* LONGUET-HIGGINS, H. C. "A computer algorithm for reconstructing a scene from two projections." *Nature* 293 (1981): 133-135.

[Paper Review]

A computer algorithm for reconstructing a scene from two projections*

[Essential Matrix]

Jin-Hyung Kim

ISL Lab seminar

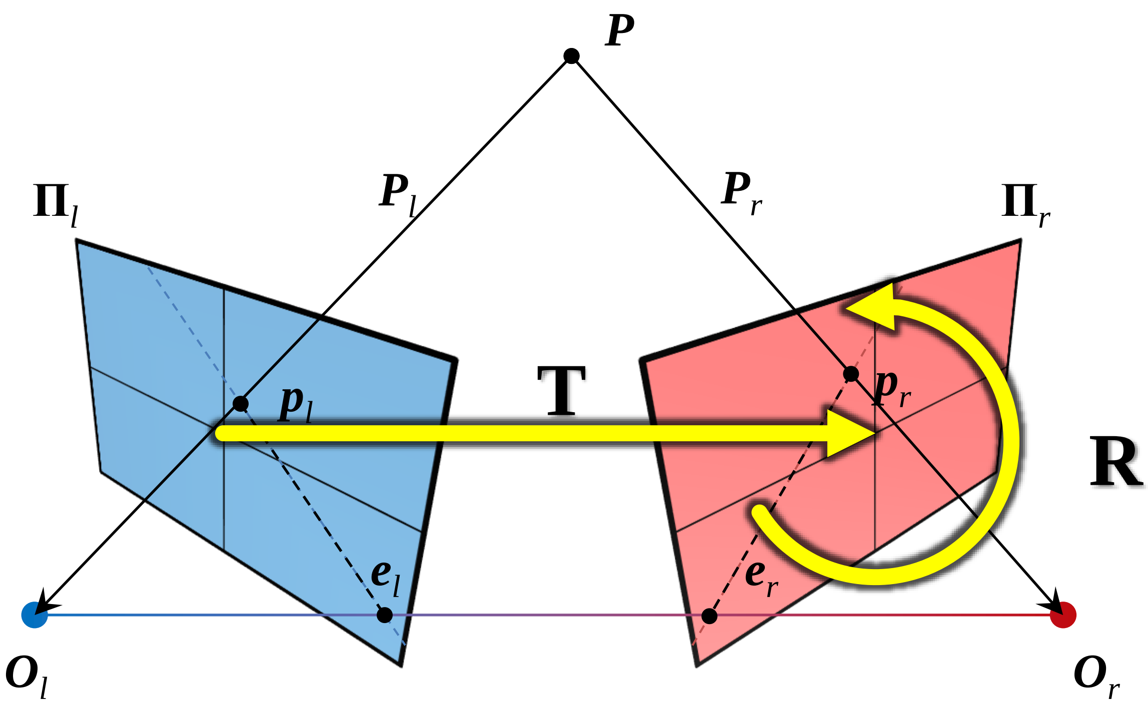
2014.10.01

Contents

ISL Image System Laboratory

Problem definition

Finding a relative camera pose between two views in physical space
(normalized image coordinate)



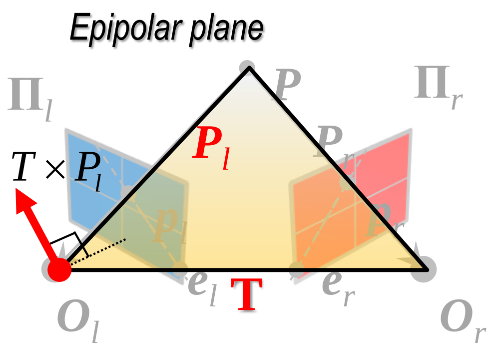
$$P_r = \mathbf{R}(P_l - \mathbf{T})$$

$$P_i = \begin{bmatrix} X_i & Y_i & Z_i \end{bmatrix}^T$$

$$p_i = \begin{bmatrix} \dfrac{X_i}{Z_i} & \dfrac{Y_i}{Z_i} & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} x_i & y_i & 1 \end{bmatrix}^T$$

Derivation



$$(P_l - \mathbf{T})^T (\mathbf{T} \times P_l) = 0$$

$$P_r = \mathbf{R}(P_l - \mathbf{T}) \Rightarrow (P_l - \mathbf{T}) = \mathbf{R}^T P_r$$

$$(\mathbf{R}^T P_r)^T (\mathbf{T} \times P_l) = 0$$

$$P_r^T \mathbf{R}[\mathbf{T}]_\times P_l = 0$$

$$p_r^T \mathbf{E} p_l = 0$$

Skew-symmetric form

$$[\mathbf{T}]_\times = \begin{bmatrix} 0 & -T_3 & T_2 \\ T_3 & 0 & -T_1 \\ T_2 & T_1 & 0 \end{bmatrix}$$

