\mathrm{Var}\left(\mathrm{Re}(X_{kl,t})\right) = \mathrm{Cov}\left(\mathrm{Re}(X_{kl,t}), \mathrm{Re}(X_{kl,t})\right) = \frac{1}{2N}e^{-2at} + \frac{1}{4aN}(1 - e^{-2at})$$

and the same equations for the imaginary part. Hence, for $1 < k < l < N$ with $a = 1/2$, we have $\mathrm{Re}(X_{kl,t}) \sim \mathcal{N}_{\mathbb{R}}(0, 1/2N)$ and $\mathrm{Im}(X_{kl,t}) \sim \mathcal{N}_{\mathbb{R}}(0, 1/2N)$.
Finally, $X_{kl,t} \sim \mathcal{N}_{\mathbb{C}}(0, 1/N)$ for $1 < k < l < N$ and $X_t \sim$ GUE(N). □

This proposition allows us to directly apply results from the first part such as Theorems 1.1.4, 1.1.5 and 1.1.10 because it ensures the solution of the matrix Langevin equation to preserve the GUE ensemble. With this proposition, we finally find a matrix-valued process which can preserve a particular distribution of matrices (Equation (1.1)) at each time t, involving that the eigenvalues joint distribution (Equation (1.4)) also hold and even that the collection of eigenvalues at a fixed time is a determinantal point process (with kernel given by Equation (1.6)).

After this brief point on SDEs, we now explain the link between random matrix SDE (1.25) and SDEs of their eigenvalues - as we have done with the Dyson Brownian motion - in the particular case where $a = 1/2$.

Random matrix SDE and associated eigenvalues SDEs

In this part, we will use quite the same notation as in [AGZ10]. One can refer to Exercice 4.3.9 in [AGZ10]. This result will be presented as a theorem.

Theorem 1.3.4 (Ornstein-Ulhenbeck process)
Let A be a $N \times N$ Hermitian matrix and $\boldsymbol{X}$ the matrix-valued process solution of the stochastic differential equation $dX_t = dH_t - \frac{1}{2}X_t\,dt$ with $\boldsymbol{H}$ an Hermitian Brownian motion and $X_0 = A$. The eigenvalues $\{\lambda_1(t), ..., \lambda_N(t)\}$ of $\boldsymbol{X}$ are almost surely disctint. Ordering $(\lambda_i(t))_{1\leq i\leq N}$ so that $\lambda_1(t) < \lambda_2(t) < ... < \lambda_{N-1}(t) < \lambda_N(t)$ for $t > 0$, they satisfy the following system of SDE :

$$d\lambda_i(t) = \frac{1}{\sqrt{N}}dw_i(t) + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{\lambda_i(t) - \lambda_j(t)}dt - \frac{1}{2}\lambda_i(t)dt \tag{1.26}$$

with $(w_i(t))_{1\leq i\leq N}$ a collection of independent real standard Brownian motions and initial conditions $(\lambda_i(0))_{1\leq i\leq N}$ the eigenvalues of A so that $\lambda_1(0) \leq \lambda_2(0) \leq ... \leq \lambda_{N-1}(0) \leq \lambda_N(0)$.

By comparing the two systems of SDEs (1.21) and (1.26), note that the drift is now formed by two terms. This second new term influence the eigenvalues trajectories by limiting their growths. It makes the process solution of SDE stronger than the classical Dyson Brownian motion in the sense that the dynamic does not only preserve the eigenvalues empirical distribution. The dynamic also preserves the eigenvalues joint distribution of the GUE - and for all t when $t \to +\infty$, not only for $t = 1$ (see Equation (1.20)).

Simulation with random matrix SDE When we represent the evolution of this process and the empirical distribution at a fixed time t, we can clearly observe the semicircule law. An example is presented on Figure 1.5, where the empirical distribution is plotted for $t_f = 1$.

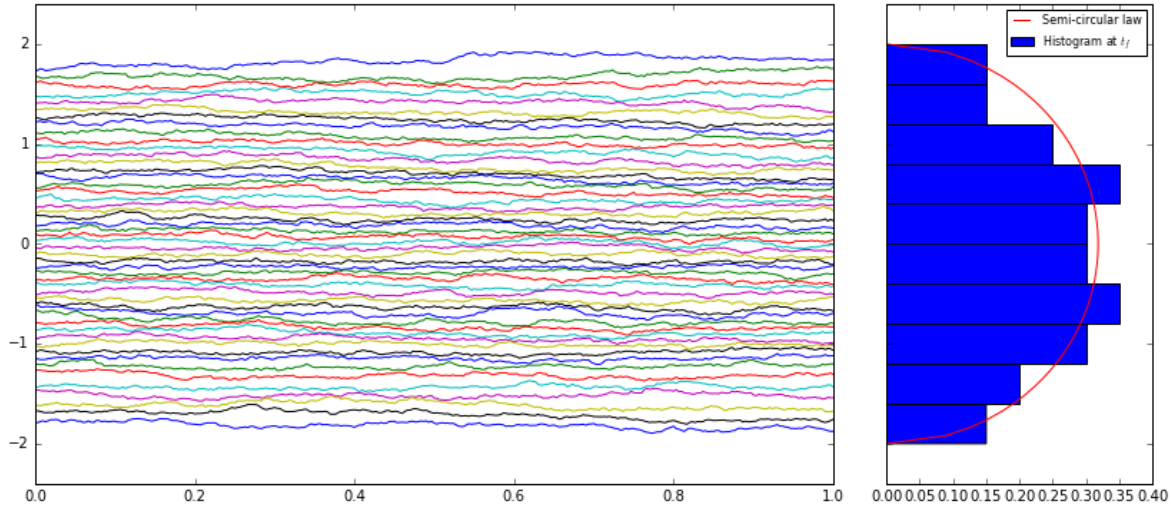

Figure 1.5 – **Eigenvalues trajectories of a Ornstein-Ulhenbeck process generated by random matrix SDE.** 50 eigenvalues, with 300 points in $[0, 1]$.

Simulation with eigenvalues SDEs As we do for the Dyson Brownian motion, we try to simulate these trajectories using the eigenvalues SDEs instead of the random matrix SDE. Results are shown on Figure 1.6.

The semicircule law seems to appear at final time t_f. However with only 50 trajectories, it is difficult to represent a valid histogram. Unfortunately, when we decide to increase the number of trajectories, it creates some "jumps" on several trajectories as Figure 1.7 shows.
This phenomenom was observed on Figure 1.4 and explained in the previous part. This can be limited by using a very thin discretisation, like on Figure 1.7 where we multiply the number of points in $[0, 1]$ by 10.

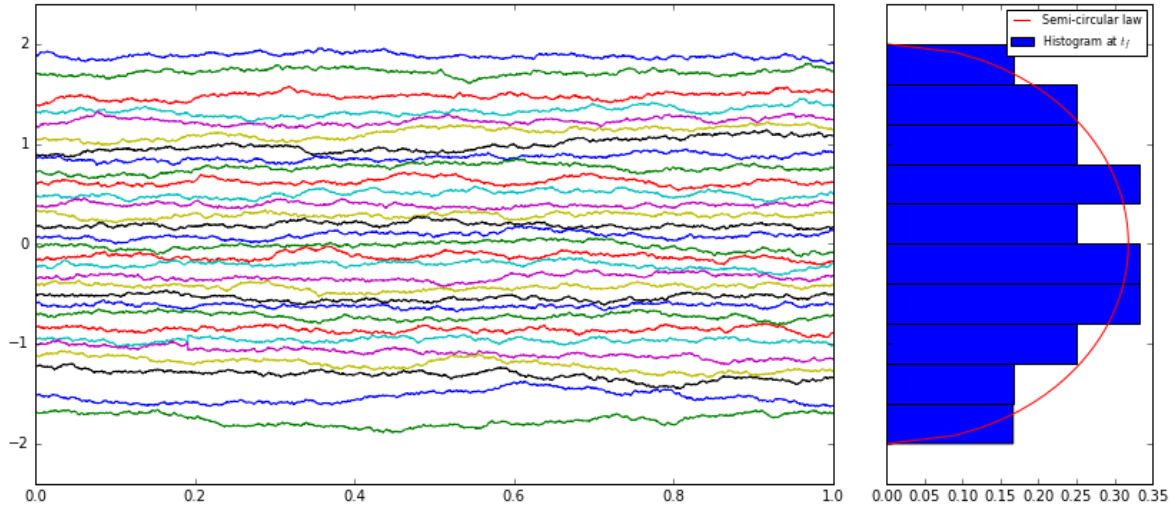

Figure 1.6 – **Eigenvalues trajectories of a Ornstein-Ulhenbeck process generated by eigenvalues SDEs.** 30 eigenvalues, with 3000 points in $[0, 1]$.

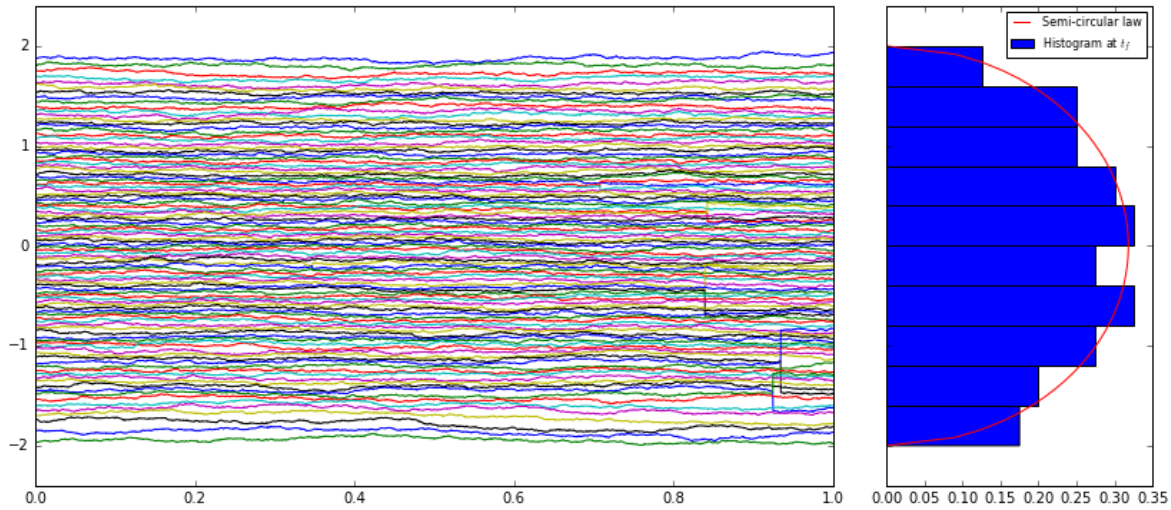

Figure 1.7 – **Eigenvalues trajectories of a Ornstein-Ulhenbeck process generated by eigenvalues SDEs.** 100 eigenvalues, with 3000 points in $[0, 1]$.

1.3.2 The Wishart process

As it was shortly introduce at the beginning of this section, we decide to present the Wishart process. This process is rather different from the Dyson Brownian motion and the Ornstein-Ulhenbeck process because it no longuer have something to do with GUE matrices. However, the Wishart process has the particularity to preserve non-negative eigenvalues trajectories. Thus, the joint distribution will be specific, and the determinant point process formed by the collection of eigenvalues at a fixed time t too.

Eigenvalues joint distribution and determinantal point process

Before explicitly defining the Wishart process, we have to present the joint distribution of the Wishart matrix ensemble, and the kernel of the determinantal point process.

Theorem 1.3.5 (Eigenvalues joint distribution and DPP kernel for Wishart matrices)
Let V be a $M \times N$ matrix with $M \leq M$ with entries $V_{ij} \sim \mathcal{N}_{\mathbb{C}}(0,1)$ for $1 \leq i \leq M$ and $1 \leq j \leq N$.
We fix the reference measure to $\mu(dx) = x^{N-M}e^{-x}\mathbb{1}_{[0,+\infty[}(x)dx$, the eigenvalues joint distribution of V^*V is given by

$$\frac{1}{Z_N}\prod_{i<j}(x_i - x_j)^2 \mu^{\otimes N}(dx)$$

with Z_N a normalisation constant.
The point process $\{\lambda_1, ...\lambda_N\}$ associated with the eigenvalues of V^*V is determinantal with kernel

$$K_N(x,y) = \sum_{i=0}^{N-1} \frac{l_i(x)l_i(y)}{||l_i||^2} x^{\frac{N-M}{2}} y^{\frac{N-M}{2}} e^{-\frac{x+y}{2}} \mathbb{1}_{[0,+\infty[\times[0,+\infty[}(x,y)$$

with $(l_i)_{0\leq i\leq N-1}$ the orthogonal polynomial with respect to the measure μ (a.k.a the Laguerre polynomials with parameter $N-M$).

Proof. Not detailed here, but very close to the proof of Theorem 1.1.10. □

Let us now introduce the Wishart process, able to preserve the Wishart matrix distribution over time.

Random matrix SDE and associated eigenvalues SDES

For this result, we refer to Exercice 4.3.8 in [AGZ10]. and present it as a theorem.

Theorem 1.3.6 (Wishart process)
Let $\boldsymbol{V}$ be a $M \times N$ matrix whose entries are independent complex Brownian motions and let V_0 be a $M \times N$ matrix with complex entries. The eigenvalues $\{\lambda_1(t), ..., \lambda_N(t)\}$ of $\boldsymbol{X}$ defined by $X_t = V_t^*V_t$ are almost surely distinct. Ordering $(\lambda_i(t))_{1\leq i\leq N}$ so that $\lambda_1(t) < \lambda_2(t) < ... < \lambda_{N-1}(t) < \lambda_N(t)$ for $t > 0$, they satisfy the following system of SDE :

$$d\lambda_i(t) = 2\sqrt{\frac{\lambda_i(t)}{N}}dw_i(t) + 2\left(\frac{M}{N} + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\lambda_j(t) + \lambda_i(t)}{\lambda_i(t) - \lambda_j(t)}\right)dt \tag{1.27}$$

with $(w_i(t))_{1\leq i\leq N}$ *a collection of independent real standard Brownian motions and initial conditions* $(\lambda_i(0))_{1\leq i\leq N}$ *the eigenvalues of* $V_0^*V_0$ *so that* $\lambda_1(0) \leq \lambda_2(0) \leq ... \leq \lambda_{N-1}(0) \leq \lambda_N(0)$.

Considering the definition of the eigenvalues $(\lambda_i(t))_{1\leq i\leq N}$, we can remark that they are define as eigenvalues of the covariance matrix of $(X_t)_{t\geq 0}$. Thus, eigenvalues of Wishart matrices are non negative.

Remark 1.3.7 (Random matrix SDE)
This theorem does not explicitly present the SDE verified by the matrix-valued process $\boldsymbol{X}$. Referring to [Gra14] - this reference will be widely used in Section 2.2, we explain it : with the conditions of Theorem 1.3.6, the process $\boldsymbol{X}$ is the solution of the random matrix SDE

$$\begin{cases} dX_t = \sqrt{\frac{1}{N}}\sqrt{X_t}dB_t + \sqrt{\frac{1}{N}}dB_t^*\sqrt{X_t} + \frac{2M}{N}Idt \\ X_0 = V_0^*V_0 \end{cases}$$

with $\boldsymbol{B}$ a $N \times N$ complex Brownian motion and V_0 be a $M \times N$ matrix with complex entries. The expression of this SDE will be more detailed in Section 2.2.4.

Simulation with the matrix-valued process For this particular simulation, we do not use the SDE presented in Remark 1.3.7, because we prefer to generate $\boldsymbol{V}$ and compute $\boldsymbol{X}$ at each step with $X_t = V_t^*V_t$. The sign of the eigenvalues (see Figure 1.8) generates a slightly different representation from the two previous examples. Due to this process, we manage to generate N Brownian motions conditioned to avoid collision, which take only positive values.

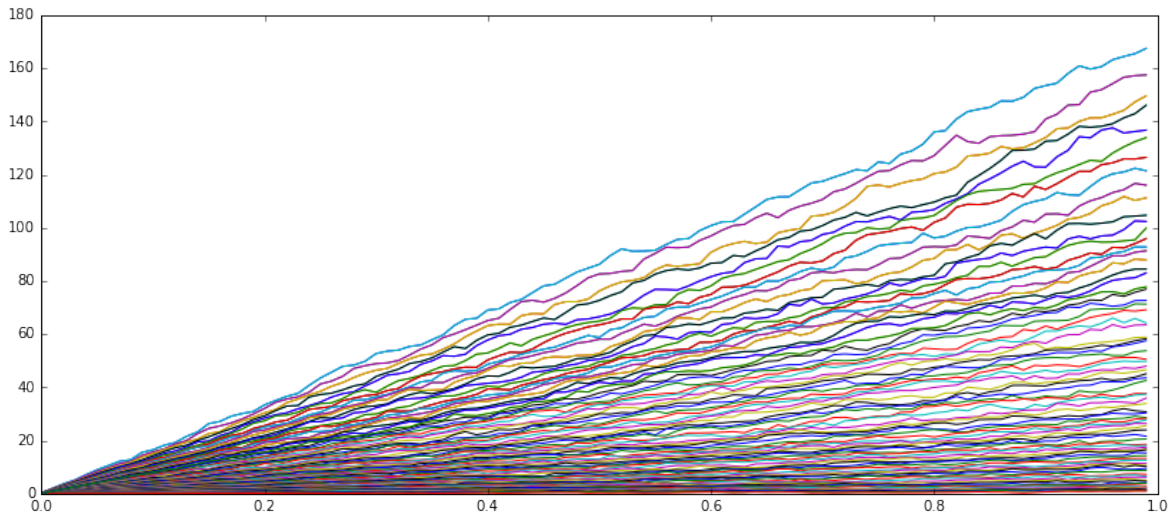

Figure 1.8 – **Eigenvalues trajectories of a Wishart process generated with the matrix-valued process.** 30 eigenvalues, with 100 points in $[0, 1]$ and $M = 25$.

Simulation with eigenvalues SDEs Unfortunately, when we try to simulate Wishart processes using the eigenvalues SDEs, we observe that it is very difficult to maintain the eigenvalues non negative. In fact, simulations fail because of the computation of the square root of a negative real. We are not able to present result of the generation of a Wishart process using eigenvalues SDEs.

At the end of this chapter of examples, we can conclude that eigenvalues processes of matrix-valued random process have the property to dynamically preserve a repulsion, by preserving a DPP. Thanks to these time processes, we have an interesting way of simulate them.
Moreover, it is clearly more robust to simulate these type of processes with associated random matrix SDEs than with systems of eigenvalues SDEs. Simulating with the system of eigenvalues SDEs was barely acceptable with the Ornstein-Ulhenbeck process but not possible with the Wishart process. Simulations from random matrices are finally more robust and easier to pratice, while they are more computationally costly and less interpretable than simulations with eigenvalues SDEs. In a perfect world, we would like to think the simulation in term of a system of eigenvalues SDEs and run it with the associated random matrix SDE.

Having a clear understanding of the link between these two aspects of the same problem is crucial and we will try to enlighten it in the following section.

Chapter 2

Matrix and eigenvalues stochastic differential equations

This chapter is mainly dedicated to give an intuition of the relations between random matrix SDE and eigenvalues SDEs. In the most simple case - the Dyson Brownian motion - this link is described by Theorem 1.2.6. A proof of this theorem is given in [Tao12] using a particular approach based on the Hadamard's formulas.
In the first section, we try to apply the same approach to prove Theorem 1.3.4 and we will see that this method is finally limited and can only give us a **heuristic** to prove this result. The second section will focus on another approach following [Gra14] which gives a general theorem to link random matrix SDE and associated system of eigenvalues SDEs in a very general case.

2.1. Hadamard formulas approach

To prove Theorem 1.3.4, following [Tao12], we first need to introduce and prove the Hadamard's first and second variation formulas.

2.1.1 Hadamard's first and second variation formulas

Let us consider a **deterministic** Hermitian matrix-valued process $\boldsymbol{A}$ with simple spectrum and with all of its coefficients $(a_{ij}(t))_{0\leq i,j\leq n}$ twice derivable functions of variable t. Then, we define $\dot{\boldsymbol{A}}$ and $\ddot{\boldsymbol{A}}$ with entries respectively defined by $\dot{a}_{ij}(t) = da_{ij}/dt$ and $\ddot{a}_{ij}(t) = d^2a_{ij}/dt^2$ for $i,j \in \{1,...,N\}$. Because $\boldsymbol{A}$ has a simple spectrum, we can clearly define N trajectories and number them from 1 to N thereby $\lambda_1(t) < \lambda_2(t) < ... < \lambda_{N-1}(t) < \lambda_N(t)$ for $t \geq 0$. It is also known that the eigenvalues processes $(\lambda_i(t))_{0\leq i\leq n}$ and the associated eigenvectors $(u_i(t))_{0\leq i\leq n}$ of $\boldsymbol{A}$ are twice derivable.

For $i \in \{1,...,N\}$, the Hadamard's first and second variation formulas are useful to link the derivative of an eigenvalue process of $\boldsymbol{A}$, denoted $(\lambda_i(t))_{t\geq 0}$ defined by $\dot{\lambda}_i(t) =$

$d\lambda_i/dt$ and $\ddot{\lambda}_i(t) = d^2\lambda_i/dt^2$ to $\dot{\boldsymbol{A}}$ and $\ddot{\boldsymbol{A}}$.

Lemma 2.1.1 (Hadamard's first variation formula)
Let $\boldsymbol{A}$ be a $N \times N$ Hermitian matrix-valued process with simple spectrum. Suppose that $\boldsymbol{A}$ is a derivable function of t. Denote by $(\lambda(t))_{t\geq 0}$ an eigenvalue of $\boldsymbol{A}$ and $u = (u(t))_{t\geq 0}$ its associated eigenvector, we have :

$$\dot{\lambda}(t) = u^* \dot{A}_t u \tag{2.1}$$

for $t \geq 0$.

Proof. See appendix A.1. □

Lemma 2.1.2 (Hadamard's second variation formula)
Let $\boldsymbol{A}$ be a $N \times N$ Hermitian matrix-valued process with simple spectrum.. Suppose that $\boldsymbol{A}$ is a twice derivable function of t. We denote by $(\lambda_i(t))_{1\leq i\leq N}$ the eigenvalues of $\boldsymbol{A}$ and $u_i = (u_i(t))_{1\leq i\leq N}$ their associated eigenvectors for $t \geq 0$. For $1 \leq i \leq N$ and $t \geq 0$, we have

$$\ddot{\lambda}_i(t) = u_i^* \ddot{A}_t u_i + 2 \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|u_j^* \dot{A}_t u_i|^2}{\lambda_i(t) - \lambda_j(t)} \tag{2.2}$$

Proof. See appendix A.2. □

Remark 2.1.3
Here one can make a very astonishing observation. The equation (2.2) in lemma 2.1.2 shows the interaction between the eigenvalues even if there is not any randomness in the theorem. Repulsion between eigenvalues seems to be more general: it is an "algebric repulsion", also remarkable for matrices satisfying Ordinary Differential Equations.

2.1.2 The Ornstein-Ulhenbeck process using Hadamard's approach

In this section, we give a **heuristic** for Theorem 1.3.4 using a Taylor expansion and the Hadamard's variation formulas (2.1) and (2.2). The following development is not rigourous and cannot establish an exact proof. However, these computations are still relevant because it gives us an intuition of the eigenvalues SDEs setting up.
Note that the ith eigenvector will be noted u_i for notation simplification but the reader must remember that u_i depends on the time i.e. $u_i = (u_i(t))_{1\leq i\leq N}$.

Heuristic of 1.3.4 using a Taylor expansion and the Hadamard variation formulas. Let $\boldsymbol{H}$ be an Hermitian Brownian motion and $\boldsymbol{X}$ be the matrix-valued process solution of the stochastic differential system:

$$dX_t = dH_t - \frac{1}{2} X_t \, dt \tag{2.3}$$

with initial condition $X_0 = A$, A a $N \times N$ Hermitian matrix.

We want to use a Taylor expansion to explain the variation of the eigenvalues with the variation of the matrix-valued process. For a twice derivable function F, a second order Taylor expansion is given by

$$F(t+dt) = F(t) + \frac{dF}{dt}(t)dt + \frac{1}{2}\frac{d^2F}{dt^2}(t)dt^2 + o(dt^2) \tag{2.4}$$

Because we present a heuristic here, we decide to apply this expansion to the function $F(t) = \lambda_i(t)$, with $(\lambda_i(t))_{t\geq 0}$ an eigenvalue of $\boldsymbol{X}$ solution of the SDE (2.3), even if the function is not twice derivable (because of the Brownian nature of this process).