Essential matrix $\mathbf{E}$

$$\mathbf{E} = \mathbf{R}[\mathbf{T}]_\times$$

$$\mathbf{E} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

- **Point-normal form of the equation of a plane**

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0$$

It defines a certain point $\mathbf{x}$ on the plane which contains point $\mathbf{a}$ and be normal to $\mathbf{n}$ vector

8-points algorithm

$$p_r{}^T \mathbf{E} p_l = 0$$

$$p_i = \begin{bmatrix} x_i & y_i & 1 \end{bmatrix}^T \qquad \mathbf{E} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

$$x_r x_l e_{11} + x_r y_l e_{12} + x_r e_{13} + y_r x_l e_{21} + y_r y_l e_{22} + y_r e_{23} + x_l e_{31} + y_l e_{32} + e_{33} = 0$$

$$\hat{\mathbf{p}}_k^T \cdot \mathbf{e} = 0$$

$$Where, \begin{cases} \hat{\mathbf{p}}_k = \begin{bmatrix} x_r^k x_l^k & x_r^k y_l^k & x_r^k & y_r^k x_l^k & y_r^k y_l^k & y_r^k & x_l^k & y_l^k & 1 \end{bmatrix}^T \\ \mathbf{e} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{21} & e_{22} & e_{23} & e_{31} & e_{32} & e_{33} \end{bmatrix}^T \end{cases}$$

Collect all corresponding pair vectors

$$\hat{\mathbf{P}}^T \cdot \mathbf{e} = 0$$

$$Where, \quad \hat{\mathbf{P}} = \begin{bmatrix} \hat{\mathbf{p}}_1^T \\ \vdots \\ \hat{\mathbf{p}}_k^T \end{bmatrix}, \quad k \geq 8$$

Homogeneous linear equation!!

Extract R, T from Essential matrix

$\mathbf{E} = \mathbf{R}[\mathbf{T}]_\times$

- Normalized translation vector $\mathbf{T}$

$$\mathbf{E}^T\mathbf{E} = [\mathbf{T}]_\times^T \mathbf{R}^T\mathbf{R}[\mathbf{T}]_\times = [\mathbf{T}]_\times^T \mathbf{I}[\mathbf{T}]_\times = [\mathbf{T}]_\times^T[\mathbf{T}]_\times$$

$$= \begin{bmatrix} \mathbf{T}_3^2 + \mathbf{T}_2^2 & -\mathbf{T}_2\mathbf{T}_1 & -\mathbf{T}_3\mathbf{T}_1 \\ -\mathbf{T}_1\mathbf{T}_2 & \mathbf{T}_3^2 + \mathbf{T}_1^2 & -\mathbf{T}_3\mathbf{T}_2 \\ -\mathbf{T}_1\mathbf{T}_3 & -\mathbf{T}_2\mathbf{T}_3 & \mathbf{T}_2^2 + \mathbf{T}_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\mathbf{T}_1^2 & -\mathbf{T}_2\mathbf{T}_1 & -\mathbf{T}_3\mathbf{T}_1 \\ -\mathbf{T}_1\mathbf{T}_2 & 1-\mathbf{T}_2^2 & -\mathbf{T}_3\mathbf{T}_2 \\ -\mathbf{T}_1\mathbf{T}_3 & -\mathbf{T}_2\mathbf{T}_3 & 1-\mathbf{T}_3^2 \end{bmatrix}$$

Sign is not determined yet.

$\hat{\mathbf{E}}^T\hat{\mathbf{E}}$ ←Might not normalized..

$$|\mathbf{T}| = \sqrt{\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2} = 1$$

$$tr\left(\hat{\mathbf{E}}^T\hat{\mathbf{E}}\right) = 2\left(\hat{\mathbf{T}}_1^2 + \hat{\mathbf{T}}_2^2 + \hat{\mathbf{T}}_3^2\right)$$

$$s = \sqrt{\frac{tr\left(\hat{\mathbf{E}}^T\hat{\mathbf{E}}\right)}{2}}$$ ←scale

$$\mathbf{E} = \frac{1}{s}\hat{\mathbf{E}}$$

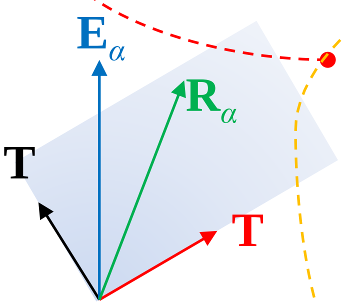
Extract R, T from Essential matrix

- Rotational matrix $\mathbf{R}$

$$\mathbf{E}_\alpha = \mathbf{T} \times \mathbf{R}_\alpha \quad (\alpha = 1, 2, 3)$$ ←row of matrices