With $F(t) = \lambda_i(t)$, we have:

$$\frac{dF}{dt}(t)dt = \dot{\lambda}_i(t)dt \qquad\qquad \frac{d^2F}{dt^2}(t)dt^2 = \ddot{\lambda}_i(t)dt^2$$

with $\dot{\lambda}_i(t)$ and $\ddot{\lambda}_i(t)$ as defined in Lemmas 2.1.1 and 2.1.2.
Considering the left-hand equation, with the *first Hadamard's formula*, we get:

$$\begin{aligned}
\dot{\lambda}_i(t)dt &= \left(u_i^* \frac{dX_t}{dt} u_i\right) \, dt \\
&= u_i^* dX_t u_i \\
&= u_i^* \left(dH_t - \frac{1}{2}X_t dt\right) u_i \\
&= u_i^* dH_t u_i - u_i^* \left(\frac{1}{2}X_t dt\right) u_i \\
&= dH_{ii}(t) - \frac{1}{2}\lambda_i(t)dt \\
\dot{\lambda}_i(t)dt &= \frac{1}{\sqrt{N}} dw_i(t) - \frac{1}{2}\lambda_i(t)dt
\end{aligned} \tag{2.5}$$

with $(w_i(t))_{t\geq 0}$ a real standart Brownian motion as defined in 1.2.1.

Now let us consider the right-hand equation. With the *second Hadamard's formula*, we get:

$$\begin{aligned}
\ddot{\lambda}_i(t)dt^2 &= u_i^* \frac{d^2X_t}{dt^2} u_i \, dt^2 + 2\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\left|u_j^* \frac{dX_t}{dt} u_i\right|^2}{\lambda_i(t) - \lambda_j(t)} \, dt^2 \\
&= u_i^* d^2X_t u_i + 2\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|u_j^* dX_t u_i|^2}{\lambda_i(t) - \lambda_j(t)}
\end{aligned} \tag{2.6}$$

The term $u_i^* d^2 X_t u_i$ is hard to explain. In [Tao12], it is considered equal to zero, without any explanation. We will follow this proof, considering the term $u_i^* d^2 X_t u_i$ negligible.
Then, we calculate $|u_j^* dX_t u_i|^2$:

$$\begin{aligned}
|u_j^* dX_t u_i|^2 &= (u_j^* dX_t u_i)(u_j^* dX_t u_i)^* \\
&= (u_j^* dX_t u_i)(u_i^* dX_t^* u_j) \\
&= (u_j^*(dH_t - X_t dt)u_i)(u_i^*(dH_t - X_t dt)^* u_j) \\
&= (u_j^*(dH_t - X_t dt)u_i)(u_i^*(dH_t^* - X_t^* dt)u_j) \\
&= (u_j^* dH_t u_i - u_j^* X_t dt u_i)(u_i^* dH_t^* u_j - u_i^* X_t^* dt u_j) \\
&= \underbrace{u_j^* dH_t u_i}_{\xi_{ji}\sqrt{\frac{dt}{N}}} \underbrace{u_i^* dH_t^* u_j}_{\xi_{ji}^*\sqrt{\frac{dt}{N}}} - \underbrace{u_j^* dH_t u_i u_i^* X_t^* dt u_j}_{=o(dt^{3/2})} \\
&\quad - \underbrace{u_j^* X_t dt u_i u_i^* dH_t^* u_j}_{=o(dt^{3/2})} + \underbrace{u_j^* X_t dt u_i u_i^* X_t^* dt u_j}_{=o(dt^2)} \\
&= |\xi_{ji}|^2 \frac{dt}{N} + o(dt^{3/2})
\end{aligned}$$

While noting ξ_{ji} the entries of dH_t. It means that:

$$\ddot{\lambda}_i(t)dt^2 = |\xi_{ji}|^2 \frac{dt}{N} + o(dt^{3/2})$$

The ξ_{ji} are i.i.d. copies of $\mathcal{N}(0,1)_{\mathbb{C}}$ so we can see that $\frac{dt}{N}\sum_{j\neq i} \frac{|\xi_{ji}|^2 - 1}{\lambda_i - \lambda_j}$ has mean zero and third moment $O(dt^3)$.

That is why (2.6) becomes:

$$\ddot{\lambda}_i(t)dt^2 = \frac{2}{N} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{\lambda_i(t) - \lambda_j(t)}\, dt \tag{2.7}$$

With (2.7) and (2.5) in (2.4), and setting $d\lambda_i = \lambda_i(t+dt) - \lambda_i(t)$ we obtain the stochastic differential equation:

$$d\lambda_i(t) = \frac{1}{\sqrt{N}} dw_i(t) + \frac{1}{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)}\, dt - \frac{1}{2}\lambda_i(t)dt$$

□

Even if we know that this proof is arguable because of the assumption we made on the function F, we can see that we find the correct equation (1.26). This heuristic shows

that the "repulsion" phenonmenon between the eigenvalues is an "algebric repulsion", not related to randomness.

However, this proof is quite heavy and implies a lot of supposition invalid in practice. Furthermore, it is very hard to adapt this proof to the Wishart case. One can imagine the difficulties to find a general link between random matrices SDE and the system of SDEs statisfied by its associated eigenvalues.

In the next section, we choose to present another proof based on [Gra14]. This approach has the advantage to explain rigourously this link however it requires more mathematical concepts to describe.

2.2. Stochastic calculus approach

As we have seen in the previous section, the approach using Taylor expansion and Hadamard's formulas on this specific type of processes cannot give a complete proof of the form of the eigenvalues SDE giving the random matrices SDE. Here we can feel the need of an other theory, to explain several terms in these equations.

For this purpose, we choose to follow the proof given in [Gra14] because it clearly highlights the dependence between the two types of SDEs even in a more general case. With this approach, we manage to have a general method to control the trajectories by controlling the random matrix SDE. Yet, we need to introduce some more complicated mathematical theory than just a Taylor expansion. Here, we must introduce Itô calculus and the Stratonovich differential notation based on the Itô integral.

2.2.1 Itô formula

Before talking about the Stratonovich integral, it is necessary to remind some fundamental ideas of Itô calculus. This will be a very short point on Itô calculus. For more details the reader can refer to the very consistent litterature on this subject (e.g. [JS05; Øks03]).
Recall that in section 1.3.1 we briefly presented the Stochastic Differential Equations (see Equation (1.22)) but we did not introduce the Itô integral. Indeed, we did not need Itô calculus previously because we only used SDEs as a description of a diffusion for the simulation. Now, as Remark 1.3.1 suggests, we introduce the Itô integral and then the Stratonovich integral for the proof of Theorem 2.2.9.
From now, we consider all process adapted to the common filtration.

Definition 2.2.1 (Itô processes)
Consider $(B(t))_{t\geq 0}$ *a Brownian motion. A Itô process is a stochastic process* $(Z(t))_{t\geq 0}$

define by

$$Z(t) = Z(0) + \int_0^t h(s)ds + \int_0^t k(s)dB(s)$$

with h and k adapted to the common filtration and almost surely defined.

For the complete definition of h and k, the reader can refer to [JS05; Le 11] or for a wide book on the subject [Øks03].
Howpublished = Université Paris-Sud, Usually, we name the process

$$t \to Z(0) + \int_0^t h(s)ds$$

the *finite variation part* of $(Z(t))_{t\geq 0}$ and

$$t \to \int_0^t k(s)dB(s)$$

the *martingale part* of $(Z(t))_{t\geq 0}$.
Moreover, by adopting the formal notation

$$dZ(t) = h(t)dt + k(t)dB(t)$$

we can introduce the very classical Itô formula.

Theorem 2.2.2 (Itô formula)
Assuming that $(Z(t))_{t\geq 0}$ is an Itô process defined by

$$Z(t) = Z(0) + \int_0^t h(s)ds + \int_0^t k(s)dB(s)$$

and $f(t,x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ a twice derivable function. Then, $Y(t) = f(t, Z(t))$ is an Itô process satisfying almost surely

$$Y(t) = f(0, Z(0)) + \int_0^t \frac{\partial f}{\partial s}(s, Z(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, Z(s))dZ(s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}k(s)^2 ds$$

Then, we can express the integration by parts formula

Theorem 2.2.3 (Integration by parts)
Considering two Itô processes $(Y(t))_{t\geq 0}$ and $(Z(t))_{t\geq 0}$ defined as

$$Y(t) = Y(0) + \int_0^t h^Y(s)ds + \int_0^t k^Y(s)dB^Y(s)$$
$$Z(t) = Z(0) + \int_0^t h^Z(s)ds + \int_0^t k^Z(s)dB^Z(s)$$

with $(B^Y(t))_{t\geq 0}$ *and* $(B^Z(t))_{t\geq 0}$ *two independent Brownian motion. We can express the product of these two processes*

$$Y(t)Z(t) = Y(0)Z(0) + \int_0^t Y(s)dZ(s) + \int_0^t Z(s)dY(s) + \int_0^t k(s)^Y k(s)^Z ds$$

Classicaly, we denote the quadratic variation of $(Y(t))_{t\geq 0}$ and $(Z(t))_{t\geq 0}$ by

$$\langle Y, Z\rangle(t) = \int_0^t k(t)^Y k(t)^Z dt$$

And with a differential notation

$$d\langle Z, Z\rangle(t) = k(t)^Y k(t)^Z dt$$

To prove Theorem 2.2.9, we need the following lemma.

Lemma 2.2.4
For $(w(t))_{t\geq 0}$ *a complex Brownian motion, the quadratic variation processes satisfy* $d\langle w, w\rangle(t) = 0$ *and* $d\langle w, \overline{w}\rangle(t) = 2dt$

Proof. Admitted, classic in the litterature [JS05; Le 11]. □

Using this short introduction of Itô calculus as a basis, we can now introduce the Stratonovich integral notation.

2.2.2 Stratonovich integral notation

In this section we do not try to give a large overview of this theory of integration because it is not the object of our study. Here, the main purpose is to give simple tools useful for the next proof to the reader.

Assuming that the Itô integral is well know, we define the Stratonovich integral:

Definition 2.2.5 (Stratonovich integral notation)
Let assume that $(X(t))_{t\geq 0}$ *and* $(Y(t))_{t\geq 0}$ *are two real Itô processes such as* $\int_0^t X(s)dY(s)$ *and* $\int_0^t Y(s)dX(s)$ *are defined (in the sense of Itô integral). We introduce the following differential notation:*

$$X(t) \circ dY(t) = X(t)dY(t) + \frac{1}{2}d\langle X, Y\rangle(t)$$

with the following sense :

$$\forall t \geq 0, \quad \int_0^t X(s) \circ dY(s) = \int_0^t X(s)dY(s) + \frac{1}{2}\langle X, Y\rangle(t)$$

Remark 2.2.6 (Matrix-valued Stratonovich integral)
Definition 2.2.5 introduces the Stratonovich integral for real Itô processes. However, our analysis will focus on matrix-valued process that is why we need to explain the extension from scalar-valued processes to matrix-valued processes in the term of the Stratonovich integral.
Considering two matrix-valued Itô processes $\boldsymbol{X}$ and $\boldsymbol{Y}$, we can define the Itô integral $\left(\int_0^t X_s dY_s\right)_{t\geq 0}$ with entries:

$$\left(\int_0^t X_s dY_s\right)_{ij} = \sum_k \int_0^t X_{ik,s} dY_{kj,s}$$

On the same subject, we can define the quadratic variation $(\langle X, Y\rangle_t)_{t\geq 0}$ with entries :

$$(\langle X, Y\rangle_t)_{ij} = \sum_k \langle X_{ik}, Y_{kj}\rangle_t \tag{2.8}$$

Now, we can define the Stratonovich integral as well

$$X_t \circ dY_t = X_t dY_t + \frac{1}{2} d\langle X, Y\rangle_t \tag{2.9}$$

with the following sense :

$$\forall t \geq 0, \quad \int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t$$

From now and to the end of this chapter, we will work with matrix valued processes.

The Stratonovich integral will allow us to considerably simplify notations and calculations in the proof of Theorem 2.2.9. For this purpose we can first remark that the Itô product can be rewrite

$$d(X_t Y_t) = dX_t \circ Y_t + X_t \circ dY_t \tag{2.10}$$

Then, with Equation (2.10), we deduce a symmetric definition of Stratonovich differential formula:

$$dX_t \circ Y_t = dX_t Y_t + \frac{1}{2} d\langle X, Y\rangle_t \tag{2.11}$$

The two equations (2.9) and (2.10) give us this equation useful for the proof of the next lemma.

Lemma 2.2.7
If $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are three matrix-valued Itô processes, we have :

$$\begin{aligned} Y_t \circ dX_t \circ Z_t &= (Y_t \circ dX_t) \circ Z_t \\ &= Y_t dX_t Z_t + \frac{1}{2} d\langle Y, X\rangle_t Z_t + \frac{1}{2} Y_t d\langle X, Z\rangle_t \end{aligned} \tag{2.12}$$

Proof. See Appendix B.1. □

In the next part, we will often consider the diagonal version of a process $\boldsymbol{X}$. Hence, we define two processes $\boldsymbol{\Lambda}$ and $\boldsymbol{O}$ by $X_t = O_t \Lambda_t O_t^*$. Then, assuming that a function $f : \mathbb{R} \to \mathbb{R}$, we write $f(X_t)$ to denote $O_t f(\Lambda_t) O_t^*$ where $f(\Lambda_t)$ is a diagonal matrix with the function f applied to each eigenvalue.

Definition 2.2.8 (Stochastic logarithm)
Using the Stratonovich notation, we define the stochastic logarithm of $\boldsymbol{O}$*:*

$$dA_t = O_t^{-1} \circ dO = O^* \circ dO$$

2.2.3 General theorem

With these notations and definitions, we are now able to introduce and prove the theorem linking random matrices SDE to eigenvalues SDE. This theorem was exposed in [Gra14] (Theorem 4).

Theorem 2.2.9
Let $\boldsymbol{B}$ *be a complex* $N \times N$ *Brownian matrix (i.e.* $B_t = B_t^1 + iB_t^2$ *where* $\boldsymbol{B}^1$ *and* $\boldsymbol{B}^2$ *are two independent real Brownian squared matrices). Suppose that* $\boldsymbol{X}$ *a process satisfying the following SDE*

$$dX_t = g(X_t) dB_t h(X_t) + h(X_t) dB_t^* g(X_t) + b(X_t) dt \tag{2.13}$$

where $g,\ h,\ b : \mathbb{R} \to \mathbb{R}$*. Let* $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$*. Then, the eigenvalues* $(\lambda_i(t))_{1 \leq i \leq N}$ *of* $\boldsymbol{X}$ *verify the following system of SDEs*

$$d\lambda_i(t) = 2g(\lambda_i(t))h(\lambda_i(t))dw_i(t) + \left(b(\lambda_i(t)) + 2 \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G(\lambda_i(t), \lambda_j(t))}{\lambda_i(t) - \lambda_j(t)} \right) dt \tag{2.14}$$

where $(w_i)_{1 \leq i \leq N}$ *is a collection of* N *independent Brownian motions.*

Note that this expression is very interesting on a simulation point of view because with a control on the random matrices SDE we can obtain control on the SDE satisfied by the associated eigenvalues. In the first part, we saw that simulations are easier and more robust when we use a matrix approach than an eigenvalue approach. Then, we understand the importance of this relation.

Using Lemma 2.2.4, we can proof a second useful lemma to prove the Theorem 2.2.9.

Lemma 2.2.10
Let $\boldsymbol{X}$ *be a matrix-valued process satisfying the SDE*

$$dX_t = g(X_t) dB_t h(X_t) + h(X_t) dB_t^* g(X_t) + b(X_t) dt$$

with $(B_t)_{t\geq 0}$ a matrix of independent complex Brownian motions. The quadratic variation between the entries is given by the formula

$$d\langle X_{ij}, X_{i'j'}\rangle_t = 2(g^2(X_t)_{ij'}h^2(X_t)_{i'j} + g^2(X_t)_{i'j}h^2(X_t)_{ij'})dt \tag{2.15}$$

Proof. See Appendix B.2. □

Then, we can start the proof of the Theorem 2.2.9.

Proof of Theorem 2.2.9. By denoting A_t the stochatic logarithm of O_t as we introduced it in the Definition 2.2.8, and apply the Itô product (2.10) to $O^*O = I$ we can observe that the matrix A is skew-Hermitian (i.e. $A^* = -A$).

Therefore, an important remark is to see that the terms on the diagonal of A are **purely imaginary**. This remark will have great consequences on the following proof.

Now applying the Itô formula to $\Lambda_t = O_t^* X_t O_t$ we get

$$d\Lambda_t = dN_t + \Lambda_t \circ dA_t - dA_t \circ \Lambda_t \tag{2.16}$$

setting $dN_t = O_t^* \circ dX_t \circ O_t$.

We can remark that the process N is a Hermitian Brownian process, so its diagonal entries are real. A_t is skew-Hermitian so $\Lambda_t \circ dA_t - dA_t \circ \Lambda_t$ is zero on the diagonal and $d\lambda_i(t) = dN_{ii,t}$.
Concerning the non-diagonal entries $i \neq j$, using (2.16) and the matrix product:

$$0 = dN_{ij,t} + (\lambda_i(t) - \lambda_j(t)) \circ dA_{ij,t}$$

Meaning that for $i \neq j$

$$dA_{ij,t} = \frac{1}{(\lambda_j(t) - \lambda_i(t))} \circ dN_{ij,t}$$

From the definition of dN_t, using 2.2.7, we have

$$\begin{aligned} dN_t &= O_t^* \circ dX_t \circ O_t \\ &= O_t^* dX_t O_t + \frac{1}{2} d\langle O_t^*, X_t\rangle O_t + \frac{1}{2} O_t^* d\langle X_t, O_t\rangle \end{aligned} \tag{2.17}$$

We choose to separate dN_t in its martingale part and its finite variation part.

Martingale part of dN_t The martingale part of dN_t is equal to the martingale part of $O_t^* dX_t O_t$ because the other two terms are finite variation terms and only contribute to the finite variation part of dN_t. By Lemma 2.2.10, we have

$$\begin{aligned} d\langle N_{ij}, N_{i'j'}\rangle_t &= 2(g^2(O_t^* dX_t O_t)_{ij'}h^2(O_t^* dX_t O_t)_{i'j} + g^2(O_t^* dX_t O_t)_{i'j}h^2(O_t^* dX_t O_t)_{ij'}) \\ &= 2(g^2(d\Lambda_t)_{ij'}h^2(d\Lambda_t)_{i'j} + g^2(d\Lambda_t)_{i'j}h^2(d\Lambda_t)_{ij'})dt \end{aligned}$$

We now study the diagonal entries of dN_t. It follows that

$$d\langle N_{ii}, N_{jj}\rangle_t = = 4\delta_{ij} g^2(\lambda_i(t))h^2(\lambda_i(t))dt \tag{2.18}$$

Finite variation part of dN_t We denote by dF_t the finite variation part of dN_t. Here, we must be careful because the two final terms of (2.17) contribute to dF_t, and the first one too.

$$\begin{aligned} dF_t &= O_t^* b(X_t) O_t + \frac{1}{2} d\langle O_t^*, X_t \rangle O_t + \frac{1}{2} O_t^* d\langle X, O \rangle_t \\ &= b(\Lambda_t) + \frac{1}{2} d\langle O^*, X \rangle_t O_t + \frac{1}{2} O_t^* d\langle X, O \rangle_t \\ &= b(\Lambda_t) + \frac{1}{2} \left(d\langle N, A \rangle_t + (d\langle N, A \rangle_t)^* \right) \end{aligned}$$

Recall that $G(x,y) = g^2(x)h^2(y) + g^2(y)h^2(x)$, using Equation (2.8) we have

$$d\langle N_t, A_t \rangle_{ij} = \sum_k d\langle N_{ik}, A_{kj} \rangle_t = 2\delta_{ij} \sum_{k \neq i} \frac{G(\lambda_i(t), \lambda_k(t))}{\lambda_i(t) - \lambda_k(t)} + d\langle N_{ij}, A_{jj} \rangle_t$$

When $i = j$, $dN_{ii,t}$ is real (because N_t is a Hermitian Brownian process) and $dA_{ii,t}$ is purely imaginary (remember that A_t is skew-Hermitian). It follows that

$$dF_{ii,t} = b(\lambda_i(t))dt + 2 \sum_{k \neq i} \frac{G(\lambda_i(t), \lambda_k(t))}{\lambda_i(t) - \lambda_k(t)} dt \tag{2.19}$$

Finally, using (2.18) and (2.19), by the Levy theorem, a collection of N independent Brownian motions $(w_i(t))_{1 \leq i \leq N}$ exists such that (2.14) holds. □

Remark 2.2.11
We can observe similarities between the heuritic in chapter 2 and the proof proposed by [Gra14]. More precisely, we can see that the "repulsion" comes from the finite variation part of dN_t, which means that it is the "smooth" part of the process that creates this phenomenon. The part with randomness is computed separately and do not create repulsion.

We can clearly remark that this second proof is more powerful and more robust. Indeed, with Theorem 2.2.9 we can treat a lot of different cases, thanks to the general form of Equation (2.13). We can especially treat the two examples introduced in Chapter 1 : the Ornstein-Ulhenbeck process and the Wishart process.

2.2.4 Application to previous examples

Let us try to apply Theorem 2.2.9 and observe that the computation is much easier an faster than with Hadamard formulas approach 2.1.

The Ornstein-Ulhenbeck process

First, we will see that it is very easy to proof Theorem 1.3.4 with Theorem 2.2.9. Thus, we set g, h and b so that

$$g(x) = h(x) = \frac{1}{\sqrt{2\sqrt{N}}}, \quad b(x) = -\frac{x}{2}$$

Then,

$$d\lambda_i(t) = 2\left(\frac{1}{\sqrt{2\sqrt{N}}}\right)^2 dw_i(t) + \left(-\frac{\lambda_i(t)}{2} + 2\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{2(\frac{1}{2\sqrt{N}})^2}{\lambda_i(t) - \lambda_j(t)}\right) dt$$

$$d\lambda_i(t) = \frac{1}{\sqrt{N}} dw_i(t) + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt - \frac{\lambda_i(t)}{2} dt$$

Just by setting the three functions g, h and b we find the system of SDEs (1.26) verifyed by the eigenvalues.
When adapting the Hadamard formulas approach to the Wishart case was very difficult and barely achievable, Theorem 2.2.9 allows us to remarkably prove Theorem 1.3.6.

The Wishart process

Concerning the Wishart process, it will be a little bit harder to prove Theorem 1.3.6 with Theorem 2.2.9, but nothing comparable to the difficulties to prove it with Hadamard formulas approach. Indeed, the issue is to find a SDE satisfyed by the matrix-valued process $X_t = V_t^* V_t$ because we want to determine g, h and b. The computation of this SDE is not straightforward.
Now, let us denote by $\mathcal{S}_N^+$ the set of all symmetric definite positive $N \times N$ matrices with distinct eigenvalues. The definition of $\boldsymbol{X}$ involve that at each time t, $X_t \in \mathcal{S}_N^+$ so we can define by $\sqrt{X_t}$ the symmetric positive square root of $\boldsymbol{X}$. As mentioned in [Gra14], the Wishart SDE for a Wishart process with a shape M is

$$\begin{cases} dX_t = \sqrt{\frac{1}{N}}\sqrt{X_t} dB_t + \sqrt{\frac{1}{N}} dB_t^* \sqrt{X_t} + \frac{2M}{N} I dt \\ X_0 = V_0^* V_0 \end{cases} \tag{2.20}$$

with $\boldsymbol{B}$ a $N \times N$ complex Brownian motion and V_0 be a $M \times N$ matrix with complex entries.

Following [Bru91], we know that if $X_t = V_t^* V_t$ is a Wishart process, then

$$dB_t = (\sqrt{X_t})^{-1} V_t^* dV_t$$

is a $N \times N$ Brownian matrix and $\boldsymbol{X}$ satisfy the SDE (2.20). Let us verify the first part of this assumption (the second is admitted) and formulate it as a lemma.

Lemma 2.2.12
Let $\boldsymbol{V}$ be a $M \times N$ matrix whose entries are independent complex Brownian motions. We consider a Wishart process $\boldsymbol{X}$, define by $X_t = V_t^* V_t$ and denote by $\sqrt{X_t}$ the symmetric positive square root of $\boldsymbol{X}$. Then, the process $\boldsymbol{B}$ define by

$$dB_t = \sqrt{X_t}^{-1} V_t^* dV_t$$

is a $N \times N$ Brownian matrix.