$$\mathbf{R}_\alpha = \mathbf{R}_\beta \times \mathbf{R}_\gamma$$ ←Orthogonality of Rotation matrix

$$\mathbf{W}_\alpha = \mathbf{E}_\alpha \times \mathbf{T}$$



$$\mathbf{R}_\alpha = a_\alpha \mathbf{T} + b_\alpha \mathbf{W}_\alpha$$ ←Linear combination of $\mathbf{T}$ and $\mathbf{W}_\alpha$

$$\mathbf{E}_\alpha = \mathbf{T} \times (a_\alpha \mathbf{T} + b_\alpha \mathbf{W}_\alpha) = b_\alpha (\mathbf{T} \times \mathbf{W}_\alpha)$$

$$\mathbf{T} \times \mathbf{W}_\alpha = \mathbf{T} \times (\mathbf{E}_\alpha \times \mathbf{T}) = \mathbf{E}_\alpha$$ ←as $\mathbf{T}$ is a unit vector $\therefore b_\alpha = 1$

$$a_\alpha \mathbf{T} + \mathbf{W}_\alpha = (a_\beta \mathbf{T} + \mathbf{W}_\beta) \times (a_\gamma \mathbf{T} + \mathbf{W}_\gamma)$$

←orthogonal to $\mathbf{T}$

$$= a_\beta \mathbf{E}_\gamma + a_\gamma \mathbf{E}_\beta + (\mathbf{W}_\beta \times \mathbf{W}_\gamma)$$

$$\mathbf{R}_\alpha = \mathbf{W}_\alpha + (\mathbf{W}_\beta \times \mathbf{W}_\gamma)$$

3D Reconstruction

- Depth scale $\mathbf{Z}$

$$x_r = \frac{\mathbf{R}_1 \cdot (P_l - \mathbf{T})}{\mathbf{R}_3 \cdot (P_l - \mathbf{T})} = \frac{\mathbf{R}_1 \cdot (p_l - \mathbf{T}/Z_l)}{\mathbf{R}_3 \cdot (p_l - \mathbf{T}/Z_l)}$$

$$Z_l = \frac{(\mathbf{R}_1 - x_r \mathbf{R}_3) \cdot \mathbf{T}}{(\mathbf{R}_1 - x_r \mathbf{R}_3) \cdot p_l}$$

$$X_l = x_l Z_l$$

$$Y_l = y_l Z_l$$

Experimental Result

- Extracting $\mathbf{T}$ and $\mathbf{R}$ from synthesized $\mathbf{E}$ using MATLAB

```
function [R1, R2, t1, t2] = RTfromE(E)

scale = sqrt(trace(E'*E)/2);
E = E/scale;
EE = E'*E;

t1(1,1) = sqrt(1-EE(1,1));
t1(2,1) = sqrt(1-EE(2,2));
t1(3,1) = sqrt(1-EE(3,3));

t2(1,1) = -sqrt(1-EE(1,1));
t2(2,1) = -sqrt(1-EE(2,2));
t2(3,1) = -sqrt(1-EE(3,3));
```

```
W1 = cross(E(1,:),t1);
W2 = cross(E(2,:),t1);
W3 = cross(E(3,:),t1);

R1(1,:) = W1 + cross(W2,W3);
R1(2,:) = W2 + cross(W3,W1);
R1(3,:) = W3 + cross(W1,W2);

W1 = cross(-E(1,:),t1);
W2 = cross(-E(2,:),t1);
W3 = cross(-E(3,:),t1);

R2(1,:) = W1 + cross(W2,W3);
R2(2,:) = W2 + cross(W3,W1);
R2(3,:) = W3 + cross(W1,W2);
```

ISL Image System Laboratory

Experimental Result

$$R = \begin{pmatrix} 0.500 & 0.000 & 0.866 \\ 0.433 & 0.866 & -0.250 \\ -0.750 & 0.500 & 0.433 \end{pmatrix} \qquad [T]_x = \begin{pmatrix} 0.000 & 0.743 & -0.557 \\ -0.743 & 0.000 & 0.371 \\ 0.557 & -0.371 & 0.000 \end{pmatrix}$$

$$E = \begin{pmatrix} 0.482 & 0.050 & -0.279 \\ -0.783 & 0.414 & 0.080 \\ -0.130 & -0.718 & 0.604 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} 0.500 & 0.000 & 0.866 \\ 0.433 & 0.866 & -0.250 \\ -0.750 & 0.500 & 0.433 \end{pmatrix} \qquad T_1 = \begin{pmatrix} 0.371 \\ 0.557 \\ 0.743 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0.116 & 0.924 & 0.365 \\ -0.093 & -0.356 & 0.930 \\ 0.989 & -0.142 & 0.045 \end{pmatrix} \qquad T_2 = \begin{pmatrix} -0.371 \\ -0.557 \\ -0.743 \end{pmatrix}$$

ISL Image System Laboratory