Proof. To prove this result, we compute the quadratic variation of $\boldsymbol{B}$ in order to use the Levy theorem (for more details, see [KS12]).
Using Lemma 2.2.4, we have:

$$\begin{aligned}
d\langle (\sqrt{X_t}^{-1} V_t^* dV_t)_{ij}, (\sqrt{X_t}^{-1} V_t^* dV_t)^*_{i'j'} \rangle &= d\langle (\sqrt{X_t}^{-1} V_t^* dV_t)_{ij}, (dV_t^* V_t (\sqrt{X_t}^{-1})^*)_{i'j'} \rangle \\
&= d\langle \sum_{k,l} \sqrt{X}^{-1}_{ij,t} V^*_{kl,t} dV_{lj,t}, \sum_{m,n} dV^*_{i'm,t} V_{mn,t} (\sqrt{X}^{-1})^*_{nj',t} \rangle \\
&= d\langle \sum_{k,l} \sqrt{X}^{-1}_{ik,t} V^*_{kl,t} dV_{lj,t}, \sum_{m,n} \overline{dV_{mi',t}} V_{mn,t} (\sqrt{X}^{-1})^*_{nj',t} \rangle \\
&= (2dt)\delta_{ji'} \sum_{k,l} \sqrt{X}^{-1}_{ik,t} \underbrace{V^*_{km,t} V_{mn,t}}_{=X_{kn,t}} (\sqrt{X}^{-1})^*_{nj',t} \\
&= (2dt)\delta_{ji'} \sum_{k} \sqrt{X}^{-1}_{ik,t} X_{kn,t} (\sqrt{X}^{-1})^*_{nj',t} \\
&= (2dt)\delta_{ji'} \sqrt{X}_{in,t} (\sqrt{X}^{-1})^*_{nj',t} \\
&= (2dt)\delta_{ji'} \sqrt{X}_{in,t} \sqrt{X}^{-1}_{j'n,t} \\
&= (2dt)\delta_{ji'}\delta_{ij'}
\end{aligned} \tag{2.21}$$

Using the Levy theorem adapted to $N \times N$ complex Brownian motion, we prove that $\boldsymbol{B}$ is a $N \times N$ complex Brownian motion.

□

Using Theorem 2.2.9, we set g, h and b:

$$g(x) = \sqrt{\frac{x}{N}}, \quad h(x) = 1, \quad b(x) = \frac{2M}{N}$$

and we prove Theorem 1.3.6.

It is not difficult to understand the power of theorem 2.2.9. We manage to prove the SDE of the Ornstein-Ulhenbeck process and the Wishart process only with identification of three simple deterministic functions. Moreover, we see that we can have

direct control on the trajectories, defining function g, h and b as we want and then go back to the random matrices SDE to perform simulation.

Now that we have a global understanding of the connection between a given random matrix SDE and the system of its eigenvalues SDEs, we will focus on a different approach. In the next part, we will see that some tridiagonal random matrix processes have the same eigenvalues joint distribution as the GUE - and generate the same DPP. Even more interesting, the eigenvalues verify the Dyson Brownian motion system of SDEs (1.21).

Chapter 3

Tridiagonal models

The main interest of this study is to simulate trajectories able to preserve a particular repulsion. Nevertheless, when it comes to simulate, the whole point is to deal with complexity. In fact, the motivation behind the last part of this study mostly comes from the conclusions in Table 1.1. This table illustrates the pros and cons to simulate with a random matrix SDE - exact probability, and robustness but high complexity - and the advantages of the simulation with the system of eigenvalues SDEs - low complexity but approximate and unstable. One can imagine a trade-off between these two simulation methods.
A recent paper [DE02] describes some tridiagonal random matrices having the same eigenvalues joint distribution as GUE matrices and other tridiagonal random matrices sharing the same eigenvalues joint distribution as Wishart matrices. Even more recently, the article [HP17] prove the existence of some tridiagonal matrix-valued process, with the same system of eigenvalues SDEs as the Dyson Brownian motion. More precisely, it covers even more general processes than just the Dyson Brownian motion by considering a general function V for the drift in the eigenvalue SDE (1.21)

$$d\lambda_i(t) = \frac{1}{\sqrt{N}} dw_i(t) + \left(\frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)} - \frac{V'(\lambda_i)}{2} \right) dt \tag{3.1}$$

Note that by choosing a null drift, we retrieve the classical Dyson Brownian motion (1.21) and by choosing a drift $V(x) = x^2/2$, we have the Ornstein-Ulhenbeck process (1.26). Recall that the Ornstein-Ulhenbeck process preserves the GUE matrix distribution.
For tridiagonal matrices, the complexity of diagonalisation is $O(N^2)$ and $O(N \ln N)$ for the fastest algorithms. Hence, using tridiagonal processes can be a significant improvement of the simulation with a random matrix SDE.

In this part, we will start by consider the static case, following [DE02]. Then, we

will try to build a "naive" dynamic model and observe the limit of this model. After that, we will briefly expose the solution of [HP17] which need an important mathematical development to describe. This study will only focus on GUE matrices and on the propagation of these matrices. For other processes, the reader can refer to the two papers cited above.

3.1. Static tridiagonal model for GUE matrices

This section will basically be a parallel of Section 1.1 and 1.2 with tridiagonal models. In fact [DE02] build a model of tridiagonal matrices sharing the same eigenvalues joint distribution as GUE matrices. They justified it by proposing an algorithm to tridiagonalise a GUE matrix which preserve the eigenvalues joint distribution of this matrix.

Our idea is to use this static model and spread it over time. In other word, we want to define a tridiagonal model having the same properties as the Dyson Brownian motion.

First, we will present the static tridiagonal model, like it was done in [DE02] and then we will see how we can naively adapt it to the dynamic case. However, this dynamic tridiagonal model will not have the correct behaviour, leading us to introduce the solution of [HP17].

3.1.1 Tridiagonal matrices and eigenvalues

Let us denote by χ_k the chi distribution so that $Z \sim \chi_k$ if $Z = \sqrt{\sum_{i=1}^{k} Y_k}$ with $Y_k^2 \sim \mathcal{N}(0, 1/2N)$ independent for $1 \leq k \leq N$.
Then, we define the tridiagonal model for the GUE.

Definition 3.1.1 (Tridiagonal model for the GUE)
Let T be a $N \times N$ symmetric matrix with all entries independent having the following distribution

$$T \sim \begin{pmatrix} \mathcal{N}(0,1/N) & \chi_{2(n-1)} & & & 0 \\ \chi_{2(n-1)} & \mathcal{N}(0,1/N) & \chi_{2(n-2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_4 & \mathcal{N}(0,1/N) & \chi_2 \\ 0 & & & \chi_2 & \mathcal{N}(0,1/N) \end{pmatrix}$$

then, T is a tridiagonal model for the GUE *and we note $T \sim$* TriGUE(N).

This matrix model does not have the same joint element distribution as GUE matrices (1.1) - refer to [DE02] for the joint element distribution of TriGUE(N) - but share the same eigenvalues joint distribution. For the GUE case, the proof is interesting and

give an tridiagonlaisation algorithm. In [DE02], the theorem and its proof is presented for the Gaussian Orthogonal Ensemble (GOE) but this matrix ensemble is not detailed in our study. That is why we adapt Theorem 2.1. of [DE02] for the GUE case.

Theorem 3.1.2 (Joint distribution of TriGUE eigenvalues)
If $G \sim$ GUE(N)*, then the reduction of* G *to a tridiagonal form shows that the matrix* T *from the* TriGUE(N) *has eigenvalues joint distribution given by* (1.4)*.*

Proof. Let us write the matrix G like

$$G = \begin{pmatrix} g_1 & x^* \\ x & B \end{pmatrix}$$

According to the definition of a GUE matrix, we have $g_1 \sim \mathcal{N}_{\mathbb{R}}(0,1)$ and x a vector of (N-1) i.i.d. of probability distribution $\mathcal{N}_{\mathbb{C}}(0,1)$. By construction, B is a GUE(N-1) matrix. Let L be any $(N-1) \times (N-1)$ unitary matrix such that

$$Lx = [||x||_2 0 \dots 0]^T = ||x||_2 \; e_1 \tag{3.2}$$

where $||.||_2$ denote the Euclidean norm in $\mathbb{R}^{N-1}$ e_1 a column vector of length $(N-1)$ with a 1 for first coefficient and 0 otherwise. Note that L depends only on x.
Then, using matrix product for block matrices, we have

$$\begin{pmatrix} 1 & 0 \\ O & L \end{pmatrix} \begin{pmatrix} g_1 & x^* \\ x & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} = \begin{pmatrix} g_1 & ||x||_2 \; e_1^* \\ ||x||_2 \; e_1 & LBL^* \end{pmatrix} \tag{3.3}$$

We remark that $||x||_2 \sim \chi_{2(N-1)}$, g_1 is unchanged and by the unitary invariance of the GUE, LGL^* is a GUE(N-1) matrix.
Proceeding by induction we create a tridiagonal model for the GUE as define in 3.1.1. Because we only perform unitary transformation on matrix G, it does not affect the eigenvalues. □

Remark 3.1.3
Note that the parameter of the χ distribution followed by $||x||_2$ is finally the number of independant Gaussians in x.

In the same time, we can represent the histogram of the eigengvalues (see Figure 3.1)), generated by a TriGUE model. This figure is very close to Figure 1.1. It can be explain mathematically by using the Wigner Theorem 1.1.4 adapted to the tridiagonal model (not detailed in this study).

With Theorem 3.1.2, we can then apply every results from Section 1.1.3, especially Theorem 1.1.10. Hence, we conclude that the collection of the GUE tridiagonal model eigenvalues is a determinantal point process, with a kernel given by Equation (1.6).

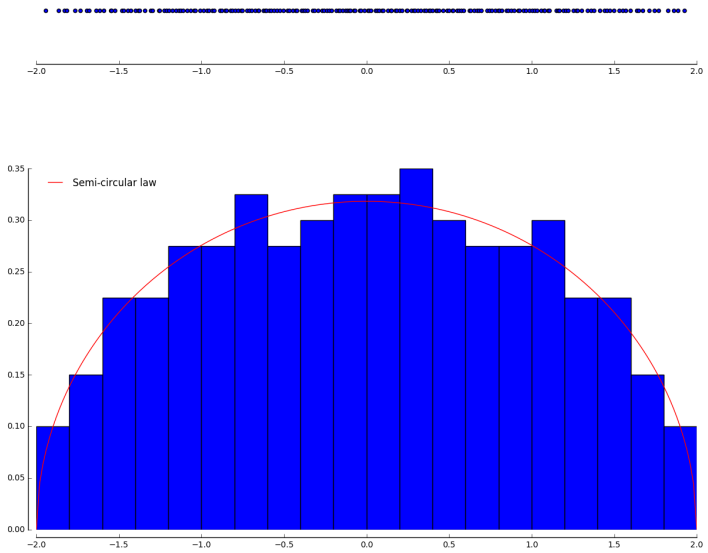

Figure 3.1 – **Eigenvalues histogram of a tridiagonal model for GUE**, generated by the matrix diagonalisation of a 200×200 tridiagonal model

To conclude on complexity, if the number of normally distributed random variable generations is the same for a classic GUE and its tridiagonal model, the diagonalisation complexity is reduced from $O(N^3)$ to $O(N^2)$ or even $O(N \ln N)$. It is an significant reduction if we need to simulate a large number of points for a determinantal point process with kernel (1.6).

Remark 3.1.4 (Generalisation of Theorem 3.1.2)
Theorem 3.1.2 can be easily apply to GOE and GSE (Gaussian Symplectic Ensemble) using approximately the same proof. However, in [DE02], this result is given for even more general ensemble, called the β-ensembles. In particuler, β-ensembles gather GOE ($\beta = 1$), GUE ($\beta = 2$) and GSE ($\beta = 4$). The proof is much more complicated for this general case, and cover the majority of the article [DE02].
Recall that the whole point of our study is to spread well known static models (e.g. GUE, Wishart). Hence, an open question raised here is to find a dynamic tridiagonal model for the global β-ensembles.

3.1.2 Naive tridiagonal model for Dyson Brownian motion

The first part of this section gives us intuitions to build a naive dynamic version of the tridiagonal model for GUE matrices. We present it, and its limits before explain the exact solution proposed in [HP17].

The first naive idea is to transform the Definition 3.1.1 into a process over time. To do that, let us first introduce Bessel processes.

Definition 3.1.5 (Bessel process)
The Bessel process of order k is the real-valued process $(S(t))_{t\geq 0}$ given by

$$S(t) = \sqrt{\sum_{i=1}^{k} B_i(t)^2} = ||B(t)||_2$$

where $||.||_2$ denotes the Euclidean norm in $\mathbb{R}^k$ and $(\boldsymbol{B}_i)_{1\leq i\leq k}$ is a k-dimensional independent Brownian motion starting at 0.

We make a parallel between the χ_k distribution and define the naive Brownian and Bessel tridiagonal process.

Definition 3.1.6 (Naive Brownian and Bessel tridiagonal process)
Let $(\boldsymbol{B}_i)_{1\leq i\leq N}$ be a collection of i.i.d. real valued standard Brownian motion and $(\boldsymbol{S}_i)_{1\leq i\leq N}$ be an independent collection of Bessel processes of order $2(N-i)$. Those two collections are also independent. The naive Brownian and Bessel tridiagonal process noted $\boldsymbol{T}$ is the random process with entries $(\boldsymbol{T}_{ij})_{1\leq i,j\leq N}$ so that

$$\boldsymbol{T}_{ij} = \begin{cases} \frac{1}{\sqrt{2N}}\,\boldsymbol{S}_i & \text{if } i = j+1 \\ \frac{1}{\sqrt{N}}\,\boldsymbol{B}_i & \text{if } i = j \end{cases}$$

and completed by symmetry.

We can now consider a random matrix SDE satisfied by a $N \times N$ matrix process $\boldsymbol{X}$, for example : $X_t = A + T_t$ with A a $N \times N$ Hermitian matrix and $\boldsymbol{T}$ a $N \times N$ Naive Brownian and Bessel tridiagonal process as defined in 3.1.6. This is the same procedure as in Section 1.2.2. Thus, we can simulate as explain on Algorithm 3 (available on [Ohl17]) and get the result shown on Figure 3.2.

Algorithm 3 Generation of naive Brownian and Bessel process

input : $A,\ N,\ n_{samples},\ t_f$
initialisation :
$X \leftarrow 0_{\mathcal{M}_N(\mathbb{R})}$
$B_{\chi^2} \leftarrow \{[0]_{k=1}^{2l}$ **for** $l = 1$ **to** $N-1\}$
$B_{diag} \leftarrow [0]_{k=1}^{N}$
$\Lambda \leftarrow \{OrderedSpectrum(A)\}$
$dt \leftarrow t_f / n_{samples}$
for t=1 **to** n_{sample} **do**
  for j=1 **to** N **do** *# fill the diagonal*
    draw $G \sim \mathcal{N}(0,1)$
    $B_{diag}(j) \leftarrow B_{diag}(j) + \sqrt{dt/N} * G$
    $X(j,j) = B_{diag}(j)$
  end for
  for j=1 **to** $N-1$ **do** *# fill the upper and lower diagonal*
    for k=1 **to** $2j$ **do**
      draw $G \sim N(0, 1/2)$
      $B_{\chi^2}(j,k) \leftarrow B_{\chi^2}(j,k) + \sqrt{dt/2N} * G$
    end for
    $X(N-j, N+1-j) = \sqrt{\sum_{k=1}^{j} B_{\chi^2}(j,k)^2}$
    $X(N+1-j, N-j) = \sqrt{\sum_{k=1}^{j} B_{\chi^2}(j,k)^2}$
  end for
  $\Lambda \leftarrow \Lambda \cup \{OrderedSpectrum(X)\}$
end for
output : Λ

This figure is close to the classical Dyson Brownian motion Figure 1.2, however, the eigenvalues trajectories seems to be less Brownian in this case, and more smooth. Mathematically, this is because this tridiagonal model does not ensure the eigenvalues to follow the system of SDEs (1.21). In fact, we only have the following result on the law of the matrix $T_t \sim \sqrt{t}\ T$ when initial condition $A = 0_{\mathcal{M}_N(\mathbb{R})}$. Hence, at $t = 1$, the eigenvalues of T_1 have their joint distribution equals to the GUE eigenvalues joint distribution.
In a nutshell, with this process we manage to obtain the desired joint distribution for the eigenvalues at a particular time, but we do not have the dynamic i.e. the system of SDEs verifyed by the eigenvalues. This remark motivates the following development in the next section.

Remark 3.1.7 (Other tridiagonal process intuition)
Another intuition to build a tridiagonal process imitating the Dyson Brownian motion could be to follow the proof of Theorem 3.1.2 dynamically. Because the matrix L

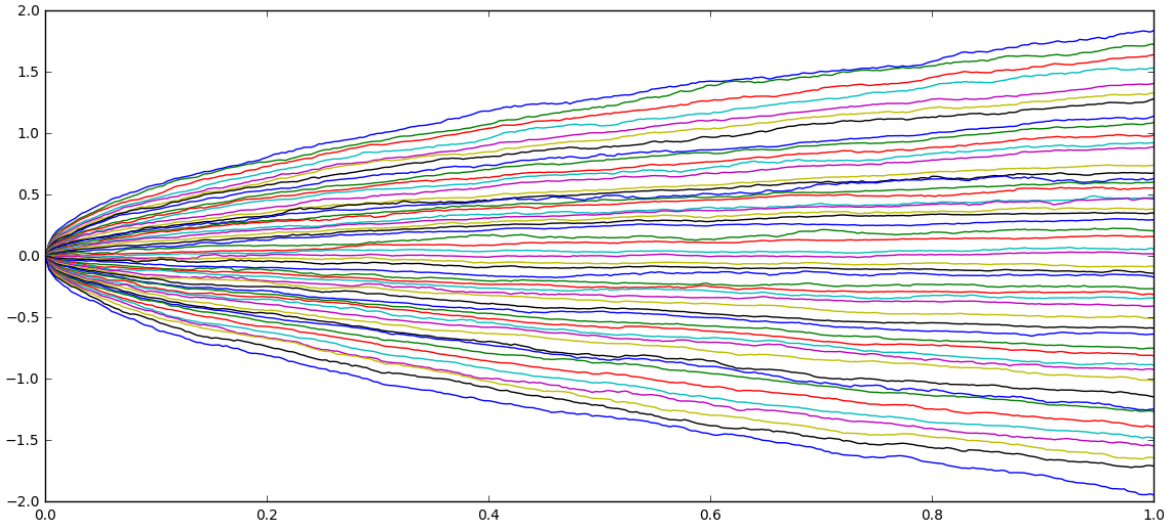

Figure 3.2 – **Eigenvalues trajectories of a naive Brownian and Bessel tridiagonal model.** 50 eigenvalues, with 300 points in $[0, 1]$ and initial condition $A = 0_{\mathcal{M}_N(\mathbb{R})}$.

only depends on x, we could build a process L_t, depending on entries of a Hermitian Brownian motion. Thus, a tridiagonal process $\boldsymbol{T}$ (not necessarily as defined in 3.1.6) could be computed from a Hermitian Brownian motion $\boldsymbol{H}$ so that $\boldsymbol{H} = \boldsymbol{LTL}^*$.
In practise, the justification is much more complicated than that, and we need to read through [HP17] to understand the whole mathematical machinery.

3.2. Exact tridiagonal model for Dyson Brownian motion

In this part, we aim to give an idea of the process to compute an exact tridiagonal model for the Dyson Brownian motion. The complete mathematical justification is long and rough, but well explained in [HP17]. Here, we will just try to explain how the principal concepts, on a simulation point of view. The goal is to simulate a matrix-valued random process which preserves the tridiagonal model for GUE. Here, we will never carry out tridiagonalisation, but we will deal with tridiagonal matrices at each time. More precisely, let us denote by $\boldsymbol{X}$ this tridiagonal process, then $X_t \sim$ TriGUE(N) for $t \geq 0$.
This operation is complex and rest on the use of discrete orthogonal polynomials defined hereafter. We only develop the desired case of the tridiagonal model for GUE but the reader should know that it was describe for more general models in [HP17]. In the following development, we try to stay as close as possible to the notation of [HP17].

Remark 3.2.1 (Dynamic tridiagonal model for β-ensembles)

In [HP17], authors give an answer to the previous Remark 3.1.4. Indeed, they develop a very general tridiagonal model which covers the β-ensembles.

3.2.1 Discrete orthogonal polynomials

Let $T \sim$ TriGUE(N), we define its coefficient so that

$$T = \begin{pmatrix} b_1 & a_1 & & & 0 \\ a_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-2} & b_{N-1} & a_{N-1} \\ 0 & & & a_{N-1} & b_N \end{pmatrix}$$

We define a sequence of polynomials $(p_k(x))_{0\leq k\leq N}$ and $\underline{p} = [p_0(x), p_1(x), \dots, p_{N-1}(x)]$to be the solution of

$$T\underline{p}(x) = x\underline{p}(x) \tag{3.4}$$

Setting $p_0(x) = 1$, Equation (3.4) describes a three term recurrence

$$\begin{aligned} b_1p_0(x) + a_1p_1(x) &= xp_0(x) \\ a_{k-1}p_{k-2}(x) + b_kp_{k-1}(x) + a_kp_k(x) &= xp_{k-1}(x) \\ a_{N-1}p_{N-2}(x) + (b_N - x)p_{N-1}(x) &= -p_N(x) \end{aligned}$$

Note that if λ is an eigenvalue of T, then $p_N(\lambda) = 0$ and its associate eigenvalue is given by $\underline{p}(\lambda)$. We now number the eigenvalues like in the previous part, so that $\lambda_1 < \lambda_2 < \cdots < \lambda_N$.
Let q_i be the spectral weight associated to λ_i by

$$q_i = \frac{1}{||\underline{p}(\lambda_i)||_2}$$

Then, we define the polynomials $p_l^{(k)}(x)$ for any $0 \leq k, l \leq N-1$

$$p_l^{(k)}(x) = \sum_{j=1}^{N} q_j^2 p_k(\lambda_j) \frac{p_l(x) - p_l(\lambda_j)}{x - \lambda_j} \tag{3.5}$$

for x not in the spectrum. By continuity, we have

$$p_l^{(k)}(\lambda_i) = q_i p_k(\lambda_i) p_k'(\lambda_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} q_j^2 p_k(\lambda_j) \frac{p_l(\lambda_i) - p_l(\lambda_j)}{\lambda_i - \lambda_j} \tag{3.6}$$

These polynomials are fundamental in the article [HP17] to describe the stability of tridiagonal models. In this work, we will only present useful matrices for the simulation of the Dyson Brownian motion tridiagonal model.

3.2.2 Useful matrices for the simulation

First, we need to define the matrix $G(x)$ so that

$$G(x)_{kl} = \begin{cases} a_{l-1}p_{l-2}(x)p_{l-1}(x) - a_l p_{l-1}(x)p_l(x) & \\ +a_{l-1}p_{l-1}(x)p_{l-2}(x) - a_l p_l(x)p_{l-1}(x) & \text{if } k = l \\ a_l(p_{l-1}(x)p_{l-1}(x) - p_l(x)p_l(x)) & \text{if } k = l+1 \\ 0 & \text{otherwise} \end{cases}$$

and completed by symmetry.

Then, we have the matrix $G^{k,l}$ which derive from the matrix G but change on the polynomials coefficients

$$G^{kl}_{ur} = \frac{1}{2}\sum_{i=1}^{N} q_i^2 p_{k-1}(\lambda_i) \begin{cases} a_{r-1}p_{r-2}(\lambda_i)p_{r-1}^{(l-1)}(\lambda_i) - a_r p_{r-1}(\lambda_i)p_r^{(l-1)}(\lambda_i) & \\ +a_{r-1}p_{r-1}(\lambda_i)p_{r-2}^{(l-1)}(\lambda_i) - a_r p_r(\lambda_i)p_{r-1}^{(l-1)}(\lambda_i) & \text{if } u = r < n \\ a_r(p_{r-1}(\lambda_i)p_{r-1}^{(l-1)}(\lambda_i) - p_r(\lambda_i)p_r^{(l-1)}(\lambda_i)) & \text{if } k = l+1 \\ 0 & \text{otherwise} \end{cases}$$

Note that G and G^{kl} are both tridiagonal symmetric matrices.
With this latter matrix, we can define the sum $\sum_{k\geq l} dP_{k,l,t}\ G^{kl}$. In our case this sum simplify and can be written

$$\sum_{k\geq l} dP_{k,l,t}\ G^{kl} = dS_t + dR_t$$

where dR_t is a lower term, which will be considered negligible for the simulations. Hence, we consider

$$\sum_{k\geq l} dP_{k,l,t}\ G^{kl} \approx dS_t \tag{3.7}$$

Still, we have an expression for dS_t

$$dS_t = \sum_{i,k,l}^{N} -\frac{q_i^2}{2}(p_{k-1}(\lambda_i)p'_{l-1}(\lambda_i) + p'_{k-1}(\lambda_i)p_{l-1}(\lambda_i))\ G^{kl}\ dt$$

With all of these definitions, we can now present the fundamental theorem that justify the algorithmic simulations.

3.2.3 Fundamental theorem and algorithm

This theorem correponds to the Theorem 18 in [HP17] and its corollary to Corollary 19. We present it in our specific case, the Dyson evolution of the GUE i.e. we set $V(x) = x^2/2$ in Equation 3.1.

Theorem 3.2.2
Let the q_i be fixed then the tridiagonal model associated to the Dyson Brownian motion

$$dX_t = \sum_{i=1}^{N} - \left(dw_i(t) + dt \left(\frac{\lambda_i(t)}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)} \right) \right) q_i^2 \; G(\lambda_i(t)) + \sum_{k\geq l} dP_{k,l,t} \; G^{kl}$$

with with $(w_i(t))_{1\leq i\leq N}$ a collection of independent real standard Brownian motions driving the Dyson Brownian motion in 3.1. Moreover, denoting by $\mathbf{\Lambda}$ the diagonal process assciated to $\boldsymbol{X}$ i.e. $\boldsymbol{X} = \boldsymbol{O\Lambda O}^t$, if $\mathbf{\Lambda}$ is a stationary process, then $\boldsymbol{X}$ will be stationary. The definitions of G, G^{kl} and the sum may be found in Section 3.2.2.

Corollary 3.2.3
If we take $X_0 \sim$ TriGUE(N), and $\mathbf{\Lambda}$ statisfies the Dyson Brownian flow 3.1 with $V(x) = x^2/2$, then $X_t \sim$ TriGUE(N) for $t \geq 0$.

This leads us to the following simplyfied Algorithm 4, the complete version is available on [Ohl17].

Algorithm 4 Eigenvalues trajectories simulation with tridiagonal model

input : A, N, $n_{samples}$, t_f
initialisation :
$dt \leftarrow t_f / n_{samples}$
$\Lambda \leftarrow \{OrderedSpectrum(A)\}$
$q \leftarrow SpectralWeights(A)$
$X \leftarrow A$
for t=1 **to** n_{sample} **do**
  draw $B \sim \mathcal{N}(0,1)$
  $p \leftarrow Polynomials(X)$
  $G \leftarrow G(X, p, \Lambda(t))$
  $dS \leftarrow S(X, p, \Lambda(t), G^{kl})$
  $dX \leftarrow \sum_{i=1}^{N} - \left(\sqrt{dt/N} B + dt \left(\frac{\Lambda(t,i)}{2} + \sum_{j\neq i} \frac{1}{\lambda(t,i) - \lambda(t,j)} \right) \right) q(i)^2 G + dS$
  $X \leftarrow X + dX$
  $\Lambda \leftarrow \Lambda \cup \{OrderedSpectrum(X)\}$
end for
output : Λ

In practise, Algorithm 4 is not easy to implement. Moreover, it does not give satisfying results because after a few increments, the diagonalisation does not converge. The approximations operated to perform simulation (e.g. Equation (3.7)) may be a fact that affect the eigenvalues trajectories generations.
Furthermore, the reader might notice the form of the SDE in Theorem 3.2.2: it is not an exact simulation of a process. Recall Table 1.1, the latter simulation is closer to the simulation with a system of eigenvalues SDEs than to the other. Let us present a new comparison table.

	Matrix SDE	Tridiagonal SDE	Eigenvalues SDEs
Type of simulation	Exact	Approximate	Approximate
# simulations $\sim \mathcal{N}$	$2N^2$	N	N
Diagonalisation	$O(N^3)$	$O(N^2)$ or $O(N \ln N)$	None
Robustness	High	Low (for this simulation)	Low ("jumps")

Table 3.1 – Comparison of the three simulation methods

We observe that the simulation with the tridiagonal model is a good trade-off between the two previous one. However, we cannot really conclude on its performance because we do not have any result to present. Some extra time is necessary to provide a complete overview of this model on a simulation scale. Moreover, note that a lot of polynomials evaluations are performed that need to be added to compute the exact complexity.

Conclusion

In this study, we manage to understand the particular repulsion phenomenon appearing in the Dyson Brownian motion and in other classical examples. Depending on the considered process, we can link the eigenvalues collection at a fixed time to a determinantal point process. These processes are precisely well-known for their repulsion property and characterised the interaction observed between the eigenvalues trajectories at a fixed time.

A big issue was to think about simulations. Matrix processes and eigenvalues trajectories are defined by stochastic differential equations and it is not obvious to find a way to properly generate these processes. Several methods came up, with their pros and cons. Simulations with a random matrix SDE are computationally costly but exact and very robust. On the contrary, simulations with eigenvalues SDEs using a Euler approximation are less costly but approximate and with low robustness.

An interesting trade-off was proposed in a very recent article, using some triadiagonal matrix-values processes. These processes were studied at the end of the master's thesis. Hence, an open question left by this work is to find a usable implementation of them. Indeed, we could get a robust way to generate a spreading determinantal point process repulsion for an affordable computational cost. Furthermore, we might use the total contribution of this new article to study tridiagonal models for general β-ensembles. This could create a significant variety of processes with specific repulsion to characterise.

References

[AGZ10] G. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*. Cambridge studies in advanced mathematics. Cambridge University Press, 2010. ISBN: 978-0-521-19452-5. URL: http://opac.inria.fr/record=b1130267.

[Bru91] Marie-France Bru. "Wishart processes". In: *Journal of Theoretical Probability* 4.4 (1991), pp. 725–751.

[DE02] Ioana Dumitriu and Alan Edelman. "Matrix models for beta ensembles". In: *Journal of Mathematical Physics* 43.11 (2002), pp. 5830–5847.

[Dys62] Freeman J Dyson. "A Brownian-motion model for the eigenvalues of a random matrix". In: *Journal of Mathematical Physics* 3.6 (1962), pp. 1191–1198.

[Gra14] P. Graczyk. *Stochastic analysis methods in Wishart theory*. 2014.

[HP17] D. Holcomb and E. Paquette. "Tridiagonal Models for Dyson Brownian Motion". In: *ArXiv e-prints* (July 2017). arXiv: 1707.02700 [math.PR].

[JS05] Monique Jeanblanc and Thomas Simon. "Elements de calcul stochastique". IRBD. 2005.

[Joh06] Kurt Johansson. "Course 1 Random matrices and determinantal processes". In: *Les Houches Summer School Proceedings* 83.C (2006), pp. 1–56. ISSN: 09248099. DOI: 10.1016/S0924-8099(06)80038-7. arXiv: 0510038 [math-ph].

[Joh01] Kurt Johansson Johansson. "Universality of the Local Spacing Distribution in Certain Ensembles of Hermitian Wigner Matrices". In: *Communications in Mathematical Physics* 215.3 (2001), pp. 683–705.

[KS12] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*. Vol. 113. Springer Science & Business Media, 2012.

[KPS12] Peter Eris Kloeden, Eckhard Platen, and Henri Schurz. *Numerical solution of SDE through computer experiments*. Springer Science & Business Media, 2012.

[Le 11] Jean-François Le Gall. "Mouvement Brownien, martingales et calcul stochastique". Université Paris-Sud. 2011.

[Ohl17] Sébastien Ohleyer. *Python code for Dyson Brownian motion*. 2017. URL: https://github.com/sohleyer/dbm.

[Øks03] Bernt Øksendal. "Stochastic differential equations". In: *Stochastic differential equations*. Springer, 2003, pp. 65–84.

[Tao12] T. Tao. *Topics in Random Matrix Theory*. Graduate studies in mathematics. American Mathematical Soc., 2012. ISBN: 9780821885079. URL: https://books.google.fr/books?id=Hjq%5C_JHLNPT0C.

[Wig55] Eugene P Wigner. "Characteristic vectors of bordered matrices with infinite dimensions". In: *Annals of Mathematics* (1955), pp. 548–564.

Appendix A

Hadamard's variation formulas

A.1. Hadamard's first variation formula

Proof of Lemma 2.1.1. Let $\boldsymbol{A}$ be a $N \times N$ Hermitian matrix-valued process with simple spectrum. Suppose that $\boldsymbol{A}$ is a derivable function of t. Denoting by $(\lambda(t))_{t\geq 0}$ an eigenvalue of $\boldsymbol{A}$ and $u = (u(t))_{t\geq 0}$ an associated eigenvector for $t \geq 0$, it is known that $\lambda(t)$ and $u(t)$ are derivable. For notation simplification, we do not write the t index for the eigenvector.

By differentiating the equations

$$A_t u = \lambda(t) u$$

We get

$$\dot{A}_t u + A_t \dot{u} = \dot{\lambda(t)} u + \lambda(t)\dot{u}$$

By left-composing with u

$$\begin{aligned} u^* \dot{A}_t u + u^* A_t \dot{u} &= u^* \dot{\lambda}(t) u + u^* \lambda(t) \dot{u} \\ &= \dot{\lambda}(t) u^* u + \lambda(t) u^* \dot{u} \\ &= \dot{\lambda}(t) + \lambda(t) u^* \dot{u} \end{aligned}$$

Moreover, remember that A_t is Hermitian: $u^* A_t = (A_t^* u)^* = (A_t u)^* = (\lambda(t) u)^* = \lambda(t) u^*$.

Thus

$$\begin{aligned} \dot{\lambda}(t) &= u^* \dot{A}_t u + u^* A_t \dot{u} - \lambda(t) u^* \dot{u} \\ &= u^* \dot{A}_t u + u^* A_t \dot{u} - u^* A_t \dot{u} \end{aligned}$$

Thus, we conclude the *Hadarmard first formula*

$$\dot{\lambda}(t) = u^* \dot{A}_t u$$

□

A.2. Hadamard's second variation formula

Proof of Lemma 2.1.2. Let $\boldsymbol{A}$ be a $N \times N$ Hermitian matrix-valued process with simple spectrum.. Suppose that $\boldsymbol{A}$ is a twice derivable function of t. Denoting by $(\lambda_i(t))_{1\leq i\leq N}$ the eigenvalues of $\boldsymbol{A}$ and $u_i = (u_i(t))_{1\leq i\leq N}$ their associated eigenvectors for $t \geq 0$, it is known that $\lambda(t)$ and $u_i(t)$ are twice derivable. For notation simplification, we do not write the t indew for the eigenvectors.

By differentiating the equations :

$$A_t u_i = \lambda_i(t) u_i$$

We get

$$\dot{A}_t u_i + A_t \dot{u}_i = \dot{\lambda}_i(t) u_i + \lambda_i(t) \dot{u}_i$$

For $j \neq i \in \{1...n\}$, by left-composing by u_j we have

$$\begin{aligned}
u_j^* \dot{A}_t u_i + u_j^* A_t \dot{u}_i - u_j^* \dot{\lambda}_i(t) u_i - u_j^* \lambda_i(t) \dot{u}_i &= 0 \\
u_j^* \dot{A}_t u_i + u_j^* A_t \dot{u}_i - \dot{\lambda}_i(t) \underbrace{u_j^* u_i}_{=0} - \lambda_i(t) u_j^* \dot{u}_i &= 0 \\
u_j^* \dot{A}_t u_i + u_j^* A_t \dot{u}_i - \lambda_i(t) u_j^* \dot{u}_i &= 0 \\
u_j^* \dot{A}_t u_i + (A_t^* u_j)^* \dot{u}_i - \lambda_i(t) u_j^* \dot{u}_i &= 0
\end{aligned}$$

Knowing that A_t is an hermitian matrix, which means that $A_t^* = A_t$

$$\begin{aligned}
u_j^* \dot{A}_t u_i + (A_t u_j)^* \dot{u}_i - \lambda_i(t) u_j^* \dot{u}_i &= 0 \\
u_j^* \dot{A}_t u_i + (\lambda_j(t) u_j)^* \dot{u}_i - \lambda_i(t) u_j^* \dot{u}_i &= 0 \\
u_j^* \dot{A}_t u_i + \lambda_j(t) u_j^* \dot{u}_i - \lambda_i(t) u_j^* \dot{u}_i &= 0 \\
u_j^* \dot{A}_t u_i + (\lambda_j(t) - \lambda_i(t)) u_j^* \dot{u}_i &= 0
\end{aligned}$$

Leading to

$$u_j^* \dot{u}_i = \frac{u_j^* \dot{A}_t u_i}{\lambda_i(t) - \lambda_j(t)} \tag{A.1}$$

The eigenvectors $(u_i)_{i\in\{1...n\}}$ form a basis of $\mathbb{C}^n$. Let $(v_i)_{i\in\{1...n\}}$ be the dual basis, thus $v_j^* u_k = \delta_{jk}$ for all $(i,j) \in \{1...n\}^2$. We have the reproducing formula

$$\dot{u_i} = \sum_{j=1}^{n} (v_j^* \dot{u_i}) u_j \tag{A.2}$$

A is self-adjoint because we suppose that A is an hermitian matrix. So we can take the eigenvectors $(u_i)_{i\in\{1...n\}}$ to be orthonormal, in which case $(v_i)_{i\in\{1...n\}}$ is identical to $(u_i)_{i\in\{1...n\}}$.

Equation (A.2) becomes

$$\begin{aligned}
\dot{u_i} &= \sum_{j=1}^{n} (u_j^* \dot{u_i}) u_j \\
\dot{u_i} &= \sum_{\substack{j=1 \\ j\neq i}}^{n} u_j^* \dot{u_i} u_j + u_i^* \dot{u_i} u_i
\end{aligned} \tag{A.3}$$

We introduce (A.1) in (A.3)

$$\dot{u_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{u_j^* \dot{A}_t u_i}{\lambda_i(t) - \lambda_j(t)} u_j + u_i^* \dot{u_i} u_i \tag{A.4}$$

Doing the same with ${\dot{u_k}}^*$

$$\begin{aligned}
{\dot{u_i}}^* &= \sum_{j=1}^{n} ({\dot{u_i}}^* u_j) u_j^* \\
{\dot{u_i}}^* &= \sum_{\substack{j=1 \\ j\neq i}}^{n} {\dot{u_i}}^* u_j u_j^* + {\dot{u_i}}^* u_i u_i^* \\
{\dot{u_i}}^* &= \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{u_i^* \dot{A}_t u_j}{\lambda_i(t) - \lambda_j(t)} u_j^* + {\dot{u_i}}^* u_i u_i^*
\end{aligned} \tag{A.5}$$

By derivating the *Hadamard first variation formula* ($\dot{\lambda_i} = u_i^* \dot{A} u_i$) and using (A.4) and

(A.5)

$$\ddot{\lambda}_i(t) = u_i^* \ddot{A}_t u_i + \dot{u_i}^* \dot{A}_t u_i + u_i^* \dot{A}_t \dot{u_i}$$

$$\ddot{\lambda}_i(t) = u_i^* \ddot{A}_t u_i + \Big(\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{u_i^* \dot{A}_t u_j}{\lambda_i(t) - \lambda_j(t)} u_j^* + \dot{u_i}^* u_i u_i^* \Big) \dot{A}_t u_i + u_i^* \dot{A}_t \Big(\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{u_j^* \dot{A}_t u_i}{\lambda_i(t) - \lambda_j(t)} u_j + u_i^* \dot{u_i} u_i \Big)$$

$$\ddot{\lambda}_i(t) = u_i^* \ddot{A}_t u_i + 2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(u_i^* \dot{A}_t u_j)(u_j^* \dot{A}_t u_i)}{\lambda_i(t) - \lambda_j(t)} + \dot{u_i}^* u_i u_i^* \dot{A}_t u_i + u_i^* \dot{A}_t u_i^* \dot{u_i} u_i$$

$$\ddot{\lambda}_i(t) = u_i^* \ddot{A}_t u_i + 2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{|u_j^* \dot{A}_t u_i|^2}{\lambda_i(t) - \lambda_j(t)} + \dot{u_i}^* u_i u_i^* \dot{A}_t u_i + u_i^* \dot{u_i} u_i^* \dot{A}_t u_i$$

$$\ddot{\lambda}_i(t) = u_i^* \ddot{A}_t u_i + 2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{|u_j^* \dot{A}_t u_i|^2}{\lambda_i(t) - \lambda_j(t)} + (\dot{u_i}^* u_i + u_i^* \dot{u_i}) u_i^* \dot{A}_t u_i \tag{A.6}$$

By derivating $u_i^* u_i = 1$ we have

$$\dot{u_i}^* u_i + u_i^* \dot{u_i} = 0 \tag{A.7}$$

Applying (A.7) in (A.6), we conclude the *Hadamard second variation formula*

$$\ddot{\lambda}_i = u_i^* \ddot{A}_t u_i + 2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{|u_j^* \dot{A}_t u_i|^2}{\lambda_i(t) - \lambda_j(t)}$$

□

Appendix B

Quadratic variations

B.1. Itô development of Stratonovich integral

Proof of Lemma 2.2.7. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be three Itô processes such that :

$$\begin{aligned} X_t &= \int_0^t H_s^X dB_s + \int_0^t K_s^X ds \\ Y_t &= \int_0^t H_s^Y dB_s + \int_0^t K_s^Y ds \\ Z_t &= \int_0^t H_s^Z dB_s + \int_0^t K_s^Z ds \end{aligned}$$

We have:

$$\begin{aligned} Y_t \circ dX_t \circ Z_t &= (Y_t \circ dX_t) \circ Z \\ &= (Y_t^* dX_t + \frac{1}{2} d\langle {}_t Y, X\rangle) \circ Z_t \\ &= (Y_t dX_t + \frac{1}{2} d\langle Y, X\rangle_t) Z_t + V_F \\ &= Y_t dX_t Z_t + \frac{1}{2} d\langle Y, X\rangle_t Z_t + V_F \end{aligned}$$

When we distribute the Stratonovich integral, we will obtain a first term: $(Y_t dX_t + \frac{1}{2} d\langle Y, X\rangle_t) Z_t$ and a second term with quadratic variation.
The second term V_F is compute by two terms: $Y_t dX_t$ and $\frac{1}{2} d\langle Y, X\rangle_t$. But $\frac{1}{2} d\langle Y, X\rangle_t$ will not have an impact when we calculate V_F because it is a finite variation term itself.
Let's focus on the term $Y_t dX_t$:

$$\int_0^t Y_s dX_s = \int_0^t Y_s K_s^X ds + \int_0^t Y_s H_s^X dB_s$$

So, we have:

$$\begin{aligned} VF &= \frac{1}{2} Y_t H_t^X H_t^Z dt \\ &= \frac{1}{2} Y_t d\langle X, Z \rangle_t \end{aligned}$$

Finally, we obtain (2.12). □

B.2. Hermitian matrices verifying a general SDE

Proof of Lemma 2.2.10. In order to compute the quadratic variation of the entries, we note

$$\begin{aligned} dX_{ij,t} &= (g(X_t) dB_t h(X_t) + h(X_t) dB_t^* g(X_t) + b(X_t) dt)_{ij} \\ &= (g(X_t) dB_t h(X_t))_{ij} + (h(X_t) dB_t^* g(X_t))_{ij} + (b(X_t) dt)_{ij} \end{aligned}$$

and

$$\begin{aligned} dX_{i'j',t} &= (g(X_t) dB_t h(X_t) + h(X_t) dB_t^* g(X_t) + b(X_t) dt)_{i'j'} \\ &= (g(X_t) dB_t h(X_t))_{i'j'} + (h(X_t) dB_t^* g(X_t))_{i'j'} + (b(X_t) dt)_{i'j'} \end{aligned}$$

When we distribute the quadratic variation, we obtain nine terms. However, terms with $b(X_t)dt$ will give a quadratic variation equals to zero. So we will have only four terms in finally.

Term 1

$$\begin{aligned} & d\langle (g(X) B h(X))_{ij}, (g(X) B h(X))_{i'j'} \rangle_t \\ &= d\langle \sum_{k,l} g(X)_{ik} B_{kl} h(X)_{lj}, \sum_{m,n} g(X)_{i'm} B_{mn} h(X_t)_{nj'} \rangle_t \end{aligned}$$

Using Lemma 2.2.4, we know that $d\langle B_{kl}, B_{mn} \rangle_t \neq 0$ if B_{kl} and B_{mn} are conjuguate. We cannot find $m,\ n,\ k$ and l to make B_{kl} and B_{mn} each other conjuguate because $\boldsymbol{B}$ is a matrix of independant complex Brownian motions. Finally, term 1 is zero.

Term 2

$$\begin{aligned} & d\langle (g(X) B h(X))_{ij}, (g(X) B^* h(X))_{i'j'} \rangle_t \\ &= d\langle \sum_{k,l} g(X)_{ik} B_{kl} h(X)_{lj}, \sum_{m,n} g(X)_{i'm} B_{mn}^* h(X)_{nj'} \rangle_t \\ &= d\langle \sum_{k,l} g(X)_{ik} B_{kl} h(X)_{lj}, \sum_{m,n} g(X)_{i'm} \overline{B_{nm}} h(X)_{nj'} \rangle_t \end{aligned}$$

Using Lemma 2.2.4, $\langle B_{kl}, \overline{B_{nm}} \rangle = 2dt$ if $k = n$ and $l = m$.
We have

$$\begin{aligned}
&d\langle (g(X)Bh(X))_{ij}, (g(X)B^*h(X))_{i'j'} \rangle_t \\
&= (2dt) \sum_{k,l} g(X)_{ik} h(X)_{lj} h(X)_{i'l} g(X)_{kj'} \\
&= (2dt) \sum_{k} g(X)_{ik} g(X)_{kj'} \sum_{l} h(X)_{i'l} h(X)_{lj} \\
&= 2(g^2(X))_{ij'} (h^2(X))_{i'j} dt
\end{aligned} \tag{B.1}$$

Term 3

$$\begin{aligned}
&d\langle (h(X)B^*g(X))_{ij}, (g(X)dBh(X))_{i'j'} \rangle_t \\
&= d\langle \sum_{k,l} h(X)_{ik} B^*_{kl} g(X)_{lj}, \sum_{m,n} g(X)_{i'm} B_{mn} h(X)_{jj'} \rangle_t \\
&= d\langle \sum_{k,l} h(X)_{ik} \overline{B_{lk}} g(X)_{lj}, \sum_{m,n} g(X)_{s'i} B_{nm} h(X)_{nj'} \rangle_t
\end{aligned}$$

Using 2.2.4, $d\langle B_{kl}, \overline{B_{nm}} \rangle = 2dt$ if $l = m$ and $k = n$.
We have

$$\begin{aligned}
&d\langle (h(X)B^*g(X))_{ij}, (g(X)Bh(X))_{i'j'} \rangle_t \\
&= (2dt) \sum_{k,l} h(X)_{ik} g(X)_{lj} g(X)_{i'l} h(X)_{kj'} \\
&= (2dt) \sum_{k} h(X)_{ik} h(X)_{kj'} \sum_{l} g(X)_{i'l} g(X)_{lj} \\
&= 2(g^2(X))_{i'j} (h^2(X))_{ij} dt
\end{aligned} \tag{B.2}$$

Term 4

$$\begin{aligned}
&d\langle (h(X)B^*g(X))_{ij}, (h(X)B^*g(X))_{i'j'} \rangle_t \\
&= d\langle \sum_{k,l} h(X)_{ik} B^*_{kl} g(X)_{lj}, \sum_{m,n} h(X)_{i'm} B^*_{mn} g(X)_{nj'} \rangle_t \\
&= d\langle \sum_{k,l} h(X)_{ik} \overline{B_{lk}} g(X)_{lj}, \sum_{m,n} h(X)_{i'm} \overline{B_{nm}} g(X)_{nj'} \rangle_t
\end{aligned}$$

For the same reason as term 1, term 4 equals to zero.

Finally, we obtain Equation (2.15). □